SUMMABILITY OF FOURIER SERIES IN $L^p (d\mu)$

BY

MARVIN ROSENBLUM

1. Introduction. Let $\mu$ be a non-negative finite Borel measure on the unit circle $C$ such that $\mu(C) > 0$. For each $p$, $1 \leq p < \infty$, let $L^p (d\mu)$ be the Banach space of $\mu$-measurable complex-valued functions $f(e^{i\phi})$ such that

$$
\|f\|_p = \left(\int |f(e^{i\phi})|^p d\mu(\phi)\right)^{1/p}
$$

$< \infty$. $\sigma$ shall be normalized Lebesgue measure on $C$. $\mathcal{P}$ and $\mathcal{P}_0$ are the classes of trigonometric polynomials of the form $\sum_n c_n e^{in\phi}$ and $\sum_{n \geq 0} c_n e^{in\phi}$ respectively. $P_r(e^{i\phi})$ shall be the Poisson kernel and $*$ the symbol of Fourier convolution. Thus if $f(e^{i\phi}) = \sum_n c_n e^{in\phi} \in \mathcal{P}$, then $(P_r * f)(e^{i\phi}) = \sum_n c_n r^n e^{in\phi} \in \mathcal{P}$. $\delta_1$, $\delta_2$, ... shall be fixed positive numbers and $K_1, K_2, ...$ absolute constants. We omit writing subscripts or use the same subscript in different contexts when we believe that no confusion can arise.

Our main concern shall be the following problems:

Problems A, B. Characterize the classes $\mathcal{P}_p$ and $\mathcal{P}_p'$ of measures $\mu$ such that

$$(1.1) \sup\{\|P_r * f\|_p: \delta < r < 1\} \leq K \|f\|_p$$

for all $f \in \mathcal{P}_0$ and $\mathcal{P}$ respectively.

We shall also be concerned with variations of problem B, where Abel summability is replaced by Fejér and several other types of summability. These problems follow a line of investigations in harmonic analysis with non-translation-invariant measures that dates back to Hardy and Littlewood [8]. Subsequent work was done by Babenko [1], Hirschman [10], Gapoškin [6], Edwards [4], Chen [3] and Helson and Szegő [9]. Our work follows up certain consequences of Helson and Szegő’s results. These authors classify the class $\mathcal{P}_{2}$ of measures $\mu$ for which

$$(1.2) \sup\|D_n * f\|_2 \leq \sum K \|f\|_2$$

for all $f \in \mathcal{P}$. $D_n$ is the Dirichlet kernel. They prove that $\mu \in \mathcal{P}_2$ if and only if

$$(1.3) \text{(i)} \mu \text{ is absolutely continuous, } d\mu = w d\sigma, \text{ and}$$

$$(1.3) \text{(ii)} w = \exp(u + v), \text{ where } u \text{ and } v \text{ are } \sigma\text{-essentially bounded real functions such that } \sigma\text{-ess } \sup |v| < \pi/2. v \to \tilde{v} \text{ is the Fourier conjugation operator.}$$

From (1.3) one can deduce that if $f \in L^2(d\mu)$, $\mu \in \mathcal{P}_2$, then the Fourier coefficients of $f$ are well-defined and the Fourier series of $f$ converges in $L^2(d\mu)$ norm to $f$. Given this, it seems reasonable to ask when the Fourier series of any $f \in L^p (d\mu)$
is Abel summable to $f$ in $L^p(d\mu)$ norm. This, in turn, leads us to problem B. Suppose $\mu \in \mathcal{B}_p$. Then the densely defined linear functionals $l_n(f) = \int (e^{i\phi}e^{-in\phi} d\sigma(\phi))$, $n = 0, \pm 1, \pm 2, \ldots$ can be shown to be bounded on $\mathcal{P}$ and thus have unique continuous extensions to all of $L^p(d\mu)$. Thus the Fourier coefficients of any $f \in L^p(d\mu)$ are well-defined. Similarly the densely defined operators $f \rightarrow P_\delta * f$, $\delta$ fixed, $0 < \delta < 1$, are bounded on $\mathcal{P}$ and thus one can speak meaningfully of the Abel means of the Fourier series of any $f \in L^p(d\mu)$. Finally (still under the assumption that $\mu \in \mathcal{B}_p$) one can deduce from (1.1) that these Abel means converge in $L^p(d\mu)$ norm to $f$.

