SOLUTIONS TO COOPERATIVE GAMES WITHOUT SIDE PAYMENTS

BY

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An extension of Von Neumann Morgenstern solution theory to cooperative games without side payments has been outlined in [1]. In this paper we revise some of the definitions given in [1] and prove that in the new theory every three-person constant sum game is solvable (see [1, Theorem 1]). Other results that were formulated in [1] had already been proved in [2]. [1; 2] are also necessary for a full understanding of the basic definitions of this paper.

1. Basic definitions. If $N$ is a set with $n$ members, we denote by $E^n$ the $n$-dimensional euclidean space the coordinates of whose points are indexed by the members of $N$. Subsets of $N$ will be denoted by $S$. If $x \in E^n$ and $i \in N$, $x^i$ will denote the coordinate of $x$ corresponding to $i$; $x^S$ will denote the set $\{x^i : i \in S\}$. The superscript $N$ will be omitted, thus we write $x$ instead of $x^N$. We write $x^S \geq y^S$ if $x^i \geq y^i$ for all $i \in S$; similarly for $>$ and $=$. $\emptyset$ denotes the empty set.

Definition 1.1. An $n$-person characteristic function is a pair $(N, v)$ where $N$ is a set with $n$ members, and $v$ is a function that carries each $S \subseteq N$ into a set $v(S) \subseteq E^n$ so that

1. $v(S)$ is closed,
2. $v(S)$ is convex,
3. $v(\emptyset) = \emptyset$,
4. if $x \in v(S)$ and $x^S \geq y^S$, then $y \in v(S)$.

Definition 1.2. An $n$-person game is a triad $(N, v, H)$, where $(N, v)$ is an $n$-person characteristic function and $H$ is a convex compact subset of $v(N)$.

We notice that this definition is not identical with that given in [1; 2]. In the first place $v$ is not assumed to be superadditive, i.e., the condition: $v(S_1 \cup S_2) \supseteq v(S_1) \cap v(S_2)$ for every pair of disjoint coalitions $S_1$ and $S_2$ is dropped. Secondly $H$ need not be a polyhedron.

2. Solutions. Let $G = (N, v, H)$ be an $n$-person game.

Definition 2.1. Let $x, y \in E^n$, $S \neq \emptyset$. $x$ dominates $y$ via $S$, written $x \succ_S y$, if $x \in v(S)$ and $x^S \geq y^S$.

Definition 2.2. $x$ dominates $y$, written $x \succ y$, if there is an $S$ such that $x \succ_S y$.

For $x \in E^n$ the following sets are defined: $\text{dom}_S x = \{y : x \succ_S y\}$ and $\text{dom} x = \{y : x \succ y\}$. Let $K \subseteq E^n$. We define $\text{dom}_S K = \bigcup_{x \in K} \text{dom}_S x$ and $\text{dom} K = \bigcup_{x \in K} \text{dom} x$.
Definition 2.3. $V$ is $K$-stable if $V = K - \text{dom } V$.

Definition 2.4. The $K$-core is the set $K - \text{dom } K$.

We use the following abbreviation: P.S.O.—the proof, which is straightforward, will be omitted.

Proposition 2.5. Every $K$-stable set contains the $K$-core. P.S.O.

Proposition 2.6. If for each $x \in K \cap \text{dom } K$ there is a $y \in K - \text{dom } K$ such that $y \not\prec x$ then the $K$-core is the only $K$-stable set. P.S.O.

We denote: $v^i = \sup_{x \in v(i)} y^i$.

Definition 2.7. $x$ is individually rational if $x^i \geq v^i$ for all $i \in N$.

Definition 2.8. $x$ is group rational if there is no $y \in H$ such that $y > x$.

We denote: $\tilde{A} = \{x: x \in H, x$ is individually rational$\}$ and $A = \{x: x \in \tilde{A}, x$ is group rational$\}$.

Proposition 2.9. $K$ is $\tilde{A}$-stable if and only if it is $A$-stable.

