SEMIGROUPS ON A HALF-PLANE([1])

BY
J. G. HORNE, JR.

Introduction. Suppose that S is a topological semigroup with identity on the closed right half-plane $\{(x,y): x \geqq 0\}$. Suppose furthermore that the set G of all elements which have inverses occupies the open half-plane. Denote the boundary of G—the Y-axis—by L. The possible multiplications for G and L were determined in [4] and [5], but whether these completely determine $G \cup L$ was left unsettled. One case in which G and L determine $G \cup L$ was given in [3]. Combining this result with the results of the present work we obtain the proposition that if L contains no nilpotent elements then G and L determine $G \cup L$ but not otherwise.

Preliminaries. We shall invariably use the word *semigroup* to mean topological semigroup. And *isomorphism* always refers to a function which is not only an algebraic isomorphism but a homeomorphism as well. We write $S \cong S'$ to signify that semigroups S and S' are isomorphic. A *homomorphism* h is only required to satisfy the identity $h(xy) = h(x)h(y)$.

Our interest is mainly in semigroups whose underlying space is a (closed) half-plane. Whenever S is such a semigroup, the following additional assumptions are tacit: S contains an identity and the set of elements having an inverse occupies the open half-plane. This latter set of elements forms the maximal subgroup of S and is referred to generically as G. The corresponding term for the complement of G (which is the maximal proper ideal of S) is L. Thus whenever we refer to a semigroup on a half-plane, we assume that G is topologically a plane, L is the boundary of G and is topologically a line. We say that G and L *determine* $G \cup L$ provided the following is true: if $G' \cup L'$ is a semigroup on a half-plane such that $G \cong G'$ and $L \cong L'$ then $G \cup L \cong G' \cup L'$.

It should probably be recalled here that a (topological) group, whose underlying space is a plane, is either isomorphic to the two dimensional vector group or to the (noncommutative) group of affine transformations of the line. The usual realization of the latter is the group on the open half-plane whose product is given by

$$(a,b)(x,y) = (ax, ay + b), \quad a,x > 0.$$

A *two sided zero* for S is an element $z \in S$ such that $zx = xz = z$ for $x \in S$.

Presented to the Society, November 17, 1961; received by the editors October 3, 1961.

([1]) Work on the paper was supported in part by NSF G–17847.

We frequently refer to such z simply as a *zero.* A zero is unique when it exists; when it is known in advance to exist it is usually denoted by 0. A *right zero* is an element z satisfying $xz = z$ for all $x \in S$. Of course, all results about right zeros have obvious duals corresponding to the dual notion of left zero. It seems unnecessary to list these or to mention this fact again.

An element x is a *nilpotent* element if x is not a (two-sided) zero but x^2 is. In particular, a semigroup which has no zero element can have no nilpotent elements.

As usual, if A,B are subsets of a semigroup then AB consists of all products ab with $a \in A$, $b \in B$. The closure of A is denoted A^-.

We shall frequently make use of the following properties of the noncommutative group on the plane. There is no harm in temporarily letting G denote this group. Presumably these properties have been known a long time and in any case are easy to verify by examining the usual realization of G. In this realization, the one-parameter subgroups of G take one of the following forms:

$$P_b = \{(x,(1-x)b) : x > 0\}$$

or

$$Q = \{(1,y) : y \text{ is real}\}.$$

The properties mentioned above are these: G has exactly one normal one-parameter subgroup Q; every pair of non-normal one-parameter subgroups is a conjugate pair. The normal subgroup Q is the only one-parameter subgroup of G such that P a one-parameter subgroup of G different from Q implies $PQ = QP = G^1$. Of course(2), for every $x \in G$, the map $y \rightarrow xyx^{-1}$ is an automorphism of G; we shall refer to it as the automorphism *induced* by x.

An isomorphism T between two commutative groups on a plane can be constructed simply by choosing pairs of one-parameter subgroups P,Q and P',Q' in each, selecting isomorphisms $f: P \rightarrow P'$; $h: Q \rightarrow Q'$ and defining $T(pq) = f(p)h(q)$. This procedure must be modified in constructing isomorphisms between two noncommutative groups on the plane: let Q, Q' denote the normal one-parameter subgroups of each. One is still free to choose any isomorphism $h: Q \rightarrow Q'$ and one is still free to select one-parameter subgroups P, P' in each with $P \neq Q$ and $P' \neq Q'$. However, no further choices can be made. For each $p \in P$ induces (by restriction to Q of the automorphism induced by p) an automorphism of Q.