Problem A leads to a generalization of Hardy spaces. Let $H^p(d\mu)$ be the class of functions $f(z) = re^{i\phi}$, holomorphic in $0 \leq r < 1$ and such that

$$
\|f\|_p = \sup \{ \int |f(re^{i\phi})|^p d\mu(\phi) \}^{1/p} : 0 \leq r < 1
$$

is finite. Let $L^p_0(d\mu)$ be the closure of $\mathcal{P}_0$ in $L^p(d\mu)$. A classical theorem [11, p. 284] states that $H^p(d\sigma)$ is vector space isomorphic and isometric to $L^p_0(d\sigma)$ under the operator $T: f(z) \rightarrow f(e^{i\phi})$. Our generalization is as follows: If $\mu \in \mathcal{B}_p$ then the operator $T$ is a vector space isomorphism mapping $H^p(d\mu)$ onto $L^p_0(d\mu)$ such that $T$ and $T^{-1}$ are bounded. If $T$ is an isometry then $\mu$ is a multiple of Lebesgue measure.

2. Solution of problem B. Let $d\mu(\phi) = w(e^{i\phi})d\sigma(\phi) + d\mu_\delta(\phi)$ be the Lebesgue decomposition of $\mu$. We shall first show that if $\mu \in \mathcal{B}_p$, then $\mu_\delta = 0$ and log $w \in L^1(d\sigma)$. These properties are incidentally shared by any $\mu \in \mathcal{B}_2$.

**Lemma 1.**

(i) $\mathcal{B}_p = \mathcal{B}_2$ for all $p$, $1 \leq p < \infty$.

(ii) If $\mu \in \mathcal{B}_p$, then log $w \in L^1(d\sigma)$.

**Proof.**

(i) is an easy consequence of the Blaschke factorization of any $f \in \mathcal{P}_0$. We shall prove (ii) by contradicting the assumption that log $w \notin L^1(d\sigma)$, while assuming that $\mu \in \mathcal{B}_p$. By [7, p. 50], \{e^{i\phi}\}_0^\infty is total in $L^2(d\mu)$, so for each positive integer $n$ there is a sequence $(h_{n,j})_{j=0}^\infty \subset \mathcal{P}_0$ such that \( \| h_{n,j}(e^{i\phi}) - e^{-in\phi} \|_2 \rightarrow 0 \).

Now fix $r$, $0 < r < 1$. From (*) and (1.1) it follows that there exists $k_n \in L^2(d\mu)$ such that $\| h_{n,j}(re^{i\phi}) - k_n(e^{i\phi}) \| \rightarrow 0$ as $j \rightarrow \infty$. But

$$
\lim_{j \rightarrow \infty} \| r^n e^{i\phi} h_{n,j}(re^{i\phi}) - 1 \| \leq \lim_{j \rightarrow \infty} K \| e^{i\phi} h_{n,j}(e^{i\phi}) - 1 \| = 0,
$$

so $k_n(e^{i\phi}) = r^{-n} e^{-in\phi}$ in $L^2(d\mu)$ norm. Thus

$$
r^{-n} \| 1 \| = \| k_n \| = \lim_{j \rightarrow \infty} \| h_{n,j}(re^{i\phi}) \| \leq \lim_{j \rightarrow \infty} K \| h_{n,j} \| = K \| 1 \|.
$$

Take $n \rightarrow \infty$ to obtain a contradiction of $\| 1 \| > 0$. Thus the proof of (ii) is complete.

**Lemma 2.** Suppose $\mu \in \mathcal{B}_p$. Then $d\mu(\phi) = |g(e^{i\phi})|d\sigma(\phi)$, where $g \in H^1(d\sigma)$ is an outer function.
Proof. Any non-negative function $w$ such that $w$ and $\log w \in L^1(\sigma)$ has a representation of the form $w(e^{i\theta}) = |g(e^{i\theta})|$, where $g$ is as above (see [2]). Hence we have only to prove that $\mu_\gamma = 0$. Let $E$ be $\sigma$-null set such that the mass of $\mu_\gamma$ is concentrated on $E$. Now, by [11, p. 276], there exists a bounded outer function $b$ such that $b(0) > 0$ and $\lim_{\gamma \to 1} b(se^{i\phi}) = 0$ for all $\phi \in E$. By Lemma 1 we may assume $\mu \in \mathcal{P}_0$. Let $0 < r, s < 1$. Then for all $f \in \mathcal{B}_0$

$$\|f(re^{i\phi}) b(se^{i\phi})\|_2^2 \leq K^2 \|f(e^{i\phi}) b(se^{i\phi})\|_2^2.$$ 

Take $s$ to 1 and obtain

$$\|f(re^{i\phi}) b(re^{i\phi})\|_2^2 \leq K^2 \|f \cdot b\|_2 = K^2 \int |f \cdot h|^2 d\sigma,$$

where $h = b \cdot g^{1/2}$ is an outer function. Next let $z = re^{i\psi}$ be any complex number with $|z| < 1$. Then, since $h$ is outer, there exists $\{f_n\} \subset \mathcal{B}_0$ with $f_n(e^{i\phi}) h(e^{i\phi}) \to (1 - te^{i(\phi - \psi)})^{-1} g^{1/2}(e^{i\phi})$ in $L^2(\sigma)$. Thus

$$\int \left| 1 - tre^{i(\phi - \psi)} \right|^{-2} d\mu_\gamma \leq K^2 \int \left| 1 - te^{i(\phi - \psi)} \right|^{-2} w(e^{i\phi}) d\sigma_\gamma.$$

