SOME CALCULATIONS OF HOMOTOPY GROUPS
OF SYMMETRIC SPACES

BY
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Introduction. We calculate the first few unstable homotopy groups of the symmetric spaces \( \Gamma_n = SO_{2n}/U_n \) and \( X_n = U_{2n}/Sp_n \) and of \( Sp_n \). The homotopy groups of \( \Gamma_n \) are needed in studying the existence of almost complex structures and knowledge of the first unstable group \( \pi_{2n-1}(\Gamma_n) \) is used in a paper of W. S. Massey [6]; in fact it was Professor Massey who first suggested to us the calculation of \( \pi_{2n-1}(\Gamma_n) \) for \( n \equiv 0 \pmod{4} \) (the other three parities of \( n \) are worked out by him), and suggested to us the use of some fibrations involving \( \Gamma_n \), or \( X_n \), and spheres. Similarly, \( X_n \) is connected with "almost quaternion" structures. We rely heavily on Kervaire’s calculations [4].

The space \( X_n \) possesses an involution \( \sigma \), induced by the involutory automorphism of \( U_{2n} \) leaving \( Sp_n \) fixed. This automorphism of \( U_{2n} \) extends to an inner automorphism of \( SO_{4n} \) and so induces a map \( \sigma \) of period two on \( \Gamma_{2n} \). We also study the effect of \( \sigma \) on homotopy groups; this is useful information, as shown in [2; 3].

The results are summarized in the following tables (the precise definition of \( \sigma \) and other notation will be given following the tables):

The groups \( \pi_{2n+r}(\Gamma_n) \):

\[
\begin{array}{cccccc}
 r \backslash n & 4k & 4k + 1 & 4k + 2 & 4k + 3 \quad (k > 0) \\
-1 & Z + Z_2 & Z_{(n-1)1} & Z & Z_{(n-1)1/2} \\
0 & Z_2 & Z_2 & 0 & 0 \\
1 & Z_{n1} + Z_2 & Z & Z_{n1} \text{ or } Z_{n1/2} + Z_2 & Z + Z_2 \\
3 & Z & & & & \\
\end{array}
\]

If \( n = 4k \) or \( 4k + 2 \), then \( \sigma \) is the identity except for the cases \( r = 1, n = 4k \) or \( 4k + 2 \). The effect of \( \sigma \) on some of the other cases is also determined.

The groups \( \pi_{4n+r}(X_n) \):

\[
\begin{array}{cc}
 r \backslash n & 2k \quad 2k + 1 \quad (k > 0) \\
0 & Z_{(2n)1} \quad Z_{(2n)1/2} \\
1 & Z_2 \\
3 & Z_2 \\
\end{array}
\]

\( \sigma = -1 \) in all cases (i.e., \( \sigma(x) = -x \)).

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The groups $\pi_{4n+r}(Sp_n)$:

<table>
<thead>
<tr>
<th>$r$</th>
<th>$n$</th>
<th>2k</th>
<th>2k + 1</th>
<th>$(k &gt; 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td>$Z_{(2n+1)!}$</td>
<td>$Z_{2(2n+1)!}$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td></td>
</tr>
</tbody>
</table>

**Notations.** $U_n$ is imbedded in $SO_{2n}$ as the subset of matrices consisting of $2 \times 2$ blocks

\[
\begin{pmatrix}
  a & b \\
  -b & a
\end{pmatrix}
\]

Let $K_{2n}$ denote the $2n \times 2n$ matrix having alternately +1, -1, down the main diagonal, and zeros elsewhere. $K_{2n}$ belongs to $SO_{2n}$ if and only if $n$ is even. Conjugation by $K_{2n}$ induces an automorphism $\sigma$ in $SO_{2n}$, and induces the complex conjugation map in $U_n$ (if the $2 \times 2$ block represents the complex number $a + ib$). The induced map in $SO_{2n}/U_n = \Gamma_n$ is also written $\sigma$. $SO_{2n}$ is imbedded in $SO_{2n+r}$ as the upper left hand block. Conjugation by $K_{2n+2}$ in $SO_{2n+2}$ maps $U_n, U_{n+1}, SO_{2n}, SO_{2n+1}$ into themselves and induces $\sigma$ in $U_n, SO_{2n}$. Denote by $\sigma$ again the induced map of $SO_{2n+1}$. The induced map $\sigma$ in $SO_{2n}/U_n = \Gamma_n, SO_{2n+1}/U_n, SO_{2n+2}/U_{n+1} = \Gamma_{n+1}$ is compatible with the natural maps

$$\Gamma_n \subset SO_{2n+1}/U_n \rightarrow \Gamma_{n+1}.$$  

The natural map $SO_{2n+1}/U_n \rightarrow SO_{2n+2}/U_{n+1} = \Gamma_{n+1}$ is 1-1 and onto (the two manifolds having the same dimension) and will be used to identify these spaces. The fibration

$$SO_{2n}/U_n \rightarrow SO_{2n+1}/U_n \rightarrow S^{2n}$$

can then be written as $\Gamma_n \rightarrow \Gamma_{n+1} \rightarrow S^{2n}$. The induced map $\sigma$ on $S^{2n}$ is of degree $(-1)^n$.