(2) Evidently the argument in [5; 3.3, p. 384] is incomplete when G is not commutative. However, as soon as it is shown that either $G_r(x)$ or $G_l(x)$ contains the normal one-parameter subgroup of G or that G^- contains a zero that argument is justified. Let Q be the normal one-parameter subgroup of G and let $g_n \rightarrow = x$. Then for each $q \in Q$, there is some $r \in Q$ so that either $g_n^{-1} q g_n \rightarrow r$ or $g_n q g_n^{-1} \rightarrow r$, and r is either constantly 1 or depends isomorphically on q. In the first case, $qx = \lim q g_n = \lim g_n g_n^{-1} q g_n = xr$. Hence either $Qx = x, xQ = x$ or $Qx = xQ$. Therefore, either Q is contained in $G_r(x)$ or $G_l(x)$ or $Qx = xQ$ and $G = G_r(x) Q = Q G_r(x)$. In the final case, $xGx = x^2G$ so $xGx = x^2$ since $G_r(x) G_l(x) = G$. Thus G has a zero.

On the other hand, every automorphism of Q is induced in this way by exactly one element of P.

Hence for each $p \in P$ there is a unique element $f(p) \in P'$ such that

$$h(pqp^{-1}) = f(p)h(q)\, f(p)^{-1} \quad \text{for all } q \in Q.$$

It is easy to show that $f: P \to P'$ is an isomorphism. Now define $T(pq) = f(p)h(q)$. Then T is a desired isomorphism. The only property of T about which there is any doubt is whether T is a homorphism. Let $x = pq$, $y = rs$ with $p, r \in P$, $q, s \in Q$. Then $xy = pqrs = prr^{-1}qrs$, so

$$T(xy) = f(pr)f(r)^{-1}\, h(q)f(r)f(s) = f(p)h(q)f(r)h(s) = T(x)\, T(y).$$

Hence T is an isomorphism.

Since $T \mid P = f$ and $T \mid Q = h$, the requirement $T(pq) = f(p)h(q)$ is necessary so that not only is this procedure sufficient for constructing an isomorphism but determines all possible isomorphisms between two noncommutative groups on a plane. It follows from this, incidentally, that all of the automorphisms of such a group are inner automorphisms.

Finally we shall need the result that no automorphism of G reverses the orientation of Q.

The theorems. In this section we prove that if $G \cup L$ is a semigroup on a half-plane without nilpotent elements then G and L determine $G \cup L$ (Theorem 3). The proof is obtained by considering the various possibilities for L. According to [4] and [5] these are (1) L has a zero 0 but $x \neq 0$ implies $x^2 \neq 0$; (2) L is a group; (3) for every $z \in L$, $Gz = z$ and $zG = L$. The theorem has already been proved for the first case in [3]. The proofs for the other two cases have some common features but only after preliminary results for the remaining two cases are obtained separately. This is done in Theorems 1 and 2. The results of these theorems are then pieced together to prove that if $G \cup L$ has no nilpotent elements and L is either a group or satisfies (3) above then G and L determine $G \cup L$.

LEMMA. 1. *If P and Q are distinct one-parameter subgroups of G and z is in the boundary of both P and Q then z is a zero for $G \cup L$.*

Proof. While it is not generally true that $PQ = G$, nevertheless PQ *generates* G. Thus, since z is a zero for P and Q, z is a zero for G and hence for $G \cup L$ (since $G \cup L = G^-$).

THEOREM 1. *Suppose L is a group. Let e be the identity of L. Then G contains two one-parameter subgroups P, Q so that*: (1) $P^- = P \cup \{e\}$, (2) *P is normal*, (3) $Q^- \cap L = \emptyset$ (i.e. *Q is closed in $G \cup L$*), *and* (4) $eQ = L$.

Proof. Let H and K denote the left and right isotropy groups of e respectively; e.g. $H = \{g \in G\colon ge = e\}$. As shown in [5, 3.3], the dimension of H is 1 so H contains a one-parameter subgroup P. Now choose a one-parameter subgroup $Q \neq P$. The choice can be made so that one of P and Q is normal so $G = PQ = QP$. Therefore there exists a sequence $p_n \in P$, $q_n \in Q$ so that $q_n p_n \to e$. Hence $q_n p_n e \to e$ so $q_n e \to e$ since $p_n e = e$. Now $L = Ge = QPe = Qe$ so the map $q \to qe$ is a homeomorphism from Q to L. Since $q_n e \to e$ it follows that $q_n \to 1$. Hence $1/q_n \to 1$ so $1/q_n\ (q_n p_n) \to e$. That is $p_n \to e$ so $e \in P^-$. According to [2], $P^- = P \cup \{e\}$.