Let $r \to 1$, so

$$\int P_\gamma(e^{i(\phi - \psi)}) d\mu_\gamma \leq K^2 \int P_\gamma(e^{i(\phi - \psi)}) w(e^{i\phi}) d\sigma_\gamma$$

for all $t, 0 \leq t < 1$ and all real $\psi$. Thus $P_\gamma(K^2 w d\sigma - d\mu)$ is a positive harmonic function in $|z| < 1$ and consequently $K^2 w d\sigma - d\mu$ is a positive measure. Thus $\mu$ is absolutely continuous. Our solution of problem A is contained in

**Theorem 1.** $\mu \in \mathcal{P}_p$ if and only if

(i) $\mu$ is absolutely continuous, $d\mu = w d\sigma$, where $w = |g|$, $g \in H^1(\sigma)$ is an outer function and

(2.1) (ii) $\int P_\gamma(e^{i(\phi - \psi)}) |g(e^{i\phi})| |g(re^{i\phi})| d\sigma_\gamma \leq K_2$ for all $r, 0 \leq r < 1$ and real $\psi$.

**Proof.** Suppose $\mu \in \mathcal{P}_p$. Then Lemmas 1 and 2 prove that (i) is true. We shall show that necessarily (ii) holds. First of all we note that for each $r, \delta < r < 1$, and real $\psi, (1 - re^{i(\phi - \psi)})^{-1} g^{-1/2}(e^{i\phi})$ is in the closed linear span of $\{e^{in\phi}\}_{n=0}^\infty$ in $L^2(d\mu)$. This is a simple consequence of the fact that $\{e^{in\phi} g^{1/2}(e^{i\phi})\}_{n=0}^\infty$ is total in $H^2(\sigma)$. Hence by (1.1)

$$\int \left| (1 - r^2 e^{i(\phi - \psi)})^{-1} g^{-1/2}(re^{i\phi}) \right|^2 w(e^{i\phi}) d\sigma_\gamma \leq K \int \left| 1 - re^{i(\phi - \psi)} \right|^{-2} d\sigma_\gamma$$

$$= K (1 - r^2)^{-1}.$$

From this and the elementary inequality $4(1 - r^2) |1 - r^2 e^{i\phi}|^{-2} \geq P_\gamma(e^{i\phi})$ we deduce that (ii) holds for $\delta < r < 1$. Then it clearly holds for all $r, 0 \leq r < 1$. 

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Conversely suppose that (i) and (ii) hold. As indicated in Lemma 1 it is sufficient to derive (1.1) for those \( f \in \mathcal{P}_0 \) that are the boundary functions of functions zerofree and holomorphic in \( |z| < 1 \), and thus we may restrict our proof to the case \( p = 1 \).

\[
\int |f(re^{i\phi})| w(e^{i\phi}) d\sigma(\phi)
\]

\[
= \int |f(re^{i\phi})g(re^{i\phi})| |g(e^{i\phi})| d\sigma(\phi)
\]

\[
\leq \int \left[ \int |f(e^{i\phi})g(e^{i\phi})| P_r(e^{i(\phi-\psi)}) d\sigma(\psi) \right] |g(e^{i\phi})| |g(re^{i\phi})| d\sigma(\phi),
\]

which by the Fubini theorem and (ii) is

\[
\leq K_2 \int |f(e^{i\phi})| w(e^{i\phi}) d\sigma(\psi).
\]

Thus (1.1) is true for \( 0 \leq r < 1 \), and the proof is complete.

3. Approximate identities. We shall set about stating some problems equivalent to problem B.

**Definition 1.** By an approx id (approximate identity) \( \{k_\lambda\} \lambda \in A \) we mean a sequence or generalized sequence of non-negative functions \( k_\lambda \) such that

(i) \( \int k_\lambda(e^{i\phi}) d\sigma(\phi) = 1 \) for all \( \lambda \in A \), and

(ii) \( \lim_{\lambda} k_\lambda * f = f \) for all \( f \in \mathcal{P} \).

**Definition 2.** Let \( k = \{k_\lambda\} \lambda \in \Lambda \) and \( k' = \{k'_\lambda\} \lambda \in \Lambda \) be approx ids. \( k \) is weaker than \( k' \) (with respect to \( L^p(dp) \)) if whenever

\[
\sup_{\lambda} \| k_\lambda * f \|_p \leq K_1 \| f \|_p \text{ for all } f \in \mathcal{P},
\]

then there exists \( K_2 \) such that

\[
\sup_{\lambda} \| k'_\lambda * f \|_p \leq K_2 \| f \|_p \text{ for all } f \in \mathcal{P}.
\]

If \( k \) is weaker than \( k' \) and \( k' \) is weaker than \( k \), then \( k \) and \( k' \) are said to be equisummable (with respect to \( L^p(dp) \)).

