THE ASYMPTOTIC DISTRIBUTION OF THE NUMBER OF ZERO FREE INTERVALS OF A STABLE PROCESS

by

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1. Introduction. Let \( \{X(t); t \geq 0\} \) be the one-dimensional symmetric stable process of index \( \alpha \), \( 0 < \alpha \leq 2 \), that is, a process with stationary independent increments whose continuous transition density \( f(t,y-x) \) relative to Lebesgue measure is given by

\[
(1.1) \quad f(t,x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-t|x|^\alpha} e^{ix\xi} d\xi.
\]

We assume throughout this paper that \( X(0) = 0 \) and that the sample functions are normalized to be right continuous and have left-hand limits everywhere.

Let us also introduce the stable subordinator, \( \{T(t) ; t \geq 0\} \), of index \( \beta \), \( 0 < \beta < 1 \), that is, a process with stationary independent and positive increment whose continuous transition density, \( g(t,y-x) \), is given by \( g(t,x) = 0 \) for \( x \leq 0 \) and by

\[
(1.2) \quad e^{-\tau \theta} = \int_0^\infty e^{-\lambda x} g(t,x) dx
\]

for \( x > 0 \). We assume \( T(0) = 0 \) and that the sample functions are normalized to be right continuous and have left-hand limits everywhere. In addition almost all sample functions of \( T \) are strictly monotone increasing. Finally we assume that the processes \( X \) and \( T \) are completely independent and are defined over the same complete probability space, \( (\Omega, \mathcal{F}, P) \). It is perhaps more reasonable to assume that \( X \) and \( T \) are defined over different (complete) probability spaces, but for notational convenience we prefer the above assumption.

Define

\[
A(\omega) = \{ t : 0 \leq t \leq 1, X(t,\omega) = 0 \text{ or } X(t-,\omega) = 0 \},
\]

\[
B(\omega) = \{ t : 0 \leq t \leq 1, T(t,\omega) = t \text{ or } T(t-,\omega) = t \text{ for some } t \}.
\]

If \( 0 < \alpha \leq 1 \) then \( A(\omega) = \{0\} \) for almost all \( \omega \), and it is not difficult to see that for general \( \alpha \), \( 0 < \alpha \leq 2 \),

\[
A(\omega) = \{ t : 0 \leq t \leq 1, X(t,\omega) = 0 \}
\]

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for almost all $\omega$. However we will not need this fact and so we omit its proof.

In view of our regularity assumptions on the sample functions of $X$ and $T$, it follows that $A(\omega)$ and $B(\omega)$ are compact subsets of $[0,1]$ for each $\omega$. Therefore the complements of $A(\omega)$ and $B(\omega)$ in $[0,1]$ are relatively open subsets of $[0,1]$, and, as such, each can be written uniquely as the disjoint union of at most countably many (relatively) open subintervals of $[0,1]$. If $\epsilon > 0$, let $N_\delta(\epsilon)$ ($N^\beta_\delta(\epsilon)$) be the number of such intervals in the complement of $A(\omega)$ ($B(\omega)$) which exceed $\epsilon$ in length. The following two theorems are the main results of the present paper.

**Theorem A.** $N_\delta(\epsilon)$ and $N^\beta_\delta(\epsilon)$ are random variables and if $\beta = 1 - 1/\alpha$, $1 < \alpha \leq 2$, then they have the same distribution for each fixed $\epsilon > 0$.

**Theorem B.** If $0 < \beta < 1$, then

\[
\lim_{\epsilon \downarrow 0} P[\Gamma(1-\beta)e^{\beta N^\beta_\delta(\epsilon)} \leq x] = G_\beta(x)
\]

where $G_\beta(x)$ is a Mittag-Leffler distribution which is uniquely determined by its moments

\[
\int_0^\infty x^n dG_\beta(x) = n! [\Gamma(1+n\beta)]^{-1}, \quad n = 0, 1, \ldots.
\]

The definition of the distribution $G_\beta$ and the fact that its moments are given by (1.4) is contained in [7]. The fact that $G_\beta$ is uniquely determined by its moments follows from the criterion on p. 110 of [4].

An immediate consequence of Theorems A and B is the following corollary.

**Corollary.** If $1 < \alpha \leq 2$, then

\[
\lim_{\epsilon \downarrow 0} P[\Gamma(1/\alpha)e^{1-1/\alpha N_\delta(\epsilon)} \leq x] = F_\delta(x) = G_{1-1/\delta}(x).
\]

Of course, in the case $\alpha = 2$ these results are well known. Moreover the above corollary should be compared with the recent result of Kesten [5]. In [5] Kesten obtains the limiting distribution of the number, $N_\delta(\epsilon)$, of intervals of positivity of $X$ in $0 \leq t \leq 1$ for all $\alpha$, $0 < \alpha \leq 2$. We would like to thank Professor Kesten for making his manuscript available to us. In particular, we owe references [4;7] to him.

