INTEGRAL CURRENTS MOD 2(1)

BY
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1. **Introduction.** In the paper *Normal and integral currents*, a great amount of progress has been made in the study of k-dimensional domains of integration in Euclidean n-space. This progress was made through the use of de Rham's (odd) currents which, in turn, led to the selection of integral currents for further study. Integral currents were shown to possess many desirable characteristics, most notable being those of "smoothness" and "compactness."

However, the fact that a current is a linear functional on the space of differential forms means that one is necessarily restricted to oriented domains of integration. Therefore, in order to consider nonoriented domains of integration and still make use of the theory of integral currents, one is led to study the quotient group

$$I_k(R^n)/2I_k(R^n)$$

where $I_k(R^n)$ denotes the k-dimensional integral currents of n-space.

The boundary operator defined on the space of currents induces a boundary operator on the quotient group; the mass M of a coset is defined as the infimum of the mass of all currents in the coset. Convergence is defined in terms of a metric which is induced by a function W, whose definition is similar to Whitney's flat norm:

$$W(\tau) = \inf\{M(\tau - \partial\sigma) + M(\sigma)\}.$$

However, due to this notion of convergence, it is convenient to consider the group

$$W_k(R^n)/2W_k(R^n)$$

where

$$W_k(R^n) = \{T\colon\ T = R + \partial S,\ R \text{ and } S \text{ rectifiable currents}\}.$$

Members of this group are called *flat classes*, and a flat class is called an *integral class* if the coset τ contains an integral current. The group $I_k(R^n)/2I_k(R^n)$ is isomorphic to the group of k-dimensional integral classes. If one makes definitions for boundary, mass, and W for flat classes analogous to the ones above, it is found that when restricted to integral classes, they agree with the definitions laid down for the group $I_k(R^n)/2I_k(R^n)$.

It is shown that rectifiable classes can be identified with Hausdorff rectifiable sets and that the mass of a rectifiable class is the Hausdorff measure of the

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corresponding set. Many of the results in *Normal and integral currents* generalize to integral classes, for example, the Eilenberg-type inequality, the deformation theorem, and the isoperimetric inequality. These basic results, combined with the geometrical character of Hausdorff rectifiable sets, establish a lower-semi-continuity theorem which, in turn, leads to a closure result in the extreme dimensions.

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2. **Definitions and notation.** The notation and terminology throughout will follow that of [3].

2.1 DEFINITION. If $U \subset R^n$, let $W_k(U) = \{T\colon\ T = R + \partial S,\ R$ and $S\ k,\ k+1$ rectifiable currents, spt $T \subset U\}$ be the group of *k-dimensional flat currents in* U. Observe, if $T \in W_k(U)$ and $M(T) < \infty$, then T is a rectifiable current by virtue of [3, (8.14)].

2.2 DEFINITION. Let

$$W_k(R^n,2) = W_k(R^n)/2W_k(R^n)$$

be the group of *k-dimensional flat classes in* R^n. Elements in $W_k(R^n,2)$ will be denoted by ρ, σ, τ, etc. If the flat class τ has a polyhedral chain with integer coefficients as a representative, then τ will be called a *polyhedral chain class*; *integral Lipschitz chain classes*, *integral classes*, and *rectifiable classes* are similarly defined. The group of k-dimensional integral classes in R^n will be denoted by $I_k(R^n,2)$.

2.3 DEFINITION. If τ is a flat class, the boundary operator ∂ is defined by

$$\partial\tau = [\partial T]^-$$

where T is a representative of τ, and where $[\partial T]^-$ denotes the coset which contains ∂T.

2.4 DEFINITION. If τ is a flat class, let

$$M(\tau) = \inf\{M(T)\colon\ T \in \tau\}$$

denote the *mass of* τ, where $M(T)$ is the mass of the current T. We will say that a flat class τ is a *normal class*, if $M(\tau) + M(\partial\tau) = N(\tau) < \infty$. By 2.1 we have: *a flat class of finite mass is a rectifiable class.*

2.5 REMARK. Making use of the fact that a flat current of finite mass is a rectifiable current, we have that if τ is a rectifiable class, then

$$M(\tau) = \inf\{M(T)\colon\ T \in \tau,\ T \text{ a rectifiable current}\}.$$

Moreover, if τ is an integral class, the fact that the rectifiable currents are in the mass closure of the integral currents implies that

$$M(\tau) = \inf\{M(T)\colon\ T\in\tau,\ T \text{ an integral current}\}.$$

2.6 Remark. If a class τ is rectifiable as well as its boundary, then there exists rectifiable currents T and R such that $T\in\tau$ and $R\in\partial\tau$. But $R\in\partial\tau$ implies the existence of a flat current S, such that

$$R = \partial T + 2S.$$

Since S is a flat current, there exists rectifiable currents X and Y such that

$$S = X + \partial Y,$$

and therefore

$$\partial T + 2\partial(Y) = \partial(T + 2Y)$$

is a rectifiable current. This, in turn, implies that $T+2Y$ is an integral current. Hence, we have *every rectifiable class whose boundary is rectifiable, is an integral class.*

By 2.4, it is now evident that *everry normal class is integral.*

2.7 Definition. If $A\subset R^n$ and r is a positive real number, let

$$S(A,r) = \{x\colon \text{ distance } (x,A) < r\}.$$

2.8 Definition. If γ is a measure over R^n, $A\subset R^n$, $\alpha(k)$ the volume of the unit k-ball, and $x\in R^n$, then

$$D_k(\gamma,A,x) = \lim_{r\to 0}\alpha(k)^{-1}r^{-k}\gamma(A\cap\{y\colon |x-y|<r\})$$

is the *k-dimensional γ density of A at x*; the *upper* and *lower densities*

$$D_k^-(\gamma,A,x) \text{ and } D\text{-}_k(\gamma,A,x)$$

are defined as the corresponding lim sup and lim inf.

3. Hausdorff k-rectifiable sets and rectifiable classes.

3.1 Definitions. If $k\leqq n$, $A\subset R^n$, then A *is k-rectifiable* if there exists a Lipschitzian function whose domain is a bounded subset of R^k and whose range is A. A is *countably k-rectifiable* if A can be expressed as a countable union of k-rectifiable sets and *Hausdorff k-rectifiable* if there is a countably k-rectifiable set B such that $H^k(A-B)=0$, where H^k denotes k-dimensional Hausdorff measure.

If A is a Hausdorff k-rectifiable set and $H^k(A)<\infty$, then A has a unique H^k approximate tangent plane at H^k almost all of its points and

$$D_k(H^k,A,x) = 1 \text{ for } H^k \text{ almost all } x\in A.$$

3.2 Rectifiable currents. It is easy to verify that a k-dimensional current T is rectifiable if and only if it has the following representation: if ϕ is a C^∞ k-form, then

$$T(\phi) = \int_E M(x)\phi(x)\cdot\alpha(x)\,dH^k(x)$$

where (i) E is a bounded, H^k measurable, countably k-rectifiable set with $H^k(E) < \infty$,

(ii) $\cdot$ denotes the scalar product; see [8, (p. 37)],

(iii) $\alpha(x)$ is an H^k approximate tangent k-vector to E at x, $|\alpha(x)| = 1$, and with α being H^k measurable,

(iv) M is integer valued and H^k integrable,

(v) $M(T) = \int_E |M(x)|\,dH^k(x)$.

In this regard, see [3, (8.16)] and [7, (7.1)].

With each rectifiable current T, one can associate the rectifiable current

$$T^*(\phi) = \int_E M^*(x)\,\phi(x)\cdot\alpha(x)\,dH^k(x)$$

where $M^*(x) \equiv M(x) \bmod 2$ and where $0 \leq M^*(x) \leq 1$. Obviously, T and T^* belong to the same coset of $W_k(R^n,2)$. If T_1 is another rectifiable current with representation

$$T_1(\phi) = \int_{E_1} M_1(x)\phi(x)\cdot\alpha_1(x)dH^k(x),$$

then, with 3.1, we have

$$T + 2T_1(\phi) = \int_{E\cup E_1} [M(x) \pm 2M_1(x)]\phi(x)\cdot\beta(x)\,dH^k(x)$$

where

$$\beta(x) = \begin{cases}\alpha(x), & x\in E,\\ \alpha_1(x), & x\in E_1 - E.\end{cases}$$

Notice in general that

$$T^* \neq (T+2T_1)^*,$$

but in view of [3, (8.16)], we do have

$$\|T^*\| = \|(T+2T_1)^*\|.$$

3.3 Definition. If τ is a rectifiable class, $T\in\tau$ is a rectifiable current, then

$$\|\tau\| = \|T^*\|$$

is the *total variation measure of* τ. From above, we can see that $\|\tau\|$ is well-defined.

3.4 The carrier of a rectifiable class. If T is a rectifiable current, then

$$T^*(\phi) = \int_D \phi(x)\cdot\alpha(x)dH^k(x)$$

where $D = \{x : M^*(x) = 1\}$. Therefore, if $B \subset R^n$ is a Borel set, then

(1) $$\|T^*\|(B) = H^k(D\cap B).$$

If we let

$$D_0 = \{x : D_k^-(\|T^*\|, R^n, x) = 1\},$$

with the aid of 3.1 and (1), it is clear that

$$H^k(D - D_0) = 0.$$

From [4, § 3], we also have $H^k(D_0 - D) = 0$, and therefore

$$T^*(\phi) = \int_{D_0} \phi(x) \cdot \alpha(x) \, dH^k(x).$$

Since $D_0 \subset \operatorname{spt} T^*$, it now follows that

$$\text{Closure } D_0 = \operatorname{spt} T^*.$$

3.5 Definition. If τ is a rectifiable class, and

$$D_0 = \{x : D_k^-(\|\tau\|, R^n, x) = 1\},$$

then we say that D_0 *is the carrier of* τ, and that *closure* D_0 *is the support of* τ. They will be written as car τ and spt τ, respectively.