The approx ids we shall consider are the

(i) Abel \( \{P_r: \delta < r < 1\} \);

(ii) generalized Abel \( \{P_{a,r}: \delta < r < 1\} \), where

\[
P_{a,r}(e^{i\phi}) = k_{a,r}(1 - r)^{2a-1} |1 - re^{i\phi}|^{-2a}; \quad a > 1/2 \text{ and } k_{a,r} \text{ is chosen so } \int P_{a,r} d\sigma = 1;
\]

(iii) moving average \( \{Q_h: 0 < h < \delta \leq \pi\} \), where \( Q_h(e^{i\phi}) = \pi/h \) if \( |\phi| \leq h \) and 0 if \( \pi \geq |\phi| > h \); and

(iv) Fejér \( \{F_n: n = N, N + 1, N + 2, \ldots\} \), where
\[ F_n(e^{i\phi}) = \frac{1}{n+1} \frac{\sin^2 \left[ \frac{1}{2}(n + 1) \phi \right]}{\sin^2(\phi/2)}, \quad \text{and} \]

\[ N \text{ is a fixed non-negative integer.} \]

We shall list some pertinent properties of the less familiar \( P_{x,r} \) later. Now it will be expedient to introduce the approx id \( \{Q_{h_j}\} \), where \( h_j = \pi(j + 1)^{-1} \) with \( j \) ranging over all sufficiently large positive integers.

**Lemma 3.** The first four approx ids listed above are all weaker than \( \{Q_{h_j}\} \).

**Proof.** This is a sequence of the inequalities

\[ (3.3) \quad P_{x,r} \geq K_3 Q_{1-r}, \quad \text{for fixed } x, \]

and \( F_j \geq K_4 Q_{h_i} \). This second inequality is easily deduced from the elementary inequalities \( |\sin \psi| \leq |\psi| \) and \( \sin \phi \geq 2/\pi \phi \) for \( \phi \in [0, \pi/2] \). (3.3) will be proved later.

**Lemma 4.** Suppose \( h \) is a fixed number in \( (0, \pi/2] \), and \( \|Q_h * f\|_p \leq K_1 \|f\|_p \) for all \( f \in \mathcal{P} \). Then

(i) \( \|1 * f\|_p \leq K_2 \|f\|_p \) for all \( f \in \mathcal{P} \) and

(ii) the linear functionals

\[ l_a(f) = \int f(e^{i\phi}) e^{-i\phi \sigma(\phi)} \, d\sigma(\phi), \quad f \in \mathcal{P}, \text{ are } L^p(du) \text{ continuous.} \]

**Proof.** Let \( A \) be the operator on \( \mathcal{P} \subset L^p(du) \) that maps any \( f \) into \( Q_h * f \). For some sufficiently large \( n \) there exists \( \epsilon > 0 \) such that \( A^* f \geq \epsilon \int f \, d\sigma \) for all non-negative \( f \in \mathcal{P} \). Thus under the assumptions of the lemma

\[ \epsilon \int |f| \, d\sigma(\int |du|^{1/p}) \leq \|A^* f\|_p \leq K_n \|f\|_p, \]

so (i) is true. (ii) is an immediate consequence of (i).

**Lemma 5.** \( \{Q_{h_j}: j \geq M\} \) is a weaker approx id than \( \{Q_h\}_{0 < h \leq \pi} \).

**Proof.** Assume \( \sup_j \|Q_{h_j} * f\|_p \leq K \|f\|_p \) for all \( f \in \mathcal{P} \). First suppose that \( 0 < h \leq h_M \), so there exists an integer \( n \geq M \) such that \( h_{n+1} < h \leq h_n \). Clearly \( h_n \leq 2h \), so \( Q_h \leq Q_{h_{n+1}} + 2Q_{h_n} \) and \( \|f * Q_h\| \leq 3K \|f\| \) for all \( f \in \mathcal{P} \).