2. **The proof of Theorem A.** Given a complete probability space $(\Omega, \mathcal{F}, P)$, a function $A$ from $\Omega$ to subsets of the real line, $R$, is said to be a random set if

(i) $A(\omega)$ is compact for almost all $\omega$,

(ii) $\{\omega: A(\omega) \subset E\} \in \mathcal{F}$ for all open subsets $E$ of $R$.

Two random sets $A$ and $B$ (not necessarily defined over the same probability space) are stochastically equivalent if for every set $E$ that is a finite union of open intervals

\[
P\{\omega: A(\omega) \subset E\} = P\{\omega: B(\omega) \subset E\}.
\]
These definitions were introduced in [2]. A random set, $A$, is contained in the closed interval $[a,b]$ if $A(\omega) \subseteq [a,b]$ for all $\omega$. If a random set $A$ is contained in $[a,b]$ then $[a,b] - A(\omega)$ is an open subset of $[a,b]$ for almost all $\omega$, and, as such, can be written uniquely as the union of at most countably many disjoint (relatively) open subintervals of $[a,b]$. If $\varepsilon > 0$, let $N_A(\varepsilon)$ be the number of such intervals whose length is greater than $\varepsilon$. Clearly $N_A(\varepsilon)$ is defined and finite for almost all $\omega$.

Theorem 2.1. If $A$ is a random set contained in $[a,b]$, then $N_A(\varepsilon)$ is a random variable. If $A$ and $B$ are stochastically equivalent random sets contained in $[a,b]$ then $N_A(\varepsilon)$ and $N_B(\varepsilon)$ have the same distribution for each $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$ be fixed, and let $k \geq 1$ be an integer. Let $E$ denote a finite disjoint union of exactly $k$ closed intervals $I_1, \ldots, I_k$ each of which has rational end points, is contained in $[a,b]$, and has length greater than $\varepsilon$. Of course, if $k$ is too large there will be no such $E$'s. Let $E_1, E_2, \ldots$ be an enumeration of all such $E$'s; then ($\emptyset$ denotes the empty set)

\begin{equation}
\{\omega: N_A(\varepsilon) \geq k\} = \bigcup_{n=1}^\infty \Delta_n,
\end{equation}

where

\begin{equation}
\Delta_n = \{\omega: E_n \cap A(\omega) = \emptyset\} = \{\omega: A(\omega) \subseteq E_n^c\}.
\end{equation}

Here $E_n^c$ is the complement of $E_n$ in $[a,b]$ and hence is a finite union of (relatively) open subintervals of $[a,b]$. Clearly this implies that $N_A(\varepsilon)$ is a random variable. Moreover

\begin{equation}
P\{\omega: N_A(\varepsilon) \geq k\} = \lim_{n \to \infty} P\left(\bigcup_{i=1}^n \Delta_i\right),
\end{equation}

and for fixed $n$ the inclusion-exclusion formula implies that

\begin{equation}
P\left(\bigcup_{i=1}^n \Delta_i\right) = \Sigma P(\Delta_i) - \Sigma P(\Delta_i \cap \Delta_j) + \ldots.
\end{equation}

Looking at a typical intersection we see that

\begin{equation}
P(\Delta_1 \cap \ldots \cap \Delta_j) = P\{\omega: A(\omega) \subseteq E_1^c \cap \ldots \cap E_j^c\}.
\end{equation}

Since $E_1^c \cap \ldots \cap E_j^c$ is a finite union of open intervals (not necessarily disjoint), it follows that if $A$ and $B$ are stochastically equivalent the left side of (2.4) is unchanged if $A$ is replaced by $B$. Thus $N_A(\varepsilon)$ and $N_B(\varepsilon)$ have the same distribution.