If G is commutative then of course P is normal. If G is not commutative then, as we have observed, G contains just one normal one-parameter subgroup and all other pairs are conjugate. Thus if P is not normal we can find a second one P' and $g \in G$ so that $P' = gPg^{-1}$. Therefore $geg^{-1} \in (P')^-$. Since L has only one idempotent and geg^{-1} is an idempotent, $geg^{-1} = e$. Thus $e \in (P')^-$. But by the preceding lemma, this implies e is a zero for G^- which is absurd. Therefore when G is not commutative P is *the* normal one-parameter subgroup of G. In any case, G contains a normal one-parameter subgroup P such that $P^- = P \cup \{e\}$.

According to what we have just seen, P is the only one-parameter subgroup of G with $e \in P^-$. Now if P' is any one-parameter subgroup of G, any point of $(P')^- \cap L$ is an idempotent. Thus P is the only one-parameter subgroup of G which is not closed in $G \cup L$. Hence for Q we may take any one-parameter subgroup different from P. Furthermore we have already seen that for such Q, $eQ = L$ so the proof of the theorem is complete.

Certain automorphisms α of a one-parameter subgroup Q satisfy the following condition: $\alpha(q)$ is between 1 and q for some (and therefore for all) $q \in Q$. We shall call such an automorphism *shrinking.*

THEOREM 2. *Suppose* $z \in L$ *implies* $Gz = z$ *and* $zG = L$. *Then for each* $z \in L$, G *contains two one-parameter subgroups* P, Q *such that* (1) $P^- = P \cup \{z\}$, (2) Q *is the normal one-parameter subgroup of* G, (3) Q *is the only closed one-parameter subgroup of* $G \cup L$ (*in particular* $Q^- \cap L = \emptyset$), (4) $zQ = L$ *and* (5) *if* $p \in P$ *then* p *is between* z *and* 1 *if and only if the automorphism induced by* p^{-1} *is shrinking.*

Proof. Let $z \in L$ be fixed. Then z is a right zero for G^- so by [2], G contains a one-parameter subgroup P such that $P^- = P \cup \{z\}$. In fact, since L contains no zero, there is only one such subgroup. We will denote it by $P(z)$.

Temporarily ignore z and let y,w be any pair of distinct elements in L. Then $y \notin wP(y)$, for if so then $y = wp$ for some $p \in P(y)$. Therefore $y(1/p) = w$. But y is a zero for $P(y)$ so $y = w$ which is a contradiction. It follows in particular that $wP(y) \neq L$.

We need even more. Since $wy = y$, both w and y belong to $(wP(y))^-$ so $wP(y)$ is not a point. Therefore $wP(y)$ is (topologically) a line in L with one end point at y. Suppose $wP(y)$ had another end point v. Then there is a sequence $x_n \in P(y)$

such that $wx_n \to v$. Now x_n can have no accumulation point p in $P(y)$. For if so, we may as well assume $x_n \to p$. Thus $1/x_n \to 1/p$ and $(wx_n)\, 1/x_n \to v\, 1/p$ so $wp = v$ and $v \in wP(y)$ which is a contradiction. If x_n has no accumulation points in $P(y)$ then either $x_n \to y$ or $1/x_n \to y$. The first possibility implies $wy = v$. But $wy = y$ so this is impossible. If $1/x_n \to y$, $w = vy$ which is also absurd. Therefore, for $y \neq w$, $wP(y)$ is an open unbounded ray eminating from y (and containing w of course).

Under the present circumstances, G is obviously not commutative. Thus let Q be the normal one-parameter subgroup of G. Evidently Q is closed in $G \cup L$. For otherwise there is an element $w \in Q^- \cap L$ so $Q = P(w)$. Thus with $y \neq w$, $P(y) \neq Q$ so $G = P(y)Q$ and $yG = yP(y)Q = yQ = yP(w)$. However, $yG = L$ while $yP(w) \neq L$. Hence Q is closed in $G \cup L$.

Conversely, suppose R is a non-normal one-parameter subgroup of G. Then for some $g \in G$, $R = gP(z)g^{-1}$. Hence $gzg^{-1} \in R^-$ so R is not closed in $G \cup L$. Thus the normal one-parameter subgroup Q of G is also the only closed one-parameter subgroup of $G \cup L$.

We have already seen that (4) holds. To prove (5) (with $P = P(z)$), let $p \in P$ and suppose first that p is between z and 1. Then $p^n \to z$. Therefore, for each $q \in Q$, $zqp^n \to z$. Since the map $q \to zq$ is a homeomorphism and since $zqp^n = zp^{-n}qp^n$, $p^{-n}qp^n \to 1$ for each $q \in Q$. As we have mentioned, no member of G induces an orientation reversing automorphism of Q. Hence for some n, $p^{-n}qp^n$ is between q and 1. For this n, the automorphism induced by p^{-n} is shrinking and, hence, so is the automorphism induced by p^{-1}.