$Sp_m$ is the subset of $U_{2m}$ of fixed points of the automorphism $\tau: A \rightarrow J^{-1} \bar{A} J$ where $\bar{A}$ denotes the complex conjugate matrix, and $J$ is the $2m \times 2m$ matrix with blocks

\[
\begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix}
\]
down the main diagonal and zeros elsewhere. Since $J \in U_{2m}$, this automorphism is homotopic to the complex conjugation automorphism $\sigma$. Extend $\tau$ to $U_{2m+1}$ by the formula

$$B \rightarrow J_1^{-1} \bar{B} J_1$$

where $J_1$ is the $(2m + 1) \times (2m + 1)$ matrix consisting of $J$ in the upper left hand block, 1 in the lower right hand corner, zeros elsewhere.
If \( \tau \) is defined on \( U_{2m+2} \) by the same formula as on \( U_{2m} \), and \( U_{2m} \to^j U_{2m+1} \to^j U_{2m+2} \) denote inclusions, then \( \tau i = i \tau \) and \( \tau j \) is homotopic to \( j \tau \).

Finally, \( \tau \) induces involutions on \( X_m = U_{2m}/Sp_m \), \( X_{m+1} = U_{2m+2}/Sp_{m+1} \), and \( U_{2m+1}/Sp_m \), and the natural maps between these spaces commute with \( \tau \) up to homotopy. Just as for \( \Gamma_n \), we have a natural homeomorphism \( U_{2m+1}/Sp_m \to X_{m+1} \),

and a fibration

\[
X_m \to X_{m+1} \to S^{4m+1}.
\]

The induced map \( \tau \) on \( S^{4m+1} \) has degree \((-1)\). In the future we shall not distinguish the various homotopic maps defined by \( \tau \).

**Calculations of the groups** \( \pi_i(\Gamma_n) \). The first unstable homotopy group of \( \Gamma_n \) is \( \pi_{2n-1}(\Gamma_n) \). For \( i < 2n - 1 \), \( \pi_i(\Gamma_n) \approx \pi_{i+1}(SO(I)) \) (I large).

For convenience, we will assume always that \( n \equiv 0 \pmod 4 \), \( n \neq 0 \), and calculate the homotopy groups of \( \Gamma_{n+r} \), \( 0 \leq r \leq 3 \).

The only difficult calculation is the following:

**Theorem.** \( \pi_{2n-1}(SO_{2n}/U_n) = \mathbb{Z} + \mathbb{Z}_2 \), with \( \sigma = \text{identity} \) (\( n \equiv 0 \pmod 4 \), \( n \neq 0 \)).

**Proof.** We need the following lemma (compare [5]):

**Lemma.** Let \( j: U_n \to SO_{2n} \) be the inclusion described above, and \( k: SO_{2n} \to U_{2n} \) the natural inclusion. Under the composite map \( kj \), a generator of the group \( \pi_{2n-1}(U_n) = \mathbb{Z} \) goes into twice a generator of \( \pi_{2n-1}(U_{2n}) = \mathbb{Z} \).

**Proof of lemma.** We will show that if \( A \) is an \( n \times n \) matrix in \( U_n \), then \( kj(A) \) is conjugate to the \( 2n \times 2n \) matrix

\[
\begin{pmatrix}
A & 0 \\
0 & A
\end{pmatrix}.
\]

Recall that the map \( j \) consists of replacing the entries \( a_{ij} = b_{ij} + (-1)^{1/2}c_{ij} \) of \( A \) by \( 2 \times 2 \) blocks. If \( M \) denotes the matrix with entries

\[
M_{ij} = \delta_{2i-1,j}, \quad \text{for } 1 \leq i \leq n,
\]

\[
= \delta_{2(i-n),j}, \quad \text{for } n < i \leq 2n,
\]

and \( N \) the matrix

\[
\frac{1}{2^{1/2}} \begin{pmatrix}
I_n & -(-1)^{1/2}I_n \\
-(-1)^{1/2}I_n & I_n
\end{pmatrix}
\]

(both are in \( U_{2n} \)) then

\[
NM(kj(A))M^{-1}N^{-1} = \begin{pmatrix}
A & 0 \\
0 & A
\end{pmatrix} \in U_{2n}.
\]

If \( i \) is the usual inclusion of \( U_n \) in \( U_{2n} \),
and
\[ i'(A) = \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix} \]

so that \( i' \) is homotopic to \( i \), then \( kj(A) \) is homotopic to \( i(A)i'(A) \) or to \( i(A)i(A) \). Thus if \( x \in \pi_{2n-1}(U_n) \) then
\[ kj(x) = i(x) + \sigma i(x). \]

But \( \sigma i(x) = i(x) \), and \( kj(x) = 2i(x) \), since \( \pi_{2n-1}(U_{2n}) \to \pi_{2n-1}(SO_{4n}) \) is a monomorphism (\( \pi_{2n}(SO_{4n}/U_{2n}) = Z_2 \) for \( n \equiv 0 \pmod{4} \)), and \( \sigma \) is inner in \( SO_{4n} \).