Let $a$ be a real number satisfying $0 \leq a < 1$, and define

\begin{equation}
A_\varepsilon(\omega) = \{t: a \leq t \leq 1, X(t,\omega) = 0\text{ or } X(t-,\omega) = 0\},
\end{equation}

\begin{equation}
B_\varepsilon(\omega) = \{t: a \leq t \leq 1, T(\tau,\omega) = t\text{ or } T(\tau-,\omega) = t\text{ for some } \tau\},
\end{equation}

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where $X$ and $T$ are the processes defined in §1. We assume from now on that
the index, $\alpha$, of $X$ satisfies $1 < \alpha \leq 2$. In particular, $A_0$ and $B_0$ are the sets $A$ and $B$
defined in (1.3). It was shown in [2, Proof of Theorem A] that $A_\alpha$ and $B_\alpha$
stochastically equivalent random sets for each $\alpha > 0$ provided $\beta = 1 - 1/\alpha$.
Thus by Theorem 2.1 we see that $N_\alpha(a,\epsilon) = N_\alpha(\epsilon)$ and $N'_\alpha(a,\epsilon) = N'_\alpha(\epsilon)$
have the same distribution for each fixed $\alpha > 0$ and $\epsilon > 0$ provided $\beta = 1 - 1/\alpha$. Since
for almost all $\omega$ the sets $A(\omega)$ and $B(\omega)$ contain points arbitrarily close to 0 (this
is an immediate consequence of Lemma 3.1 of [2]), it follows that $N_\alpha(a,\epsilon) \rightarrow N_\alpha(\epsilon)$
and $N'_\alpha(a,\epsilon) \rightarrow N'_\alpha(\epsilon)$ as $\alpha \rightarrow 0$ for almost all $\omega$. In fact for $\alpha$ sufficiently small
(depending on $\epsilon$ and $\omega$) we have $N_\alpha(a,\epsilon,\omega) = N_\alpha(\epsilon,\omega)$ and similarly for $N'_\alpha$. Thus
$N_\alpha(\epsilon)$ and $N'_\alpha(\epsilon)$ have the same distribution if $\beta = 1 - 1/\alpha$, and Theorem A is
proved.

3. First passage times. In this section we give a preliminary calculation that will
be needed in the proof of Theorem B. Let $T = \{T(t); t \geq 0\}$ be the stable
subordinator of index $\beta$, $0 < \beta < 1$, and we assume in this section that $T(0) = 0$.
Let us recall the Ito representation of $T$ (see [3] or [6, §37]). In the present
case ($T(t)$ strictly increasing) this is especially simple. For fixed $\omega$ let $p(dt,dx,\omega)$
be the measure on $[0,\infty) \times (0,\infty)$ defined by the relationship that
\[
p((t_1,t_2],[x_1,x_2],\omega) = \int_{t_1}^{t_2} x \cdot p([0,t], dx, \omega),
\]
where in this case the integral is just the countable sum of the jumps of $T(\tau,\omega)$
on the interval $0 \leq \tau \leq t$. Finally the random variable $p([t,s],dx)$, $s > t$, is in-
dependent of $\mathcal{B}_{t-}$, the $\sigma$-algebra generated by $\{T(\tau): \tau < t\}$, and if $A_1, \ldots, A_n$
are disjoint Borel subsets of $\{(t,x): t \geq 0, x > 0\}$ which are at a positive distance
from the $t$-axis, then \[\int_{A_1} p(dt,dx), \ldots, \int_{A_n} p(dt,dx)\] are independent random
variables.

If $u > 0$ define
\[
S(u,\omega) = \inf \{t: T(t,\omega) \geq u\}.
\]
Since $T(t)$ is strictly increasing, $S(u)$ is continuous and nondecreasing. It is the
first passage time of $T$ past $u$. We now state the main result of this section.

**Theorem 3.1.** For each $u > 0$ and integer $k \geq 0$ we have
\[
E(S(u)^k) = k! \Gamma(1 + \beta k)^{-1} u^{\beta k}.
\]
Proof. For \( \lambda > 0 \) and \( s > 0 \), define
\[
H_\lambda(\lambda,s) = E \int_0^\infty e^{-\lambda S(u)} e^{-su} du.
\]
Now \( T(t,\omega) \) is a sum of jumps and so if we let \( t_n \) be the places where \( T(t) \) jumps and \( I_n = [T(t_n^-), T(t_n)] \) then \( \bigcup I_n = [0, \infty) \) since \( T(0) = 0 \) and \( T(t) \to \infty \) as \( t \to \infty \). Of course, the \( t_n \) depend on \( \omega \). For notational convenience let us write \( T^*(t) \) for \( T(t^-) \). Thus
\[
\int_0^\infty e^{-\lambda S(u)} e^{-su} du = \frac{1}{s} \sum_n e^{-\lambda t_n} \left[ e^{-sT^*(t_n)} - e^{-sT(t_n)} \right]
\]
\[
= \frac{1}{s} \sum_n e^{-\lambda t_n} e^{-sT^*(t_n)} \left[ 1 - e^{-s[T(t_n^-) - T^*(t_n)]} \right]
\]
\[
= \frac{1}{s} \int_0^\infty \int_0^\infty e^{-\lambda t} e^{-sT^*(t)} \left[ 1 - e^{-sx} \right] p(dt, dx).
\]
But \( T^*(t) \) and \( p(dt, dx) \) are independent and
\[
E[\exp(-sT^*(t))] = E[\exp(-sT(t))] = e^{-t\beta}
\]
for fixed \( t \). Therefore (these manipulations are easily justified by first approximating the integral by a sum and then passing to the limit)
\[
H_\lambda(\lambda,s) = \frac{1}{s} \beta \int_0^\infty e^{-\lambda t} e^{-sT^*(t)} \frac{\beta dt dx}{\Gamma(1 - \beta) x^{1+\beta}}
\]
\[
= s^{\beta-1} \int_0^\infty e^{-\lambda t} e^{-t\beta} dt.
\]
Differentiating with respect to \( \lambda \) and letting \( \lambda \to 0 \) we obtain (the interchange of limit procedures is easily justified)
\[
\int_0^\infty E[S(u)^k] e^{-su} du = s^{\beta-1} \int_0^\infty t^k e^{-t\beta} dt
\]
\[
= k! s^{-1-\beta}.
\]
Inverting this Laplace transform we obtain Theorem 3.1.

