ON COMMUTATIVE ALGEBRAS OF DEGREE TWO

BY

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Let \( \mathfrak{A} \) be a simple, commutative, power-associative algebra of degree 2 over an algebraically closed field \( \mathbb{F} \) of characteristic not equal to 2, 3 or 5. The degree of \( \mathfrak{A} \) is defined to be the number of elements in the maximal set of pairwise orthogonal idempotents in \( \mathfrak{A} \). This algebra has a unit element \( 1 \) [1, Theorem 3]. The algebras \( \mathfrak{A} \) of characteristic zero were considered by Kokoris [8] and found to be Jordan algebras. Kokoris also gave examples of algebras \( \mathfrak{A} \) that were not Jordan [6]. This left the problem of determining those algebras \( \mathfrak{A} \) that are not Jordan algebras.

Since \( 1 = e + f \) where \( e \) and \( f \) are primitive orthogonal idempotents, we have a decomposition \( \mathfrak{A} = \mathfrak{A}(1) + \mathfrak{A}(1/2) + \mathfrak{A}(0) \) where \( x \in \mathfrak{A}(\lambda) \) if and only if \( ex = e. \) We have \( \mathfrak{A}(\lambda) = \mathfrak{A}(1-\lambda); \mathfrak{A}(\lambda)\mathfrak{A}(1/2) \subseteq \mathfrak{A}(1-\lambda) + \mathfrak{A}(1/2) \) for \( \lambda = 1,0; \) and \( \mathfrak{A}(1) = e\mathfrak{F} + \mathfrak{N}_1, \mathfrak{A}(0) = f\mathfrak{F} + \mathfrak{N}_0 \) where \( \mathfrak{N}_1 \) and \( \mathfrak{N}_0 \) are nilideals of \( \mathfrak{A}(1) \) and \( \mathfrak{A}(0) \) respectively. If \( \mathfrak{A}(\lambda)\mathfrak{A}(1/2) \subseteq \mathfrak{A}(1/2) \) for \( \lambda = 1,0 \) we say that \( e \) is a stable idempotent. If \( \mathfrak{A}(\lambda)\mathfrak{A}(1/2) \subseteq \mathfrak{A}(1/2) + \mathfrak{N}_{1-\lambda} \) for \( \lambda = 1,0 \) we say that \( e \) is a nilstable idempotent.

The results of Albert extend the characteristic zero case to include algebras of characteristic \( p \neq 2,3,5 \) for which every idempotent is stable [2]. He also characterized those algebras of characteristic \( p \neq 2,3,5 \) that have at least one stable idempotent [3; 4]. Recently Kokoris announced [9] that every simple, flexible, power-associative algebra over an algebraically closed field of characteristic \( \neq 2,3 \) that is of degree two and in which every idempotent is nilstable is a J-simple algebra.

It is the purpose of this paper to fill in the remaining gap by giving a characterization of those algebras \( \mathfrak{A} \) that have an idempotent that is not nilstable. An example is also given of an algebra \( \mathfrak{A} \) that does not have a stable idempotent.

1. Let \( \mathfrak{A} \) be an algebra that is simple, commutative, power-associative, of degree two and whose base field \( \mathbb{F} \) is an algebraically closed field of characteristic \( p \neq 2,3,5. \) Let \( e \) be a primitive idempotent of \( \mathfrak{A} \) that is not nilstable. Since \( \mathfrak{A} \) is power-associative we have \( x^2x^2 = x^4 \) for all \( x \in \mathfrak{A} \) and the linearization of this identity

\[
P(x,y,s,t) = 4(xy)(st) + 4(xs)(yt) + 4_xt(y)s_y)
\]

\[
(1) \quad - x\left[y(xts) + t(xys) + s(xt)ight] - y\left[x(yst) + s(xyt) + t(xys)ight] - s\left[y(xyt) + y(xts) + t(xys)ight] - t\left[x(yst) + y(xts) + s(xt)ight] = 0.
\]

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We will use \( \mathcal{G} \) to represent the space \( \mathbb{A}(1) + \mathbb{A}(0) \), \( a_j \) to represent the \( \mathbb{A}(\lambda) \)-component of \( a \), \( a_{10} \) to represent the \( \mathbb{C} \)-component of \( a \), and \( z \) to represent \( e-f \). We will make frequent use of some of the results of Albert on commutative power-associative algebras; namely, results (5), (6), (7), (8) of [1]. We state them as

\[
(2) \quad \left[ g(xy)_{1/2} \right]_{1/2} = \left[ \left( gx_{\lambda} \right)_{1/2} y_{\lambda} \right]_{1/2} + \left[ \left( gy_{\lambda} \right)_{1/2} x_{\lambda} \right]_{1/2},
\]

\[
(3) \quad \left[ g(xy)_{1-\lambda} \right]_{1-\lambda} = 2 \left[ \left( gx_{\lambda} \right)_{1/2} y_{1-\lambda} \right]_{1-\lambda} + 2 \left[ \left( gy_{\lambda} \right)_{1/2} x_{1-\lambda} \right]_{1-\lambda},
\]

\[
(4) \quad \left[ \left( gx_{\lambda} \right)_{1/2} y_{1-\lambda} \right]_{1/2} = \left[ \left( gy_{\lambda} \right)_{1/2} x_{1-\lambda} \right]_{1/2},
\]

\[
(5) \quad \left( gx_{\lambda} \right)_{1-\lambda} y_{1-\lambda} = 2 \left[ \left( gy_{1-\lambda} \right)_{1/2} x_{1-\lambda} \right]_{1-\lambda},
\]

where \( \lambda = 1, 0; g \in \mathbb{A}(1/2) \) and \( x \) and \( y \) are in \( \mathcal{G} \).

Two other relations

\[
(6) \quad 2 \left[ \left( x_{1/2} g \right)_{1/2} \right]_1 + \left[ \left( x_{1/2} g \right)_{1-\lambda} g \right]_1 = x_{1/2} g, \]

\[
(7) \quad \left( x_{1} g \right)_{1/2} = \left( x_{0} g \right)_{1/2} \quad \text{implies} \quad \left( x_{1/2} g \right)_{1/2} = \left( x_{0} g \right)_{1/2}
\]

for \( x \) and \( g \) as above will be useful. The first of these is obtained from \( P(x, e, g, g) = 0 \) while the second can be derived from (2) and (4).

**Theorem 1.** \( \mathcal{C} \) is an associative subalgebra of \( \mathbb{A} \) with an element \( c \in \mathcal{C} \) such that there is a \( \omega \in \mathbb{A}(1/2) \) with \( z(c\omega) = 1 \), \( (c_1\omega)_{1/2} = (c_{0\omega})_{1/2} \) and \( (c_1^2\omega)_0 = -2c_0 \).

**Proof.** It is easily seen that the subset \( \mathcal{I} \) of \( \mathbb{A}(1) \) consisting of all elements of the form \( (a_0g)_1 \) is an ideal of \( \mathbb{A}(1) \) where \( g \in \mathbb{A}(1/2) \) and \( a_0 \) is a fixed element of \( \mathbb{A}(0) \) because by (5) we have

\[
\left[ a_1 (a_0 g)_1 \right]_1 = 2 \left[ a_0 (b_1 g)_1 \right]_1.
\]

The additive property of an ideal is immediate.

We now let \( b_1, d_1 \) be elements of \( \mathbb{A}(1) \), \( g \in \mathbb{A}(1/2) \) and \( a_0 \in \mathbb{A}(0) \) with \( (a_0 g)_1 = a_1 \). If we consider only the \( \mathbb{A}(1) \)-components of each of the terms in \( P(b_1, d_1, g, a_0) = 0 \) we get

\[
2(b_1 d_1) a_1 = b_1 (d_1 a_1) + d_1 (b_1 a_1).
\]

If \( b_1 \) is also in \( \mathcal{I} \) we can interchange \( a_1 \) and \( b_1 \) to get

\[
a_1 (d_1 b_1) = 2b_1 (d_1 a_1) - d_1 (b_1 a_1).
\]

Therefore

\[
a_1 (d_1 b_1) = (a_1 d_1) b_1.
\]

Hence \( \mathcal{I} \) is associative.

It has been shown [1, Lemma 11] that if \( (a_0 g)_1 \in \mathcal{I} \) for all \( a_0 \in \mathbb{A}(0) \) and \( g \in \mathbb{A}(1/2) \) then \( (a_1 g)_0 \in \mathbb{A}(0) \) for all \( a_1 \in \mathbb{A}(1) \) and \( g \in \mathbb{A}(1/2) \). From this result and the assumption that \( e \) is not nilstable we can conclude that there is an element \( c_0 \in \mathbb{A}(0) \) and an element \( g \) in \( \mathbb{A}(1/2) \) such that \( (c_0 g)_1 \) is nonsingular. If \( b_1 \) is the inverse of \( (c_0 g)_1 \) in \( \mathbb{A}(1) \) then \( c_0 (2b_1 g)_1 = b_1 (c_0 g)_1 = e \). We may also conclude that \( \mathbb{A}(1) = \mathcal{I} \) is associative. In a similar manner we obtain the result that \( \mathbb{A}(0) \) is associative.

If we take \( c_0 \in \mathbb{A}(0) \) and \( \omega \in \mathbb{A}(1/2) \) such that \( (c_0 \omega)_{1} = e \) and let \( 2c_1 = (c_0^2 \omega)_{1/2} = [4(c_0(c_0 \omega))_{1/2}] \), then we can quote the results of Kokoris [7, Lemma 4 and Identity 29] that \( (c_1 \omega)_{1/2} = -f \) or \( 0 \) and \( (c_1 \omega)_{1/2} = (c_0 \omega)_{1/2} \). No generality will be lost if we also assume that \( c_0 \) is nilpotent because \( \mathbb{A}(0) = f \mathcal{G} + \mathbb{N}_0 \) and
(c_0 w)_i = [(c_0 + c_0 w)_i] for any \( x \in \mathbb{R} \). To complete the proof of the theorem it remains only to show that \( (c_1 w)_0 \neq 0 \). We assume that \( (c_1 w)_0 = 0 \). If we examine the \( \mathfrak{H}_l(1) \)-components of the terms of the relation \( P(c_0, c_0, w, w) = 0 \) we get
\[
8[(c_0 w)^2 + 8[(c_0 w)^2]_i] = 4[c_0 [w(w c_0)_1]_1]_1 + 2[w(c_0 w)_{1/2}]_1 + 4[w(c_0 w)_{1/2}]_1.
\]
Using this relation together with (2), (6), (7) and \( (c_0 w)_1 = e \), we get
\[
6e + 8[(c_1 w)^2]_i = 2w^2 c_1^2 - 2w(c_1^2 w)_{0}.
\]
But \( (c_1^2 w)_0 = 4[c_1(c_1^2 w)_{1/2}]_0 = 4[c_1(c_0 w)_{1/2}]_0 = 2c_0(c_1 w)_0 = 0 \). Therefore either \( [(c_1 w)^2]_{1/2} \) or \( w^2 c_1^2 \) must be nonsingular. If we again use (1) with \( P(c_1, c_1, w, w) = 0 \) and examine the \( \mathfrak{H}_l(0) \)-components of the resulting terms we get
\[
8[(c_0 w)^2]_0 = 2c_0^2 w^2.
\]
But then \( [(c_0 w)^2]_{1/2} \) is nilpotent. Since \( (c_0 w)^2]_{1/2} = \alpha 1 + n \) where \( n \in \mathfrak{H}_l + \mathfrak{R}_0 \) [1, Lemma 10] we must also have \( [(c_1 w)^2]_{1/2} \), nilpotent. Now by (6) we have
\[
2[(c_0 w)_{1/2}]_1 = 2[(c_1 w)_{1/2}]_1 = - [(c_1 w)_{0} w]_1 + c_1 w^2 = c_1 w^2.
\]
But \( 2[(c_0 w)_{1/2}]_1 = - [(c_0 w)_{1/2}]_1 + c_0 w^2 = c_0 w^2 \) is nilpotent. Therefore \( c_1 w^2 \) and \( c_1^2 w^2 \) are nilpotent. We have arrived at a contradiction. Hence \( (c_1 w)_0 = -f \) and the theorem is proved.