Proof. Let $K$ be $\tilde{A}$-stable. We show firstly that (1) $\tilde{A} - A \subset \text{dom } K$. If $x \in \tilde{A} - A$ then there is a $y_0 \in \tilde{A}$ such that $y_0 > x$. Define $f(y) = \min_{i \in N} (y^i - x^i)$. Since $f$ is continuous and $\tilde{A}$ is compact $f$ receives its maximum in $\tilde{A}$ at a point $z$, which must be in $A$. By 1.2 $z \in v(N), f(z) \geq f(y_0)$; therefore $z > x$. We have that $z \not\prec y$ and if $w \not\prec z$ then $w \not\prec x$. If $z \in K$ then $x \in \text{dom } K$. If $z \in \text{dom } K$ then there is a $w_0 \in K$ such that $w_0 \not\prec z$ and therefore $w_0 \not\prec x$, so $x \in \text{dom } K$. From (1) it follows that $\tilde{A} - \text{dom } K = A - \text{dom } K$ and therefore $K$ is $A$-stable. Now, let $K$ be $\tilde{A}$-stable. If $x \in \tilde{A} - A$ we define $z$ as before and we see that $z \in A \subset K \cup \text{dom } K$ implies that $x \in \text{dom } K$. We conclude that (1) holds and therefore $K = A - \text{dom } K = \tilde{A} - \text{dom } K$, i.e., $K$ is $\tilde{A}$ stable.

Definition 2.10. A solution of $G$ is an $A$-stable set.

If $G$ has a solution we say that $G$ is solvable.

Theorem 2.11. Every two person game has a unique solution, consisting of all of $A$. P.S.O.

Definition 2.12. $G$ is constant-sum if $H$ is contained in a plane

\[ \sum_{i \in N} x^i = e. \]

3. Three-person constant sum games.

I. Auxiliary lemmas. We use the following abbreviations: 3-P.C.G. — three-person constant sum game, W. L. G. — without loss of generality.

Let $G = (N, v, H)$ be 3-P.C.G. We denote the members of $N$ by the first three positive integers and set $S_i = N - \{i\}$ for $i = 1, 2, 3$. Let $x \in H$. We denote: $\sum_{i=1}^3 x^i = e$ and $L = \{y: \sum_{i=1}^3 y^i = e\}$. We have that $\tilde{A} = \{x: x \in H, x^i \geq v^i, i = 1, 2, 3\} = A$. So $A$ is a convex compact subset of $L$. Domination between
points of $A$ is possible only via the $S_i$, i.e., if $x, y \in A$ and $x \preceq_{S_i} y$ then $S$ is one of the $S_i$. For a subset $B$ of $L$ and $i \in \mathbb{N}$ the following sets are defined: $B^i = B \cap v(S_i)$, $\tilde{B}^i = B^i \cap \text{dom}_{S_i}^i B^i$ and $b^i = B^i - \tilde{B}^i$.

**Lemma 1.1.** If $B$ is convex then $\tilde{B}^i$ is convex.

**Proof.** $B^i = B \cap v(S_i)$ is convex. If $x_1, x_2 \in \tilde{B}^i$ then there are $y_1, y_2 \in B^i$ such that $y_j \preceq_{S_i} x_j$ for $j = 1, 2$. If $0 < t < 1$, $x = tx_1 + (1 - t)x_2$ and $y = ty_1 + (1 - t)y_2$ then $x, y \in B^i$ and $y \preceq_{S_i} x$, so $x \in \tilde{B}^i$.

We remark that $A^i$ is convex and compact, $\tilde{A}^i$ is convex and $\tilde{A}^i$ is compact.

Let $x \in L$ and $\varepsilon > 0$. The set $\{y : y \in L, \sum_{i=1}^{3} (y^i - x^i)^2 < \varepsilon^2\}$ is denoted by $S(x, \varepsilon)$. $x$ is an interior point of a subset $B$ of $L$ if there is an $\varepsilon > 0$ such that $S(x, \varepsilon) \subseteq B$.

**Lemma 1.2.** If $B \subseteq L$ is convex and $K = \tilde{B}^i \cap \tilde{B}^j \neq \emptyset$, $i \neq j$, then $K$ contains an interior point.

**Proof.** W.L.G. $i = 1$ and $j = 2$. We show firstly that $K \neq \emptyset$ implies that $B$ contains an interior point. If $B$ has no interior points then there are points $x_1$ and $x_2$ such that every $y \in B$ can be written as $y = tx_1 + (1 - t)x_2$, $-\infty < t < \infty$. Let $x \in K$. $x = t_0x_1 + (1 - t_0)x_2$. There are $y_l = t_lx_1 + (1 - t_l)x_2$, $y_l \preceq_{S} x$ for $l = 1, 2$. We have $y_1^2 > x_1^2$ and $y_1^2 > x_2^2$, i.e., $t_1x_1 + (1 - t_1)x_2 > t_0x_1 + (1 - t_0)x_2^2$ and $t_1x_1 + (1 - t_1)x_2 > t_0x_1^2 + (1 - t_0)x_2$.