It is an easy consequence of this theorem that the distribution function of \( S(u) \) is \( G_\beta(u^{-\beta}x) \) where \( G_\beta \) is defined in (1.4).

4. Proof of Theorem B. Let \( M(\varepsilon) \) be the number of jumps of \( T(t) \) in the interval \( 0 \leq t \leq S(1) \) of length greater than \( \varepsilon \). Thus if \( Q(dt, \varepsilon) = p(dt, (\varepsilon, \infty)) \) where \( p \) is the Poisson measure for \( T(t) \), then \( M(\varepsilon) = Q([0, S_1], \varepsilon) \). Of course, \( S_1 = S(1) \)
is defined in (3.2). Clearly $0 \leq M(\varepsilon) - M^*_{\varepsilon}(\varepsilon) \leq 1$ for almost all $\omega$. We will prove that
\begin{equation}
(4.1) \quad E\{[\Gamma(1 - \beta)\varepsilon]^k M(\varepsilon)]^k \} \rightarrow k! \left[\Gamma(1 + \beta k)\right]^{-1}
\end{equation}
as $\varepsilon \rightarrow 0$ for each integer $k \geq 0$. From this the corresponding relation with $M(\varepsilon)$ replaced by $N^*_\varepsilon(\varepsilon)$ follows easily, and then Theorem B is a consequence of the moment convergence theorem for distributions \cite[p. 115]{4}. Thus the proof of Theorem B reduces to the proof of (4.1). Note that $M(\varepsilon) \leq \lceil \varepsilon^{-1} \rceil + 2$, where $\lceil \cdot \rceil$ is the greatest integer function, and so all moments of $M(\varepsilon)$ exist.

It will be convenient to consider the subordinator $T(t)$ starting not only at 0 but also at any $x \geq 0$. We will write $P_x$ and $E_x$ for probabilities and expectations when $T(0) = x$, and, as is usual in the general theory of Markov processes, $E_T\{\}$ stands for the evaluation of the function $E_x\{\}$ at the point $x = T(t)$. Let $G(t,x,A)$ be the transition probability function for $T(t)$, and $U(x,A) = \int_0^\infty G(t,x,A) dt$ be the potential kernel. Since $U(x,A)$ is just the expected amount of time $T(t)$ spends in $A$ when $T(0) = x$, we have
\begin{equation}
U(x,[0,y]) = E_x(S(y)) = E_0(S(y - x))
\end{equation}
provided $y > x$. Thus Theorem 3.1 implies
\begin{equation}
U(x,dy) = \begin{cases}
\beta \left[\Gamma(1 + \beta)\right]^{-1}(y - x)^{\beta - 1} dy & \text{if } y > x, \\
0 & \text{if } y \leq x.
\end{cases}
\end{equation}

We begin our calculations with several lemmas. Recall that $T^*(t) = T(t-)$. and $\mathcal{B}$ is the $\sigma$-algebra generated by $T(\tau)$ for $\tau \leq t$.