3.6 Remark. If τ is a rectifiable class, we can associate with it, in a unique manner, a Hausdorff k-rectifiable set: the carrier of τ. It is clear that

$$M(\tau) = M(T^*) = H^k(\text{car } \tau) \text{ and } \operatorname{spt} \tau = \operatorname{spt} T^*,$$

where T is a rectifiable representative of τ.

If E is a bounded Hausdorff k-rectifiable set with finite k-measure, we can associate with it a rectifiable current T:

$$T(\phi) = \int_E \phi(x) \cdot \alpha(x) dH^k(x)$$

where $\alpha(x)$ is an H^k approximate tangent k-vector to E at x with $|\alpha(x)| = 1$. This definition of T depends upon the two possible choices for the H^k approximate tangent k-vector; that is, the choice of either $\alpha(x)$ or $-\alpha(x)$. However, the rectifiable class which is defined by T is uniquely determined. In the sequel, no distinction will be made between a Hausdorff k-rectifiable set and the rectifiable class it defines.

3.7 Theorem. *If τ is an integral class, $\varepsilon > 0$, and U an open set containing* spt τ, *then there exists an integral current $T \in \tau$ such that*

(i) spt $T \subset U$,

(ii) $M(T) < M(t) + \varepsilon$.

Proof. Since spt τ is compact, there exists $\eta > 0$ so that

$$S(\operatorname{spt} \tau, \eta) \subset U.$$

Let R be an integral representative of τ with

$$M(R) < M(\tau) + \varepsilon.$$

By [3, (3.10) and (8.14)], there exists $\eta_0 < \eta$ such that

$$R \cap S(\operatorname{spt} \tau, \eta_0) \text{ is an integral current.}$$

Letting $V = S(\operatorname{spt}\tau, \eta_0)$ we have

$$R = R \cap V + R \cap (R^n - V),$$

and therefore

$$R \cap (R^n - V) \text{ is an integral current.}$$

But from 3.4, we see that $R \cap (R^n - V)$ can be expressed in the form

$$R \cap (R^n - V) = 2S,$$

where S is integral. Hence, letting

$$T = R \cap V,$$

the conclusion of the theorem follows.

3.8 COROLLARY. *If τ is an integral class, then*

$$\operatorname{spt}\tau = \bigcap \{\operatorname{spt} T: \quad T \text{ integral}, \quad T \in \tau\}.$$

3.9 DEFINITION. If τ is a rectifiable class and A is a Borel subset of R^n, let

$$\tau \cap A = [T \cap A]^-$$

where T is a rectifiable representative of τ. This is well-defined in view of the following fact: if $2R$ is a rectifiable current and if R is a flat current, then R is rectifiable. If τ is a k-dimensional rectifiable class, then

$$\|\tau\|(A) = M(\tau \cap A) = H^k(\operatorname{car}\tau \cap A).$$

3.10 DEFINITION. If $U \subset R^n$, let

$$I_k(U, 2) = I_k(R^n, 2) \cap \{\tau: \operatorname{spt}\tau \subset U\}$$

and

$$I_*(U, 2) \quad \text{the corresponding chain complex.}$$

Let $W_*(R^n, 2)$ be the chain complex associated with $W_k(R^n, 2)$, $k = 0, 1, \dots n$.

3.11 INDUCED MAPS. An infinitely differentiable map

$$f: R^m \to R^n$$

induces a homomorphism

$$f_\#: W_*(R^m, 2) \to W_*(R^n, 2)$$

such that $\partial \circ f_\# = f_\# \circ \partial$. If $\tau \in W_k(R^m, 2)$, then

$$f_\#(\tau) = [f_\#(T)]^-, \text{ where } T \in \tau.$$

If τ is k-dimensional rectifiable class, then

$$\operatorname{spt} f_\#(\operatorname{spt}\tau) \subset f(\operatorname{spt}\tau) \text{ and}$$

$$M[f_\#(\tau)] \leqq \|\tau\|(|Df|).$$

This can be seen by selecting a rectifiable current $T \in \tau$ such that

$$M(T) = M(\tau),$$

$$\operatorname{spt} T = \operatorname{spt} \tau.$$

3.12 LOCALLY LIPSCHITIZIAN MAPS. See [3, (3.5)]. If U and V are open subsets of R^m and R^n, then each locally Lipschitizian map

$$f: U \to V$$

induces a chain map

$$f_\#: I_*(U,2) \to I_*(V,2)$$

defined for $\tau \in I_k(U,2)$ by

$$f_\#(\tau) = [f_\#(T)]^-$$

where $T \in \tau$ is an integral current with $\operatorname{spt} T \subset U$. With the aid of 3.7, it is easy to verify

$$\operatorname{spt} f_\#(\tau) \subset f(\operatorname{spt} \tau).$$

If f has Lipschitz constant λ on a neighborhood of $\operatorname{spt} \tau$, then

$$M[f_\#(\tau)] \leqq \lambda^k M(\tau).$$

3.13 DEFINITION. If I denotes the unit interval and τ is an integral class, then

$$I \times \tau = [I \times T]^-$$

where T is an integral representative of τ.

If f and g are locally Lipschitzian maps of U into R^n, h is the linear homotopy from f to g, $\tau \in I_k(U,2)$, λ is the Lipschitz constant of f and g on a neighborhood N of $\operatorname{spt} \tau$, and $|f(x) - g(x)| < \varepsilon$ for $x \in N$, then

$$M[h_\#(I \times \tau)] \leqq \varepsilon \lambda^k M(\tau),$$

and

$$\partial h_\#(I \times \tau) + h_\#(I \times \partial \tau) = g_\#(\tau) - f_\#(\tau).$$

This follows from 3.7 and [3, (3.6)].

3.14 CONES. The cone $z\tau \in I_{k+1}(R^n,2)$ with vertex $z \in R^n$ and base $\tau \in I_k(R^n,2)$ is defined by

$$z\tau = h_\#(I \times \tau)$$

where h: $R \times R^n \to R^n$, $h(t,x) = (1-t)z + tx$ is the linear homotopy from the constant map $R^n \to \{z\}$ to the identity map of R^n. We have

$$\partial z\tau + z\partial\tau = \tau \text{ if } k \geqq 1,$$

and from [3, (9.2)] and 3.7,

$$M(z\tau) \leqq rM(\tau)/k + 1$$

where $r = \sup\{|x - z| : x \in \operatorname{spt}\tau\}$.

4. The theorems in this section are generalizations of some of the basic results in *Normal and integral currents*. See [3, (3.9), (3.10), (5.5), (6.1)].

4.1 THEOREM. *Suppose $\tau \in I_k(R^n,2)$, μ is a real-valued function on R^n with Lipschitz constant ξ, $A_r = \{x : \mu(x) > r\}$ for $r \in R$, and $-\infty \leqq a < b \leqq +\infty$; then*:

(i) $\int^*_{[a,b]} M[\partial(\tau \cap A_s) - (\partial\tau) \cap A_s]\,ds \leqq \xi\,\|\tau\|\,(\{x : a < \mu(x) \leqq b\})$ *where "$\int^*$" denotes upper integral,*

(ii) *$\tau \cap A_s$ is integral for L_1 almost all s,*

(iii) $\operatorname{spt}[\partial(\tau \cap A_s) - (\partial\tau) \cap A_s] \subset (\operatorname{spt}\tau) \cap \{x : \mu(x) = s\}$ *for L_1 almost all s.*

Proof. Let $T \in \tau$ be an integral representative and observe that $T \cap A_s$ is integral for L_1 almost all s; (see [3, (3.10), (8.14)]). This establishes (ii).

From 3.7, we can find a sequence of integral currents T_i such that $T_i \in \tau$ and

$$\bigcap \{\operatorname{spt} T_i : i = 1,2,\ldots,\} = \operatorname{spt}\tau.$$

Since

$$\partial(T_i \cap A_s) - (\partial T_i) \cap A_s \in \partial(\tau \cap A_s) - (\partial\tau) \cap A_s$$

for all s and $i = 1,2,\ldots$, it follows that

$$\operatorname{spt}[\partial(\tau \cap A_s) - (\partial\tau) \cap A_s] \subset \operatorname{spt}[\partial(T_i \cap A_s) - (\partial T_i) \cap A_s].$$

But [3, (3.9), (3.10)] implies, for each i,

$$\operatorname{spt}[\partial(T_i \cap A_s) - (\partial T_i) \cap A_s] \subset \operatorname{spt} T_i \cap \{x : \mu(x) = s\}$$

for L_1 almost all s. Now (iii) follows.

If $T \in \tau$ is an integral current and $s \in R$, then

$$M[\partial(\tau \cap A_s) - (\partial\tau) \cap A_s] \leqq M[\partial(T \cap A_s) - (\partial T) \cap A_s]$$

and therefore

$$\int^*_{[a,b]} M[\partial(\tau \cap A_s) - (\partial\tau) \cap A_s] \leqq \int_a^b M[\partial(T \cap A_s) - (\partial T) \cap A_s]ds.$$

Letting I denote the integral on the left, $T_0 \in \tau$ an integral current, $A = \{x : a < \mu(x) \leqq b\}$, one sees from [3, (3.10)] that

$$\begin{aligned} I &\leqq \inf\{\xi M(T \cap A) : T \in \tau,\ T \text{ integral}\} \\ &\leqq \inf\{\xi M(T_0 + 2R) \cap A : R \in I_k(R^n)\} \\ &\leqq \inf\{\xi M[T_0 \cap A + (2R) \cap A] : R \in I_k(R^n)\} \\ &\leqq \inf\{\xi M(T_0 \cap A + 2R) : R \text{ a rectifiable current}\} \\ &\leqq \xi M(\tau \cap A) \\ &\leqq \xi\|\tau\|(A), \text{ by 2.5 and 3.9.} \end{aligned}$$

The proof of the theorem is complete.

REMARK. Once we have read Theorem 5.14, it becomes clear that

$$M[\partial(\tau \cap A_s) - (\partial\tau) \cap A_s]$$

is a lower semi-continuous function and therefore L_1 integrable.