**Theorem 2.** There is an isomorphism \( T \) between \( \mathfrak{H}_l(1) \) and \( \mathfrak{H}_l(0) \) such that for \( b_1 \in \mathfrak{H}_l(1) \), \( T(b_1) \) is the unique element of \( \mathfrak{H}_l(0) \) satisfying \( (c_1 w)_{1/2} = [(T(b_1) w)_{1/2} \). The subset \( \mathcal{B} \subset \mathbb{C} \) of all elements of the form \( b_1 + T(b_1) \) is an associative subalgebra of \( \mathbb{C} \) isomorphic to both \( \mathfrak{H}_l(0) \) and \( \mathfrak{H}_l(1) \).

**Proof.** We use \( c_1, c_0 \) and \( w \) as in Theorem 1. If we consider only the \( \mathfrak{H}_l(1/2) \)-components of the terms in \( P(c_0, b_1, w, w) = 0 \) we get
\[
8[(c_0 w)_{1/2} (b_1 w)_0]_{1/2} + 4(b_1 w)_{1/2} = 2[w(b_1 + [(b_1 w)_{1/2} c_0])_{1/2}]_1 + 2[w(c_0 w)_{1/2} b_1 + (b_1 w)_0 c_0]_{1/2} + 2[c_0 [w(w b_1)_{0}]_{1/2}]_{1/2} + (b_1 w)_{1/2}.
\]
Using (5) and (2) on the terms \( [(c_0 w)_{1/2} b_1]_0 \), \( [(b_1 w)_{1/2} c_0]_1 \), and \( c_0 [w(w b_1)_{0}]_{1/2} \) this relation reduces to
\[
[(c_0 w)_{1/2} (b_1 w)_0]_{1/2} = \{w[(c_0 w)_{1/2} b_1]_0 \} \{w[(b_1 w)_{1/2} c_0]_1 \} \{c_0 [w(w b_1)_{0}]_{1/2} \}.
\]
We now consider the \( \mathfrak{H}_l(1/2) \)-component of each term in \( P(c_1, b_1, w, w) = 0 \). We have
\[
-4(b_1 w)_{1/2} + 8[(c_1 w)_{1/2} (b_1 w)_0]_{1/2} = 2[w((c_1 b_1) w)_{1/2} + (b_1 c_1) w)_{1/2} + b_1 (c_1 w)_{1/2}, w]_{0} \}_{1/2} = (b_1 w)_{1/2} = 2[c_1 [w(w b_1)_{0}]_{1/2} b_1, w]_{1/2}.
\]
This relation together with (2) and (4) gives us
\[
2[(c_1 w)_{1/2} (b_1 w)_0]_{1/2} = (b_1 w)_{1/2} + \{w[(c_1 b_1) w]_0 \}_{1/2}. \text{ But } [(c_0 w)_{1/2} (b_1 w)_0]_{1/2} = \{w[(c_0 w)_{1/2} b_1]_0 \} \quad \text{and } (c_0 w)_{1/2} = (c_1 w)_{1/2}. \text{ Therefore } (b_1 w)_{1/2} = \{2[(c_1 w)_{1/2} b_1]_0 - [(c_1 b_1) w]_0 \} = -2[(c_1 w)_{1/2} b_1]_0. \text{ We can now define } T(b_1) = -2[(b_1 w)_{1/2} c_1]_0 \text{ to be the element } b_0 \text{ in } \mathfrak{H}_l(0) \text{ such that } (b_0 w)_{1/2} = (b_0 w)_{1/2}. \text{ To show that } T \text{ is well-defined we assume } a_0 = -a_0 = 0 \text{ by (5). Therefore } (b_0 w)_{1/2} = (b_0 w)_{1/2} \text{ implies } b_0 = b_0. \text{ Simply by changing the signs of } c_1 \text{ and } c_0 \text{ and interchanging } 1 \text{ and } 0 \text{ we can get a similar result for } \mathfrak{H}_l(0); \text{ i.e., for every } b_0 \in \mathfrak{H}_l(0) \text{ there is a unique } b_1 = 2([(b_0 w)_{1/2} c_1]_0 \} \text{ such that } (b_0 w)_{1/2} = (b_1 w)_{1/2}. \text{ Therefore } T \text{ is onto } \mathfrak{H}_l(0) \text{ and is a } 1-1 \text{ correspondence between } \mathfrak{H}_l(1) \text{ and } \mathfrak{H}_l(0).
Now if $a$ and $b$ are elements of $\mathcal{B}$ as defined in the theorem we have, with the help of (2) and (4), that

$$[w(a_1,b_1)]_{1/2} = [(wb_1)a_1 + (wa_1)b_1]_{1/2}$$

$$= [(wb_0)a_1 + (wa_0)b_1]_{1/2} = [(wa_1)b_0 + (wb_1)a_0]_{1/2}$$

$$= [(wa_0)b_0 + (wb_0)a_0]_{1/2} = [w(a_0,b_0)]_{1/2}.$$

Therefore $T(a_1,b_1) = a_0b_0$ and $ab = a_1b_1 + a_0b_0 \in \mathcal{B}$. Clearly $\mathcal{B}$ is closed under addition and scalar multiplication.

Define $S(b) = be$ for every $b \in \mathcal{B}$. It follows immediately from the above results that $S$ is a 1–1 correspondence of $\mathcal{B}$ onto $\mathcal{A}_e(1)$. From the definition we have $S(ab) = (ab)e = (ae)(be) = S(a)S(b)$ and $S(a + b) = S(a) + S(b)$ for all $a$ and $b$ in $\mathcal{B}$. Therefore $\mathcal{B}$ and $\mathcal{A}_e(1)$ are isomorphic as rings and hence as algebras. In the same manner we show that $\mathcal{B}$ is isomorphic to $\mathcal{A}_e(0)$. We have shown also that $T$ is an isomorphism. The associativity of $\mathcal{B}$ follows from that of $\mathcal{C}$.

From the definition of $\mathcal{B}$ it is clear that $c = c_1 + c_0$ is in $\mathcal{B}$. From $P(w,w,w,z) = 0$ it follows that $w^2$ is in $\mathcal{B}$. Theorem 2 also implies that $\mathcal{C} = \mathcal{B} + \mathcal{B}z$.

**Theorem 3.** The mapping $b \mapsto D(b) = (bw)z$ is a derivation of $\mathcal{B}$ into $\mathcal{B}$ such that $D(c) = 1$.

**Proof.** Let $a$ and $b$ be arbitrary elements of $\mathcal{B}$. Then $[(ab)w]_{1/0} = [(ab)_1w]_0 + [(ab)_0w]_1 = [(a_1b_1w)_0 + [(a_0b_0w)_1 = 2[a_1(b_1w)]_{1/2} + [b_1(a_1w)]_{1/2} + 2[a_0(b_0w)]_{1/2} + [b_0(a_0w)]_{1/2} + a_1(b_0w)_1 + b_1(a_0w)_1 + a_0(b_1w)_1 + b_0(a_1w)_1 = b_0(axw)_{1/2} + a_0(bxw)_{1/2} + bx(a_0w)_{1/2} + ax(b_0w)_{1/2} = b(axw)_{1/2} + a(bxw)_{1/2}$ by (3), (5) and the definition of $\mathcal{B}$. If this relation is multiplied by $z$ we have $D(ab) = aD(b) + bD(a)$ and $D$ is a derivation on $\mathcal{B}$ into $\mathcal{B}$.

To show that $D(b)$ lies in $\mathcal{B}$ for $b = b_1 + b_0$, an element of $\mathcal{B}$, we need several identities; the first of which is obtained from $P(b_0,w,w,c_1) = 0$. We get $8[(b_0w)_1(wc_1)_1]_0 + 8[(b_0w)_1(wc_1)_1]_0 = 2(b_0c_0)w^2 + 2[(c_1(b_0w)_1]_0$ after the usual simplifications using (2), (5), (6) and $(c_1w)_{1/2} = (c_0w)_{1/2} - 8[(b_0w]^2/(wc_1)_1]_0 = 2(b_0c_0)w^2 + 2[(c_1(b_0w)_1]_0$ after the usual simplifications. We consider $P(b_0,w,w,c_0) = 0$ next to get

$$-3(b_0c_0)w^2 + 8[(b_0w)_1/(wc_1)_1]_0 + 6[(b_0w)_1(c_0w)_{1/2}]_0$$

$$= -2[(b_0c_0w)_{1/2}]_0.$$
By successively applying to this relation the three identities above in the order we obtained them we get
\[ 6T[(b_0w)_0] = -12[(b_0w_1)(c_1w_1)_,0] - 24[(b_0w_1)(c_0w_2)_,0] + 6c_1b_0w_2^2 \]
\[ = 4[(b_0c_0w_1)_1/2w_0] - 8[(b_0w_1)(c_0w_2)_,0] = -6(b_1w_0). \]
Therefore we have
\[ D(b)z = (bw)_10z = (b_0w)_1 - (b_1w)_0 \in \mathfrak{B}. \]
The fact that \( D(c) = 1 \) follows immediately from the definition of \( c \).