**Lemma 1.** Let $g$ be a bounded Baire function with compact support; then
\begin{equation}
\int_0^\infty E_x[g(T^*_t)] dt = \int_0^\infty E_x[g(T_t)] dt = \frac{\beta}{\Gamma(1 + \beta)} \int_x^\infty (y - x)^{\beta - 1} g(y) dy.
\end{equation}

**Proof.** Since $T^*(t) = T(t)$ for almost all $\omega$ for each fixed $t$, we have using (4.2)
\begin{equation}
\int_0^\infty E_x[g(T^*_t)] dt = \int_0^\infty E_x[g(T_t)] dt
\end{equation}
\begin{equation}
= \int_0^\infty dt \int G(t,x,dy) g(y) = \frac{\beta}{\Gamma(1 + \beta)} \int_x^\infty (y - x)^{\beta - 1} g(y) dy.
\end{equation}
It will be convenient to introduce the following conventions. Let $0 \leq a < b < \infty$; then $\int_a^b$ and $\int_a^\infty$ are the integrals over the sets $(a,b]$ and $(a,\infty)$, respectively. Also let
\begin{equation}
h(u) = \begin{cases}
1 & \text{if } u \leq 1, \\
0 & \text{if } u > 1.
\end{cases}
\end{equation}
Lemma 2. Let $\phi$ be a bounded non-negative continuous function; then
\[ E_x \int_0^\infty h(T_u^*)\phi(T_u^*)Q(du,\varepsilon) = \lambda E_x \int_0^\infty h(T_u^*)\phi(T_u^*)du, \]
where $\lambda = \left[ e^\beta \Gamma(1 - \beta) \right]^{-1}$.

Proof. Since $T_u^* \leq 1$ if and only if $u \leq S_1$, we see that $\int_0^\infty h(T_u^*)\phi(T_u^*)Q(du,\varepsilon) \leq M(\varepsilon)\sup \phi \leq (\lceil \varepsilon^{-1} \rceil + 2)\sup \phi$, and $\int_0^\infty h(T_u^*)\phi(T_u^*)du \leq S_1 \sup \phi$. Thus both integrals exist for almost all $\omega$ and have finite expectations. The result now follows by a straightforward approximation argument making use of the easily verified facts that (i) $T_u^*$ and $Q([u,v),\omega)$, $v > u$, are independent, (ii) $h(T_u^*)\phi(T_u^*)$ is left continuous in $u$, and (iii) $E_x Q(du,\varepsilon) = E_x P(du,(\varepsilon,\infty)) = \lambda du$.

Lemma 3. For each $\varepsilon > 0$ let $Y^\varepsilon = \{Y^\varepsilon_u; u \geq 0\}$ be a stochastic process defined on the sample space of $\{T(t); t \geq 0\}$ with right continuous sample functions. Let $H_x(u,\omega)$ be a fixed version of $E_x\{Y^\varepsilon_u|\mathcal{B}_u\}$ defined for $u \geq 0$ and all $\omega$ which is assumed to satisfy $H_x(u,\omega) = g(T_u) + O(\gamma(\varepsilon))$ where $g$ is a bounded continuous function and the $O$-term is uniform in $u$, $\omega$, and $x$. Finally assume that $0 \leq Y^\varepsilon_u(\omega) \leq M_\varepsilon < \infty$ for all $u$, $\omega$. Under these conditions
\[ E_x \int_0^\infty h(T_u^*)Y^\varepsilon_uQ(du,\varepsilon) = E_x \int_0^\infty h(T_u^*)g(T_u)Q(du,\varepsilon) + O(e^{-\beta}\gamma(\varepsilon)) \]
where the $O$-term is uniform in $x$.

Proof. Let $0 < b < \infty$ and let $Nt_j = jb/N$, $j = 0, 1, \ldots, N$, for each integer $N > 0$. Let $Z(u) = h(T_u^*)$, then since $Z(u)$ is left continuous and $Y(u) = Y^\varepsilon_u$ is right continuous we have for each fixed $\varepsilon > 0$ that
\[ J(b) = E_x \int_0^b Z(u)Y^\varepsilon_uQ(du,\varepsilon) \]
\[ = \lim_{N \to \infty} \sum_{j=0}^{N-1} E_x \{Z(Nt_j)Y(Nt_{j+1})Q(\Delta Nt_j,\varepsilon)\} \]
where $\Delta Nt_j = (Nt_{j+1} - Nt_j)$. But $Z(Nt_j)$ and $Q(\Delta Nt_j,\varepsilon)$ are measurable and so using our fixed version of $E_x\{Y^\varepsilon_u|\mathcal{B}_u\}$ we have
\[ J(b) = \lim_{N \to \infty} E_x \sum_{j=0}^{N-1} Z(Nt_j)[g(T(Nt_{j+1})) + O(\gamma(\varepsilon))]Q(\Delta Nt_j,\varepsilon) \]
\[ = E_x \int_0^b h(T_u^*)g(T_u)Q(du,\varepsilon) + O(\gamma(\varepsilon)) E_x \int_0^b h(T_u^*)Q(du,\varepsilon). \]
Using Lemmas 1 and 2 we see that
\[ E_x \int_0^\infty h(T_u^*)Q(du,\varepsilon) = \lambda E_x \int_0^\infty h(T_u^*)du = \lambda [\Gamma(1 + \beta)]^{-1}(1 - x)^\beta h(x) = O(\varepsilon^{-\beta}) \]
uniformly in $x$. Thus letting $b \to \infty$ in the above expression for $J(b)$ we obtain Lemma 3.