Conversely, let $p \in P$ and suppose that the automorphism induced by p^{-1} is shrinking. Hence for fixed $q \neq 1$ and all n, $p^{-n}qp^n$ is bounded between 1 and q. Therefore the set $zp^{-n}qp^n = zqp^n$, n a positive integer, is a bounded subset of L contained in zqP. According to the result in the third paragraph of this proof, the component of $P - \{1\}$ which is unbounded in $G \cup L$ is sent, by the map $p \to zqp$, onto the component of $zqP - \{zq\}$ which is unbounded in L. Therefore p^n belongs to the bounded component of $P - \{1\}$. That is, p is between z and 1 and the proof of the theorem is complete.

Suppose $G \cup L$ and $G' \cup L'$ are semigroups on a half-plane each of which satisfies the hypothesis of the previous theorem. For $z \in L$, let P,Q be one-parameter subgroups satisfying the conclusions of that theorem. Let z', P', Q' be corresponding objects in $G' \cup L'$. Let $h: Q \to Q'$ be an isomorphism. According to results in the section on preliminaries, if $f: P \to P'$ is determined by the equation $h(pqp^{-1}) = f(p)\,h(q)f(p)^{-1}$ for all $q \in Q$ and $T(pq) = f(p)\,h(q)$ then $T: G \to G'$ is an isomorphism. It follows from Theorem 2 that if we define $f(z) = z'$ then $f: P^- \to (P')^-$ is an isomorphism. The only property of f which is questionable is its continuity at z. But by part (5) of Theorem 2, if p is in the bounded component of $P - \{1\}$ then p^{-1} induces a shrinking automorphism of Q. Hence $q' \to$

$h(ph^{-1}(q')p^{-1})$ is a shrinking automorphism of Q'. But this is the same as the mapping $q' \to f(p)q'f(p)^{-1}$. Hence $f(p)$ is in the bounded component of $P' - \{1\}$. The continuity of f at z follows immediately. This leaves us just one step away from proving that G and L determine $G \cup L$ in this case.

LEMMA 2. *Suppose L contains an element z and G contains two one-parameter subgroups such that $P^- = P \cup \{z\}$, $Q^- \cap L = \emptyset$ and $zQ = L$. If $x_n = p_n q_n$ with $p_n \in P^-$, $q_n \in Q$ then $x_n \to z$ if and only if $p_n \to z$ and $q_n \to 1$.*

Proof. *If* $p_n \to z$ and $q_n \to 1$ then obviously $x_n \to z$. Since L is a line and $zQ = L$, the map $q \to zq$ is a homeomorphism from Q to L. Hence if $q_n \nrightarrow 1$ then $zq_n \nrightarrow z$. Since $zp_n = z$, $zq_n = zp_n q_n$ so $zp_n q_n \nrightarrow z$. Finally, suppose $p_n \nrightarrow z$. There is either a subsequence of p_n which converges to some $p \in P$ or $1/p_n \to z$. In either case $p_n q_n \nrightarrow z$. For suppose so. In the first case, we may as well suppose $p_n \to p$. Then $1/p_n\,(p_n q_n) \to 1/p\,z$ so $q_n \to z$. This is a contradiction since $Q^- \cap L = \emptyset$. If $1/p_n \to z$ and $p_n q_n \to z$ the same contradiction results. Thus if $p_n q_n \to 1$ then $p_n \to z$ and $q_n \to 1$.

THEOREM 3. *Suppose $G \cup L$ is a semigroup on a half-plane. If L contains no nilpotent element then G and L determine $G \cup L$.*

Proof. As we have already mentioned, the only cases for which the theorem remains to be proved are these: (1) L is a group or (2) $z \in L$ implies $Gz = z$ and $zG = L$.

Let $G' \cup L'$ be a second semigroup on a half-plane with $G \cong G'$ and $L \cong L'$. Whether L satisfies (1) or (2), there is, by Theorems 1 and 2, an element $z \in L$ and two one-parameter subgroups P, Q in G such that $P^- = P \cup \{z\}$, $Q^- \cap L = \emptyset$ and $zQ = L$. There is an element $z' \in L'$ and one-parameter subgroups P', Q' in G' having corresponding properties in $G' \cup L'$. In case L is a group then P is normal so, as we have seen, every isomorphism $f: P \to P'$ has an extension to an isomorphism from G to G'. We choose one which can be extended to an isomorphism from P^- to $(P')^-$. In case L satisfies (2), Q is normal and by the remarks preceding Lemma 2, there exists an isomorphism $f: P^- \to (P')^-$ and $h: Q \to Q'$ so that if $T(pq) = f(p)h(q)$ then $T: G \to G'$ is an isomorphism. Hence, whether L satisfies (1) or (2) there is an isomorphism $T: G \to G'$ so that $T|P$ has an extension to an isomorphism from P^- to $(P')^-$.