If \( \pi \geq h > h_M \), then Lemma 4(i) guarantees that

\[ \|f * Q_h\| \leq K_2 \|f\| \text{ for all } f \in \mathcal{P}. \]

**Lemma 6.** Suppose \( k(e^{i\phi}) \) is an even \( \sigma \)-absolutely continuous function and put \( k_1(e^{i\phi}) = \frac{d}{d\phi} k(e^{i\phi}) \). Suppose further that

(i) \( |k(e^{i\phi})| \leq K_1, \int |k_1(e^{i\phi})| \, d\sigma(\phi) \leq K_2, \) and

(ii) \( \sup \{\|Q_h * f\|_p: 0 < h \leq \pi\} \leq K_3 \|f\|_p \) for all \( f \in \mathcal{P} \). Then

\[ \|k * f\|_p \leq (K_1 K_3 + 2K_2 K_3) \|f\|_p \text{ for all } f \in \mathcal{P}. \]
Proof. Assume (i), (ii) and let $0 < \psi \leq \pi$. Then
\[ k(e^{i\psi}) - k(e^{i\theta}) = -2 \pi \int_{\theta}^{\psi} k_1(e^{i\phi}) \, d\sigma(\phi) = -2 \int_{0}^{\pi} Q_\phi(e^{i\phi}) \phi \, k_1(e^{i\phi}) \, d\sigma(\phi), \]
so
\[ \| (k - k(e^{i\theta})) \ast f \|_p \leq 2 \int_{0}^{\pi} Q_\phi \ast f \|_p \phi \, k_1(e^{i\phi}) \, d\sigma(\phi) \leq 2K_3K_2 \| f \|_p. \]
Finally
\[ \| k \ast f \|_p \leq \| (k - k(e^{i\theta})) \ast f \|_p + |k(e^{i\theta})| \| 1 \ast f \|_p \leq (2K_2K_3 + K_1K_3) \| f \|_p \text{ for all } f \in \mathcal{S}. \]

Theorem 2. The Abel, generalized Abel, Fejér and moving average approximate identities are equisummable with respect to $L^p(d\mu)$.

Proof. In view of Lemmas 3 and 5 we have only to show that \( \{Q_h\}_{0 \leq h < \pi} \) is weaker than the generalized Abel and Fejér approx ids. The argument in [11, p. 155] shows $P_{a, r}$ satisfies the hypotheses of Lemma 6, as does a dominant of the Fejér kernel.

4. Generalized Abel approximate identities. We list here some properties of the functions $P_{a, r}$ defined in (3.1). From [5, p. 81] we have
\[ F(\alpha, \alpha, 1, r^2) = \int |1 - r e^{i\phi}|^{-2x} \, d\sigma(\phi), \quad \alpha > \frac{1}{2}, \]
where $F$ is the hypergeometric function. Since
\[ \lim_{r \to 1} (1 - r)^{2x-1} F(\alpha, \alpha, 1, r^2) = \frac{\Gamma(2\alpha - 1)}{(\Gamma(\alpha))^2} \]
[12, p. 299] it follows that for fixed $\alpha$, $\infty > \alpha > 1/2$, $\{k_{a, r}: 0 \leq r < 1\}$ is bounded away from 0 and $\infty$.

Now we prove (3.3). If $|\phi| \leq 1 - r \leq 1$, then
\[ P_{a, r}(e^{i\phi}) = k_{a, r}(1 - r)^{2x-1} [(1 - r)^2 + 4r \sin^2 \phi/2]^z \]
\[ \geq k_{a, r}(1 - r)^{2x-1} [(1 - r)^2 + \phi^2]^z \geq KQ_{1-r}(e^{i\phi}). \]

The inequalities
\[ K |1 - r e^{i\phi}|^{-1} \geq |1 - r^2 e^{i\phi}|^{-1} \] and
\[ K |1 - r^3 e^{i\phi}|^{-1} \geq |1 - r^2 e^{i\phi}|^{-1} \geq K_2 \| 1 - re^{i\phi} \|_1^{-1} \] are also easily demonstrated.

Of course $P_r = P_{1, r}$. A paraphrase of the proof of (2.1) of Theorem 1 shows that the condition
\[ \int P_{a, r}(e^{i(\phi - \psi)}) \, |g(e^{i\phi})| \, |g(re^{i\phi})| \, d\sigma(\phi) \leq K \]
is necessary for $\mu \in \mathcal{M}$ if $\alpha$ (fixed) is $> 1/2$. We shall use (4.4) later.
5. Solution of problem B. We introduce the notation $f_h$ for $Q_h * f$, so $f_h(e^{i\theta}) = (\pi/h) \int_{-h}^{h} w(e^{i(\theta + h)}) \, d\sigma(\psi)$. We first treat problem B for the easy case $p = 1$.