We are now ready to begin the proof of (4.1). Since $t \leq S_1$ if and only if $T^*(t) \leq 1$ we may write ($Q$ assigns no mass to 0, so $\int_0^\infty$ equals $\int_0^\infty$)

$$M(e) = \int_0^\infty h(T^*(u))Q(du, e).$$

Using Lemma 2 with $\phi \equiv 1$ and then Lemma 1 we obtain

$$E_x(M(e)) = E_x\{p([0, S_1] \cap (e, \infty))\} = \frac{\lambda}{\Gamma(1 + \beta)}(1 - y)^\beta h(y).$$

Setting $y = 0$ and recalling that $\lambda = [\epsilon^\beta T(1 - \beta)]^{-1}$ we have established (4.1) when $k = 1$.

In order to fix the ideas let us consider the case $k = 2$ before proceeding to the general case. Now

$$E_0(M(e) \cdot M(e)) = 2E_0 \int_0^\infty h(T^*_t)Q(dt, e) \int_0^\infty h(T^*_u)Q(du, e)$$

$$= E_0 \int \int h(T^*_t)h(T^*_u)Q(dt, e)Q(du, e).$$

But for each $\omega, Q$ is a purely discrete measure assigning mass one to each point of a countable set (depending on $\omega$) and $h^2 = h$. Therefore the second term in (4.5) reduces to $E_0(M(e)) = O(e^{-\beta})$ by (4.4). Let us consider the first term.

Define $Y^*_t = \int_0^\infty h(T^*_u)Q(du, e)$; then $Y^*_t$ is right continuous since the integral is over the open interval $(t, \infty)$, and is bounded by $M(e) \leq [e^{-1}] + 2$. Since $T(t)$ is a Markov process we note that

$$E_x\{Y^*_t \mid \mathcal{B}_t\} = E_{T(t)} \int_0^\infty h(T^*_u)Q(du, e)$$

for almost all $\omega(P_x)$. Thus using Lemmas 2 and 1 we can take as our fixed version of $E_x\{Y^*_t \mid \mathcal{B}_t\}$ the expression $\lambda q_1(T_t)$ where it is understood that $T(0) = x$, and $q_1(x) = [\Gamma(1 + \beta)]^{-1}(1 - x)^\beta h(x)$. Therefore the hypotheses of Lemma 3 are satisfied with $\gamma(e) = 0$. Applying Lemma 3 with $x = 0$ the first term in (4.5) becomes

$$2\lambda E_0 \int_0^\infty h(T^*_t)q_1(T_t)Q(dt, e)$$

$$= 2\lambda E_0 \int_0^\infty h(T^*)q_1(T^*)Q(dt, e) - 2\lambda E_0 \int_0^\infty h(T^*)[q_1(T^*) - q_1(T)]Q(dt, e).$$
Call the first term above \( J_1 \) and the second \( J_2 \). Using Lemmas 2 and 1, and then the definition of \( q_1 \), we find
\[
J_1 = 2\lambda^2 \beta [\Gamma(1 + \beta)]^{-1} \int_0^1 y^{\beta-1} q_1(y) dy
= 2\lambda^2 [\Gamma(1 + 2\beta)]^{-1}.
\]
Concerning \( J_2 \), we note that \( q_1 \) is bounded by \( c = [\Gamma(1 + \beta)]^{-1} \) and satisfies
\[
0 \leq q_1(x) - q_1(x + y) \leq c y^\beta
\]
for all \( x \geq 0 \) and \( y \geq 0 \). Moreover there can be at most one jump of \( T(t) \) of magnitude greater than one in the interval \( 0 \leq t \leq S_1 \), and so
\[
|J_2| \leq 4c\lambda + 2\lambda c E_0 \int_0^{S_1} (T_t - T^*_t)^\beta p(dt, (e, 1]).
\]
But this last integral is just \( E_0 \int_0^1 x^\beta p([0, S_1], dx) \), and (4.4) implies that
\[
E_0 \{ p([0, S_1], dx) \} = \beta [\Gamma(1 + \beta) \Gamma(1 - \beta)]^{-1} x^{-\beta-1} dx.
\]
Thus using a simple approximation procedure we have that
\[
|J_2| \leq c_1 \lambda + c_2 \lambda \int_0^1 x^\beta x^{-\beta-1} dx = O(e^{-\beta} |\log e|).
\]
Combining these estimates we finally obtain \( E_0(M(e)^2) = 2\lambda^2 [\Gamma(1 + 2\beta)]^{-1} + O(e^{-\beta} |\log e|) \), and this implies (4.1) in the case \( k = 2 \).