Finally, \( i: \pi_{2n-1}(U_n) \to \pi_{2n-1}(U_{2n}) \) is an isomorphism, and the conclusion of the lemma follows. Q.E.D. for the lemma.

Next we consider the exact sequence
\[ \pi_{2n-1}(SO_{2n-1}) \to \pi_{2n-1}(SO_{2n}) \to \pi_{2n-2}(SO_{2n-1}) = Z_2, \]

namely (see [4]),
\[ 0 \to Z \to Z + Z \to Z \to Z_2 \to 0. \]

Let \( x \) generate \( \pi_{2n-1}(SO_{2n-1}) \), \( y \) and \( z \) generate \( Z + Z = \pi_{2n-1}(SO_{2n}) \) and \( \pi_{2n-1}(S^{2n-1}) = Z \).

Let \( T: S^{2n-1} \to SO_{2n} \) be the characteristic map [7, §23], and \( R \) the automorphism of period 2 in \( SO_{2n} \) leaving \( SO_{2n-1} \) pointwise fixed and inducing a map \( R \) of degree 1 in \( S^{2n-1} \). If \( s \in S^{2n-1} \), \( s = p(A) \) for \( A \in SO_{2n} \), then \( T(s) = AR(A)^{-1} \). Hence \( RT(s) = T(s)^{-1} \) and \( RT(t_{2n-1}) = -T(t_{2n-1}) \).

Also \( PT(t_{2n-1}) = 2T(t_{2n-1}) \) generates the image of \( p \) in \( \pi_{2n-1}(S^{2n-1}) \). Thus \( \pi_{2n-1}(SO_{2n}) \) is the direct sum of Image \( i \) and the subgroup generated by \( T(t_{2n-1}) \), so we may take \( y = i(x) \), \( z = T(t_{2n-1}) \) and so \( R(z) = -z \), \( R(y) = y \). We note that under \( k: SO_{2n} \to U_{2n} \) \( z \) maps into zero, since \( R \) becomes inner in \( U_{2n} \), and \( \pi_{2n-1}(U_{2n}) = Z \), (for, \( k(z) = RK(z) = kR(z) = -k(z) \)).

Now consider the (commutative) diagram
\[
\begin{array}{ccc}
\pi_{2n-1}(U_n) & \to & \pi_{2n-1}(SO_{2n}) \\
\downarrow{\rho'} & & \downarrow{\rho} \\
\pi_{2n-1}(S^{2n-1}) & & \\
\end{array}
\]

We may choose the generator \( x \) of \( \pi_{2n-1}(U_n) \) so that \( p'(x) = (n - 1)! \cdot \tau_{2n-1} \) (since \( \pi_{2n-2}(U_{n-1}) = \mathbb{Z}_{(n-1)!} \)), and, if \( j(x) = ry + sz \) then \( s = (n - 1)!/2 \), (since \( p(z) = 2t_{2n-1} \), \( p(y) = 0 \), \( p' = p' \)).

Next we show that \( r = 2 \); for, under
\[ \pi_{2n-1}(U_n) \to \pi_{2n-1}(SO_{2n}) \to \pi_{2n-1}(U_{2n}), \]
kj(x) = \( k(ry + (n - 1)!/2z) = rk(y) = \) twice a generator of \( \pi_{2n-1}(U_{2n}) \); how-
ever $k(y)$ is a generator, and $k$ is onto (since $k$ followed by the isomorphism $\pi_{2n-1}(U_{2n}) \to \pi_{2n-1}(U_{2n+1})$ equals the composition of $\pi_{2n-1}(SO_{2n}) \to \pi_{2n-1}(SO_{2n+1})$, which is an epimorphism, and $\pi_{2n-1}(SO_{2n+1}) \to \pi_{2n-1}(U_{2n+1})$, which is also an epimorphism since the stable group $\pi_{2n-1}(U_{2n+1}/SO_{2n+1})$ is zero) so that $r = 2$.

Thus the cokernel of $j$ is isomorphic to $\mathbb{Z} + \mathbb{Z}_2$. However $\pi_{2n-1}(SO_{2n}) \to \pi_{2n-1}(\Gamma_n)$ is onto, since $\pi_{2n-2}(U_n)$ is zero. Thus $\pi_{2n-1}(\Gamma_n)$ is isomorphic to the cokernel of $j$; further, $\sigma = \text{id}$ on it, since $\sigma = \text{id}$ on $\pi_i(SO_{2n})$ for even $n$.

This concludes the proof.

The values for $\pi_{2n+1}(\Gamma_{n+1})$, $\pi_{2n+3}(\Gamma_{n+2})$, $\pi_{2n+5}(\Gamma_{n+3})$ are computed in [6], and it only remains to determine the value of $\sigma$ on these groups (we do not settle the case $\pi_{2n+1}(\Gamma_{n+1})$).