4.2 THE DEFORMATION THEOREM. The notation used in the proof of the following theorem will be found in [3, (5.5)].

THEOREM. *If $\tau \in I_k(R^n,2)$ and $\varepsilon > 0$, then there exist*

$$\pi \in I_k(R^n,2), \quad \rho \in I_k(R^n,2), \quad \sigma \in I_{k+1}(R^n,2)$$

with the following properties:

(1) $\tau = \pi + \rho + \partial\sigma$.

(2) $$\frac{M(\pi)}{\varepsilon^k} \leqq 7n^k \left[\binom{n}{k} \frac{M(\tau)}{\varepsilon^k} + \binom{n}{k-1} \frac{M(\partial\tau)}{\varepsilon^{k-1}} \right],$$

$$\frac{M(\partial\pi)}{\varepsilon^{k-1}} \leqq 7n^{k-1} \binom{n}{k-1} \frac{M(\partial\tau)}{\varepsilon^{k-1}},$$

$$\frac{M(\rho)}{\varepsilon^k} \leqq 19n^k \binom{n}{k-1} \frac{M(\partial\tau)}{\varepsilon^{k-1}},$$

$$\frac{M(\sigma)}{\varepsilon^{k+1}} \leqq 13n^{k+1} \binom{n}{k} \frac{M(\tau)}{\varepsilon^k}.$$

(3) $\operatorname{spt} \pi \cup \operatorname{spt} \sigma \subset \{x : \textit{distance}(x, \operatorname{spt} \tau) \leqq 3n\varepsilon\}$,
$\operatorname{spt} \partial\pi \cup \operatorname{spt} \rho \subset \{x : \textit{distance}(x, \operatorname{spt} \partial\tau) \leqq 3n\varepsilon\}$.

(4) *π is a polyhedral chain of $\mu_\varepsilon(C')$ (with integer coefficients).*

(5) *In case τ is an integral Lipschitz chain, so are π, ρ, and σ.*

(6) *In case $\partial\tau$ is an integral Lipschitz chain, so is ρ.*

Proof. We may assume that $\varepsilon = 1$. Given $\eta > 0$, with the aid of 3.7 we can select integral currents $T$and X_2 such that $T \in \tau$ and

$$M(T) < M(\tau) + \eta,$$

$$M(\partial T + 2X_2) < M(\partial\tau) + \eta,$$

$$\operatorname{spt} T \subset S(\operatorname{spt} \tau, \eta),$$

$$\operatorname{spt}(\partial T + 2X_2) \subset S(\operatorname{spt} \partial\tau, \eta).$$

In case τ is an integral Lipschitz chain, there will exist an integral current X_0 suchthat $T + 2X_0$ is an integral Lipschitz chain. In case $\partial\tau$ is an integral Lipschitz chain, there will exist an integral current X_1 with a similar property. Of course, if τ is an integral Lipschitz chain, one may take $X_1 = \partial X_0$. Let us agree to say that $X_0 = 0$ if and only if τ is not an integral Lipschitz chain, and similarly for X_1. By [3, (5.2), (5.3), (5.4)], there exists $a \in A$ so that T, ∂T, X_0, X_1, X_2, $\sigma_k \circ \tau_a$

and σ_n are μ admissible and ∂T, X_1, X_2, $\sigma_k \circ \tau_a$, and $\sigma_{k-1} \circ \tau_a$ are v admissible; also a can be chosen so that

$$\|T\| \; [(\mu_k \circ \tau_a)^{-k}] \leqq 6 \binom{n}{k} M(T),$$

$$\|\partial T\| \; [(\mu_{k-1} \circ \tau_a)^{-k+1}] \leqq 6 \binom{n}{k-1} M(\partial T),$$

$$\|\partial T + 2X_2\| [(\mu_{k-1} \circ \tau_a)^{-k+1}] \leqq 6 \binom{n}{k-1} M(\partial T + 2X_2),$$

$$\|\partial 2X_2\| \; [(\mu_{k-2} \circ \tau_a)^{-k+2}] \leqq 6 \binom{n}{k-2} M(\partial 2X_2),$$

$$\|X_i\| \; [(\mu_{k-i} \circ \tau_a)^{-k+1} \leqq 6 \binom{n}{k-i} M(X_i),$$

$$\|\partial X_i\| [(\mu_{k-i-1} \circ \tau_a)^{-k+i+1}] \leqq 6 \binom{n}{k-i-1} M(\partial X_i), \quad i = 0 \text{ or } 1,$$

$$\tau_a \; [\operatorname{spt}(T + 2X_0)] \subset R^n - C''_{n-k-1},$$

$$\tau_a \; [\operatorname{spt}(\partial T + 2X_1)] \subset R^n - C''_{n-k}.$$

We may now use [3, (4.2)] to define

$$\begin{aligned}
P_1 &= (\sigma_k \circ \tau_a)_\#(T + 2X_0), \\
P_2 &= H_v(\sigma_k \circ \tau_a, \; \sigma_{k-1} \circ \tau_a) \; (\partial T + 2X_1 + 2X_2), \\
Q_1 &= H_u(\sigma_k \circ \tau_a, \; \sigma_n)(\partial T + 2X_1 + 2X_2), \\
S &= H_u(\sigma_k \circ \tau_a, \; \sigma_n) \; (T + 2X_0), \\
P &= P_1 + P_2, \quad Q = Q_1 - P_2,
\end{aligned}$$

and find that T and $P + Q + \partial S$ are in the same coset. Therefore, by taking $\pi = P^-$, $\rho = Q^-$, and $\sigma = S^-$, (1), (4), (5), and (6) are established. Moreover,

$$M(P_1^-) \leqq \|T\| \; (\mu^{-k}) \leqq 6n^k \binom{n}{k} \; [M(\tau) + \eta],$$

$$M(P_2^-) \leqq \|\partial T + 2X_2\| \; (nv^{-k+1}) \leqq 6n^k \binom{n}{k-1} \; [M(\partial\tau) + \eta],$$

$$M(Q_1^-) \leqq \|\partial T + 2X_2\| \, (2n\mu^{-k+1}) \leqq 12n^k \binom{n}{k-1} \, [M(\partial\tau) + \eta],$$

$$M(S^-) \leqq \|T\|(2n\mu^{-k}) \leqq 12n^{k+1} \binom{n}{k} [M(\tau) + \eta],$$

$$M(\partial P^-) \leqq \|\partial T + 2X_2\| \, (v^{-k+1}) \leqq 6n^{k-1} \binom{n}{k-1} \, [M(\partial\tau) + \eta],$$

which establishes (2) because of the arbitrariness of η. (3) follows from similar reasoning.

4.3 LEMMA. *If $\tau \in I_k(R^n, 2)$, $\varepsilon > 0$, and $T \in \tau$ is an integral current with the property that $M(T) < M(\tau) + \varepsilon$, then*

$$\|T\|(B) \leqq \|\tau\|(B) + 3\varepsilon$$

where B is any Borel set in R^n.

Proof. If $\phi \in E^k(R^n)$, then by 3.2,

$$T(\phi) = \int_F M(x)\phi(x) \cdot \alpha(x) dH^k(x)$$

where F is a countably k-rectifiable set. It may be assumed that $M \geqq 0$. Letting $E = \{x : M(x) \text{ is odd}\}$, $E_0 = E \cap \{x : M(x) \geqq 3\}$, $E_1 = E \cap \{x : M(x) = 1\}$, we have

$$E_0 \cap E_1 = 0,$$
$$E_0 \cup E_1 = E,$$

and from 3.4 and 3.6,

$$H^k(E) = M(\tau).$$

Since $M(T) < M(\tau) + \varepsilon$, it follows that

$$\int_{F-E} M(x) dH^k(x) + \int_E M(x) dH^k(x) < H^k(E) + \varepsilon$$

and in particular,

$$\int_E M(x)dH^k(x) = \int_{E_0} M(x)dH^k(x) + \int_{E_1} M(x)dH^k(x) = \int_{E_0} M(x)dH^k(x) + H^k(E_1)$$
$$\leqq H^k(E_0) + H^k(E_1) + \varepsilon.$$

Therefore, we have

$$\int_{F-E} M(x)dH^k(x) \leqq \varepsilon, \qquad \int_{E_0} M(x)dH^k(x) \leqq H^k(E_0) + \varepsilon$$

with the second inequality implying that $H^k(E_0) < \varepsilon$ and therefore

$$\int_{E_0} M(x)dH^k(x) \leqq 2\varepsilon.$$

If B is a Borel set in R^n, then [3, (8.16)], 3.4, and 3.6 imply

$$\|T\|(B) = \int_{(F-E)\cap B} M(x)dH^k(x) + \int_{E_0 \cap B} M(x)dH^k(x) + H^k(E_1 \cap B)$$
$$\leqq H^k(E_1 \cap B) + \varepsilon + 2\varepsilon$$
$$\leqq H^k(E \cap B) + 3\varepsilon = \|\tau\|(B) + 3\varepsilon.$$

4.4 LEMMA. *If U is an open set in R^m, $\tau \in I_k(U,2)$, f and g Lipschitzian maps of U into R^n with Lipschitz constant λ, then,*

$$M[f_\#(\tau) - g_\#(\tau)] \leqq 2\lambda^k \|\tau\|(\{x : f(x) \neq g(x)\}).$$

Proof. If $\varepsilon > 0$, 3.7 implies the existence of an integral current $T \in \tau$ such that $\operatorname{spt} T \subset U$ and $M(T) < M(\tau) + \varepsilon$. Then [3, (3.6)] and 4.3 imply that

$$\begin{aligned} M[f_{\#}(\tau) - g_{\#}(\tau)] &\leqq M[f_{\#}(T) - g_{\#}(T)] \\ &\leqq 2\lambda^k \|T\| (\{x : f(x) \neq g(x)\}) \\ &\leqq 2\lambda^k[\| \tau \| (\{x : f(x) \neq g(x)\}) + 3\varepsilon]. \end{aligned}$$

Since ε is arbitrary, the conclusion follows.