**Theorem 4.** If \( a \) and \( b \) are elements of \( \mathfrak{B} \) then \( [(wa)_1/2b]_1/2 = [w(ab)]_1/2, \]
\[ [(wa)_1/2b]_10 = (wb)_10a \quad \text{and} \quad (wa)_1/2(wb)_1/2 \in \mathfrak{B}. \]

**Proof.** By (2) and (4) and the definition of \( \mathfrak{B} \) we have
\[ w(ab)_1/2 = 2[(wa)_1/2b]_1/2 + 2[(wb)_1/2a]_1/2 = [(wa)_1/2b]_1/2 + [(wb)_1/2a]_1/2. \]
By (5) we have
\[ [(wa)_1/2b]_10 = 2[(wa)_1/2b]_1/2 + 2[(wa)_1/2b]_1/2 = (wb)_10a + (wb)_10a = (wb)_10a. \]
Now use \( P(w, w, a, b) \) to get
\[ 4w^2ab + 8(wa)(wb) = 2w[(ab)w + (aw)b + (bw)a] + a[w^2b + 2w(wb)] + b[w^2a + 2w(wa)]. \]

**Corollary.** If \( a \in \mathfrak{C} \) and \( b \in \mathfrak{B} \) then \( [(wa)_1/2b]_1/2 = [w(ab)]_1/2. \)

**Proof.** We can write \( a = a' + a''z \) where \( a' \) and \( a'' \) are in \( \mathfrak{B} \). Since
\[ [(a''z)w]_1/2 = [(a''zw)]_1/2 = 0 \]
we have
\[ [(wa)_1/2b]_1/2 = [(wa')_1/2b]_1/2 = [w(ab)]_1/2. \]

We now define \( \mathfrak{G} \) to be the set of all \( g \in \mathfrak{U}(1/2) \) such that \( (gc)_10 \) is in \( \mathfrak{B} \).

**Theorem 5.** \( \mathfrak{U}(1/2) \) is the direct sum of the two subspaces \( (w\mathfrak{B})_1/2 \) and \( \mathfrak{G} \). Moreover \( (\mathfrak{G}a)_1/2 \subseteq \mathfrak{G} \), \( (\mathfrak{G}(az))_1/2 \subseteq (w\mathfrak{B})_1/2 \), and \( [(w\mathfrak{B})_1/2(az)]_1/2 \subseteq \mathfrak{G} \), for all \( a \in \mathfrak{B} \).

**Proof.** If \( g \) is any element of \( \mathfrak{U}(1/2) \), let \( (gc)_10 = a + a'z \) where \( a \) and \( a' \) are in \( \mathfrak{B} \). Since
\[ [(a'w)_1/2c]_10 = a'z \]
we have
\[ [(g - (a'w)_1/2)c]_10 = a[g - (a'w)_1/2] \]
equal to the sum of an element of \( \mathfrak{G} \) and an element of \( (w\mathfrak{B})_1/2 \). If \( h \) lies in both \( (w\mathfrak{B})_1/2 \) and \( \mathfrak{G} \) then \( (hc)_10 \) lies in \( \mathfrak{B}z \) and \( \mathfrak{B} \). Hence \( (hc)_10 = 0 \). But
\[ [(wa)_1/2c]_10 = az. \]
Therefore if \( h = (wa)_1/2 \) then \( a = (w)_1/2 = 0 \) and \( h = 0 \). Hence \( \mathfrak{U}(1/2) \) is the direct sum of \( \mathfrak{G} \) and \( (w\mathfrak{B})_1/2 \).

Since \( D(c)^2 = 2c \), the \( \mathfrak{U}(1/2) \)-components of the terms obtained from
\[ P(c, c, w, g) = 0 \quad \text{with} \quad g \in \mathfrak{G} \]
yield the relation
\[ 8[(cw)_1/2(cg)_10]_1/2 = 2c[(w(cg)_10)_1/2 + w(c^2g)_10 + 2w[c(cg)_10]_1/2 + 2w[c(cg)_1/2 + 6g(cz)]_1/2]. \]
Using this relation, Theorem 4 and the property that \((c_1 g)_{10} \in \mathfrak{B}\) it is easily seen that \([g(c_2)]_{1/2} \in (w\mathfrak{B})_{1/2}\). Therefore \([g(c_2)]_{1/2} \in \mathfrak{B}\). But

\[
[g(c_2)]_{1/2} = \left[ (g_{c_1})_{1/2} c_1 - (g_{c_0})_{1/2} c_0 + (g_{c_1})_{1/2} c_0 \right]_{10}
\]

\[
= [(1/4)(c_1^2 g) - (1/4)(c_0^2 g) - (1/2)(g_{c_1})_{10} c_0 + (1/2)(g_{c_0})_{1/2} c_1]_{10}
\]

\[
= -(1/4)(c_2^2 g)_{10} z + (1/2)(c_2)_{10} c_{10}.
\]

Therefore since \((c_1 g)_{10} \in \mathfrak{B}\) we also have \((c_1^2 g)_{10} \in \mathfrak{B}\) and \((c_2 g)_{1/2} \in \mathfrak{B}\). Therefore \((c_1^2 g)_{1/2} \in \mathfrak{B}\) is in \(\mathfrak{B}\). We now examine the \(\mathfrak{A}(1/2)\)-components of the terms resulting from \(P(a_1, c_1 w, g) = 0\). With the help of (3) and (4) we get

\[
[2(a_1 w)_{1/2} (c_1 g)_{1/2} + 2(a_1 w)_{1/2} (c_1 g)_{0} + 2(c_1 w)_{1/2} (a_1 g)_{0} + 2(c_1 w)_{0} (a_1 g)_{1/2}]_{1/2}
\]

\[
= \{w[(a_1 c_1) g]_{0} + g[(a_1 c_1 w)]_{1/2}\}_{1/2}.
\]

Interchanging the subscripts 1 and 0 we obtain

\[
[2(a_0 w)_{1/2} (c_0 g)_{1/2} + 2(a_0 w)_{1/2} (c_0 g)_{0} + 2(c_0 w)_{1/2} (a_0 g)_{0} + 2(c_0 w)_{0} (a_0 g)_{1/2}]_{1/2}
\]

\[
= \{w[(a_0 c_0) g]_{1} + g[(a_0 c_0 w)]_{1/2}\}_{1/2}.
\]

But

\[
\{g[(a_0 c_0 w)]_{1/2}\}_{1/2} = \{2g[(a_0 c_0 w)_{1/2} + c_0 (a_0 w)_{1/2}]_{1/2}\}_{1/2}
\]

\[
= \{2g[(a_0 (c_1 w)_{1/2} + g[(a_1 (c_0 w)]_{1/2}\}_{1/2}
\]

\[
= \{g[c_1 (a_0 w)]_{1/2}\}_{1/2}.
\]

Therefore

\[
[2(a_0 w)_{1/2} (c_0 g)_{1/2} + 2(a_0 w)_{1/2} (c_0 g)_{0} + 2(c_0 w)_{1/2} (a_0 g)_{0} + 2(c_0 w)_{0} (a_0 g)_{1/2}]_{1/2}
\]

\[
= \{g[c_1 (a_0 w)]_{1/2} + w[(a_0 c_0) g]_{1} + [a_1 g]_{1/2}\}_{1/2}.
\]

Again consider only the \(\mathfrak{A}(1/2)\)-components of the terms resulting from \(P(a_0, c_1 w, g) = 0\). This relation together with (2), (3) and (4) gives us

\[
[2(a_0 w)_{1/2} (c_1 g)_{0} + 2(a_0 w)_{1/2} (c_0 g)_{1/2} + 2(a_0 g)_{1} (c_1 w)_{1/2} + 2(c_1 w)_{0} (a_0 g)_{1/2}]_{1/2}
\]

\[
= \{g[c_0 (a_1 w)]_{0} + g[c_0 (a_1 g)]_{1/2} + w[c_0 (a_1 g)]_{1}\}_{1/2}.
\]

Interchanging the subscripts 0 and 1 in (10) we obtain

\[
[2(a_1 w)_{1/2} (c_0 g)_{1} + 2(a_1 w)_{0} (c_0 g)_{1/2} + 2(a_1 g)_{0} (c_0 w)_{1/2} + 2(c_0 w)_{1} (a_1 g)_{1/2}]_{1/2}
\]

\[
= \{g[c_0 (a_1 w)]_{1/2} + g[c_0 (a_1 g)]_{0} + w[c_0 (a_1 g)]_{0}\}_{1/2}.
\]