Extend T to $G \cup L$ by defining $T(zq) = z'T(q)$. It is obvious that T is one-to-one on all of $G \cup L$. To show that T is a homomorphism we treat the two cases separately. Suppose first that L is a group. Then z is the identity of L and its only idempotent. Therefore since qzq^{-1} is an idempotent in L, $qzq^{-1} = z$ for all $q \in Q$. Now let $x, y \in G^-$. Suppose first that $x = zq$, $q \in Q$, and $y = pq'$ with $p \in P^-$, $q' \in Q$. Then $xy = zqpq' = qzpq' = qzq'$ since z is a zero for P. Thus $T(xy) = z'T(qq') = z'T(q)\,T(q')$. Similarly, since z is a zero for P, $T(x)\,T(y) =$

$= z'T(q)T(q')$ so $T(xy) = T(x)T(y)$. It is just as easy to show that $T(yx) = T(y)T(x)$. Since these are the only cases not already covered by the fact that T is a homomorphism on G, $T(xy) = T(x)\,T(y)$ for $x,y \in G \cup L$.

Now suppose L satisfies (2) and let $x = zq$, $y = pq'$ with $p \in P$, q, $q' \in Q$. We have $qpq' = p'q''$ for some $p' \in P$, $q'' \in Q$, so $xy = zqpq' = zp'q'' = zq''$. Thus $T(xy) = z'T(q'')$. Moreover, T is a homomorphism on G, so $T(qpq') = T(p')\,T(q'')$ and hence $T(x)T(y) = z'T(q)T(p)\,T(q') = z'T(qpq') = z'T(p'q'') = z'T(p')\,T(q'') = z'T(q'')$. Therefore $T(xy) = T(x)\,T(y)$. On the other hand $T(yx) = T(pq'zq) = T(zq')$ since z is a right zero for L. Thus $T(yx) = z'T(q) = T(p)\,T(q')z'T(q)$ since z' is a right zero for L', so $T(yx) = T(y)\,T(x)$. If $x,y \in L$ it follows similarly that $T(xy) = T(x)\,T(y)$. Hence $T : G \cup L \to G' \cup L'$ is also a homomorphism in this case.

It follows that whether L satisfies (1) or (2), T is a one-to-one homomorphism from $G \cup L$ onto $G' \cup L'$. The continuity of T and T^{-1} come essentially from Lemma 2. First suppose $x_n \to z$. Then $x_n = p_n q_n$ with $p_n \in P^-$, $q_n \in Q$. By that lemma, $p_n \to z$ and $q_n \to 1$. Since T is continuous on P^-, $T(p_n) \to T(z)$. Hence $T(x_n) \to z'$ and T is continuous at z. The continuity of T at other points of L follows from the continuity of T at z together with the facts that $L = zQ$, T is homomorphism and that both $G \cup L$ and $G' \cup L'$ are topological semigroups. All of these arguments are equally applicable to T^{-1}. Therefore $T : G \cup L \to G' \cup L'$ is an isomorphism.

The counter-examples. In this section we show that if $G \cup L$ is a semigroup on a half-plane which contains a nilpotent element then G and L do not determine $G \cup L$. It would be nice to have a technique for taking this hypothesis and intrinsically constructing two ways of putting G and L together. So far we have not found such a technique. What we do instead is to enumerate the various possibilities for G and L under these conditions. An example of each possibility is given in [5]. In each case we modify this example and show that the modification is not isomorphic to the original. Actually, in one case we exhibit an uncountable number of distinct examples and have no doubt that a little extra effort would yield as many examples of the other cases.

If $G \cup L$ contains a nilpotent element then L contains a zero 0 and an element $x \neq 0$ such that $x^2 = 0$. Let A and B denote the components of $L - \{0\}$. According to [4] and [5] there are then only the following three possibilities for G and L: (1) $A^2 = \{0\}$, B is a group and G is commutative: (2) $L^2 = \{0\}$ and G is commutative; (3) $L^2 = \{0\}$ and G is not commutative. We shall say that $G \cup L$ is of type 1, 2 or 3 according as G and L satisfy (1), (2) or (3).