For any integer $e$, we have an exact sequence

$$\pi_{2e-1}(SO_{2e}) \to \pi_{2e-1}(\Gamma_e) \to \pi_{2e-2}(U_e) = 0.$$ 

Hence it suffices to determine $\sigma$ on $\pi_{2e-1}(SO_{2e})$. If $e$ is even, $\sigma = \text{id}$. If $e = n + 3$, the exact sequence

$$\pi_{2n+5}(SO_{2n+6}) \to \pi_{2n+5}(S^{2n+5}) \to \pi_{2n+4}(SO_{2n+5})$$

or,

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

and the fact that $\sigma$ on $S^{2n+5}$ has degree $-1$, shows that $\sigma = -1$ on $\pi_{2n+5}(SO_{2n+6})$ and also on $\pi_{2n+5}(\Gamma_{n+3})$.

If $e = n + 1$, $\pi_{2n+1}(SO_{2n+2})$ is $\mathbb{Z} + \mathbb{Z}_2$ and $\sigma$ sends the generator of $\mathbb{Z}$ into its negative or its negative $+$ the element of order two.

We note for future use that $\sigma = -1$ on $\pi_{4k}(U_{2k})$ and $\pi_{4k}(U_{2k-1})$, and $\sigma = +1$ on $\pi_{4k+2}(U_{2k+1})$: for the exact sequence

$$\pi_{4k+1}(S^{4k+1}) \to \pi_{4k}(U_{2k}) \to \pi_{4k}(U_{2k+1}) = 0$$

and the fact that $\sigma$ has degree $-1$ on $S^{4k+1}$, shows that $\sigma = -1$ on $\pi_{4k}(U_{2k})$.

Also, under inclusion $\pi_{4k}(U_{2k-1})$ maps monomorphically into $\pi_{4k}(U_{2k})$. Since $\sigma = +1$ on $S^{4k+3}$, $\sigma = +1$ on $\pi_{4k+2}(U_{2k+1})$.

The rest of the groups $\pi_i(\Gamma_n)$ now follow; we denote by $n$ always a positive integer $\equiv 0 \mod 4$.

1. $\pi_{2n}(\Gamma_n) = \mathbb{Z}_2 + \mathbb{Z}_2$, $\sigma = \text{id}$.

Proof. The exact sequence

$$\pi_{2n+1}(S^{2n}) \to \pi_{2n}(\Gamma_n) \to \pi_{2n}(\Gamma_{n+1}) \to \pi_{2n}(S^{2n})$$

or

$$\mathbb{Z}_2 \to \pi_{2n}(\Gamma_n) \to \mathbb{Z}_2 \to \mathbb{Z}$$
shows that \( \pi_{2n}(\Gamma_n) \) has order 2 or 4. In the exact sequence

\[
\pi_{2n}(U_n) \xrightarrow{i} \pi_{2n}(SO_{2n}) \xrightarrow{p} \pi_{2n}(\Gamma_n) \xrightarrow{\partial} \pi_{2n-1}(U_n)
\]

the image of \( i \) is cyclic, hence 0 or \( Z_2 \). But \( \partial \) is zero, hence \( \pi_{2n}(\Gamma_n) \) has order 4 or 8. Finally \( \pi_{2n}(\Gamma_n) = Z_2 + Z_2 + Z_2 \), and \( \sigma = +1 \) (since \( \sigma = +1 \) on \( \pi_{2n}(SO_{2n}) \)).

We note also that since \( i \) has image \( Z_2 \), \( \partial : \pi_{2n+1}(\Gamma_n) \to \pi_{2n}(U_n) \) has cokernel \( Z_2 \), i.e., image of \( \partial \) is \( 2Z_{n+1} \).

2. \( \pi_{2n+1}(\Gamma_n) = Z_{n+1} + Z_2 \).

**Proof.** From the exact sequence

\[
\pi_{2n+2}(S^{2n}) \to \pi_{2n+1}(\Gamma_n) \to \pi_{2n+1}(\Gamma_{n+1}) \to \pi_{2n+1}(S^{2n})
\]

we see that \( \pi_{2n+1}(\Gamma_n) \) has order \( \leq 2(n!) \).

From the exact sequence

\[
\pi_{2n+1}(U_n) \to \pi_{2n+1}(SO_{2n}) \to \pi_{2n+1}(\Gamma_n) \to \pi_{2n}(U_n)
\]

and the remark at the end of 1, we get

\[
Z_2 \to Z_{n+1} \to Z_2.
\]

Thus \( \pi_{2n+1}(\Gamma_n) \) has order at least \( 2(n!) \), therefore exactly \( 2(n!) \). Furthermore it is not a cyclic group since image of \( P = Z_2 + Z_2 \) is not cyclic. Thus \( \pi_{2n+1}(\Gamma_n) = Z_{n+1} + Z_2 \). Since \( \sigma = -1 \) on \( \pi_{2n}(U_n) \) \( \sigma \) is, at least, different from the identity on \( \pi_{2n+1}(\Gamma_n) \).