4.5 LEMMA. *If U is an open subset of R^m, $\tau \in I_k(U,2)$, f and g Lipschitzian maps from U into R^n with Lipschitz constant λ and such that $|f(x) - g(x)| \leqq \alpha$ for $x \in U$, $A \supset B$, $f|B \cap \operatorname{spt}\tau = g|B \cap \operatorname{spt}\tau$, $\operatorname{spt}\tau \subset A$, and if h is the linear homotopy from f to g, then*

$$M[h_{\#}(I \times \tau)] \leqq \alpha\lambda^k \|\tau\| (A - B).$$

Proof. Given $\varepsilon > 0$, let $T \in \tau$ be an integral current such that $M(T) < M(\tau) + \varepsilon$ and spt $T \subset U$. Then from [3, (3.6)] and 4.3, we have

$$\begin{aligned} M[h_{\#}(I \times \tau)] &\leqq M[h_{\#}(I \times T)] \\ &\leqq \alpha\lambda^k \|T\| [R^n - (\operatorname{spt}\tau \cap B)] \\ &\leqq \alpha\lambda^k(\|\tau\| [R^n - (\operatorname{spt}\tau \cap B)] + 3\varepsilon) \\ &\leqq \alpha\lambda^k \|\tau\|(A - B) + \varepsilon 3\alpha\lambda^k. \end{aligned}$$

But ε is arbitrary, and therefore the conclusion follows.

4.6. With the help of the preceding lemmas and Theorem 4.2, the proof of the following theorem is very similar to that of [3, (6.1)]. For details, see [9].

THEOREM. *Suppose $A \supset B$, U and V are neighborhoods of A and B in R^n, $f : U \to A$ and g: $V \to B$ are retractions, Γ is a compact subset of A, $a > 0$, $b > 0$, f has Lipschitz constant ξ on $\{x$: distance $(x, \Gamma) \leqq a\}$, g has Lipschitz constant η on $\{x$: distance $(x,B) \leqq b\}$. If $\chi \in I_k(\Gamma,2)$, spt $\partial\chi \subset B$ and $C_1\|\chi\|(A-B) \leqq [\inf\{b,(\eta + 2)^{-1}a\}]^k$, then there exists $\psi \in I_{k+1}(A,2)$ such that*

$$\begin{aligned} &\operatorname{spt}(\chi - \partial\psi) \subset B, \\ &M(\psi)^{k/k+1} \leqq C_2\xi(\eta + 2)^k\|\chi\| (A - B), \\ &M(\partial\psi) \leqq C_3(\eta + 2)^k\|\chi\|(A - B). \end{aligned}$$

Here,

$$C_1 = 2^{-k}8^{k+1}n^{2k} \binom{n + 1}{k},$$

$$C_2 = \left[(13n^{k+1} \binom{n}{k} + 4n)^k 8n^k \binom{n + 1}{k} 2^{-k}\right]^{1/k+1},$$

$$C_3 = 3 + 19n^k \binom{n}{k - 1}.$$

4.7 COROLLARY. *If* $\chi \in I_k(\Gamma,2)$ *with*

$$\partial\chi = 0 \text{ and } C_1 M(\chi) \leqq a^k,$$

then there exist $\psi \in I_{k+1}(A,2)$ *such that*

$$\partial\psi = \chi \text{ and } M(\psi)^{k/k+1} \leqq 2^k C_2 \xi^k M(\chi).$$

In case A *is convex, one can take* a *arbitrarily large and* $\xi = 1$.

4.8 COROLLARY. *If* $\tau \in I_n(R^n,2)$, *then*

$$M(\tau)^{n-1/n} \leqq C_4 M(\partial\tau).$$

Here $C_4 = 2^{n-1}C_2$ *and* C_2 *is as in* 4.6 *with* $k = n-1$.

Proof. Let $\chi = \partial\tau$, $A = R^n$ to obtain $\psi \in I_n(R^n,2)$ with $\partial\psi = \partial\tau$ and $M(\psi)^{n-1/n} \leqq C_4 M(\partial\tau)$. Furthermore $\psi - \tau \in I_n(R^n,2)$ and $\partial(\psi - \tau) = 0$, hence $\psi - \tau = 0$.

The following theorem is a generalization of a result due to Federer, see [6], and since the proof is very similar to [6], it will not be included here. The reader is referred to [9] for details.

4.9 THEOREM. *If* A *and* B *are compact Lipschitz neighborhood retracts in* R^n *with* $A \supset B$, *and* k *is a positive integer then there exists a positive number* r *with the following two properties*:

(1) *For each* $\chi \in \partial I_{k+1}(A,2) + I_k(B,2)$ *there exists a* $\psi \in I_{k+1}(A,2)$ *such that* $\operatorname{spt}(\chi - \partial\psi) \subset B$ *and*

$$M(\psi)^{k/k+1} \leqq r\,\|\chi\|\,(A - B).$$

(2) *For each* $\tau \in I_{k+1}(A,2)$ *there exists a* $\sigma \in I_{k+1}(A,2)$ *such that* $\operatorname{spt}\partial\sigma \subset B$ *and*

$$M(\tau - \sigma)^{k/k+1} \leqq r\,\|\partial\tau\|(A - B).$$

5. **Flat convergence.** One of the difficulties encountered in the study of flat classes is that of finding a suitable topology. For example, imposing the relative topology on $W_k(R^n)$ and then the quotient topology on $W_k(R^n, 2)$ proves to be unsatisfactory for our purposes because the group $2W_k(R^n)$ is not closed.

We introduce here a real-valued, subadditive function on $W_k(R^n,2)$ which is very similar to Whitney's flat norm [8, Chapter V]. This induces a pseudometric on $W_k(R^n,2)$ with respect to which the boundary operator is continuous. It is not known whether this pseudometric is actually a metric on $W_k(R^n,2)$ but it is known to be true on the subgroup of rectifiable classes (5.9).

5.1 DEFINITION. If $T \in W_k(R^n)$, let

$$W(T) = \inf\{M(T - \partial S) + M(S)\colon\ S \in W_{k+1}(R^n)\}.$$

5.2 ELEMENTARY PROPERTIES OF W.

(1) If $T \in W_k(R^n)$, then

$$W(T) = \inf\{M(R) + M(S)\colon\ T = R + \partial S,\quad R \in W_k(R^n),\quad S \in W_{k+1}(R^n)\}.$$

(2) If $T \in W_k(R^n)$, then

$$M(T) \geqq W(T) \geqq W(\partial T).$$

The first inequality follows by taking $S = 0$ in 5.1. To prove the second, given $\varepsilon > 0$, choose S so that

$$M(T - \partial S) + M(S) < W(T) + \varepsilon.$$

Let $R = T - \partial S$ and obtain

$$M(\partial T - \partial R) + M(R) = M(T - \partial S) < W(T) + \varepsilon.$$

(3) If $T \in W_n(R^n)$, then $W(T) = M(T)$.

(4) If T, S $W_k(R^n)$, then

$$W(T + S) \leqq W(T) + W(S).$$

(5) If $T \in W_k(R^n)$, then

$$W(T) = \inf\{W(T - \partial S) + W(S): S \in W_{k+1}(R^n)\}.$$

This follows from (2) and (4).

(6) If T is a k-dimensional rectifiable current, then

$$\begin{aligned} W(T) &= \inf\{M(T - \partial S) + M(S): S \in I_{k+1}(R^n)\}, \\ &= \inf\{M(R) + M(S): T = R + \partial S, R \text{ rectifiable}, S \text{ integral}\}, \end{aligned}$$

because of (1) and (2) above, 2.1, and the fact that $M(T) < \infty$.

(7) If T is a k-dimensional integral current, then

$$W(T) = \inf\{M(R) + M(S): T = R + \partial S, R \text{ integral}, S \text{ integral}\}.$$

(8) If $T \in W_k(R^n)$, $p: R^n \to P$ is an orthogonal projection of R^n onto the m-plane P, then

$$W[p_\#(T)] \leqq W(T)$$

because

$$M[p_\#(T) - \partial p_\#(S)] + M[p_\#(S)] \leqq M(T - \partial S) + M(S).$$

(9) $W(T) = 0$ if and only if $T = 0$, for flat T.

(10) If T is a flat current, then $W(T) \geqq F(T)$ where F denotes the norm in [3].

REMARK. From (7) above and [3, (8.13), (7.1)], we have the following fact: If A is a compact subset of R^n, $T_i \in I_k(A)$ for $i = 1, 2, \cdots$, $\sup\{N(T_i): i = 1, 2, 3, \cdots\} < \infty$, then

$$\lim_{i \to \infty} T_i = T \text{ if and only if } \lim_{i \to \infty} W(T_i - T) = 0.$$

5.3 LEMMA. *Suppose A is an additive group and F a real-valued, non-negative, subadditive function on A such that $F(a) = F(-a)$ for $a \in A$. If*

$$\lim_{n,m\to\infty} F(a_n - a_m - b_{n,m}) = 0,$$

then there exists a sequence b_n, where the b_n are finite sums of the $b_{n,m}$ such that

$$\lim_{n,m\to\infty} F[(a_n + b_n) - (a_m + b_m)] = 0.$$

Proof. Let $n_1 < n_2 < n_3 \ldots$ be a sequence of integers such that

$$F(a_n - a_m - b_{n,m}) < 2^{-i} \text{ for } n, m \geqq n_i.$$

Define $b_n = 0$ for $1 \leqq n \leqq n_1$ and let

$$b_{n_i+j} = \sum_{p=1}^{i-1} b_{n_p,n_{p+1}} + C_{n_i,n_i+j}$$

where $j = 1,2,3,\ldots$ but with the restriction $n_i < n_i + j \leqq n_{i+1}$, and where

$$C_{n_i,n_i+j} = \begin{cases} b_{n_i,n_i+j}, & n_i + j < n_{i+1}, \\ b_{n_i,n_{i+1}}, & n_i + j = n_{i+1}. \end{cases}$$

Now assume $k > i$, $n_k + l < n_{k+1}$, and $n_i + j < n_{i+1}$. Then,

$$\begin{aligned}
&F\,[(a_{n_i+j} + b_{n_k i+j}) - (a_{n_k+l} + b_{n_k+l})] \\
&= F\left[\left(a_{n_i+j} + \sum_{p=1}^{i-1} b_{n_p,n_{p+1}} + b_{n_i,n_i+j}\right) - \left(a_{n_k+l} + \sum_{p=1}^{k-1} b_{n_p,n_{p+1}} + b_{n_k,n_k+l}\right)\right] \\
&\leqq F\left[\left(a_{n_i+j} + \sum_{p=1}^{i-1} b_{n_p,n_{p+1}} + b_{n_i,n_i+j}\right) - \left(a_{n_i} + \sum_{p=1}^{i-1} b_{n_p,n_{p+1}}\right)\right] \\
&\quad + F\left[\left(a_{n_i} + \sum_{p=1}^{i-1} b_{n_p,n_{p+1}}\right) - \left(a_{n_k} + \sum_{p=1}^{k-1} b_{n_p,n_{p+1}}\right)\right] \\
&\quad + F\left[\left(a_{n_k} + \sum_{p=1}^{k-1} b_{n_p,n_{p+1}}\right) - \left(a_{n_k+l} + \sum_{p=1}^{k-1} b_{n_p,n_{p+1}} + b_{n_k,n_k+l}\right)\right] \\
&\leqq (2^{-i}) + (2^{-i} + 2^{-i-1} + \cdots + 2^{-k+1}) + (2^{-k}) \\
&\leqq (2^{-i}) + 2(2^{-i}) = 3 \cdot 2^{-i}.
\end{aligned}$$

The cases when $n_k + l = n_{k+1}$ and $n_i + j = n_{i+1}$ are treated similarly, and therefore we have established that the sequence $(a_n + b_n)$ is fundamental.