We now subtract the sum of identities (10) and (11) from the sum of the identities (8) and (9) and use the facts that \((a_1 w)_{1/2} = (a_0 w)_{1/2}, (c_1 w)_{0} = -f\) and \((c_0 w)_{1} = e\). We have \(2[(a_2 w)]_{0} (c_2 g)_{1/2} + (a_0 g) - 2(a_1 g) - g[(a_1 c_1) w]_{0} + g[c_0 (a_1 w)]_{1/2}\) is in \((w\mathfrak{B})_{1/2}\). Therefore
\[-2D(a)[(cz)g]_{1/2} + (a_0g) - 2(a_1g) - 2g[(a_1c_1w)_{1/2}]_0
\]
\[-2g[c_1(a_1w)]_{1/2} + 2g[a_1(c_0w)]_{1/2} \cdot 1/2
\]
\[= -2D(a)[(cz)g]_{1/2} + (a_0g) - 2(a_1g)
\]
\[= -2D(a)[(cz)g]_{1/2} - 2(a_1g)_{1/2}
\]
\[is \in \mathfrak{B}. \text{Since } [(cz)g]_{1/2} \in (w\mathfrak{B})_{1/2} \text{ we have } [(az)g]_{1/2} \in (w\mathfrak{B})_{1/2}. \text{ To show that}
\]
\[(ga)_{1/2} \in \mathfrak{G} \text{ for } a \in \mathfrak{B} \text{ we consider}
\]
\[[(ga)_{1/2}c]_{10} = [(ga_1)_{1/2}c_1 + (ga_0)_{1/2}c_0 + (ga_1)_{1/2}c_0]_{10}
\]
\[= 2[(ga_0)_{1/2}c_1 + [g(az)]_{1/2}c_1 + 2(ga_1)_{1/2}c_0 - [g(az)]_{1/2}c_0]_{10}
\]
\[= [(gc_0)_{1/2}c_0 + (gc_0)_{1/2}c_1 + g(az)_{1/2}(cz)]_{10}
\]
\[= (gc)_{10}a + [(g(az)]_{1/2}(cz)]_{10}.
\]
Since \((gc)_{10} \in \mathfrak{B} \) so is \((gc)_{10}a). \text{ Also since } [(az)g]_{1/2} \in \mathfrak{B}_{1/2} \text{ we have}
\[[(ga)_{1/2}(cz)]_{10} \in \mathfrak{B}.
\]
Hence \([(ga)_{1/2}c]_{10} \in \mathfrak{B} \text{ and } (ga)_{1/2} \in \mathfrak{G}. \text{ Finally if we take } a, b \text{ and } h \in \mathfrak{B}
\]
we have \{[(wa_0)_{1/2}(bz)]_{1/2}h_1 \cdot 0 = [(wa_0)_{1/2}b_1]_{1/2}h_1 \cdot 0
\]
\[= [(wb_1)_{1/2}a_0]_{1/2}h_1 - (1/4)(wh_1)_{1/2}a_0b_0 \cdot 0 = [(wb_0)_{1/2}a_0]_{1/2}h_1 - (1/4)(wh_1)_{1/2}a_0b_0 \cdot 0
\]
\[= (1/4)(wb_1)_{1/2}a_0b_0 - (1/4)(wh_1)_{1/2}a_0b_0 = 0. \text{ Similarly } [(wa_1)_{1/2}(b)z]_{1/2}h_1 = 0. \text{ By}
\]
\[taking h = c \text{ we can see that the } (w\mathfrak{B})_{1/2} \text{ component of } [(wa)_{1/2}(b)z]_{1/2} \text{ is 0.}
\]
Hence \([(wa)_{1/2}(b)z]_{1/2} \in \mathfrak{G}.
\]
THEOREM 6. \([(w\mathfrak{B})_{1/2}(\mathfrak{B}z)]_{1/2} = 0.
\]
Proof. Let a be a nilpotent element of \(\mathfrak{A}_c(1). \text{ There exists a } \lambda \in \mathfrak{F} \text{ such that}
\[d = a + \lambda c \text{ has the property that } (d_0w)_1 \text{ is a nonsingular element } b_1 \text{ of } \mathfrak{A}_c(1).\text{Then}
\]
\[d(2b_1^{-1}w)_{1/2} = b_1^{-1}(dw)_{1/2} = e. \text{ If we let } b \text{ be the unique element of } \mathfrak{B} \text{ whose}
\]
\(\mathfrak{A}_c(1)-\text{component is } b_1 \text{ we have by the isomorphism established in Theorem 2 that}
\]
\[d(b^{-1}w)_{1/2} = b^{-1}D(d)z = z. \text{ For these elements } d \in \mathcal{C} \text{ and } (wb^{-1})_{1/2} \in \mathfrak{A}_c(1/2)
\]
we get a \(\mathfrak{B} \subseteq \mathcal{C} \text{ such that } \mathfrak{B} + \mathfrak{B}z = \mathcal{C} \text{ and where } \mathfrak{B} \text{ has the properties described}
\text{ for } \mathfrak{B} \text{ in Theorems 2–5. Let } t + sz \in \mathfrak{B}z \text{ where } t \text{ and } s \in \mathfrak{B}. \text{ We have}
\]
\[[(wb^{-1})_{1/2}(t + sz)]_{1/2} = 0. \text{ Therefore } (wb^{-1})_{1/2} + [(wb^{-1})_{1/2}(sz)]_{1/2} = 0. \text{ Since}
\]
\[[(wb^{-1})_{1/2}(sz)]_{1/2} \in \mathfrak{G} \text{ we must have } (wb^{-1})_{1/2} = 0 \text{ and } b^{-1} t = 0. \text{ Therefore}
\]
\[t = 0 \text{ and } \mathfrak{B}z \subseteq \mathfrak{B}z. \text{ If } \mathfrak{B}z \text{ is a proper subset of } \mathfrak{B}z \text{ then } \mathfrak{B} \text{ is a proper subset of } \mathfrak{B}.
\text{ But this would imply that } \mathfrak{B} + \mathfrak{B}z \text{ is a proper subset of } \mathcal{C} \text{ which is a contradiction.}
\text{ Therefore we must have } \mathfrak{B}z = \mathfrak{B}z \text{ and } [(wb^{-1})_{1/2}(\mathfrak{B}z)]_{1/2} = 0. \text{ Now let } \mathfrak{G} \text{ be the}
\text{ subset of } \mathfrak{B} \text{ of all elements } s \text{ such that } [(ws)_{1/2}(\mathfrak{B}z)]_{1/2} = 0. \text{ Let } x, y \in \mathfrak{B}. \text{ The re-}
lation \( P(y, x, w, z) = 0 \) yields \( ([wx](yz)]_{1/2} + [(wy)(xz)]_{1/2} = 0. \) Let \( t \in \mathcal{B} \), \( s \) and \( s' \in \mathcal{G} \). Then we get \( ([w(ss')]_{1/2}(tz)]_{1/2} = - ([w(t)_{1/2}(ss')z] = 0 \) from \( P(tw, s, s'z) = 0. \) Hence \( \mathcal{G} \) is a subalgebra of \( \mathcal{B} \). If we let \( b_{-1} = \alpha + \sigma \) where \( b \) is as described above and \( n \) is a nilpotent element of \( \mathcal{B} \) and \( \alpha \in \mathcal{B} \), then \( n \in \mathcal{G} \) and hence every power of \( n \) is in \( \mathcal{G} \). But \( b \) is the sum of a multiple of the identity and a linear combination of powers of \( n \). Hence \( b = \lambda + D(a) \in \mathcal{G} \) and the derivative of every element of \( \mathcal{B} \) is in \( \mathcal{G} \). Now \( a \in \mathcal{B} \) implies \( a = D(ca) - cD(a) \). Since \( D(ca) \), \( c = (1/2)D(c^2) \) and \( D(a) \) are in \( \mathcal{G} \) we have \( \mathcal{B} \subseteq \mathcal{G} \) and \( [(wB)_{1/2}(Bz)]_{1/2} = 0 \).

At this point we have obtained partial results on the multiplications of \( \mathcal{A} \). However, the chief remaining gap in the characterization of \( \mathcal{A} \) lies with the products involving elements of \( \mathcal{G} \). To facilitate the determination of these products we shall introduce some symbols \( Q_g \), \( \phi_g \), \( k_g \), \( f_g \), and \( h_g \) on \( \mathcal{B} \) into \( \mathcal{B} \) for every \( g \in \mathcal{G} \) by letting

\[
(12) \quad [g(bz)]_{1/2} = [wQ_g(b)]_{1/2},
\]
\[
(13) \quad (gb)_{10} = h_g(b) + k_g(b)z,
\]
\[
(14) \quad [g(wb)]_{1/2} = f_g(b) + \phi_g(b)z
\]

for every \( b \in \mathcal{B} \). In our subscripts we abbreviate \((ga)_{1/2}\) to \( ga \).

From (2) and (3) and the definition of \( \mathcal{G} \) we have

\[
([ga]_{1/2}(bz)]_{1/2}c) = ([ga]_{1/2}b_1]_{1/2}c_0) - ([ga]_{1/2}b_0]_{1/2}c_0)
\]
\[
= \{2[[ga_1]_{1/2}b_1]_{1/2}c_0 + [(wQ_g(a)]_{1/2}b_1]_{1/2}c_0
\]
\[
- 2[[ga_1]_{1/2}b_0]_{1/2}c_0 - [(wQ_g(a)]_{1/2}b_1]_{1/2}c_0
\]
\[
= (1/2)(gc_0)a_1b_1 + (1/2)b_1Q_g(a) - (1/2)b_1Q_g(a)
\]
\[
- 2\{[(gb_0)_{1/2}a_1]_{1/2}c_0\}
\]
\[
= (1/2)(gc_0)a_1b_1 - 2\{[(gb_1)_{1/2}a_1]_{1/2}c_0\}
\]
\[
+ 2\{[(wQ_g(b)]_{1/2}a_1]_{1/2}c_0\}
\]
\[
= a_1Q_g(b).
\]

Now \([ga]_{1/2}(bz)]_{1/2} = [wQ_{w}(b)]_{1/2}c\) and therefore \([wQ_{w}(b)]_{1/2}c)]_{10} = Q_{w}(b)z\). Hence

\[
Q_{w}(b) = aQ_g(b).
\]

Consider \( h_{w}(a) + k_{w}(a)z = ([gb]_{1/2}a_1]_{10} = ([gb]_{1/2}a_1]_{10} + (gb)_{1/2}a_0)_1 = 2[[gb_0]_{1/2}a_1]_{10} + [(wQ_g(b)]_{1/2}a_1]_{10} + 2[[gb_1]_{1/2}a_0]_1 - [(wQ_g(b)]_{1/2}a_0]_1 = b_0(ga_1)_0 + b_1(ga_0)_1 + Q_g(b)[(az)w]_{10} = bh_g(a) + bzk_g(a) - Q_g(b)D(a).\)

From this relation we obtain

\[
h_{w}(a) = bh_{w}(a) - Q_{w}(b)D(a),
\]
\[
k_{w}(a) = bk_{w}(a).
\]
We now consider the \( C \)-components of the terms of \( P(a, a, g, z) = 0 \). We have

\[ 3ahg(a)z + 3akg(a) - 5hga(a)z - 5kga(a) = Q_g(a)D(a)z - h_g(a^2)z - k_g(a^2) \]

If we equate \( B \)-components and \( Bz \)-components we have

\[
\begin{align*}
(18) & \quad k_g(a^2) = 2ak_g(a), \\
(19) & \quad h_g(a^2) = 2ah_g(a) - 4Q_g(a)D(a)
\end{align*}
\]

by using (16) and (17).

We have proved that \( k_g \) is a derivation for every \( g \in G \). We shall now prove that \( Q_g \) is a derivation for every \( g \in G \). We have

\[
[wQ_g(ab)]_{1/2} = [g(abz)]_{1/2} - [g(ab)w]_{1/2}
\]

\[
= [(ga_1)_{1/2}b_1 + (gb_1)_{1/2}a_1 - (ga_0)_{1/2}b_0 - (gb_0)_{1/2}a_0]_{1/2}
\]

\[
= [(ga_1)_{1/2}b_1 + (gb_0)_{1/2}a_1 + (wQ_g(b))_{1/2}a_1 - (ga_0)_{1/2}b_0
\]

\[
- (gb_1)_{1/2}a_0 + (wQ_g(b))_{1/2}a_0]_{1/2}
\]

\[
= [(ga_0)_{1/2}b_1 + (gb_0)_{1/2}a_1 + (wQ_g(b))_{1/2}a_1
\]

\[
- (ga_1)_{1/2}b_0 + (wQ_g(a))_{1/2}b_0 - (gb_1)_{1/2}a_0 + (wQ_g(b))_{1/2}a_0]_{1/2}
\]

\[
= [(ga_0)_{1/2}b_1 - (gb_1)_{1/2}a_0 + (gb_0)_{1/2}a_1 - (ga_1)_{1/2}b_0
\]

\[
+ wQ_g(ab) + wQ_g(b)a]_{1/2}.
\]

By (4) we have \( (wQ_g(ab))_{1/2} = [wQ_g(a)b + Q_g(b)a]_{1/2} \). Therefore

\[
(20) \quad Q_g(ab) = Q_g(a)b + Q_g(b)a.
\]

Next, we consider the \( B \)-components of the terms of \( P(g, a, bz, z) = 0 \) to get

\[
4[(ga)b]_{1/2} = [3g(ab) + (gb)a]_{1/2}.
\]

However

\[
[(ga)b]_{1/2} = [2(ga_0)b_1 + (wQ_g(a))b_1 + 2(ga_1)b_0 - (wQ_g(a))b_0]_{1/2}
\]

\[
= 2[(gb_1)a_0 + (gb_0)a_1]_{1/2}
\]

\[
= [(gb)b_0 + (wQ_g(b))b_0 + (gb)a_1 - (wQ_g(b))a_1]_{1/2}
\]

\[
= [(gb)a]_{1/2}.
\]

If we combine the above two relations we have

\[
(21) \quad [(ga)b]_{1/2} = [g(ab)]_{1/2}.
\]

A similar computation using \( P(w, w, a, z) = 0 \) and \( P((wa)_{1/2}, w, a, z) = 0 \) gives us

\[
(22) \quad w(wa)_{1/2} = w^2a + D^2(a)
\]

\[
(23) \quad (wa)_{2/2} = w^2a^2 + 2aD^2(a) - D(a)D(a).
\]

If we consider the \( (wB)_{1/2} \)-components of the terms of \( P(z, (aw)_{1/2}, w, g) = 0 \) we have \( [wQ_g(w^2a) + wQ_g(D^2(a)) + w(af_g(1)) + w\phi_g(a)]_{1/2} = 0 \). By letting \( a = 1 \) we get

\[
(24) \quad \phi_g(1) = -\frac{1}{2} Q_g(w^2).
\]
Therefore

\( \phi_g(a) = \frac{1}{2} a Q_g(w^2) - Q_g(a w^2) - Q_g(D^2(a)) \).