**Lemma 7.** $\mu \in \mathcal{B}_1$ if and only if

(i) $d\mu = w \, d\sigma$ and

(ii) $w_{h} \leq K \, w$ a.e. for each $h$, $0 < h \leq \pi$. In fact, for fixed $h$, $0 < h \leq \pi$

$$\sup \{ \| Q_h \ast f \|_1 : \| f \|_1 = 1, \ f \in \mathcal{P} \} = \sigma\text{-ess sup} \ (w_{h}/w).$$

**Proof.** $\mathcal{B}_1 \subset L_1$ so (i) is certainly necessary. In fact $\log w \in L^1(d\sigma)$ so $w$ vanishes on no set of positive measure. The following statements are equivalent to the statement $\mu \in \mathcal{B}_1$:

(a) $L_1(f) = \int f w_{h} \, d\sigma$ is a bounded linear functional on $L^1(d\mu)$;

(b) $L_2(f) = \int f(w_{h}/w) \, d\sigma$ is a bounded linear functional on $L^1(d\sigma)$;

(c) $\| L_2 \| = \sigma\text{-ess sup} \ (w_{h}/w)$.

This set of equivalences proves the lemma.

For any $p > 1$ we define $q$ by $p^{-1} + q^{-1} = 1$. By considering the adjoint operator of $A : f \to Q_h * f$ we will prove the following duality result.

**Theorem 3.** Let $p > 1$. $\mu \in \mathcal{B}_p$ if and only if

(i) $d\mu = w \, d\sigma$, with

(ii) $w^{1-q} \in L^1(d\sigma)$, and

(iii) $\sup \{ \int [Q_h * f]^{q} w^{1-q} \, d\sigma : 0 < h \leq \pi \} \leq K \int |f|^{q} w^{1-q} \, d\sigma$ for all $f \in \mathcal{P}$.

**Proof.** Suppose $\mu \in \mathcal{B}_p$. Then $\mu \in \mathcal{B}_p$, so (i) holds and $\log w \in L^1(d\sigma)$. We see from Lemma 4 that $1 : f \to \int f \, d\sigma$ is an element of the adjoint space $L^q(d\mu)$ of $L^p(d\mu)$. Thus there exists $h \in L^q(d\mu)$ with $\int f \, d\sigma = \int f w \, d\sigma$ for all $f \in \mathcal{P}$. Necessarily $h \, w = 1$, so $\infty > \int |h|^{q} w \, d\sigma = \int w^{1-q} \, d\sigma$, which proves (ii). To show that (iii) holds we consider the adjoint operator $A^*$ of $A$. Since $\| A^* \| = \| A \|$ we have

$$\int |w^{-1} [Q_h * (hw)]|^{q} w \, d\sigma \leq \| A \|^q \int |h|^{q} w \, d\sigma$$

for all $h \in \mathcal{P}$. Put $f = hw$ to obtain (iii).

Conversely suppose (i), (ii), (iii) hold. Then one can interchange the roles of $p$ and $q$ to derive (1.1).

It should be noted that the moving average approx identity in (iii) may be replaced by any of the approx ids of Theorem 2 without affecting the validity of the proof.

Our solution of problem B is stated in

**Theorem 4.** $\mu \in \mathcal{B}_p$ if and only if

(i) $\mu$ is absolutely continuous, $d\mu = w \, d\sigma$,

(ii) $w^{1-q} \in L^1(d\sigma)$ if $p > 1$, and

(iii) $Q_h * (w_{h}/w)^{q-1} \leq K$ for all $h$, $0 < h \leq \pi$, if $p > 1$, or (5.1) holds if $p = 1$. 

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Proof. Lemma 7 takes care of the case when $p = 1$, so assume that $p > 1$. Then (i), (ii) follow from Theorem 3. We shall defer the rather involved proof of the necessity of (iii) until later.

Conversely, suppose (i), (ii), (iii) hold and let $f \in \mathcal{D}$, $0 < h \leq \pi$. Then

$$\int |Q_h * f|^q w^{1-q} d\sigma = \int |Q_h * (f w^{-1/p} w w^{1/p})|^q w^{1-q} d\sigma,$$

which by the Hölder inequality is $\leq \int |Q_h * (|f|^q w^{-q})| \cdot [w_h/|w|]^{q-1} d\sigma$. By the Fubini theorem and (iii) this is $\leq K \int |f|^q w^{1-q} d\sigma$. Thus Theorem 3 guarantees that $\mu \in \mathcal{B}_p$.

We note in passing that due to the duality Theorem 3 we can replace (iii) by the condition

$$(iii') \quad Q_h * [(w^{-1-q})h/|w|]^{p-1} \leq K.$$

Our task now is to prove that (iii) is a necessary condition for $\mu \in \mathcal{B}_p$. We thus assume that $\mu \in \mathcal{B}_p$, $p > 1$, so $d\mu = |g| d\sigma$ as in Theorem 1.