5.4 THEOREM. *If $T \in W_k(R^n)$, then there exist $T_i \in I_k(R^n)$ such that*

$$\lim_{i\to\infty} W(T_i - T) = 0.$$

Moreover, if A is a compact subset of R^n and

$$\lim_{i,j\to\infty} W(T_i - T_j) = 0$$

where $T_i \in I_k(A)$, then there exists $T \in W_k(A)$ such that

(i) $\lim_{i\to\infty} T_i = T$,

(ii) $\lim_{i\to\infty} W(T_i - T) = 0$.

Proof. If $T \in W_k(R^n)$, then $T = R + \partial S$ where R and S are rectifiable. Hence, there exist integral currents R_i and S_i such that

$$\lim_{i\to\infty} M(R_i - R) = 0, \quad \lim_{i\to\infty} M(S_i - S) = 0;$$

therefore, by 5.2 (2), (4)

$$\lim_{i\to\infty} W[(R_i + \partial S_i) - T] = 0.$$

Since $W(T_i) \geqq F(T_i)$ (see 5.2, (10)) there exists a current T with support contained in A such that

$$\lim_{i\to\infty} T_i = T.$$

It will now be proved that $T \in W_k(A)$.

Case I. Assume that T_i is an integral cycle, for $i = 1,2,3 \cdots$. Since $\lim_{i,j\to\infty} W(T_i - T_j) = 0$, by 5.2 (6) there exist integral currents $R_{i,j}$ such that $M(T - T_j - \partial R_{i,j}) + M(R_{i,j}) \to 0$, and such that the supports of the $R_{i,j}$ are contained in a fixed compact set K. Choose $z \in R^n$ to construct cones

$$z(T_i - T_j - \partial R_{i,j})$$

which have the property that

$$\partial z(T_i - T_j - \partial R_{i,j}) = T_i - T_j - \partial R_{i,j}$$

and

$$\lim_{i,j\to\infty} M[z(T_i - T_j - \partial R_{i,j})] = 0.$$

Letting S_i and $S_{i,j}$ be integral currents such that $\partial S_i = T_i$, support of S_i contained n some fixed compact set for $i = 1,2,\ldots,$

$$S_{i,j} = z(T_i - T_j - \partial R_{i,j}) + R_{i,j},$$

we have that

$$S_{i,j} = S_i - S_j - C_{i,j}$$

where $C_{i,j}$ is an integral cycle for $i,j = 1,2,3,\ldots$.

Since

$$\lim_{i,j\to\infty} M(S - S_j - C_{i,j}) = 0,$$

by 5.3 there exist cycles C_i such that

$$\lim_{i,j\to\infty} M[(S_i + C_i) - (S_j + C_j)] = 0,$$

and therefore, there exists a rectifiable current S such that

$$M(S_i + C_i - S) \to 0 \quad \text{and} \quad \lim_{i\to\infty} (S_i + C_i) = S.$$

Hence, $W[\partial(S_i + C_i) - \partial S] \to 0$ which implies $W(T_i - \partial S) \to 0$ and therefore $T = \partial S$.

Case II. The general case. As in Case I, we have the existence of integral currents $R_{i,j}$ with supports contained in a fixed compact set K and such that

$$M(T_i - T_j - \partial R_{i,j}) + M(R_{i,j}) \to 0.$$

But 5.3 supplies us with integral cycles S_i so that

$$\lim_{i,j\to\infty} M[(T_i + S_i) - (T_j + S_j)] = 0,$$

$$\operatorname{spt} S_i \subset K.$$

Hence, there exists a rectifiable current R with

$$M(T_i + S_i - R) \to 0 \text{ and } \lim_{i\to\infty} T_i + S_i = R.$$

Since $W(S_i - S_j) \to 0$, Case I supplies us with a rectifiable current G so that $W(S_i - \partial G) \to 0$. This implies

$$\lim_{i\to\infty} W[T_i - (R - \partial G)] = 0,$$

$$\lim_{i\to\infty} T_i = R - \partial G,$$

and therefore

$$T = R - \partial G.$$

5.5 Definition. If $\tau \in W_k(R^n, 2)$, let

$$W(\tau) = \inf\{M(\tau - \partial\sigma) + M(\sigma) \colon \sigma \in W_{k+1}(R^n, 2)\}.$$

5.6 Remark. One can easily check to find properties (1) through (8) of 5.2 valid when applied to classes. We have in addition for flat classes τ,

$$W(\tau) = \inf\{W(T) \colon T \in \tau\}.$$

However, it is not known whether $\tau = 0$ is equivalent to $W(\tau) = 0$ for flat classes τ. In view of the above property this is equivalent to determining whether the group $2W_k(A)$ is closed under W-convergence, where A is a compact subset of R^n. We are able to show that if τ is a rectifiable class and $W(\tau) = 0$, then $\tau = 0$, (Theorem 5.9).

5.7 Lemma. *Suppose μ is a real-valued function on R^n with Lipschitz constant ξ, $A_r = \{x \colon \mu(x) > r\}$ for $r \in R$. If τ_i are rectifiable classes such that $\lim_{i\to\infty} W(\tau_i) = 0$, then for L_1 almost all $r > 0$, there exists a subsequence $\{\tau_{i_k}\}$ (which depends upon r) such that*

$$\lim_{k\to\infty} W[\tau_{i_k} \cap A_r] = 0.$$

Proof. From 5.6 and 5.2 (6), we have the existence of rectifiable classes ρ_i and integral classes σ_i such that

(1) $$\tau_i = \rho_i + \partial\sigma_i \text{ and } M(\rho_i) + M(\sigma_i) \to 0.$$

For each i, 4.1 implies that

$$\int^* M[\partial(\sigma_i \cap A_r) - (\partial\sigma_i) \cap A_r]dr \leqq \xi M(\sigma_i);$$

therefore, letting

(2) $$f(r) = \liminf_{i\to\infty} M[\partial(\sigma_i \cap A_r) - (\partial\sigma_i) \cap A_r]$$

we have from Fatou's lemma that

$$\int^* f(r)dr = 0.$$

Hence, we now have from (2) that for L_1 almost all $r > 0$, there exists a subsequence such that

$$\lim_{k\to\infty} M\,[\partial(\sigma_{i_k} \cap A_r) - (\partial\sigma_{i_k}) \cap A_r] = 0.$$

Since $M(\sigma_i) \to 0$, we have that

$$\lim_{k\to\infty} W[(\partial\sigma_{i_k}) \cap A_r] = 0,$$

and now the conclusion follows from (1).

5.8 Lemma. *Suppose τ is a nonzero k-dimensional rectifiable class. Then, at $\|\tau\|$ almost all $x \in R^n$, we have the following: if $p: R^n \to P_x$ is the orthogonal projection of R^n onto P_x, the H^k approximate tangent plane to* car τ *at x and, if K_x^r is an open n-cube of side length $2r$ with center at x and with one of its k-faces parallel to P_x, then there exists $r_0(x) > 0$ such that $p_\#(\tau \cap K_x^r) \neq 0$* for *$r < r_0(x)$.*

Proof. Since car τ is a bounded, Hausdorff k-rectifiable set with finite H^k measure, [3, (8.16)] supplies us with a countable family F of k-dimensional proper regular submanifolds of class 1 of R^n such that, for H^k almost all $x \in$ car τ, the H^k approximate tangent k-vector to car τ at x is tangent to some member of F. Letting G_i be the members of F, we may assume that each G_i is bounded and has finite H^k measure. If we let $\alpha(x)$ denote a unit H^k approximate tangent k-vector to car τ at x, then with the aid of [4, § 3], the following sets have $\|\tau\|$ measure zero:

(1) $B_0 =$ car $\tau \cap \{x : \alpha(x)$ does not exist or is not tangent to G_i for some $i\}$,

$B_i =$ car $\tau \cap \{x : \overline{D}_k(\|\tau\|, R^n - G_i, x) \neq 0$, for $x \in G_i\}$, $i = 1,2,3,\cdots$,

$C_i =$ car $\tau \cap \{x : \overline{D}_k(\|G_i\|, R^n -$ car $\tau, x) \neq 0$ for $x \in$ car $\tau\}$, $i = 1,2,3,\cdots$, where the above densities have been computed with respect to cubes as stated in the lemma. Let $B = \operatorname{car}\tau - B_0 - \bigcup_{i=1}^{\infty} B_i - \bigcup_{i=1}^{\infty} C_i$ and choose $x \in B$. Now there exists a k-plane P_x which is the H^k approximate tangent plane to $\operatorname{car}\tau$ at x and is tangent to some G_i, say G. Given $0 < \varepsilon < 1$, there exists $r_0(x) > 0$ such that for $r < r_0(x)$,

(i) by (1), $M(\tau \cap K_x^r - G \cap K_x^r) < \varepsilon\beta(k)r^k$ where $\beta(k)$ is the volume of a k-cube of side length 2

(ii) the map $p \,|\, G \cap K_x^r$ is univalent and maps onto $P_x \cap K_x^r$, and therefore

$$M[p_{\#}(\tau \cap K_x^r) - p_{\#}(G \cap K_x^r)] < \varepsilon\beta(k)r^k,$$

which implies that $P_{\#}(\tau \cap K_x^r) \neq 0$ for $r < r_0(x)$.