The example II.c of [5] is an example of type 1. This example was defined in terms of the function $\phi(x) = 1 + \log^2 x$ as follows:

$$(a,b)(x,y) = (ax, by\ \phi(ax)/\ \phi(a)\phi(x)) \quad \text{if } a,x > 0\ ,$$

$$(0,b)(x,y) = (0, by\ \phi(x)) \quad \text{if } x > 0\ ,$$

$$(0,b)(0,y) = (0,0)\ .$$

This semigroup occupies the closed first quadrant. The origin is a zero, (1,1) is the identity, etc. All that is required in order that the above definition of product lead to a topological semigroup is that ϕ be a function such that if $f(a,x) = \phi(ax)/\phi(a)\phi(x)$ then f satisfies (i) $f(a,x)$ is bounded for $a,x > 0$; (ii) $\lim f(a,x) = 0$ as $a,x \to 0$; (iii) $\lim f(a,x)$ exists and is positive as $a \to 0$ and $x \to x_0 > 0$. Any such semigroup will be of type 1.

For given ϕ, denote the group that results on the open first quadrant by G_ϕ and denote the corresponding semigroup on the closed quadrant by G_ϕ^-. In place of $1 + \log^2 x$ one can take $\phi(x) = 1 + |\log x|^k$ for any $k > 0$; (unfortunately for our purposes all of these semigroups turn out to be isomorphic). We shall take $\phi(x) = 1 + |\log x|$

Let ψ be defined by

$$\psi(x) = 1 + \cot x \quad \text{if } 0 < x \leqq 1$$

and

$$\psi(x) = 1 \quad \text{if } 1 < x.$$

By considering various cases and using the fact that $\cot ax < \cot a$ if $a,x < 1$, it is not difficult to show that the function

$$f(a,x) = \psi(ax)/\psi(a)\psi(x)$$

has properties (i)–(iii) above.

We shall show that G_ϕ^- and G_ψ^- are not isomorphic by obtaining a description of an arbitrary isomorphism $S: G_\phi \to G_\psi$ and showing that no such S can be extended to be continuous and one-to-one on G_ϕ^-.

First observe that the map $T(x,y) = (x,(xy)/\phi(x))$ is an isomorphism from G_ϕ onto the group G on the first quadrant whose multiplication is ordinary coordinate-wise multiplication. Similarly, if $U(x,y) = (x,(xy)/\psi(x))$ then $U: G_\psi \to G$ is an isomorphism. Hence, if $S: G_\phi \to G_\psi$ is an isomorphism then $S^* = UST^{-1}$ is an automorphism of G. Furthermore, if there is to be any hope of extending S to be an isomorphism on G_ϕ^-, it is necessary that $S(Q) = Q$, where Q is the set $\{(1,y): y$ is real$\}$. For Q forms the only one-parameter subgroup of both G_ϕ and G_ψ which has a nonzero idempotent in its boundary. This means that $S^*(Q) = Q$ also. There is the further requirement that S^* map the subgroup of G on $\{(x,x): x > 0\}$ onto a subgroup of the form $\{(x,x^t): x > 0\}$ for some t. Thus there are numbers $s \neq 0$, t, u such that $S^*(x, y) = (x^s, x^{st})(1,\ (y/x)^u) = (x^s, y^u x^v)$ where $v = st - u$. By calculating U^{-1} and recalling that $S = U^{-1}S^*T$, we see that $S(x,y) = (x^s, x^w y^u \phi(x^s)/\psi(x)^u)$ for some number w. Now

$$\lim x^w \phi(x^s)/\psi(x)^u = \lim\ x^w(1 + |\log x^s|)/(1 + \cot x)^u \ \text{ as } \ x \to 0.$$

But this latter limit is either 0 or infinite for every choice of $w,s \neq 0$ and u. Therefore lim $S(x,y)$ as $x \to 0$ is not a one-to-one function of y in any case, so S can not be extended to be an isomorphism between G_ϕ^- and G_ψ^-. We thus have distinct examples of semigroups on a half-plane of type 1.

Example II.b of [5] is an example of a semigroup of type 2. It is defined on the right half-plane $\{(x,y):x \geqq 0\}$; multiplication is given by the rule

$$(a,b)(x,y) = (ax,ay + bx) \quad \text{if } a,x \geqq 0.$$

A way of obtaining other examples of this type is the following: Let ϕ be a homeomorphism from the space of real numbers onto itself. Define $h(x,y) = (x,x\phi(y))$. Then h is a homeomorphism of the open half-plane Γ onto itself. Let the preceding multiplication be denoted by juxtaposition and the resulting group on the open half-plane by G. Define

$$(a,b) * (x,y) = h(h^{-1}(a,b)h^{-1}(x,y)).$$

Then $(\Gamma,*)$ is a group and h^{-1} is an isomorphism from $(\Gamma,*)$ onto G. We shall show that for certain ϕ the operation $*$ can be extended to Γ^- so that $(\Gamma^-,*)$ is a semigroup of type 2 but $(\Gamma^-,*)$ is not isomorphic to G^-.