**Lemma 8.** If $0 \leq r < 1$,

$$\int P_r(e^{i(x-\phi)}) (1 - r^2 e^{i(x-\phi)})^{-1} g^{-1/p} (e^{i\phi}) d\psi = J_1(e^{i\phi}) + J_2(e^{i\phi}),$$

where

$$J_1(e^{i\phi}) = (1 - r e^{i(x-\phi)})^{-1} g^{-1/p} (r e^{i\phi})$$

and

$$J_2(e^{i\phi}) = - (1 - r^2 e^{i(x-\phi)})^{-1} (1 - r^3 e^{i(x-\phi)})^{-1} g^{-1/p} (r^2 e^{i\phi}).$$

**Proof.** Use the partial fraction expansion of $P_r$. If $z = r e^{i\phi}$, $z* = r e^{-i\phi}$, $\zeta = r^2 e^{i\phi}$, then

$$P_r(e^{i(x-\phi)}) (1 - \zeta e^{-i\phi})^{-1} = [(1 - z e^{-i\phi})^{-1} - (1 - \zeta e^{-i\phi})^{-1}] e^{i\phi} \cdot (z - \zeta)^{-1} + z e^{i\phi} (1 - z e^{i\phi})^{-1} (1 - \zeta e^{-i\phi})^{-1}.$$

Since (1.1) holds, necessarily

$$(1 - r^2)^{p-1} \int |J_1 + J_2|^{p} w d\sigma \leq K_1 (1 - r^2)^{p-1} \int |1 - r^2 e^{i(x-\phi)}|^{-p} d\psi,$$

which is $\leq K_2$ by (4.1). In addition, (4.4) guarantees that $(1 - r^2)^{p-1} \int |J_1|^{p} w d\sigma \leq K_3$, so by the Minkowski inequality $(1 - r^2)^{p-1} \int |J_2|^{p} w d\sigma \leq K_4$. But this implies that

$$r^p (1 - r^2)^{2p-1} \int |1 - re^{i(x-\phi)}|^p w(\phi) d\sigma(\phi) \leq K_4 |g(r e^{i\phi})|.$$

By invoking (4.2), (4.3) and replacing $r^2$ by $r$ we obtain

**Lemma 9.** Suppose $\mu \in \mathcal{B}_p$, $p > 1$. Then $d\mu = w d\sigma$ and

$$(5.3) \quad \int P_r(e^{i(x-\phi)}) w(\phi) d\sigma(\phi) \leq K_5 |g(r e^{i\phi})|$$

for all $r, \frac{1}{2} \leq r < 1$. 

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It is an open question whether Lemma 9 is valid if "P_p,r" is replaced by "P_r". Our preoccupation with the generalized Abel approx ids is, of course, in anticipation of inequality (5.3). With (5.3) we can wrap up the proof of Theorem 4. For, if \( r \geq \frac{1}{2} \)

\[
(*) \quad P_{p,r} \ast \left[ \left( (P_{p,r} \ast w) / w \right) \right]^{q - 1} \leq K_3 P_{p,r} \ast \left| (P_r \ast g) / w \right|^{q - 1}
\]

\[
(**) \quad = K_3 P_{p,r} \ast \left| g^{1-q} / (P_r \ast g^{1-q}) \right| .
\]

By Theorem 3 we know that

\[
\sup_r \int |P_r \ast f|^{q} \, w^{1-q} \, d\sigma \leq K \int |f|^q \, w^{1-q} \, d\sigma
\]

for all \( f \in \mathcal{F} \), thus this relation is true for all \( f \in \mathcal{F}_0 \). Thus from (4.4) with \( g \) replaced by \( g^{1-q} \) we deduce that (**) is \( \leq K_6 \). An application of (3.3) to (*) derives (5.2) for all \( h \) with \( |h| \leq 1/2 \). The inequality (5.2) is obviously true for \( h (\leq \pi) \) bounded away from 0, so the proof of Theorem 4 is complete.

6. Examples and concluding remarks. It is an easy matter to show that \( \mathcal{D}_p \subset \mathcal{B}_p \). Let \( f \in \mathcal{F} \). Then

\[
P_r \ast f = (1 - r) \sum_{\alpha=0}^{\infty} (D_{\alpha} \ast f) r^n,
\]

so, if \( \mu \in \mathcal{D}_p \)

\[
\| P_r \ast f \| \leq (1 - r) \sum_{\alpha=0}^{\infty} \| D_{\alpha} \ast f \| \, r^n
\]

\[
\leq K (1 - r) \sum_{\alpha=0}^{\infty} \| f \| \, r^n = K \| f \|,
\]

and thus \( \mu \in \mathcal{B}_p \).

Babenko [1] has shown that the measures \( w_\alpha(e^{i\phi}) \, d\sigma(\phi) = |\phi|^\alpha \, d\sigma, \quad -\pi < \phi \leq \pi, \quad -1 < \alpha < p - 1 \geq 0 \) are in \( \mathcal{D}_p \). Thus they are also in \( \mathcal{B}_p \).

The following theorem indicates how certain types of local conditions on \( w \) are sufficient to guarantee that \( \mu \in \mathcal{D}_p \).