5.9 THEOREM. *If τ is a k-dimensional rectifiable class and if $W(\tau) = 0$, then $\tau = 0$.*

Proof. Since τ is rectifiable, there exist integral classes σ_i such that $M(\tau - \partial\sigma_i) + M(\sigma_i) \to 0$. Choose $x \in \operatorname{car}\tau$ which satisfies the conditions of 5.8. Then, for L_1 almost all $r > 0$, 5.7 supplies a subsequence (depending on r) such that

$$\lim_{k\to\infty} W[(\partial\sigma_{i_k}) \cap K_x^r] = 0,$$

and therefore

$$\lim_{k\to\infty} M[p_{\#}((\partial\sigma_{i_k}) \cap K_x^r)] = 0$$

where p and K_x^r are as in 5.8. But

$$\lim_{k\to\infty} M[p_{\#}(\tau \cap K_x^r) - p_{\#}((\partial\sigma_{i_k}) \cap K_x^r)] = 0$$

which implies $p_{\#}(\tau \cap K_x^r) = 0$ for L_1 almost all $r > 0$, and now the conclusion of the theorem follows from 5.8.

5.10 COROLLARY. *If T_i are flat currents and T a rectifiable current with*

$$\lim_{i\to\infty} W(2T_i - T) = 0,$$

then $T/2$ is a rectifiable current.

Proof. See 5.6.

5.11 THEOREM. *If A is a compact subset of R^n, τ_i are rectifiable classes with supports contained in A for $i = 1,2,3,\ldots$, and $\lim_{i,j\to\infty} M(\tau_i - \tau_j) = 0$ then there exists a rectifiable class τ with support contained in A such that*

$$\lim_{i\to\infty} M(\tau_i - \tau) = 0.$$

Proof. Since $M(\tau_i - \tau_j) \to 0$, there exist rectifiable currents S_i and $Q_{i,j}$ such that $S_i \in \tau_i$ and $\lim_{i,j\to\infty} M(S_i - S_j - 2Q_{i,j}) = 0$. Therefore, from 5.3, we can find rectifiable currents R_i such that $R_i \in \tau_i$ and $\lim_{i,j\to\infty} M(R_i - R_j) = 0$; moreover, if we let $T_i = R_i \cap A$, we have the existence of a rectifiable current T so that

$$\lim_{i\to\infty} M(T_i - T) = 0.$$

Since $T_i \in \tau_i$, the conclusion follows if we let $\tau = T^{\sim}$.

5.12 Lemma. *If A is a compact subset of R^n, $\tau_i \in I_k(A,2)$ for $i = 1,2,3,\cdots$, and $\lim_{i,j\to\infty} W(\tau_i - \tau_j) = 0$, then there exists $\tau \in W_k(R^n,2)$ such that*

$$\lim_{i\to\infty} W(\tau_i - \tau) = 0.$$

Moreover, if $\tau \in W_k(R^n,2)$, then there exist $\tau_i \in I_k(R^n,2)$ such that

$$\lim_{i\to\infty} W(\tau_i - \tau) = 0.$$

Proof. By reasoning similar to that in the proof of 5.11, we can find integral currents R_i such that $R_i \in \tau_i$ and $\lim_{i,j\to\infty} W(R_i - R_j) = 0$. Let S be a closed n-ball which contains A and such that $R_i \cap S$ is integral for $i = 1,2,\cdots$. Note that $R_i = R_i \cap S + 2S_i$ where S_i is integral. Hence, letting p be the radial retraction of R^n onto S, we have that

$$p_\#(R_i) = R_i \cap S + 2p_\#(S_i)$$

and therefore, if we let $T_i = p_\#(R_i)$ we find that $T_i \in \tau_i$, $\operatorname{spt} T_i \subset S$. Now we can apply 5.4 to find a flat current T such that $W(T_i - T) \to 0$. The first part of the lemma now follows by taking $\tau = T^{\sim}$.

The proof of the second part is similar to the proof of the analogous statement 5.4.

5.13 Theorem. *If A is a compact subset of R^n, $\tau_i \in I_k(A,2)$ for $i = 1,2,3,\cdots$, and $\sup\{N(\tau_i) : i = 1,2,3,\cdots\} < \infty$, then there exists a subsequence $\{\tau_{i_j}\}$ and a $\tau \in W_k(R^n,2)$ such that*

$$\lim_{j\to\infty} W(\tau_{i_j} - \tau) = 0.$$

Proof. Let

$$\lambda = \sup\{13n^{k+1}\binom{n}{k}M(\tau_i) + 19n^k\binom{n}{k-1}M(\partial\tau_i) : i = 1,2,\cdots\}.$$

For $\varepsilon > 0$, one may use 4.2 to obtain sequences of integral classes π_i, ρ_i, σ_i whose supports are contained in

$$\{x : \text{distance }(x,A) \leqq 3n\varepsilon\}$$

and satisfy the conditions

$$\tau_i = \pi_i + \rho_i + \partial\sigma_i, \quad \pi_i \text{ a chain of } \mu_\varepsilon(C'),$$
$$M(\pi_i) \leqq \lambda(1+\varepsilon), \quad M(\rho_i) \leqq \lambda\varepsilon, \quad M(\sigma_i) \leqq \lambda\varepsilon.$$

The subcomplex of $\mu_\varepsilon(C')$ consisting of all cells within $3n\varepsilon$ of A is finite and therefore, one may replace the given sequences by subsequences such that, for all positive integers i and j,

$$M(\pi_i - \pi_j) = 0.$$

Hence,

$$\tau_i - \tau_j = (\pi_i - \pi_j + \rho_i - \rho_j) + \partial(\sigma_i - \sigma_j),$$

$$M(\rho_i - \rho_j) \leqq 2\lambda\varepsilon \ , \quad M(\sigma_i - \sigma_j) \leqq 2\lambda\varepsilon.$$

This process may be applied successively with $\varepsilon = (1/2), (1/2)^2, (1/2)^3, \cdots$, each time yielding subsequences of the preceding. Then Cantor's diagonal process supplies sequences whose ith terms belong to the subsequences corresponding to $\varepsilon = 1/2, (1/2)^2, \cdots, (1/2)^i$. The Cantor sequence has the following property: there exist integral classes χ_i and ψ_i whose supports are contained in

$$\{x : \text{distance}(x,A) \leqq n3^{1-i}\}$$

for which

$$\tau_i - \tau_{i+1} = \chi_i + \partial\psi_i,$$

$$M(\chi_i) \leqq \lambda 2^{1-i} \ , \quad M(\psi_i) \leqq \lambda 2^{1-i}.$$

Thus, the Cantor sequence is fundamental in the W-sense and therefore, the conclusion of the theorem follows from 5.12.

REMARK. This argument is essentially used in [3, (7.1)].

5.14 THEOREM. *Suppose τ_i and τ are k-dimensional rectifiable classes for $i = 1,2,3, \cdots$. If*

$$\lim_{i \to \infty} W(\tau_i - \tau) = 0$$

then

$$\liminf_{i \to \infty} M(\tau_i) \geqq M(\tau).$$

Proof. After passing to a suitable subsequence, we may assume that

$$\lim_{i \to \infty} M(\tau_i) = L < \infty,$$

for if not, the theorem would trivially follow. Therefore, again passing to a subsequence if necessary, we may assume the existence of a non-negative Radon measure μ over R^n such that $\|\tau_i\|$ converges weakly to μ.

As in the proof of 5.8, G_i will denote the elements of a countable family F of k-dimensional proper regular submanifolds of class 1 of R^n such that, for H^k almost all $x \in \operatorname{car}\tau$, the H^k approximate tangent k-vector to $\operatorname{car}\tau$ at x is tangent to some member of F. Also, $\alpha(x)$ will denote a unit H^k approximate tangent k-vector to $\operatorname{car}\tau$ at x and as in 5.8, $B \subset \operatorname{car}\tau$ will be a set with the following properties:

(i) $H^k(\operatorname{car}\tau - B) = 0$.

(1) (ii) For each $x \in B$, $\alpha(x)$ exists and is tangent to some G_i.

(iii) $D_k^-(\|\tau\|,\ R^n - G_i,x) = D_k^-(\|G_i\|,\ R^n - \operatorname{car}\tau,x) = 0$ for $x \in B$ and for $i = 1,2,3,\cdots$.

It will be shown that $D_k^-(\mu,R^n,x) \geqq 1$ for $x \in B$. From [4, § 3], this will imply that

$$\mu(R^n) \geqq H^k(B) = M(\tau),$$

and since $L = \mu(R^n)$, the desired conclusion will follow.

The proof is by contradiction: assume there exists a point $x \in B$ and a number $\varepsilon > 0$ such that

$$D_k^-(\mu,R^n,x) < 1 - \varepsilon.$$

If we let P denote the H^k approximate tangent k-plane to $\operatorname{car}\tau$ at x, p the orthogonal projection of R^n onto P, and G a member of F such that $\alpha(x)$ is tangent to G at x, then there exists a number $r_0 > 0$ with the following properties:

(i) if $r \leqq r_0$ and if $f_r = p^{-1} \mid P \cap S(x,r)$, then the mapping $f_r : P \cap S(x,r) \to U_r \subset G$ is of class C^1, univalent, and

$$|f_r(y) - y| < \varepsilon r(12 \cdot 4^{k+1} \cdot a^k)^{-1}$$

where $a = \max(1, \text{Lipschitz constant of } f_{r_0})$,

(ii) $M[\tau \cap S(x,r) - G \cap S(x,r)] < \varepsilon\alpha(k)6^{-1}r^k$ for $r \leqq r_0$, from (1),

(2) (iii) $U_r \cap S(x,r) = G \cap S(x,r)$ for $r \leqq r_0$,

(iv) $\mu[S(x,r)] < (1-\varepsilon)\alpha(k)r^k$ for $r \leqq r_0$.