Since the one-parameter subgroups of G can be shown to have one of the two forms

$$R_t = \{(x,(t-1)x\log x):x > 0\} \text{ for some } t > 0$$

or

$$S = \{(1,y): \quad y \text{ is real}\},$$

the one-parameter subgroups of $(\Gamma,*)$ have one of the forms

$$P_t = \{(x,x\phi((t-1) \quad \log x)):x > 0\}$$

or

$$Q = \{(1,y): \quad y \text{ is real}\}.$$

It turns out that we may suppose $\phi(0) = 0$, so $P_1 = \{(x,0): x > 0\}$. Suppose T: $(\Gamma,*) \to G$ is an isomorphism. Since S and Q are the only one-parameter subgroups of $(\Gamma,*)$ and G respectively which are closed in Γ and G^-, $T(S) = Q$ if T is to have an extension to Γ^- to be an isomorphism. However, T can map P_1 into any one of the groups R_t. For each $(x,y) \in \Gamma$ we can write $(x,y) = (x,0)*$ $(1,\phi((1/x)\phi^{-1}(y/x)))$. The only isomorphisms from Q to S are of the form $(1,y) \to$ $(1,u\phi^{-1}(y))$ for some $u \neq 0$. The only isomorphisms from P_1^- to P_t^- are of the form $(x,0) \to (x,(t-1)\log x)^s = (x^s,sx^s(t-1)\log x)$ for some $s > 0$. Therefore there are numbers s,t,u, with $s > 0$, $u \neq 0$, such that

$$\begin{aligned} T(x,y) &= (x^s,sx^s(t-1)\log x)(1,(u/x)\phi^{-1}(y/x)) \\ &= (x^s,ux^{s-1}\phi^{-1}(y/x) + sx^s(t-1)\log x). \end{aligned}$$

Obviously T can not be extended to an isomorphism of $(\Gamma^-,*)$ if

$$\lim x^{s-1}\ \phi^{-1}\ (y/x) \text{ as } x \to 0$$

either fails to exist or is not a one-to-one function of y.

For this purpose, let ϕ be a homeomorphism of the reals onto themselves so that (i) $\phi(y) = \log y$ if $y > e$, and (ii) $\phi(-y) = -\phi(y)$ for all y. For such ϕ, the above limit fails to exist for every choice of s so that T has no extension to Γ^- to be one-to-one and continuous.

Of course, it remains to show that the operation $*$ can be extended to Γ^- so that $(\Gamma^-,*)$ is a (topological) semigroup of the desired type.

A detailed definition of $*$ on Γ is the following:

$$(a,b)*(x,y) = (ax, ax\phi(a\phi^{-1}(y/x) + x\phi^{-1}(b/a)), a,x > 0.$$

It is straightforward to show that $\lim ax\phi(a\phi^{-1}(y/x) + x\phi^{-1}(b/a))$ as $a \to 0$ exists and is bx for fixed $x > 0$, y and b.

Thus define

$$(0,b)*(x,y) = (x,y)*(0,b) = (0,bx)$$

and

$$(0,b)*(0,y) = (0,0).$$

The operation $*$ has now been defined for all pairs in Γ^-. It is clear that $(\Gamma^-,*)$ will be of type 2 if indeed it is a topological semigroup. The only feature of this fact which is not immediate is the continuity of $*$ at such points as $((0,b), (0,y))$ and $((0,b), (x,y))$ with $x > 0$. (Associativity everywhere will follow from continuity and associativity on Γ.)

Proof of the continuity of $*$ involves the consideration of several cases. We take two of the more involved.

For one case, we must show that

$$\lim ax\phi(a\phi^{-1}(v/x) + x\phi^{-1}(w/a)) = 0 \text{ as } (a,w,x,v) \to (0,b,0,y).$$

If $(0,b,0,y)$ is approached by a sequence of points so that the term $a\phi^{-1}(v/x) + x\phi^{-1}(w/a)$ is bounded then this limit is obviously 0. *If* $(0,b,0,y)$ is approached so that this term is unbounded then $|ax\phi(a\phi^{-1}(v/x) + x\phi^{-1}(w/a))|$ is eventually bounded by either

$$ax \log |a \operatorname{sgn} v \exp(|v|/x) + x \operatorname{sgn} w \exp(|w|/a)|$$

or

$$ax \log |a\phi^{-1}(v/x) + x \operatorname{sgn} w \exp(|w|/a)|.$$

(There is a third possibility with the roles of a,v,x and x,w,a interchanged, but this need not be treated separately.) Now each of these in turn is bounded by

$$ax \log 2 \max(a \exp(|v|/x), x \exp(|w|/a)),$$

which evidently has limit zero as $a \to 0$. This yields the desired result.