3. \( \pi_{2n+2}(\Gamma_{n+1}) = 0 = \pi_{2n+6}(\Gamma_{n+3}) \).

**Proof.** Let \( m = n + 1 \) or \( n + 3 \). The exact sequence

\[
\pi_{2m+1}(S^{2m}) \to \pi_{2m}(SO_{2m}) \to \pi_{2m}(SO_{2m+1}) \to \pi_{2m}(S^{2m})
\]

reduces to

\[
Z_2 \to Z_4 \to Z_2 \to 0.
\]

Consider next

\[
\pi_{2m+1}(S^{2m+1}) \xrightarrow{\partial} \pi_{2m}(U_m) = Z_{m!}
\]

\[
Z_4 = \pi_{2m}(SO_{2m}) \xrightarrow{j} \pi_{2m}(SO_{2m+1}) = Z_2.
\]

Here \( \partial' \) is onto, since \( \pi_{2m}(SO_{2m+2}) = 0 \) for \( 2m \equiv 2 \) or 6 mod 8; hence \( k \) is onto. However \( k \) factors:

\[
\pi_{2m}(U_m) = Z_{m!}
\]

\[
\pi_{2m}(SO_{2m}) \xrightarrow{j} \pi_{2m}(SO_{2m+1})
\]

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Since \( \pi_{2m}(SO_{2m}) = Z_4 \) and \( j \) is onto, the fact that \( k \) is onto implies that \( e \) is also onto. Finally,

\[
\pi_{2m}(U_m) \xrightarrow{e} \pi_{2m}(SO_{2m}) \rightarrow \pi_{2m}(\Gamma_m) \rightarrow \pi_{2m-1}(U_m) \rightarrow \pi_{2m-1}(SO_{2m})
\]
gives

\[
0 \rightarrow \pi_{2m}(\Gamma_m) \rightarrow Z = \pi_{2m-1}(U_m) \rightarrow \pi_{2m-1}(SO_{2m}).
\]

However \( \pi_{2m-1}(U_m) = Z \rightarrow \pi_{2m-1}(SO_{2m}) = Z \) or \( Z + Z_2 \) is a monomorphism, since \( \pi_{2m-1}(\Gamma_m) \) is finite for \( m \equiv 1 \) or \( 3 \mod 4 \). Hence \( \pi_{2m}(\Gamma_m) = 0 \) if \( m = n + 1 \) or \( n + 3 \).

4. \( \pi_{2n+4}(\Gamma_{n+2}) = Z_2 \).

**Proof.** From the exact sequence

\[
\pi_{2n+5}(S^{2n+4}) = Z_2 \rightarrow \pi_{2n+4}(\Gamma_{n+2}) \rightarrow \pi_{2n+4}(\Gamma_{n+3})
\]

and \( \pi_{2n+4}(\Gamma_{n+3}) = \pi_{2n+3}(SO) = 0 \), we see that \( \pi_{2n+4}(\Gamma_{n+2}) = Z_2 \) or \( 0 \). From

\[
\pi_{2n+4}(U_{n+2}) \rightarrow \pi_{2n+4}(SO_{2n+4}) \rightarrow \pi_{2n+4}(\Gamma_{n+2})
\]

\[
Z_{(n+2)!} \rightarrow Z_2 + Z_2 \rightarrow \pi_{2n+4}(\Gamma_{n+2})
\]

we see that \( \pi_{2n+4}(\Gamma_{n+2}) \) is not zero, hence is \( Z_2 \).

5. \( \pi_{2n+3}(\Gamma_{n+1}) = Z, \pi_{2n+3}(\Gamma_n) = Z \), with \( \sigma = \) identity on both.

**Proof.** In the exact sequence

\[
\pi_{2n+4}(SO_{2n+2}) \xrightarrow{i} \pi_{2n+4}(SO_{2n+3}) \rightarrow \pi_{2n+4}(S^{2n+2}) \xrightarrow{\partial} \pi_{2n+3}(SO_{2n+2}),
\]

namely,

\[
Z_{12} \xrightarrow{i} Z_2 \rightarrow Z_2 \xrightarrow{\partial} Z.
\]

\( \partial \) is zero, hence \( i \) is zero. Thus the composite map

\[
j: \pi_{2n+4}(SO_{2n+2}) \xrightarrow{i} \pi_{2n+4}(SO_{2n+3}) \rightarrow \pi_{2n+4}(SO_{2n+4})
\]

is also zero.