We have $t_1x_1 + (1 - t_1)x_2 > t_0x_1^2 + (1 - t_0)x_2$. So $(t_1 - t_0)(x_1^2 - x_2^2) > 0$ and $(t_1 - t_0)(x_1^2 - x_2^2) > 0$.

Therefore $\text{sgn}(x_2^3 - x_3^3) = \text{sgn}(x_1^3 - x_2^3)$. In the same way $y_2 \preceq_{S} x$ implies that $\text{sgn}(x_1^3 - x_2^3) = \text{sgn}(x_1^3 - x_2^3)$.

So the three differences $x_1 - x_2$ have the same sign, which is impossible since $\sum_{k=1}^{3} x_1 = \sum_{k=1}^{3} x_2$. Now, let $z$ be an interior point of $B$ and $y \in K$. For small positive $t$ the points $tz + (1 - t)y$ are interior points of $K$.

**Lemma 1.3.** If $x \in O = \bigcap_{i=1}^{3} \tilde{A}^i$ then $x$ is an interior point of $O$.

**Proof.** There are $y_j \preceq_{S} x$ for $j = 1, 2, 3$. We have: $y_1^2 > x_1^2$, $y_1^2 > x_2^2$, $y_1^2 > x_2^2$, $y_1^2 > x_1^2$, $y_1^2 > x_2^2$, $y_1^2 > x_2^2$ and $y_1^2 > x_2^2$. There exist $0 < t_k < 1$ such that $z_k = t_ky_1 + (1 - t_k)y_k$ satisfy $z_k^2 = x_k^2$, $k = 2, 3$. Since $z_1^2 > x_1^2$ and $z_2^2 < x_2^2$ there is a $0 < t < 1$ such that $x = t_1z_1 + (1 - t_1)z_2$. So $x$ is an interior point of the convex hull of $\{y_1, y_2, y_3\}$ and therefore of $A$. But if $x \in O$ is an interior point of $A$ then $x$ is also an interior point of $O$.

**Lemma 1.4.** If $B \subseteq L$ is convex, $x_1, x_2 \in B$, $x_1^i = x_2^i$, $x_1^i < x_2^i$, $x_1^i > x_2^i$ and $y$ satisfies $y^i = x_2^i$, $y^i = x_1^i$ and $y^i = x_1^i + x_2^i - x_2^i$ then: $y \notin B$ if and only if $B \cap \{z : z_{S_k} \geq y_{S_k}\} = \emptyset$. P.S.O.

**Lemma 1.5.** If $x \in \tilde{A}^i$ then there is a $y \in a^i$ such that $y \preceq_{S_i} x$ and for every $\varepsilon > 0$, $S(y, \varepsilon) \cap \tilde{A}^i \neq \emptyset$.

**Proof.** Define $f(z) = \min_{y \in S_i}(z^i - x^i)$. $f$ receives its maximum in $A^i$ at a point...
which must be in a'. Since \( x \in \hat{A}' \), \( f(y) > 0 \) and therefore \( y \gtrsim_{s_1} x \). If \( 0 \leq t < 1 \) then \( ty + (1 - t)x \) is in \( \hat{A}' \), therefore for every \( \varepsilon > 0 \) \( S(y, \varepsilon) \cap \hat{A}' \neq \emptyset \).

If \( x, y \in L \) then the set \( \{ z : z = tx + (1 - t)y, 0 \leq t \leq 1 \} \) is denoted by \([xy]\) and is called an interval. \( x \) and \( y \) are called the ends of \([xy]\). \( [xy] = \{y\} \cdot (xy) = [xy] - \{x\} \cdot [xy] = [xy] - \{y\} \cdot (xy) = [xy] - (\{x\} \cup \{y\}) \). For \( i = 1, 2, 3 \) the following sets are defined: \( D^i = A^i \cap A^k \), where \( S_i = \{j, k\} \), and \( F^i = \{x : x \in D^i, x^i \geq y^i \) for every \( y \in D^j \}. D^i \) is an interval.