In proving that $\lim at\phi(a\phi^{-1}(v/t) + t\phi^{-1}(w/a)) = bx$ as $(a,w,t,v) \to (0,b,x,y)$ with $x > 0$, it is best to consider the cases $b = 0$ and $b \neq 0$ separately. If $b = 0$, one has an argument somewhat like the preceding to show that the limit is 0. If $b \neq 0$, there are the various combinations of the signs of b and y to consider (the case $y = 0$ presents no problem now since $t \to x > 0$). For the case $b > 0$, $y < 0$ the following inequality eventually holds and easily yields the desired conclusion:

$$at \log (t/2) \exp (w/a) \leqq at\phi(a\phi^{-1}(v/t) + t\phi^{-1}(w/a)) \leqq at \log t \exp (w/a).$$

With this much of the pattern established it is straightforward to complete the proof that $*$ is continuous at all pairs of points in Γ^-. Therefore $(\Gamma^-,*)$ is a semigroup of type 2 which is not isomorphic to G^-.

Examples of type 3 are obtained by defining, for each $\alpha > 0$ (and $\neq 1$).

$$(a,b)(x,y) = (ax, a^\alpha y + bx) \text{ if } a,x \geqq 0.$$

Let G_α denote the resulting group on the open half-plane and let G_α^- denote the semigroup which results on the closed half-plane. G_2^- is example II.b of [5;4.6.1].

Let $P_0 = \{(x,0) : x > 0\}, Q = \{(1,y) : y \text{ is real}\}$ and $P(b) = \{(u,bu - bu^\beta) : u > 0\}$ for b real. Then P_0 is a one-parameter subgroup and Q is the normal subgroup of G_α. The groups $P(b)$ comprise the non-normal one-parameter subgroups of G_β while Q is the normal one-parameter subgroup of G_β. Suppose $T : G_\alpha \to G_\beta$ is an isomorphism. Thus $T(1,y) = (1,sy)$ for some $s \neq 0$, and $T(P_0) = P(b)$ for some b. Thus if $(x,0) \in P_0$, $T(x,0) = (u,bu - bu^\beta)$ for some u.

Since $(x,0)^{-1} = (x^{-1},0)$ and $(u,bu\text{-}bu^\beta)^{-1} = (u^{-1},b/u - b/u^\beta)$ we have (with $t = \alpha - 1$)

$$T(1,x^t y) = T((x,0)(1,y)(x,0)^{-1}) = T(x,0)(1,sy)T(x,0)^{-1} = (1,s\, u^{\beta-1}y).$$

Setting $y' = x^t y$ in the previous result yields

$$T(1,y') = (1,s\, u^{\beta-1}y'(1/x)^t).$$

However $T(1,y') = (1,sy')$. Therefore $u^{\beta-1} = x^{\alpha-1}$ or $x = u^v$ where

$$v = (\beta - 1)/(\alpha - 1).$$

Substituting this in the original expression for $T(x,0)$ yields

$$T(x,0) = (x^v, (1 - x^t)\, bx^v)$$

with $v = (\alpha - 1)/(\beta - 1)$ and $t = \alpha - 1$.

Hence

$$T(x,y) = T((x,0)(1,y/x^\alpha) = (x^v, (1 - x^t)\, bx^v)(1,sy/x^\alpha)).$$

If this last multiplication is carried out in G_β it is found that

$$T(x,y) = (x^v, syx^w + bx^v - bx^{\beta v}),$$

with $v = (\alpha - 1)/(\beta - 1)$, $s \neq 0$, $w = (\alpha - \beta)/(\beta - 1)$ and b an arbitrary real number.

Such T is typical of all isomorphisms from G_α to G_β. Obviously no such function can be extended to G_α^- to be one-to-one and continuous unless $\alpha = \beta$. Thus, if $\alpha \neq \beta$, G_α^- and G_β^- are not isomorphic, yet every G_α^- is a semigroup of type 3.

References

1. J. G. Horne, Jr., *Real commutative semigroups on the plane*, Pacific J. Math. **11** (1961), 981–997.

2. ———, *On one-parameter subgroups in semigroups in the plane*, Michigan Math. J. **9** (1962), 177–186.

3. ———, *Real commutative semigroups on the plane*. II, Trans. Amer. Math. Soc. (to appear).

4. P. S. Mostert, *Plane semigroups*, Trans. Amer. Math. Soc. **103** (1962), 320–328.

5. P. S. Mostert and A. L. Shields, *Semigroups with identity on a manifold*, Trans. Amer. Math. Soc. **91** (1959), 380–389.

University of Georgia,
Athens, Georgia