From (15) and (25) we have

\( \phi_{g(a)}(b) = a \phi_g(b) \).

We now wish to express \( h_g \) in terms of \( Q_g \) and \( D \). We examine the \( 2x \)-components of \( P(w, g, c, a) = 0 \) and use (21) and (26) to get

\[
3 \phi_g(c) a + 3 \phi_g(a) c + 3 h_g(a) + 3 D(h_g(c)) + 3 D(a) h_g(c) + 3 c D(h_g(a)) - 4 h_g(D(a)) = 3 \phi_g(1) c a + D(h_g(c a)) + h_g(a) + h_g(c) a + h_g(c) c + 3 \phi_g(c a) - 3 h_g(D(a)) - c h_g(D(a)) + \phi_g(D(a)).
\]

We simplify this relation using (25), (16) and the linearized form of (19) to get

\[
-3 Q_g(D^2(a)) c + 3 h_g(a) = -3 Q_g(D^2(c)) - Q_g(c) D^2(a) - 3 D(Q_g(c)) D(a) - 3 D(Q_g(a)) - 3 h_g(c) D(a) - 7 Q_g(D(a)).
\]

Since \( Q_g \) and \( D \) are derivations we have

\( \phi_{g(a)} = -3 D(Q_g(c)) D(a) - 3 D(Q_g(a)) - 3 h_g(c) D(a) - 4 Q_g(c) D^2(a) \).

If we let \( a = c \) in (27) we get \( h_g(c) = -D(Q_g(c)) \). Therefore (27) simplifies to

\( 3 h_g(a) = -3 D(Q_g(c)) D(a) - 3 D(Q_g(a)) - 4 Q_g(c) D^2(a) \).

We substitute the values obtained from (28) in \( h_g(a c) = c h_g(a) + a h_g(c) - 2 Q_g(a) - 2 Q_g(c) D(a) \), a linearized form of (19), to get

\( Q_g(a) = Q_g(c) D(a) \).

If we use this relation in (28) we obtain

\( h_g(a) = -D(Q_g(c)) D(a) - 2 Q_g(c) D^2(a) \).

We now investigate the behaviour of \( f_g \). Consider the \( 2x \)-components of the terms of \( P(w b 1/2, g, a, z) = 0 \). We have

\( 2 f_g(b) a = f_{g(a)}(b) + f_g(ab) - b D(k_g(a)) - b k_g(D(a)) - D(a) k_g(b) \)

and when \( b = 1 \)

\( 2 f_g(1) a = f_{g(1)}(1) + f_g(a) - D(k_g(a)) - k_g(D(a)) \).

We define a new mapping \( T_g \) on \( \mathcal{B} \) into \( \mathcal{B} \) for each \( g \) by

\( T_g(a) + f_g(1) a - f_{g(1)}(1) + D(k_g(a)) \).

This definition together with (32) gives us \( f_g(a) = f_g(1) a + T_g(a) + k_g(D(a)) \) and \( f_{g(1)} = f_g(1) a + T_g(a) + D(k_g(a)) \). Now \( f_{g(a)} b = -f_g(ab) + 2 f_g(b) a + b D(k_g + k_g D(a)) + k_g(b) D(a) \) and \( f_g(b) = -f_{g(1)}(1) a + 2 f_{g(1)}(1) b + a D(k_g + k_g D(b)) + D(a) k_g(b) \) by (31) and (32). Substituting the values for \( f_g(ab), f_g(b), f_{g(1)}(1) \) and \( f_{g(1)}(1) \) expressed in terms of \( T_g \) in these relations and simplifying we have
(34) \[ T_g(ab) = T_g(a)b + T_g(b)a \]

and

(35) \[ f_g(b) = f_g(1)ab + T_g(b)a - bT_g(a) + ak_g(D(b)) + bD(k_g(a)) - k_g(a)D(b). \]

It follows readily that

(36) \[ T_g(b) = aT_g(b) - D(b)k_g(a). \]

We have already shown that \( \phi_g(a) = Q_g(c)\left[\frac{1}{2}aD(w^2) - D(w^2a) - D(a)\right] \). We also have that \( P(g,g,(aw)_{1/2};z) = 0 \) implies \( [g\phi_g(a)]_{1/2} = 0 \). If we let \( a = c^3 \) we have \( \phi_g(c^3) = Q_g(c)\left[-\frac{1}{2}c^3D(w^2) - 3c^2D(w^2) - 6\right] \). Since the second factor on the right-hand side is nonsingular we have \( [gQ_g(c)]_{1/2} = 0 \). Multiplying by \( cz \) and considering the \((w^{3/2})_{1/2}\)-component we get

(37) \[ Q_g(c)^2 = 0. \]

Similarly we have

(39) \[ Q_g(c)k_g(a) = 0. \]

Now consider the element \( w' = [w - wD(Q_g(c))]_{1/2} + g \) of \( \mathcal{G}_{1/2} \). We have \( (c_2w')_0 = -f \). By Theorem 1 and its proof, \( c_2 - (1/2)(c_2w')_0 \) is an element \( c \) in \( \mathcal{C} \) such that \( (aw')z = 1 \). Also \( (c_2w')_0 = -2c_0 - 4(Q_g(c))_0 \). Therefore \( (aw')z = \left[\left[c + Q_g(c) - Q_g(c)z\right](aw')\right]_0 = 1 - 2D(Q_g(c))^2 - 2D(Q_g(c))^2z - 2Q_g(c)D^2(Q_g(c))z + k_g(Q_g(c)) - 2Q_g(c)D^2(Q_g(c)) + k_g(Q_g(c))z \). Simple properties of derivations and the fact that \( Q_g(c)^2 = 0 \) gives us \( (aw')z = 1 + k_g(Q_g(c)) + k_g(Q_g(c))z \). Therefore

(40) \[ k_g(Q_g(c)) = 0. \]

We also have from (35) and (36) that

(41) \[ T_g(Q_g(c)) = f_g(1)Q_g(c) \text{ and } T_g(b)Q_g(c) = 0 \]

for every \( b \in \mathcal{B} \).

For \( w' \) and \( c' = c + Q_g(c) - Q_g(c)z \) we have a corresponding \( \mathcal{B}' \) and \( \mathcal{B}'z \) as described in Theorem 2. To determine these two subspaces we let \( a + bz \) be an element of \( \mathcal{C} \) with \( a, b \in \mathcal{B} \) and such that the \( 1/2 \)-component of \( w'(a + bz) \) is 0. We obtain \( wa - wD(Q_g(c))a + ga + wQ_g(b))_{1/2} = 0 \). Therefore \( a[1 - D(Q_g(c))] = -Q_g(c)D(b) \). Solving for \( a \) we have \( a = -D(b)Q_g(c) \). Since \( \mathcal{B}' + \mathcal{B}'z = \mathcal{C} \), we can conclude from the above result that \( \mathcal{B}'z \) consists of all elements of the form \( a - Q_g(a)z \). We note that the \( \mathcal{C} \)-component of the element \( (a - Q_g(a)z)w' \) must be an element of \( \mathcal{B}'z \) by Theorem 3. If we calculate this element we obtain

\[
D(a)z - D(a)D(Q_g(c))z + Q_g(c)D^2(a) + k_g(a)z - D(Q_g(a))D(Q_g(c)) + D(Q_g(c))^2z - D(a)z.
\]

In order for this element to be in \( \mathcal{B}'z \) we must have \( Q_g(c)D^2(a) + D(Q_g(c))^2D(a) = Q_g(c)D[D(a) - D(a)D(Q_g(c)) + k_g(a) + D(Q_g(c))^2D(a)] \) by the definition of \( \mathcal{B}'z \). Therefore
We also have

\[ Q_g(c)k_t(b) = 0 \]

for any \( t \in \mathcal{G} \) and any \( b \in \mathcal{B} \) since

\begin{align*}
- k_t(Q_g(c))b &= - k_t(Q_g(c)) - k_t(Q_g(c))b = - k_t(Q_g(c))b = 0.
\end{align*}

We define \( t' \) to be the 1/2-component of

\[ w[- D(Q_g(c))D(Q_g(c)) + Q_g(c)D^2(Q_g(c)) - k_t(Q_g(c))] + t \]

for \( t \in \mathcal{G} \). Then the \( \mathcal{C} \)-component of \( (c + Q_g(c) - Q_g(c)z)t' \) is

\[ -D(Q_t(c)) - D(Q_t(c))D(Q_g(c)) - 2Q_t(c)D^2(Q_g(c)) = 0 \]

since \( Q_t(c)D^2(Q_g(c)) + 2D(Q_g(c))D(Q_t(c)) + Q_g(c)D^2(Q_t(c)) = 0 \) and

\[ 2D(Q_g(c))D(Q_g(c))D(Q_g(c)) = - Q_t(c)D^2(Q_g(c))D(Q_g(c)) = 3Q_g(c)Q_g(c)D^3(Q_g(c)) = 0. \]

Hence \( t' \) is in \( \mathcal{G}' \). We now compute \( D' \) and \( Q' \). We have simply that

\begin{align*}
D' &: a - Q_g(a)z \to D(a) - D(Q_g(c))D(a) + D(Q_g(c))^2D(a) + k_g(a) \\
Q' &: c + Q_g(c) - Q_g(c)z \to Q_t(c) + Q_t(c)D(Q_g(c)) - Q_g(c)D(Q_t(c))z.
\end{align*}

Therefore

\[ D'Q' = c + Q_g(c) - Q_g(c)z \to D(Q_t(c)) + D(Q_g(c))D(Q_g(c)) \\
&+ Q_t(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_t(c)) + k_t(Q_t(c)) - Q_g(c)D^2(Q_g(c))z. \]

By (30) and (44) we have

\[ D(Q_t(c)) + D(Q_t(c))D(Q_g(c)) + Q^2(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_t(c)) + k_t(Q_t(c)) \\
= D(Q_t(c)) + D(Q_t(c))D(Q_g(c)) + 2Q_t(c)D^2(Q_g(c)) - k_t(Q_t(c)). \]

Therefore \( Q_t(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_t(c)) = 2Q_t(c)D^2(Q_g(c)) \) and

\[ Q_t(c)D^2(Q_g(c)) = - D(Q_g(c))D(Q_t(c)). \]