Next consider the commutative diagram

\[
\begin{array}{ccc}
\pi_{2n+4}(SO_{2n+2}) & \xrightarrow{e} & \pi_{2n+4}(\Gamma_{n+1}) \\
\downarrow j & & \downarrow k \\
\pi_{2n+4}(SO_{2n+4}) & \xrightarrow{e'} & \pi_{2n+4}(\Gamma_{n+2})
\end{array}
\]

Image of \( k = \) Image of \( kp \) (since \( p \) is onto) but \( kp = p'j = 0 \), so \( k = 0 \). Finally, the exact sequence

\[
\pi_{2n+4}(\Gamma_{n+1}) \xrightarrow{e} \pi_{2n+4}(\Gamma_{n+2}) \rightarrow \pi_{2n+4}(S^{2n+2})
\]

\[
\rightarrow \pi_{2n+3}(\Gamma_{n+1}) \rightarrow \pi_{2n+3}(\Gamma_{n+2}) \rightarrow \pi_{2n+3}(S^{2n+2})
\]

becomes
Thus $\pi_{2n+3}(\Gamma_{n+1})$ is a subgroup of $\pi_{2n+3}(\Gamma_{n+2}) = \mathbb{Z}$, of index two. Since $\sigma = +1$ on $\pi_{2n+3}(\Gamma_{n+2})$, $\sigma = +1$ also on $\pi_{2n+3}(\Gamma_{n+1})$. The exact sequence $\pi_{2n+4}(S^{2n}) = 0 \to \pi_{2n+3}(\Gamma_n) \to \pi_{2n+3}(\Gamma_{n+1}) \to \pi_{2n+3}(S^{2n})$ shows that $\pi_{2n+3}(\Gamma_n) = \mathbb{Z}$ with $\sigma = +1$.

6. $\pi_{2n+7}(\Gamma_{n+3}) = \mathbb{Z} + \mathbb{Z}^2$, $\sigma =$ identity.

Proof. In the exact sequence

$$\pi_{2n+7}(SO_{2n+6}) \to \pi_{2n+7}(\Gamma_{n+3}) \to \pi_{2n+7}(U_{n+3})$$

$p$ is zero [4, Theorem 1], so $i$ is an isomorphism.

Writing $\Gamma_{n+4} = SO_{2n+7}/U_{n+3}$, $\Gamma_{n+3} = SO_{2n+6}/U_{n+3}$, we have a commutative diagram

\[
\begin{array}{ccc}
\pi_{2n+7}(SO_{2n+6}) & \to & \pi_{2n+7}(\Gamma_{n+3}) \\
\downarrow i & & \downarrow j \\
\pi_{2n+7}(SO_{2n+7}) & \to & \pi_{2n+7}(\Gamma_{n+4}) \\
\downarrow p & & \downarrow \delta \\
\pi_{2n+7}(S^{2n+6}) & = & \pi_{2n+7}(S^{2n+6})
\end{array}
\]

$p_1$, $p'$ are monomorphisms, since $\pi_{2n+3}(U_{n+1}) = 0$, and $i$ is an isomorphism, thus $j$ is a monomorphism. Since $\pi_{2n+7}(\Gamma_{n+4}) = \mathbb{Z} + \mathbb{Z}^2$, the subgroup $\pi_{2n+7}(\Gamma_{n+3})$ is either $\mathbb{Z}$ or $\mathbb{Z} + \mathbb{Z}^2$.

From the exact sequence

$$\pi_{2n+7}(SO_{2n+7}) \to \pi_{2n+7}(\Gamma_{n+4}) \to \pi_{2n+7}(U_{n+3})$$

and the fact that image of $\delta' = 2Z_{(n+3)}$, we see that under $p'$, a generator $u$ of $\pi_{2n+7}(SO_{2n+7})$ maps into $((n + 3)!/2)x + y$, where $x, y$ generate $\mathbb{Z}, \mathbb{Z}_2$ in $\pi_{2n+7}(\Gamma_{n+4}) = \mathbb{Z} + \mathbb{Z}_2$. From the diagram

\[
\begin{array}{ccc}
\pi_{2n+7}(SO_{2n+7}) & \xrightarrow{\epsilon'} & \pi_{2n+7}(\Gamma_{n+4}) \\
\downarrow p & & \downarrow \delta \\
\pi_{2n+7}(S^{2n+6}) & = & \mathbb{Z}_2
\end{array}
\]

where $p = 0$ (as remarked at the beginning of the proof) we have $qp'(u) = p(u) = 0$, but

$$qp'(u) = q((n + 3)!/2x + y) = q(y)$$

(since $(n + 3)!/2$ is even), so finally $q(y) = 0$ and the element $y$ of order 2 is in the image of $j$: $\pi_{2n+7}(\Gamma_{n+3}) \to \pi_{2n+7}(\Gamma_{n+4})$. Thus $\pi_{2n+7}(\Gamma_{n+3})$ has an element of order 2, and must be $\mathbb{Z} + \mathbb{Z}_2$. $\sigma = +1$ on it since $\sigma = +1$ on $\pi_{2n+7}(\Gamma_{n+4})$. This concludes the proof of 6.
7. \( \pi_{2n+5}(\Gamma_{n+2}) = Z_{(n+2)!} \) or \( Z_{(n+2)!/2} + Z_2 \).