In order to attack the general case it will be necessary to introduce some notation. As above let \( q_1(x) = E_0 \int_0^1 h(T^*_t) dt = [\Gamma(1 + \beta)]^{-1} (1 - x)^\beta h(x) \), and define
\[
q_n(x) = E_0 \int_0^1 h(T^*_t) q_{n-1}(T^*_t) dt
\]
for \( n \geq 2 \). Let us show by induction that
\[
q_n(x) = [\Gamma(1 + n\beta)]^{-1} (1 - x)^n h(x).
\]
This is true when \( n = 1 \), assuming it for \( n - 1 \) and using Lemma 1 we find
\[
q_n(x) = \frac{\beta}{\Gamma(1 + \beta)} \int_x^\infty (y - x)^{\beta-1} q_{n-1}(y) dy
= [\Gamma(1 + n\beta)]^{-1} (1 - x)^n h(x),
\]
and so (4.8) is established.

Secondly letting \( q_0(x) \equiv 1 \) we define
\[
L_n(t, \omega) = \lambda^{n-1} E_T(t) \int_0^\infty h(T^*_u) q_{n-1}(T_u) Q(du, \omega)
\]
for \( n \geq 1 \). Of course, \( L_n \) depends on \( \epsilon \). Next we show that
\( L_n(t) = \lambda^n q_n(T_t) + O(e^{-\alpha_1 n} |\log \epsilon|) \)

for \( n \geq 1 \), where the \( O \)-term is uniform in \( t \) and \( \omega \). If \( n = 1 \), applying Lemma 2 and the definition of \( q_1 \), we have

\[
L_1(t, \omega) = E_{T(t)} \int_0^\infty h(T_u^\omega)Q(du, \epsilon) = \lambda q_1(T_t),
\]

which certainly implies (4.10) when \( n = 1 \). For \( n \geq 2 \) we can write

\[
L_n(t) = \lambda^{n-1} E_{T(t)} \int_0^\infty h(T_u^\omega)q_{n-1}(T_u^\omega)Q(du, \epsilon)
\]

\[- \lambda^{n-1} E_{T(t)} \int_0^\infty h(T_u^\omega) [q_{n-1}(T_u^\omega) - q_n(T_u^\omega)]Q(du, \epsilon)
\]

\[= J_1 - J_2.\]

From Lemma 2 and (4.7) we see that \( J_1 = \lambda^n q_n(T_t) \). Concerning \( J_2 \), we note \( q_{n-1} \) is bounded, say by \( M \), and that \( 0 \leq q_{n-1}(x) - q_n(x + y) \leq c_y \) if \( n \geq 2 \), since \( 0 < \beta < 1 \). Thus exactly as in the case \( k = 2 \) we obtain

\[
J_2 \leq 2\lambda^{n-1} M + \lambda^{n-1} E_{T(t)} \int_\epsilon^1 x^\beta p([0, S_1], dx).
\]

But from (4.4) with \( y = T(t) \) we see that

\[
E_{T(t)} p([0, S_1], dx) = c_1 (1 - T_t) h(T_t) x^{-1} dx,
\]

and so \( J_2 = O(e^{-\alpha_1 n} |\log \epsilon|) \) uniformly in \( t \) and \( \omega \) since \( (1 - T_t) h(T_t) \leq 1 \).

Combining these results we obtain (4.10).