**Theorem 5.** Let \( w \in L^1(d\sigma) \) and suppose that for each point \( \phi_0 \in \mathcal{C} \) there exists some measure \( v (e^{i\phi}) \, d\sigma(\phi) \in \mathcal{B}_p \) and constants \( \delta_1, \delta_2 \) such that

\[
0 < \delta_1 v(e^{i\phi}) \leq w(e^{i\phi}) \leq \delta_2 v(e^{i\phi})
\]

for all \( \phi \) in a neighborhood of \( \phi_0 \). Then \( w d\sigma \in \mathcal{B}_p \).

**Proof.** This follows from Theorem 4 and the Borel-Lebesgue theorem. It is clear that if (5.2) holds for all sufficiently small \( h > 0 \), and since \( w^{1-q} \in L^1(d\sigma) \), necessarily (5.2) holds for all \( h, 0 < h \leq \pi \).

When applying Theorem 5 the functions \( w_\alpha \) of above or any positive constant function are eligible \( v \)'s.
It should be noted that if \( f \in L^2(\mu) \) where \( \mu \in \mathcal{B}_p \), then

\[
\int |f| \, d\sigma = \int |f| \, w^{1/p} w^{-1/p} \, d\sigma \\
\leq \|f\|_p \cdot \left( \int w^{1-q} \, d\sigma \right)^{1/q} < \infty,
\]

so \( f \in L^1(\sigma) \). Thus the Fourier coefficients and Abel means which are obtained by completion as described in the introduction coincide with the Fourier coefficients and Abel means for \( L^1(\sigma) \) functions. The story is different for \( \mu \in \mathcal{B}_p \). Any \( f \in H^p(\mu), \mu \in \mathcal{B}_p \), is holomorphic in \( |z| < 1 \), in fact \( f \cdot g^{1/p} \in H^p(\sigma) \). Thus \( f \) is of Nevanlinna class \([11, p. 271]\). Thus the Fourier coefficients and Abel means are those of Fourier power series. Furthermore \( f \) need not be in \( L^1(\sigma) \).

**Theorem 6.** The functions \( w_n(e^{i\phi}) = |\phi|^\alpha, \, -\pi < \phi \leq \pi, \, \alpha > -1 \) are such that \( w_n \, d\sigma \in \mathcal{B}_p \).

**Proof.** If \( -1 < \alpha < n - 1 > 0 \), \( w_n \in \mathcal{B}_n \subset \mathcal{B}_p = \mathcal{B}_p \).

Local conditions similar to those of Theorem 5 can be imposed on \( w \) that are sufficient to guarantee that \( \mu \in \mathcal{B}_p \).

As observed in the introduction \( T \) is bounded and has a bounded inverse. This is so because \( \|f\|_p \) and \( \|f\|_p \) can be viewed as being equivalent norms for \( L^p(d\mu) \). We next show that if \( T \) is an isometry then necessarily \( \mu \) is a multiple of Lebesgue measure. This is a consequence of the following

**Theorem 7.** Let \( p > 0 \). Suppose that for some \( r, \, 0 \leq r < 1 \)

(6.1)

\[
\|P_r \ast f\|_p \leq \|f\|_p
\]

for all \( f \in \mathcal{B}_0 \). Then \( \mu \) is a multiple of Lebesgue measure.

**Proof.** Normalize \( \mu \) so \( \int d\mu = 1 \). Assume (6.1) holds for some \( p > 0 \). Then it holds for all \( p > 0 \) and upon letting \( p \to 0 \) we obtain

\[
\exp \int \log |f(re^{i\phi})| \, d\mu(\phi) \leq \exp \int \log |f| \, d\mu(\phi)
\]

so

\[
\int (\log |f|) \cdot (P_r \ast d\mu) \, d\sigma \leq \int \log |f| \, d\mu(\phi)
\]

for all \( f \in \mathcal{B}_0 \). By putting \( f(z) = \exp \pm (1 + se^{iz})/(1 - se^{-iz}), \, 0 \leq s < 1 \) we see that

\[
P_s \ast (P_r \ast d\mu) = P_s \ast d\mu,
\]

so

\[
P_{ss} \ast d\mu = P_s \ast d\mu,
\]

and by comparing Fourier expansions we see that \( \mu = \sigma \).

In closing we suggest that the following problems merit further study:

(i) Find representation theorems for the elements of \( \mathcal{B}_p \) and \( \mathcal{B}_p \).
(ii) Solve problems A and B for wider classes of approximate identities;
(iii) Characterize the measures $\mu$ for which $f \to \sup_{r} |P_{r} * f|$ is a bounded operation.
(iv) Extend the results to several variables and more general groups.

REFERENCES


UNIVERSITY OF VIRGINIA,
CHARLOTTESVILLE, VIRGINIA