Replacing \( t \) by \((ct)_{1/2}\) we have \( cQ_t(c)D^2(Q_g(c)) = -cD(Q_t(c))D(Q_g(c)) - Q_g(c)D(Q_t(c)) \) and therefore

\[ Q_t(c)D(Q_g(c)) = 0. \]

We now examine the \( \mathcal{B} \)-components of the terms of \( P(g, t, a, z) = 0 \) for \( g, t \in \mathcal{G} \) and \( a \in \mathcal{B} \). We have

\[ m(1, a) + m(a, 1) = 2m(1, 1)a + 2D(Q_t(c))D(D(Q_g(c))D(a)) \\
+ 2D(Q_g(c))D(D(Q_t(c))D(a)) + (k_gk_t + k_tk_g)(a) \]
where \( m(a, b) \) denotes the \( \mathfrak{B} \)-component of \((ga)_{1/2} \cdot (tb)_{1/2}\). Since \( m(a, b) \) does depend on \( g \) and \( t \) also, we will use \( m_a(a, b) \) for \( m(a, b) \) when there is any chance of confusion. Replacing \( t \) by \((tb)_{1/2}\) in (49) we obtain

\[
m(1, ab) + m(a, b) = 2m(1, b)a + 2bD(Q_g(c))D(D(Q_g(c))D(a))
\]

(50)

\[+ 2bD(Q_g(c))D(D(Q_g(c))D(a)) + 2D(Q_g(c))D(b)D(a)
\]

\[+ k_g(b)k_t(a) + b(k_gk_t + k_tk_g)(a).\]

Define

\[
S_g(a) = m(1, a) - m(1, 1)a - 2D(Q_g(c))D(D(Q_g(c))D(a)) - k_gk_t(a)
\]

(51)

for all \( a \in \mathfrak{B} \). If \( g = t \) the right-hand side of (51) reduces to identity (49) with \( g = t \).

Therefore \( S_g \) is identically zero. A simple linearization gives us

\[
S_g = -S_tg.
\]

Substituting (51) into (50) and letting \( a = b \) we have

\[
m(a, a) + k_gk_t(a^2) = 2S_g(a)a + m(1, 1)a^2 + 4aL_qL_r(a) + 2ak_gk_t(a)
\]

where \( L_q = D(Q_g(c))D \) and \( L_r = D(Q_t(c))D \) are derivations. Interchanging \( g \) and \( t \) in this result and subtracting gives us

\[
2S_g(a^2) + 2L_qL_r(a^2) - 2L_rL_q(a^2) + (k_gk_t - k_tk_g)(a^2)
\]

\[= 4S_g(a)a + 4a(L_qL_r - L_rL_q)(a) + 2a(k_gk_t - k_tk_g)(a).\]

Since both \( L_qL_r - L_rL_q \) and \( k_gk_t - k_tk_g \) are derivations this relation reduces to

\[
S_g(a^2) = 2aS_g(a).
\]

Hence \( S_g \) is a derivation of \( \mathfrak{B} \) into \( \mathfrak{B} \).

We can now replace (50) by

\[
m(a, b) = m(1, 1)ab + aS_g(b) - bS_g(a) + 2aL_qL_r(b) + 2bL_rL_q(a)
\]

(53)

\[- 2L_q(a)L_r(b) + ak_gk_t(b) + bk_tk_g(a) - k_gk_t(a)k_t(b).\]

By setting \( g = t, a = 1 \) and \( b = Q_g(c) \) in (53) we have

\[
m_{g, t}(1, 1)Q_g(c) = 0.
\]

An examination of the \((w\mathfrak{B})_{1/2}\)-components of the terms of \( P(g, g, g, z) = 0 \) gives us

\[
Q_g(c)D(m_{g, t}(1, 1)) = 0.
\]

Finally we compute \( P((ga)_{1/2}, (tb)_{1/2}, w, z) = 0 \) to get

\[
n_{g, t}(a, b) = -aQ_g(f_g(1)b - T_t(b) + D(k_t(b)) - bQ_t(f_g(1)a - T_g(a) + D(k_g(a))
\]

(56)

where \( n_{g, t}(a, b) \) is the \( \mathfrak{B} \)-component of \((ga)_{1/2} \cdot (tb)_{1/2}\). Now \( P(g, g, (wa)_{1/2}, z) = 0 \) Therefore \( n_{g, t}(1, 1)a + 2Q_g(f_g(1)a) + 2Q_g(T_g(a)) = 0 \). From (56) with \( g = t \) and \( a = b = 1 \) we have

\[
Q_g(T_g(a)) = -Q_g(a)f_g(1).
\]

2. In the previous section we expressed the multiplications of \( \mathfrak{A} \) in terms of constants and derivations. In this section we use these multiplicative properties to construct a simple power-associative algebra of degree two from an associative algebra.
Let \( B \) be an associative, commutative algebra over a field \( \mathbb{F} \) of characteristic \( p > 5 \). Also assume that \( B \) has a single nonzero idempotent \( 1 \) that is a unity quantity.

Let \( B_0, \ldots, B_{n-1} \) be \( n \) homomorphic images of the vector space \( B \). We let \( \mathcal{L} \) be a sum of these \( n \) vector spaces, but not necessarily the vector space direct sum. We let \( zB \) be a one-dimensional module over \( B \). Clearly \( zB \) is a vector space over \( \mathbb{F} \) and we form the vector space direct sum \( \mathcal{A} = B + \mathcal{L} + zB \). We now extend the multiplication of \( B \) to \( \mathcal{A} \) in such a way that \( \mathcal{A} \) remains a commutative, power-associative algebra. First we define

\[
(za)(zb) = (zb)(za) = ab,
\]

\[
1x = x,
\]

\[
zy = 0
\]

for every \( a \) and \( b \) in \( B \), every \( x \) in \( \mathcal{A} \) and every \( y \) in \( \mathcal{L} \). The element \( e = (1/2)(1 + z) \) is an idempotent. We have already defined sufficient multiplicative properties to determine an idempotent decomposition of \( \mathcal{A} \). Clearly \( \mathcal{L} \subseteq \mathcal{A}_s(1/2) \) and \( B + Bz \subseteq \mathcal{A}_s(1) + \mathcal{A}_s(0) \). The second part of this statement follows by consideration of \( a + bz = (c + cz) + (d - dz) \) with \( 2c = a + b \) and \( 2d = a - b \). For each of the vector spaces \( B_i \) and the corresponding homomorphism of \( B \) onto \( B_i \) we define \( (g_i b)_{1/2} \) to be the image of \( b \). Since this notation is consistent with that of the decomposition of \( \mathcal{A} \) with respect to \( e \) we will allow the confusion of the two notations.

In order to complete our definitions of the multiplications of \( \mathcal{A} \) we choose elements \( b_{ij} \) and \( b_i \) of \( B \) and derivations \( D_{ij} \) and \( D_i \) on \( B \) into \( B \) for \( i, j = 0, 1, \ldots, n - 1 \) with the following restrictions:

\[
D_{ij} = -D_{ji}, \quad b_{ij} = b_{ji}, \quad b_0 = 0
\]

for all values of \( i \) and \( j \) and

\[
b_i b_j = (b_i + b_j) b_{ij} = 0, \quad b_i D_0(b_j) = (b_i + b_j) D_0(b_{ij}) = D_j(b_j b) + D_j(b_i b) = 0,
\]

\[
b_j D_0(b_i) + b_i D_0(b_j) = b_j D_i(b) = 0,
\]

\[
(b_i g_j + b_j g_i)_{1/2} = 0, \quad b_i b_0 D_0 = -b_i D_0 D_{0j}
\]

for all \( i \) and \( j \) different from 0 and all \( b \in B \). We now define

\[
(g_i a)_{1/2} b = [g_i(ab)]_{1/2} - D_0(ab) D_0(b) - 2b_0 a D_0^2(b) + a D_i(b) z,
\]

\[
(g_i a)_{1/2} (bz) = -[(g_i a)_{1/2} b] z + [g_i [D_0(b) b_i]]_{1/2},
\]

\[
(g_i a)_{1/2} (g_j b)_{1/2} = abb_i + a D_{ij}(b) - b D_{ij}(a) + a D_j D_i(b) + b D_j D_i(a)
\]

\[
- D_j(b) D_i(a) + 2a L_j L_i(b) + 2b L_j L_i(a) - 2L_j(b) L_i(a)
\]

\[
+ a b_i [D_0(b) - b_0 b - D_0 D_i(b)] z \cdot b j D_0 [D_0(a) - b_0 a - D_0 D_i(a)] z
\]
where \( L_i = D_0(b_j)D_0 \), \( i,j = 0, \ldots, n-1 \), and \( a \) and \( b \in \mathcal{B} \). Since we did not restrict \( L \) to be a direct sum of subspaces it is necessary to assume that our multiplications in \( \mathfrak{A} \), as defined above, are well-defined. We place two additional assumptions on \( \mathfrak{A} \). If \( \mathfrak{D} \) is the set of derivations consisting of \( D_i \) and \( D_{ij} \) for all \( i \) and \( j \) we assume, in the terminology of Albert [3], that \( \mathcal{B} \) is \( \mathfrak{D} \)-simple; i.e., there is no nontrivial ideal \( \mathfrak{I} \) of \( \mathcal{B} \) such that \( \mathfrak{I} \) is \( \mathfrak{D} \)-admissible. The second assumption is that for every element \( g \) in \( \mathfrak{L} \) there is a \( t \) in \( \mathfrak{L} \) such that \( gt \) is not zero.

**Theorem 7.** Every commutative, power-associative, simple algebra of degree two over an algebraically closed field \( \mathfrak{F} \) of characteristic \( p \neq 2,3,5 \) is an algebra of the type described above.

**Proof.** We choose a set of elements \( g_1, \ldots, g_{n-1} \) in \( \mathfrak{G} \) such that every element of \( \mathfrak{G} \) is expressible in the form \( \sum (g_ia_i)_{1/2} \) where \( a_i \in \mathcal{B} \). We translate the notation of §1 to the notation of this section by letting \( \mathfrak{L} = \mathfrak{A}_{e}(1/2) \), \( g_0 = w \), \( D_0 = D \), \( b_{00} = w^2 \), \( b_{0i} = f_{gi}(1) \), \( D_{0i} = T_{gi} \), \( D_i = k_{gi} \), \( b_i = Q_{gi}(c) \), \( b_{ij} = m_{gi,j}(1,1) \) and \( D_{ij} = S_{gi,j} \) where \( i,j \neq 0 \). Identities (25)-(57) give us the relations (61)-(65).