**Proof.** From the exact sequence

\[
\pi_{2n+6}(\Gamma_{n+3}) \to \pi_{2n+6}(S^{2n+4}) \to \pi_{2n+5}(\Gamma_{n+2}) \to \pi_{2n+5}(\Gamma_{n+3})
\]

and \( \pi_{2n+6}(\Gamma_{n+3}) = 0 = \pi_{2n+4}(\Gamma_{n+3}), \pi_{2n+4}(\Gamma_{n+2}) = Z_2 \) we get

\[
0 \to Z_2 \to \pi_{2n+5}(\Gamma_{n+2}) \to \pi_{2n+5}(\Gamma_{n+3}) = Z_{(n+2)!/2} \to 0;
\]

further, \( \sigma = -1 \) on \( \pi_{2n+5}(\Gamma_{n+3}) \), so \( \sigma \neq 1 \) on \( \pi_{2n+5}(\Gamma_{n+2}) \).

**The groups** \( \pi_i(X_m) \) **and** \( \pi_i(S_{p_m}) \). For \( i < 4k, \pi_i(X_k) = \pi_{i+2}(SO_i), l \) large, \( m \) will denote an even integer, \( \geq 2 \). The involution \( \tau \) described above will be denoted by \( \sigma \) here.

1. \( \pi_{4m}(X_m) = Z_{(2m)!}, \) with \( \sigma = -1 \). \( \pi_{4m+1}(X_m) = Z_2. \)

**Proof.** From the exact sequence

\[
\pi_{4m-1}(S_{p_m}) \to \pi_{4m-1}(U_{2m}) \to \pi_{4m-1}(X_m)
\]

\[
Z \to Z \to Z_2
\]

\( i \) is a monomorphism.

Hence the sequence

\[
\pi_{4m}(S_{p_m}) \to \pi_{4m}(U_{2m}) \to \pi_{4m}(X_m) \to \pi_{4m-1}(S_{p_m}) \to
\]

becomes \( 0 \to Z_{(2m)!} \to \pi_{4m}(X_m) \to 0. \) Thus \( \pi_{4m}(X_m) = Z_{(2m)!}; \) and \( \sigma = -1 \) on it, since \( \sigma = -1 \) on \( \pi_{4m}(U_{2m}). \)

The exact sequence

\[
\pi_{4m+1}(S_{p_m}) \to \pi_{4m+1}(U_{2m}) \to \pi_{4m+1}(X_m) \to \pi_{4m}(S_{p_m})
\]

\[
0 \to Z_2 \to \pi_{4m+1}(X_m) \to 0
\]

shows \( \pi_{4m+1}(X_m) = Z_2. \)

2. \( \pi_{4m+4}(X_{m+1}) = Z_{(2(m+1))!/2}, \) with \( \sigma = -1. \)

**Proof.** From the fibrations

\[
U_{2m+2} \to U_{2m+3} \to S^{4m+5}
\]

\[
X_{m+1} \to X_{m+2} \to S^{4m+5}
\]

we get the diagram

\[
\pi_{4m+5}(U_{2m+3}) = Z \to \pi_{4m+5}(S^{4m+5}) \to \pi_{4m+4}(U_{2m+2})
\]

\[
\pi_{4m+5}(X_{m+2}) = Z \to \pi_{4m+5}(S^{4m+5}) \to \pi_{4m+4}(X_{m+1})
\]

\[
\pi_{4m+4}(S_{p_{m+1}}) = Z_2.
\]
\(\partial, \partial'\) are onto since \(\pi_{4m+4}(U_{2m+3}) = 0 = \pi_{4m+4}(X_{m+2})\), \(\partial_1\) is onto since \(\pi_{4m+4}(U_{2m+3}) = 0\), so that if \(u\) generates \(\pi_{4m+5}(U_{2m+3})\), \(p_1(u) = 2v\), where \(v\) generates \(\pi_{4m+5}(X_{m+2})\). If \(w\) is a generator of \(\pi_{4m+5}(S^{4m+5})\) then
\[p'p_1(u) = p(u) = (2m + 2)!w\]
so \(2p'(v) = (2m + 2)!w\). or \(p'(v) = [(2m + 2)!/2]w\), and it is clear that \(\pi_{4m+4}(U_{2m+2}) \to \pi_{4m+4}(X_{m+1})\) is onto, with kernel \(Z_2\).

Since \(\sigma = -1\) on \(\pi_{4m+4}(U_{2m+2})\), \(\sigma = -1\) on \(\pi_{4m+4}(X_{m+1})\) also.

3. \(\pi_{4m+6}(Sp_m+1) = \mathbb{Z}_{2(2m+3)!}\), \(\pi_{4m+2}(Sp_m) = \mathbb{Z}_{(2m+1)!}\).

**Proof.** Consider the fibrations \(Sp_m+2/Sp_m+1 = S^{4m+7} = U_{2m+4}/U_{2m+3}\) and the associated diagram
\[
\begin{array}{ccc}
\pi_{4m+7}(Sp_m+2) & \xrightarrow{i} & \pi_{4m+7}(U_{2m+4}) \\
\downarrow & & \downarrow \\
\pi_{4m+7}(U_{2m+4}) & \xrightarrow{j} & \pi_{4m+7}(S^{4m+7}) \\
\downarrow & & \downarrow \\
\pi_{4m+7}(U_{2m+2}) & \xrightarrow{k} & \pi_{4m+7}(S^{4m+7}) \\
\end{array}
\]
\(\partial, \partial'\) are onto since \(\pi_{4m+6}(Sp_m+2) = 0 = \pi_{4m+6}(U_{2m+4})\). The groups \(\pi_{4m+7}(Sp_m+2), \pi_{4m+7}(U_{2m+4}), \pi_{4m+7}(S^{4m+7})\) are all \(\mathbb{Z}\), with generators \(x, y, z\).