Finally we will show by induction that

\[
(4.11) \quad L_n(t) = E_{T(t)} \int_0^\infty h(T_u^\omega)Q(du, \epsilon) + \int_0^\infty \cdots \int_0^\infty h(T_u^\omega)Q(du, \epsilon) + O(e^{-\alpha_1 n} |\log \epsilon|)
\]

provided \( n \geq 1 \), where again the \( O \)-term is uniform in \( t \) and \( \omega \). Denote the first term on the right-hand side on (4.11) by \( J_n(t) \). If \( n = 1 \), (4.11) is immediate from the definition of \( L_1 \). Assume (4.11) for \( n - 1 \). Define

\[
Y^\epsilon(t_1) = \int_{t_1}^\infty h(T_u^\omega)Q(du, \epsilon) + \int_{t_2}^\infty h(T_u^\omega)Q(du, \epsilon).
\]

Clearly \( Y^\epsilon(t_1) \) is right continuous since the integral is over \((t_1, \infty)\), and \( 0 \leq Y^\epsilon(t_1) \leq ([\epsilon^{-1}] + 2)^{n-1} \). Also since \( T(t) \) is a Markov process it is easy to see that for each \( y \) a version of \( E_y (Y^\epsilon(t_1) | \mathcal{F}_{t_1}) \) is given by (\( T \) starts from \( y \))

\[
E_{T(t_1)} \int_0^\infty h(T_u^\omega)Q(du, \epsilon) + \int_0^\infty \cdots \int_0^\infty h(T_u^\omega)Q(du, \epsilon)
\]

\[= L_{n-1}(t_1) + O(e^{-\alpha_1 (n-2)} |\log \epsilon|)
\]

\[= \lambda^{n-1} q_{n-1}(T_t) + O(e^{-\alpha_1 (n-2)} |\log \epsilon|)
\]
where we have used the induction hypothesis and (4.10). Applying Lemma 3 whose hypotheses are satisfied we find that

\[ J_n(t) = E_{T(t)} \int_0^\infty h(T_{i1}^*) Y^*(t_1) Q(dt_1, \varepsilon) \]

\[ = \lambda^{n-1} E_{T(t)} \int_0^\infty h(T_{i1}^*) q_{n-1}(T_{i1}) Q(dt_1, \varepsilon) + O(\varepsilon^{-(n-1)\beta} |\log \varepsilon|), \]

which is (4.11) in view of the definition (4.9) of \( L_n(t) \).

We are now ready to prove (4.1) for general values of \( k \geq 2 \). Assume (4.1) for \( 0, 1, \ldots, k - 1 \); then

\[ E_0(M(\varepsilon)^k) = E_0 \left( \int_0^\infty h(T_u^*) Q(du, \varepsilon) \right)^k \]

(4.12)

\[ = k! E_0 \int_0^\infty h(T_u^*) Q(dt_1, \varepsilon) \int_t^{t_1} \cdots \int_t^{t_{k-1}} h(T_u^*) Q(dt_k, \varepsilon) \]

+ error term.

Now the error term consists of a finite sum of integrals over lower dimensional hyperplanes obtained by identifying a subset of the variables \( t_1, \ldots, t_k \). Exactly as in the case \( k = 2 \) each such integral reduces to \( E_0(M(\varepsilon)^{k-n+1}) \) if \( n \) of variables are identified, \( n = 2, \ldots, k \). Thus by the induction hypothesis the error term is \( O(\varepsilon^{-(k-1)\beta}) \). But arguing exactly as in the proof of (4.11) and making use of (4.11) and (4.10) the first term, \( J \), on the right-hand side of (4.12) is just

\[ J = k! \lambda^{k-1} E_0 \int_0^\infty h(T_u^*) q_{k-1}(T_u) Q(du, \varepsilon) + k! O(\varepsilon^{-(k-2)\beta}|\log \varepsilon|) E_0 \int_0^\infty h(T_u^*) Q(du, \varepsilon). \]

Writing \( q_{k-1}(T_u) = q_{k-1}(T_u^*) - [q_{k-1}(T_u^*) - q_{k-1}(T_u)] \) and using an argument that is by now familiar we find

\[ J = k! \lambda^{k-1} E_0 \int_0^\infty h(T_u^*) q_{k-1}(T_u^*) du + O(\varepsilon^{-(k-1)\beta}|\log \varepsilon|). \]

Finally combining this with (4.7) and (4.8) we have

\[ E_0(M(\varepsilon)^k) = k! \lambda^k [\Gamma(1 + \beta k)]^{-1} + O(\varepsilon^{-(k-1)\beta}|\log \varepsilon|) \]

and this implies (4.1). Thus Theorem B is, at long last, established.

Note added in proof. The results of this paper and those of [2] are valid for general stable processes of index \( \alpha \), \( 1 < \alpha \leq 2 \). The proofs in the general case are exactly the same as in the symmetric case once Lemma 3.1 of [2] is established for general stable processes \( X \) of index \( \alpha \). However, a careful examination of the proof of this lemma in the symmetric case (which goes back to Kac) reveals that exactly the same argument works in the general case. See also a forthcoming paper of C. J. Stone in the Illinois Journal of Mathematics.
REFERENCES


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