If \( \mathfrak{I} \) is a nontrivial ideal of \( \mathfrak{B} \) that is \( \mathfrak{D} \)-admissible then if \( a \in \mathfrak{I} \) we have \( Q_{g}(a), f_{g}(b), \phi_{g}(a), \phi_{g}(b), f_{g}(a)m_{g,a}(a,b) \) and \( n_{g,a}(a,b) \in \mathfrak{I} \). This is sufficient to guarantee that \( \mathfrak{I} + \mathfrak{I} + (w\mathfrak{I})_{1/2} + (w\mathfrak{I})_{1/2} \) is a proper ideal of \( \mathfrak{A} \). Since this contradicts the simplicity of \( \mathfrak{A} \) we have that \( \mathfrak{B} \) is \( \mathfrak{D} \)-simple.

Let \( (wa)_{1/2} + g \) be an element of \( \mathfrak{A}_{e}(1/2) \) such that there is no element \( t \) in \( \mathfrak{A}_{e}(1/2) \) such that \( (wa)_{1/2} + gt \neq 0 \). Choosing \( t \) to be successively \( w, (wc)_{1/2} \) and \( (wc^2)_{1/2} \) and considering only the \( \mathfrak{B} \)-components of the resulting terms we have \( w^2a + D^2(a) + f_{g}(1) = w^2ac + cD^2(a) - D(a) + f_{g}(1)c + T_{g}(c) = w^2ac^2 + cD^2(a) + 2a - 2cD(a) + f_{g}(1)c^2 + 2T_{g}(c) = 0 \). Eliminating \( w^2 \) from these equations we have \( -D(a) + T_{g}(c) = 2a - cD(a) + cT_{g}(c) = 0 \). Hence \( a = 0 \) and \( f_{g}(1) = T_{g}(c) = 0 \). If we multiply \( g \) by \( (wb)_{1/2} \) for \( b \in \mathfrak{B} \) we have \( f_{g}(b) = \phi_{g}(b) = 0 \) by our assumption on \( g \). By a previous result we had that \( Q_{g}(c) \) was a multiple of \( \phi_{g}(c^3) \). Hence \( Q_{g}(c) = 0 \). Now \( f_{g}(b) = T_{g}(b) + k_{g}(D(b)) = 0 \) for all \( b \in \mathfrak{B} \). If we substitute \( bc \) for \( b \) we have \( cT_{g}(b) + c\phi_{g}(D(b)) + k_{g}(b) = 0 \). Therefore \( k_{g}(b) = 0 \). We now have that \( \mathfrak{g} = \{ (ag)_{1/2} : a \in \mathfrak{B} \} \). With this choice of \( g \) and for any \( b \in \mathfrak{B} \) we have \( f_{g}(b) = 0 \) by (35) and \( \phi_{g}(b) = 0 \) since \( Q_{g}(c) = aQ_{g}(c) \). Also \( m_{g,a}(a,b) = aS_{g,a}(b) - bS_{g,a}(a) \). But by the assumption on \( g \) and (51) we have \( S_{g,a} = 0 \). Therefore \( m_{g,a}(a,b) = 0 \) for all \( a \) and \( b \in \mathfrak{B} \). Combining this result with (56) we have \( (ga)_{1/2}t = 0 \) for all \( a \in \mathfrak{B} \) and all \( t \in \mathfrak{A}_{e}(1/2) \). Therefore the ideal generated by \( g \) is \( \{ (ag)_{1/2} : a \in \mathfrak{B} \} \). This contradicts the assumption of simplicity of \( \mathfrak{A} \). Hence for each \( x \in \mathfrak{A}_{e}(1/2) \) there is an element \( t \) in \( \mathfrak{A}_{e}(1/2) \) such that \( xt \neq 0 \).

**Theorem 8.** An algebra \( \mathfrak{A} \) over a field \( \mathfrak{F} \) of characteristic \( p \neq 2,3,5 \) as described in identities (58)-(65) is a commutative, power-associative, simple algebra.

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Proof. It follows readily from the definition of $\mathfrak{A}$ that $\mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2}$ is a subalgebra of $\mathfrak{A}$. We shall show that this subalgebra is power-associative by examining $P(x, y, s, t)$ for various values in $\mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2}$. If $P(x, y, s, t) = 0$ for all possible choices of the variables $x, y, s$ and $t$ in $\mathfrak{B}$, $\mathfrak{B}z$ or $(g_0\mathfrak{B})_{1/2}$ we have $\mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2}$ power-associative. We examine the powers of $x = a + g_0$ for $a \in \mathfrak{B}$. We have $x^2 = a^2 + b_{00} + (ag_0)_{1/2} + 2D_0(a)z$, $x^3 = a^3 + 2ab_{00} - D_0^2(a) + 5aD_0(a)z + D_0(b_{00})z + [(2a^2 + b_{00})g_0]_{1/2}$ and $x^2x^2 = x^3x$. The proof of this result depends on the properties

\begin{align*}
(a(bz))z &= (ab)z, \\
(az)(bz) &= ab, \\
(bz)(g_0a)_{1/2} &= -aD_0(b), \\
b(g_0a)_{1/2} &= [(ab)g_0]_{1/2} + aD_0(b)z, \\
(g_0a)_{1/2}(g_0b)_{1/2} &= abb_{00} + aD_0^2(b) + bD_0^2(a) - D_0(a)D_0(b).
\end{align*}

(66)

If $d \in \mathfrak{B}$ and if we replace $D_0$ by $dD_0$, $b_{00}$ by $b_{00}d^2 + 2dD_0(d) - D_0(d)^2$ and $g_0$ by $(g_0d)_{1/2}$ we see that relations similar to those expressed in (66) hold. Therefore we can conclude that $a + (g_0d)_{1/2}$ has a unique fourth power.

Next we investigate the fourth powers of $x = az + g_0$. We have $x^2 = a^2 + b_{00} - 2D_0(a)$, $x^3 = a^3 + b_{00}az + D_0(b_{00})z - 2D_0^2(a)z + a^2 + b_{00} - [2D_0(a)g_0]_{1/2}$ and $x^2x^2 = x^3x$. Again the only multiplicative properties used were those expressed in (66). Therefore $az + (g_0b)_{1/2}$ has a unique fourth power for all $a$ and $b \in \mathfrak{B}$. It is easily seen that $\mathfrak{B} + \mathfrak{B}z$ is associative. Hence $a + bz$ has a unique fourth power. The assumption on the characteristic and simple linearizations of these three fourth powers we have obtained give us the result that $P(x, y, s, t) = 0$ provided that in any evaluation the four values $x, y, s$, and $t$ are chosen from only two of the three subspaces $\mathfrak{B}$, $\mathfrak{B}z$ and $(g_0\mathfrak{B})_{1/2}$. This leaves us those choices of $x, y, s$ and $t$ for which $x \in \mathfrak{B}$, $y \in \mathfrak{B}z$, $s \in (g_0\mathfrak{B})_{1/2}$ and $t$ is arbitrary. Because of the linearization process we need only consider $P(a, bz, (g_0d)_{1/2}, a)$, $P(a, bz, (g_0d)_{1/2}, bz)$ and $P(a, bz, (g_0d)_{1/2}, (g_0d)_{1/2})$. Straightforward computations, which we omit, show that each of these relations is zero. Therefore $\mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2}$ is power-associative.

Now let $g = \sum (g_0a_i)_{1/2}$ where $a_i \in \mathfrak{B}$. The index $i$, or indices $i$ and $j$, of this summation and all subsequent ones will run from 1 to $n - 1$. Define

\begin{align*}
b_g &= \sum a_i b_i, \\
D_g &= \sum a_i D_i, \\
b_{0g} &= \sum a_i b_{0i} - \sum D_0(a_i) + \sum D_0 D_i(a_i), \\
D_{0g} &= \sum a_i D_{0i} - \sum D_i(a_i)D_0, \\
b_{gg} &= \sum b_{ij}a_i a_j + 2 \sum a_i D_{ij}(a_j) + 4 \sum a_i L_i L_i(a_i) \\
&\quad - \sum D_i(a_i)D_j(a_j).
\end{align*}

(67)
From (62) and (67) we have
\[ b_\varphi^2 = b_\varphi b_{\varphi \varphi} = b_\varphi D_0(b_{\varphi \varphi}) = b_\varphi D_\varphi(b) = D_\varphi(b_\varphi) = b_\varphi D_\varphi D_\varphi(b) = 0. \] (68)
\[ b_\varphi b_{\varphi 0} D_0(a) = -b_\varphi D_0 D_{\varphi 0}(a), \]
\[ (g b_\varphi)_{1/2} = 0. \]

From (65) we have \((g a)_{1/2}(g a)_{1/2} = b_{\varphi \varphi} + 2a D_\varphi^2(a) - D_\varphi(a)^2 + 4 \sum a_i L_i a_j L_j(a) - 2 \sum a_i L_i(a) a_j L_j(a). \) Now \( \sum a_i L_i(a) = \sum a D_0(b_i) D_0(a) = D_0(b_\varphi) D_0(a) - \sum b_i D_0(a_i) D_0(a). \) Therefore \( \sum a_i L_i(a) a_j L_j(a) = L_\varphi(a)^2 \) where \( L_\varphi = D_0(b_\varphi) D_0. \) Also
\[ \sum a_i L_i a_j L_j(a) = \sum L_i a_j L_j(a) - \sum b_i D_0(a_i) D_\varphi a_j L_j(a) \]
\[ = L_\varphi^2(a) - \sum D_0(b_i) D_0(b_j) a_i D_0(a_j) D_0(a) - \sum b_i D_0^2(b_i) a_j D_0(a_i) D_0(a) \]
\[ = L_\varphi^2(a). \]

Therefore
\[ (g a)^2_{1/2} = b_{\varphi \varphi} + 2a D_\varphi^2(a) - D_\varphi(a)^2 + 4a L_\varphi^2(a) - 2L_\varphi(a)^2. \] (69)

We also have
\[ b(g a)_{1/2} = g(a b)_{1/2} - D_0(ab_\varphi) D_0(b) - 2b_\varphi a D_0^2(b) + a D_\varphi(b)z, \]
\[ (b z)(g a)_{1/2} = g(a D_0(b) b_\varphi)_{1/2} - [(g a)_{1/2} b]z \]
for all \( a \) and \( b \) in \( \mathfrak{B}. \)

We now let \( g_0 = g_0 + g \) and \( a' = a - b_\varphi D_0(a)z \) for \( a \in \mathfrak{B}. \) We define a derivation \( D'_0(a') = [D_0(a) + D_0(b_\varphi)^2 D_0(a) + D_\varphi(a)] \) and let \( t = b_{00} + 2b_\varphi - b_\varphi D_0(b_{00})z + b_{\varphi \varphi} - 2b_\varphi D_0(b_{\varphi \varphi})z. \) Now \( (D_0 + D_0(b_\varphi)^2 D_0 + D_\varphi)^2 = (D + D_\varphi)^2 + 2L_\varphi^2. \) Therefore
\[ a' D'_0^2(a') = a (D_0 + D_\varphi)^2(a) + 2a L_\varphi^2(a) - b_\varphi [a D_0^2(a) + a D_0 D_\varphi(a) + D_0(a) D_\varphi^2(a)]z \]
since \( 3b_\varphi D_0 L_\varphi^2(a) = 3 b_\varphi D_0^3(b_\varphi) D_0(a) = -3 D_0(b_\varphi) D_0(b_\varphi) D_\varphi^2(b_\varphi) D_0(a) = 2 D_0(b_\varphi) b_\varphi D_0^3(b_\varphi) D_0(a) = 0. \) Also \([D_0 + D_0(b_\varphi)^2 D_0 + D_\varphi(a)]^2 = [(D_0 + D_\varphi(a))^2 + 2L_\varphi^2 - 2b_\varphi D_0 D_\varphi D_0(a) D_0(a) z. \)