Choose $r_0/2 < t < r_0$ and let

$$E = [r_0/4,\ r_0/2] \cap \{r : \mu(\{y : \text{distance } (y,x) = r\}) = 0\}$$

and note that

$$L_1([r_0/4,\ r_0/2] - E) = 0.$$

Since $W(\tau_i - \tau) \to 0$, there exist rectifiable classes ρ_i and integral classes σ_i such that

$$\tau_i - \tau = \rho_i + \partial\sigma_i \text{ and } M(\rho_i) + M(\sigma_i) \to 0.$$

Therefore, by 4.1,

$$\lim_{i \to \infty} \int_E^* M[\partial(\sigma_i \cap S(x,r)) - (\partial\sigma_i) \cap S(x,r)]dr = 0$$

and thus, by Fatou's lemma, there exists a set $F \subset E$ such that $L_1(E - F) = 0$ and for each $r \in F$, there is a subsequence (which depends on r) such that

$$M[\partial(\sigma_i \cap S(x,r)) - (\partial\sigma_i) \cap S(x,r)] < \varepsilon 3^{-1}\alpha(k)(r_0/4)^k \tag{3}$$

for all i of the subsequence. If i is the identity map $i : P \cap S(x,r_0) \to P \cap S(x,r_0)$, h the linear homotopy from i to f_{r_0},

$$\lambda = h_\#[I \times \partial(P \cap S(x,t))], \text{ and } \zeta = h_\#[I \times (P \cap S(x,t))],$$

then from 3.13, we have

$$\partial\zeta = U_t - P \cap S(x,t) - \lambda,$$

$$M(\zeta) \leqq \varepsilon\alpha(k)(12 \cdot 4^{k+1})^{-1} r_0^{k+1}, \tag{4}$$

$$M(\lambda) \leqq \varepsilon\alpha(k)(12 \cdot 4^{k+1})^{-1} r_0^k.$$

From 4.1,

$$\int_F^* M[\partial(\zeta \cap S(x,r)) - (\partial\zeta) \cap S(x,r)]\, dr \leqq \varepsilon\alpha(k)\ (12 \cdot 4^{k+1})^{-1} r_0^{k+1}$$

and therefore, there exists $r_1 \in F$ such that,

$$L_1(F) M[\partial(\zeta \cap S(x,r_1)) - (\partial\zeta) \cap S(x,r_1)] \leqq \varepsilon\alpha(k)(12 \cdot 4^{k+1})^{-1} r_0^{k+1}$$

and

$$M(\chi) \leqq \varepsilon \cdot 12^{-1} \cdot \alpha(k)(r_0/4)^k \leqq \varepsilon \cdot 12^{-1} \cdot \alpha(k) r_1^k \tag{5}$$

where

$$\chi = \partial(\zeta \cap S(x,r_1)) - (\partial\zeta) \cap S(x,r_1).$$

Since $r_1 < t$ and $\partial\zeta = U_t - P \cap S(x,t) - \lambda$, we have

$$\partial(\zeta \cap S(x,r_1)) = G \cap S(x,r_1) - P \cap S(x,r_1) - (\lambda \cap S(x,r_1) + \chi)$$

from (iii) of (2). Abbreviating $\zeta_0 = \zeta \cap S(x,r_1)$ and $\lambda_0 = \lambda \cap S(x,r_1) + \chi$, one obtains from (4) and (5)

$$\partial\zeta_0 = G \cap S(x,r_1) - P \cap S(x,r_1) - \lambda_0,$$

$$M(\lambda_0) \leqq \varepsilon \cdot 6^{-1} \cdot \alpha(k) r_1^k$$

But with the aid of (ii) of (2),

$$\tau \cap S(x,r_1) = P \cap S(x,r_1) + \partial\zeta_1 + \lambda_1$$

where

$$\zeta_0 = \zeta_1,\ \lambda_1 = \lambda_0 + \tau \cap S(x,r_1) - G \cap S(x,r_1),$$

$$M(\lambda_1) \leqq \varepsilon \cdot 3^{-1} \cdot \alpha(k) r_1^k.$$

Since $\partial[\partial(\sigma_i \cap S(x,r_1) + \phi_1) - P \cap S(x,r_1)] = \partial[P \cap S(x,r_1)]$, one finds that

$$M[\partial(\sigma_i \cap S(x,r_1) + \zeta_1) - P \cap S(x,r_1)] \geqq \alpha(k) r_1^k, \text{ for } i = 1,2,\cdots.$$

Letting $\psi_i = \partial(\sigma_i \cap S(x,r_1)) - (\partial\sigma_i) \cap S(x,r_1)$, we have

$$\begin{aligned} &\partial(\sigma_i \cap S(x,r_1)) + \partial\zeta_1 - P \cap S(x,r_1) \\ &\quad = (\partial\sigma_i) \cap S(x,r_1) + \psi_i + \partial\zeta_i - P \cap S(x,r_1) \\ &\quad = (\partial\sigma_i) \cap S(x,r_1) + \psi_i + \tau \cap S(x,r_1) - P \cap S(x,r_1) - \lambda_1 - P \cap S(x,r_1) \end{aligned}$$

$$= \tau_i \cap S(x,r_1) - \tau \cap S(x,r_1) - \rho_i \cap S(x,r_1) + \psi_i + \tau \cap S(x,r_1) - \lambda_1$$

$$= \tau_i \cap S(x,r_1) - \rho_i \cap S(x,r_1) + \psi_i - \lambda_i$$

and therefore for $i = 1,2,3,\cdots$,

$$M[\tau_i \cap S(x,r_1)] + M[\rho_i \cap S(x,r_1)] + M(\psi_i) + M(\lambda_1) \geqq \alpha(k)r_1^k.$$

But (3) implies

$$M(\psi_i) \leqq \varepsilon \cdot 3^{-1} \cdot \alpha(k)r_1^k \text{ for all } i \text{ of a suitable subsequence,}$$

$$M[\rho_i \cap S(x,r_1)] < \varepsilon \cdot 3^{-1} \cdot \alpha(k)r_1^k \text{ for large } i,$$

$$M(\lambda_1) < \varepsilon \cdot 3^{-1} \cdot \alpha(k)r_1^k.$$

Hence,

$$M[\tau_i \cap S(x,r_1)] \geqq (1-\varepsilon)\alpha(k)r_1^k \text{ for all } i \text{ of some subsequence,}$$

or

$$\|\tau_i\|[S(x,r_1)] \geqq (1-\varepsilon)\alpha(k)r_1^k \text{ for all } i \text{ of some subsequence,}$$

which implies

$$\mu[S(x,r_1)] \geqq (1-\varepsilon)\alpha(k)r_1^k,$$

a contradiction to (iv) of (2).

REMARK. We may now regard the integrand in 4.1 as an L_1 integrable function.

5.15 COROLLARY. *Suppose τ_i and τ are integral classes for $i = 1,2,\cdots$. If*

$$\lim_{i\to\infty} W(\tau_i - \tau) = 0,$$

then

$$\liminf_{i\to\infty} N(\tau_i) \geqq N(\tau).$$

Proof. This is evident since $W(\tau) \geqq W(\partial\tau)$.

6. **Extreme dimensions.** In this section, it is shown that 0, 1, and n-dimensional integral classes in R^n are, in a sense, orientable and this fact is then used to establish a compactness theorem in the extreme dimensions.

6.1 LEMMA. *If $\tau \in I_k(R^n, 2)$ and $\varepsilon > 0$, then there exists a polyhedral chain π (with integer coefficients) in R^n and a diffeomorphism f of class 1 mapping R^n onto R^n, such that*

f and f^{-1} have Lipschitz constant $1+\varepsilon$,

$|f(x) - x| < \varepsilon$ for $x \in R^n$.

$f(x) = x$ wherever distance $(x, \operatorname{spt} \tau) \geqq \varepsilon$,

$\operatorname{spt} \pi \subset \{x : \text{distance } (x, \operatorname{spt} \tau) \leqq \varepsilon\}$,

$N[\pi - f_{\#}(\tau)] < \varepsilon$.

Proof. Choose an integral current $T \in \tau$ such that spt $T \subset S(\operatorname{spt}\tau, \varepsilon/2)$ and apply [3, (8.22)] to obtain a class 1 diffeomorphism f and a polyhedral chain P such that

$$f \text{ and } f^{-1} \text{ have Lipschitz constant } 1 + \varepsilon/2,$$
$$|f(x) - x| < \varepsilon/2 \text{ for } x \in R^n,$$
$$f(x) = x \text{ whenever distance } (x, \operatorname{spt} T) \geqq \varepsilon/2,$$
$$\operatorname{spt} P \subset \{x : \text{distance } (x, \operatorname{spt} T) \leqq \varepsilon/2\},$$
$$N[P - f_{\#}(T)] < \varepsilon/2.$$

The conclusion now follows by taking $\pi = [P]^{-}$.

6.2 Lemma. *For each integral class τ there exists a sequence of polyhedral chains π_i (with integer coefficients) such that*

$$\operatorname{spt} \pi_i \subset \{x : \textit{distance } (x, \operatorname{spt} \tau) \leqq i^{-1}\}$$
$$\lim_{i \to \infty} W(\pi_i - \tau) = 0,$$
$$\lim_{i \to \infty} N(\pi_i) = N(\tau).$$

Proof. Given $\varepsilon > 0$, use 6.1 to find f and π such that

$$N(\pi) < N[f_{\#}(\tau)] + \varepsilon \leqq (1 + \varepsilon)^k N(\tau) + \varepsilon,$$
$$\operatorname{spt} \pi \subset \{x : \text{distance}(x, \operatorname{spt} \tau) \leqq \varepsilon\}.$$

If h is the linear homotopy from f to the identity map of R^n, then

$$\begin{aligned} W[\tau - f_{\#}(\tau)] &\leqq M[h_{\#}(I \times \tau)] + M[h_{\#}(I \times \partial\tau)] \\ &\leqq \varepsilon(1 + \varepsilon)^k M(\tau) + \varepsilon(1 + \varepsilon)^{k-1} M(\partial\tau), \end{aligned}$$

hence,

$$\begin{aligned} W(\pi - \tau) &\leqq M[\pi - f_{\#}(\tau)] + W[f_{\#}(\tau) - \tau] \\ &\leqq \varepsilon[1 + (1 + \varepsilon)^k N(\tau)]. \end{aligned}$$

Now, by appealing to 5.15, the lemma is established.