From
\[\pi_{4m+7}(Sp_m+2) \xrightarrow{i} \pi_{4m+7}(U_{2m+4}) \to \pi_{4m+7}(X_{m+2}) = \mathbb{Z}_2 \to \pi_{4m+6}(Sp_m+2) = 0\]
we see that \(i(x) = 2y\), so that \(p'i(x) = p(x) = 2p'(y) = 2[(2m+3)!]z\). Hence \(\pi_{4m+6}(Sp_m+1) = \mathbb{Z}_{2(2m+3)!}\) and
\[0 \to \mathbb{Z}_2 \xrightarrow{i} \pi_{4m+6}(Sp_m+1) \to \pi_{4m+6}(U_{2m+3}) \to 0\]
is an exact sequence.

For \(\pi_{4m+2}(Sp_m)\) we use the diagram
\[
\begin{array}{ccc}
\pi_{4m+3}(Sp_m+1) & \xrightarrow{i} & \pi_{4m+3}(U_{2m+2}) \\
\downarrow & & \downarrow \\
\pi_{4m+3}(U_{2m+2}) & \xrightarrow{j} & \pi_{4m+3}(S^{4m+3}) \\
\downarrow & & \downarrow \\
\pi_{4m+3}(U_{2m+1}) & \xrightarrow{k} & \pi_{4m+3}(U_{2m+1}). \\
\end{array}
\]
Again \(\partial, \partial'\) are epimorphisms since \(\pi_{4m+2}(Sp_m+1) = 0 = \pi_{4m+2}(U_{2m+2})\). \(i\) is actually an isomorphism since \(\pi_{4m+3}(X_{m+1}) = 0\), so \(j\) is also an isomorphism.

4. \(\pi_{4m+8}(Sp_m+1) = \mathbb{Z}_2 = \pi_{4m+8}(Sp_m+1)\).

**Proof.** Consider the diagram
\[
\begin{array}{ccc}
\pi_{4m+8}(U_{2m+3}) & \xrightarrow{i} & \pi_{4m+8}(X_{m+2}) \\
\downarrow & & \downarrow \\
\pi_{4m+8}(X_{m+2}) & \xrightarrow{j} & \pi_{4m+8}(S^{4m+8}) \\
\downarrow & & \downarrow \\
\pi_{4m+8}(U_{2m+4}) & \xrightarrow{k} & \pi_{4m+8}(U_{2m+4}). \\
\end{array}
\]
\(\delta\) is onto since \(\pi_{4m+8}(U_{2m+3}) = 0\), and \(p'\) is an isomorphism, by 1. \(i\) is a mono-
morphism with cokernel $Z_2[4, \text{p. 164}], so p$ is a monomorphism with cokernel $Z_2 = \pi_{4m+7}(S_{p+1})$.

From the exact sequence

$$\pi_{4m+9}(S_{p+2}) \to \pi_{4m+9}(S^{4m+7}) \xrightarrow{\partial} \pi_{4m+8}(S_{p+1}) \to \pi_{4m+8}(S_{p+2})$$

and the (stable) values $\pi_{4m+9}(S_{p+2}) = 0 = \pi_{4m+8}(S_{p+2})$ we see that

$$\partial: \pi_{4m+9}(S^{4m+7}) \xrightarrow{\cong} \pi_{4m+8}(S_{p+1}).$$

5. $\pi_{4m+9}(X_{m+1}) = Z_2$.

Proof. In the homotopy sequence of the fibration $X_{m+2}/X_{m+1} = S^{4m+5}$, we have $\pi_{4m+9}(S^{4m+5}) = 0 = \pi_{4m+10}(S^{4m+5})$ and $\pi_{4m+9}(X_{m+2}) = Z_2$ (from 1).

6. $\pi_{4m+3}(S_{p+1}) = Z_2$.

Proof. We have the commutative diagram

$$\begin{array}{ccc}
\pi_{4m+4}(U_{2m+1}) & \xrightarrow{p} & \pi_{4m+4}(X_{m+1}) \\
\downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
\pi_{4m+4}(U_{2m+2}) & \xrightarrow{p'} & \pi_{4m+4}(X_{m+1}) \\
\end{array}$$

Since $\pi_{4m+4}(X_{m+1}) / Z_2(2m+2)_{1/2}$ is finite, $p'$ is an epimorphism; $i$ is a monomorphism with cokernel $Z_2$, hence $p = p'i$ has cokernel $Z_2$. But $\partial$ is an epimorphism since $\pi_{4m+3}(U_{2m+1}) = 0$, so $\pi_{4m+3}(S_{p+1}) = Z_2$.

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