We have, using these results, that \( (g_0 a')_{1/2} = (g_0 a)_{1/2} = b_{00} a^2 + 2a D_0^2(a) - D_0(a)^2 + 2a b_{00} D_0(a) + 2a D_0 D_0(a) - D_\varphi(a) D_0(a) - ab_\varphi a D_0(b_{00})z - 2b_{000} D_0(a) b_\varphi a - 2ab_\varphi D_0^2(a)z + b_{\varphi \varphi} a^2 + 4a L_\varphi^2(a) - 2L_\varphi(a)^2 + 2a D_\varphi^2(a) - D_\varphi(a)^2 - 2b_\varphi D_0 D_\varphi(b_{00})^2 z - 2ab_\varphi b_{00} D_\varphi(a) + 2ab_\varphi D_0 D_0(b_{00})(a) - 2b_\varphi a D_0 D_\varphi D_0(a) = t(a^2)^2 + 2a D^2(a') - D'(a')^2 + 2b_\varphi b_{00} a D_0(a) + 2b_\varphi a D_0 D_0(b_{00}) D_0(a) = t(a^2)^2 + 2a D^2(a') - D'(a')^2. \) Since \( t = g_0^2 \) we have
\[ (g_0 a')_{1/2} = g_0^2 + 2a' D_0^2(a) - D_0(a')^2. \] (71)

From (68) and (70) we have
\[ a'(b'z) = (a'b')z = (ab)'z, \]
\[ (a'z)(b'z) = a'b = (ab)', \]
\[ (b'z)(g_0a')_{1/2} = -a'D_0(b'), \]
\[ b'(g_0a')_{1/2} = [(a'b')g_0]_{1/2} + a'D_0(b')z. \]

If \( \mathcal{B}' \) is the set of all elements of the form \( a' \) where \( a \in \mathcal{B} \) then \( \mathcal{B}' + \mathcal{B}'z + (g_0\mathcal{B}')_{1/2} \) is a subalgebra with multiplications similar to those expressed in (66). Hence we can conclude that this subalgebra is power-associative and that \( a' + b'z + (g_0d')_{1/2} \) has a unique fourth power for every \( a', b', d' \in \mathcal{B}' \). But \( \mathcal{C} = \mathcal{B}' + \mathcal{B}'z \).

Therefore \( a + bz + (g_0d + gd)_{1/2} \) has a unique fourth power for every \( a, b, d \in \mathcal{B} \) and every \( g \). If \( d \) is nonsingular then \( d \) can be absorbed in the coefficients \( a_i \) of \( g_i \) in the expression for \( g \). Hence \( a + bz + (g_0d)_{1/2} + g \) has a unique fourth power if \( d \) is nonsingular. We can restate this as \( x = g_0 + a(a + bz) + \beta(g_0d)_{1/2} + yg \) has a unique fourth power for \( d \) a singular element of \( \mathcal{B}, a, b \in \mathcal{B}, g = \Sigma(a_i^j g_j)_{1/2} \) and \( a, \beta \in \mathcal{I} \). The characteristic is sufficiently high so that the attached polynomials of the expression \( x^2x^2 - x^4 \) are all zero [6]. The sum of those polynomials with a coefficient \( x^i\beta^j x^k \) where \( i + j + k = 4 \) is of course also equal to zero. But by replacing \( x, \beta \) and \( y \) by 1 in this sum we get \( y^2y^2 - y^4 = 0 \) where \( y = (a + bz + (g_0d)_{1/2} + g) \). Hence any element of \( \mathcal{A} \) has a unique fourth power and \( \mathcal{A} \) is power-associative.

To complete the proof it remains only to show the simplicity of \( \mathcal{A} \). Let \( \mathcal{I} \) be a proper ideal of \( \mathcal{A} \) with the nonzero element \( a + bz + t \) where \( a, b \in \mathcal{B} \) and \( t \in \mathcal{I} \). Since \( z\mathcal{I} \subseteq \mathcal{I} \) we have \( az + b \in \mathcal{I} \). Now multiply \( az + b \) by \( g_0 \) to get \( (ag_0)_{1/2} + D_0(a)z - D_0(b) \in \mathcal{I} \). By the above \( (ag_0)_{1/2} \in \mathcal{I} \). Multiplying this element by \( cz \) we get \( a \in \mathcal{I} \) and therefore \( b, t, D(a) \) and \( D(b) \in \mathcal{I} \). Let \( \mathcal{P} \) be the set of all elements of \( \mathcal{B} \) that are in \( \mathcal{I} \). Clearly, \( \mathcal{P} \) is a proper ideal of \( \mathcal{B} \). Since \( \mathcal{P} \mathcal{L} \subseteq \mathcal{I} \) and \( (\mathcal{P}\mathcal{L})_{1/2} \subseteq \mathcal{I} \) it can be easily shown that \( \mathcal{P} \) is D-admissible. Hence \( \mathcal{B} = 0 \) and the only nonzero elements that could be in \( \mathcal{I} \) are of the form \( t \) where \( t \in \mathcal{I} \). But by the assumption on \( \mathcal{A} \) there is an \( x \in \mathcal{I} \) such that \( gx \neq 0 \). Since \( gx \in \mathcal{B} + \mathcal{B}z \) and \( \mathcal{I} \cap (\mathcal{B} + \mathcal{B}z) = 0 \) we must have \( \mathcal{I} = 0 \). Therefore \( \mathcal{A} \) is simple.

To further characterize the algebra \( \mathcal{A} \) and its subalgebra \( \mathcal{B} \) we quote a result of Harper [5, Theorem 1].

**Theorem 9.** Let \( \mathcal{B} \) be a commutative, associative algebra with unity 1 over an algebraically closed field \( \mathbb{F} \), and let \( \mathcal{B} \) be D-simple relative to a set of derivations of \( \mathcal{B} \) over \( \mathbb{F} \). Then \( \mathcal{B} = \mathbb{F}[1, x_1, \ldots, x_n] \) is an algebra with generators \( x_1, \ldots, x_n \) over \( \mathbb{F} \) which are independent except for the relations \( x_1^2 = \ldots = x_n^p = 0 \) where \( p \) is the characteristic of \( \mathbb{F} \).

3. Let \( p \) be a prime \( \neq 2, 3, 5 \) and let \( \mathcal{B} \) be the associative commutative algebra of all polynomials \( \sum_{i=0}^{p-1} c_i \) in \( c \) with \( c^p = 0 \) and \( c^0 = 1 \), the identity of \( \mathcal{B} \). Let \( \mathcal{L} = \{ (g_0a)_{1/2} : a \in \mathcal{B} \} \). Then \( \mathcal{A} = \mathcal{B} + \mathcal{B}z + (g_0\mathcal{B})_{1/2} \). Let \( b_{00} = 0 \) and \( D_0 \) be ordinary polynomial differentiation; i.e., \( D_0(c) = 1 \). Assume that \( u = a + bz \)}
+ \left( g_0d \right)^{1/2}, \text{ where } a, b, d \in \mathfrak{B}, \text{ is an idempotent of } \mathfrak{A} \text{ that is not in } \mathbb{C}. \text{ Then }
\begin{align*}
a^2 + b^2 + 2dD_0(d) - D_0(d)^2 - 2dD_0(b) + 2abz + 2dD_0(a)z + 2\left( g_0(da) \right)^{1/2} \\
= a + bz + \left( g_0d \right)^{1/2}, \text{ Therefore } d(2a - 1) = 0 \text{ and } 2ab + 2dD_0(a) = b. \text{ If } d = 0 \text{ then } u \in \mathbb{C}. \text{ By our assumptions } d \neq 0 \text{ and we must have } 2a - 1 \text{ is singular. Therefore we can write } a = 1/2 + c's \text{ where } s \text{ is a nonsingular element of } \mathfrak{B} \text{ and } t \geq 1. \text{ We have } dc' = 0 \text{ and } c'b + tc^{-1}d = 0. \text{ Hence } c^{t+1}b = 0. \text{ Since }
(73) \quad a^2 + b^2 + 2dD_0(d) - D_0(d)^2 - 2dD_0(b) = a
\end{align*}

it follows that \( a^{t+1} = ac^{t+2} \). But this implies that \( c^{t+1} = 2c^{t+1} \). Hence \( t + 1 \geq p \).

Assume \( t = p - 1 \); then \( c^{p-1}b = c^{p-2}d \). Now if \( b = \sum_0^{p-1} \beta c^i \) and \( d = \sum_0^{p-1} \alpha c^i \) then we must have \( \alpha_0 = 0 \) and \( \beta_0 = \alpha_1 \). From \( (73) \) we must also have \( \beta_0^2 - \alpha_1^2 = 1/4 \) which is a contradiction. Therefore \( t + 1 > p \) and \( a = 1/2 \).

Let \( x' = a' + b'z + \left( g_0d' \right)^{1/2} \) be an arbitrary element of \( \mathfrak{A} \). By considering the product \( x'u \) we see that a necessary and sufficient condition that \( x' \in \mathfrak{A}_u(1) \) is that \( 2a'd = d' \).

(74) \quad 2ba' + 2D_0(a')d = b'.

The correspondence \( a' \rightarrow a' + 2a'bz + 2D_0(a')dz + 2\left[ g_0(a'd) \right]^{1/2} \) is clearly a 1–1 correspondence between \( \mathfrak{B} \) and \( \mathfrak{A}_u(1) \) preserving the vector space operations. Therefore \( \mathfrak{A}_u(1) \) is of dimension \( p \).

If \( u \) is a stable idempotent then Albert has shown \([3; 4]\) that \( \mathfrak{A} = \mathfrak{A}_u(1) + \mathfrak{A}_u(0) + w\mathfrak{C}' + \mathfrak{C} \) where \( \mathfrak{C}' = \mathfrak{A}_u(1) + \mathfrak{A}_u(0) \) and \( w\mathfrak{C}' + \mathfrak{C} = \mathfrak{A}_u(1/2) \). Albert also showed that the dimensions of \( \mathfrak{A}_u(1), \mathfrak{A}_u(0) \) and \( w\mathfrak{C}' \) are all equal. Therefore \( \mathfrak{C} = 0 \). A further result of Albert’s is that \( \mathfrak{A}_u(1) + \mathfrak{A}_u(0) + w\mathfrak{C}' \) is associative. This implies that \( \mathfrak{A} \) is a simple, associative algebra and hence we must have \( c = 0 \).

We can conclude that our example contains no stable idempotents.

**Bibliography**


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