6.3 Theorem. *If $k = 0,1$, or n, and $\tau \in I_k(R^n, 2)$, then there exists an integral current $T \in \tau$ such that*

$$N(T) = N(\tau).$$

Proof. If π is a k-dimensional polyhedral chain for $k = 0,1,n$, then it is easy to verify that there exists a polyhedral chain $P \in \pi$ such that

$$N(P) = N(\pi).$$

6.2 supplies a sequence of polyhedral chains π_i such that

$$\operatorname{spt}\pi_i \subset \{x : \operatorname{distance}(x, \operatorname{spt}\tau) \leqq i^{-1}\},$$
$$W(\pi_i - \tau) \to 0 \text{ and } N(\pi_i) \to N(\tau).$$

Let $P_i \in \pi_i$ be a polyhedral chain such that $N(P_i) = N(\pi_i)$, for $i = 1,2,3,\cdots$. Since the P_i are N-bounded [3, (8.13)] implies the existence of an integral current T such that, for a suitable subsequence,

$$\lim_{i\to\infty} P_i = T,$$

and by the remark in 5.2,

$$\lim_{i\to\infty} W(P_i - T) = 0.$$

Now, 5.6 and 5.9 imply that $T \in \tau$ and therefore

$$N(\tau) \leqq N(T).$$

But, $N(\tau) = \lim_{i\to\infty} N(\pi_i) = \lim_{i\to\infty} N(P_i) \geqq N(T)$ and hence, $N(\tau) = N(T)$.

6.4 Corollary. *Suppose A is a compact subset of R^n, $k = 0,1,n$, $\tau_i \in I_k(A,2)$ and* $\sup\{N(\tau_i)\colon i = 1,2,3,\cdots\} < \infty$. *Then, for a suitable subsequence, there exists $\tau \in I_k(A,2)$ such that*

$$\lim_{i\to\infty} W(\tau_i - \tau) = 0.$$

Proof. For each i, use 6.3 to find an integral current $T_i \in \tau_i$ such that

$$N(T_i) = N(\tau_i) \quad \text{for} \quad i = 1,2,3,\cdots .$$

By [3, (8.13)], one can find a subsequence and an integral current T such that

$$\lim_{i\to\infty} T_i = T,$$

and by the remark in 5.2,

$$\lim_{i\to\infty} W(T_i - T) = 0.$$

By taking $\tau = [T]^-$, the conclusion follows from 5.6.

6.5 Corollary. *If $\tau \in I_{n-1}(R^n,2)$ is a cycle, then there exists an integral current $T \in \tau$ such that,*

$$N(\tau) = N(T).$$

Proof. Use 3.14 to find $\sigma \in I_n(R^n,2)$ such that $\partial\sigma = \tau$. By 6.3 there exists an integral current $S \in \sigma$ such that $N(S) = N(\sigma)$. Now let $T = \partial S$ to establish the conclusion.

6.6 Theorem. *Suppose A is a compact subset of $R^n, \tau_i \in I_{n-1}(A,2)$ with* $\sup\{N(\tau_i)\colon i = 1,2,3,\cdots,\} < \infty$. *Then, for a suitable subsequence, there exists a rectifiable class τ such that*

$$\lim_{i\to\infty} W(\tau_i - \tau) = 0.$$

Proof. From 5.13, we know there exists a subsequence such that

$$\lim_{i,j\to\infty} W(\tau_i - \tau_j) = 0,$$

and therefore, there exist integral classes ρ_{ij} such that

$$M(\tau_i - \tau_j - \partial\rho_{ij}) + M(\rho_{ij}) \to 0 \text{ as } i,j \to \infty .$$

The ρ_{ij} can be chosen so that their supports are contained in some fixed compact set. Now 5.3 implies the existence of integral cycles σ_i such that

$$\lim_{i,j\to\infty} M[(\tau_i + \sigma_i) - (\tau_j + \sigma_j)] = 0$$

and hence, by 5.11, we have the existence of a rectifiable class ρ such that

$$\lim_{i\to\infty} M[(\tau_i + \sigma_i) - \rho] = 0.$$

The σ_i are N-bounded and therefore, by 6.5 and reasoning similar to that in 6.4, by passing to another subsequence, we have an integral cycle σ such that

$$\lim_{i\to\infty} W(\sigma_i - \sigma) = 0.$$

This implies that

$$\lim_{i\to\infty} W[\tau_i - (\rho - \sigma)] = \lim_{i\to\infty} W[(\tau_i + \sigma_i - \rho) - (\sigma_i - \sigma)] = 0,$$

and therefore, the conclusion follows by taking

$$\tau = \rho - \sigma.$$

6.7 Remark. Suppose A is a compact subset of R^n and assume the following statement to be true: If $\sigma_i \in I_k(A,2)$ are cycles with $\sup\{N(\sigma_i): i = 1,2,3,\cdots,\} < \infty$; then, for a suitable subsequence, there exists an integral class σ such that

$$\lim_{i\to\infty} W(\sigma_i - \sigma) = 0.$$

Then, the following general statement holds: If $\tau_i \in I_k(A,2)$ and

$$\sup\{N(\tau_i): i = 1,2,3,\cdots,\} < \infty;$$

then, for a suitable subsequence, there exists a rectifiable class τ such that

$$\lim_{i\to\infty} W(\tau_i - \tau) = 0.$$

This follows from the proof of 6.6.

6.8 Remark. Some results have been obtained that leads one to believe that the assumption stated in 6.7 is, in fact, true. The validity of this assumption would then lead to a closure result for k-dimensional integral classes, where k is arbitrary. See 6.6. These results will be stated here and their proofs will appear at a later date.

If $\tau \in W_k(R^n,2)$, let

$$L(\tau) = \inf\{\liminf_{i\to\infty} M(\tau_i): \tau_i \in I_k(R^n,2), \partial\tau_i = 0, W(\tau_i - \tau) \to 0\}$$

and let τ_i be a sequence of integral cycles for which

$$\lim_{i\to\infty} W(\tau_i - \tau) = 0 \text{ and } \lim_{i\to\infty} M(\tau_i) = L(\tau).$$

Assuming that $L(\tau) < \infty$, we have the existence of a non-negative Radon measure μ over R^n such that, after passing to a suitable subsequence, $\|\tau_i\|$ converges weakly to μ. It can be shown that

$$D_k^-(\mu,R^n,x) > 0$$

for μ almost all $x \in R^n$. Let

$$A_3 = \{x\colon\ D_k^-(\mu,R^n,x) = \infty\}$$

and assume that $\mu(A_3) = 0$. Then, from [4, § 9] we know that R^n can be decomposed into four μ measurable sets A_1, A_2, A_3, A_4 such that:

(1) $R^n = A_1 \cup A_2 \cup A_3 \cup A_4$,

(2) $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = 0$,

(3) A_1 is a countably k-rectifiable set and at each point x of A_1, there exists a μ approximate tangent k-plane to A_1 at x,

(4) either $\mu(A_2) = 0$ or A_2 contains no k-rectifiable subset B for which $\mu(B) > 0$,

(5) $L_k[p(A_2)] = 0$ for almost all orthogonal projections of R^n onto R^k,

(6) $H^k(A_3) = 0$ since $\mu(R^n) < \infty$ and by [4, (3.1)]. By our assumption, $\mu(A_3) = 0$,

(7) $\mu(A_4) = 0$.

If, in addition to these facts, we assume that $H^k(A_1) < \infty$ and $\mu(A_2) = 0$, then it can be shown that there exists a Hausdorff k-rectifiable set $A \cap E \subset A_1$ such that $\mu(A_1 - A \cap E) = 0$ and

$$\lim_{i\to\infty} W(\tau_i - A \cap E) = 0.$$

Since the Hausdorff k-rectifiable set $A \cap E$ is to be identified with a k-dimensional rectifiable class, the desirable closure result is obtained under the stated assumptions.

In the case of the Plateau problem, a little more is known. Suppose $\sigma \in I_{k-1}(R^n,2)$ is a cycle and let

$$\Omega(\sigma) = \inf\{M(\tau)\colon\ \tau \in I_k(R^n,2),\quad \partial\tau = \sigma\}.$$

Suppose $\{\tau_i\}$ is a sequence of integral classes such that $\partial\tau_i = \sigma$ and $\lim_{i\to\infty} M(\tau_i) = \Omega(\sigma)$. By passing to a subsequence, we may assume the existence of a non-negative Radon measure μ such that $\|\tau_i\|$ converges weakly to μ. Then, it can be shown that

(i) for all $x \notin \operatorname{spt}\sigma$,

$$D_k^-(\mu,R^n,x) \leqq \Omega(\sigma)[\alpha(k)r^k]^{-1}$$

where $r = \text{distance }(x, \operatorname{spt}\sigma)$,

(ii) for μ almost all $x \in R^n - \operatorname{spt}\sigma$,

$$D_{-k}(\mu, R^n, x) \geqq [k \cdot \alpha(k)^{1/k} \cdot 2^{k-1} \cdot C_2]^{-k}$$

where C_2 is as in 4.6 with k replaced by $k-1$.

These results are analogous to known theorems concerning the Plateau problem; cf. E. R. Reifenberg, Acta Math. **104** (1960), 1–92 and [3, (9.13)]. By employing the methods described above, it is hoped that the limiting measure μ will provide an integral class as a solution.

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