**Lemma 1.6.** Let \( S_k = \{i, j\} \). If \( x^i \) receives its maximum in \( F^k \) at a point \( a \), then:

\[ a \not\in a^j \text{ if and only if } A^j \supset A^i. \]

**Proof.** W.L.G. \( i = 2 \) and \( j = 3 \). If \( A^2 \supset A^3 \) then \( a \in F^1 \subset D^1 \subset A^2 \subset A^3 \). If \( a \notin A^3 \) then there is an \( x \in A^3 \) such that \( x \gtrsim_{s_2} a \). There is an \( \varepsilon > 0 \) such that \( U = S(x, \varepsilon) \subset A \subset \hat{A}^3 \). Now we show that if \( y \in A^2 \) then \( y^1 \leq a^1 \). If there is a \( z \in A^2 \) such that \( z^1 > a^1 \) then for a small positive \( tu = tz + (1 - t)a \) satisfies \( u^1 > a^1 \) and \( u \in U \). So we have \( u \in D^1 \) and \( u^1 > a^1 \) which is impossible. Next we show that \( y \in A^2 \) implies that \( y^2 \leq a^2 \). Suppose that there is a \( z \in A^2 \) such that \( z^2 > a^2 \). If \( z^1 = a^1 \) then for a small positive \( tu = tz + (1 - t)a \) satisfies \( u \in U \), \( u^1 = a^1 \) and \( u^2 > a^2 \). So we have that \( u \in F^1 \) and \( u^2 > a^2 \) which is impossible. If \( z^1 < a^1 \) then there is a \( 0 < t < 1 \) such that \( w = tz + (1 - t)a \) satisfies \( w^1 = a^1 \) and \( w^2 > a^2 \). Therefore \( w \in A^2 \), but this is impossible as we have already shown.

We have shown that every \( y \in A^2 \) satisfies \( y^1 \leq a^1 \). Since \( a \in \text{dom}_{S_2} x \) we have \( A^2 \subset A \cap \text{dom}_{S_3} \ x \subset \hat{A}^3 \).

The sets \( \{A^1, A^2\}, \{A^2, A^3\} \) and \( \{A^3, A^1\} \) will be called pairs.

**Definition 1.7.** The pair \( \{A^i, A^j\} \) intersects maximally if:

1. \( A^1 \cap A^j \neq \emptyset \).
2. \( A^1 \cap A^i \neq \emptyset \).

The number of pairs that intersect maximally will be denoted by \( m(G) \).

**Lemma 1.8.** Let \( i \neq j \) and \( A^i \cap A^j \neq \emptyset \). \( A^i \not\supset \ A^j \) and \( A^j \not\supset A^i \) if and only if \( \{A^i, A^j\} \) intersects maximally.

**Proof.** W.L.G. \( i = 2 \) and \( j = 3 \). If \( A^2 \supset A^3 \) or \( A^3 \supset A^2 \) then \( a^2 \cap a^3 = \emptyset \) and therefore \( \{A^2, A^3\} \) does not intersect maximally. Now suppose that \( A^2 \not\supset A^3 \) and \( A^3 \not\supset A^2 \). Let \( x^2 \) and \( x^3 \) receive their maxima in \( F^1 \) at the points \( a \) and \( \beta \) respectively. By I.6: \( A^2 \not\supset \ A^3 \) implies that \( a \in a^3 \) and \( A^2 \not\supset A^3 \) implies that \( \beta \in a^2 \). We have (1) that \( F^1 \subset a^2 \cup a^3 \) and \( F^1 \cap a^k \neq \emptyset \), \( k = 2,3 \). Since \( F^1 \) is connected and \( F^1 \cap a^2 \) and \( F^1 \cap a^3 \) are closed we must have \( (F^1 \cap a^2) \cap (F^1 \cap a^3) = F^1 \cap a^2 \cap a^3 \neq \emptyset \).

From the proof of I.8 we can conclude that: (1.9) if \( \{A^i, A^j\} \) intersects maximally then \( F^k \cap a^1 \cap a^j \neq \emptyset \) where \( \{k\} = N - \{i, j\} \).
Lemma 1.10. If $i \neq j, x, y \in A^i, x \neq y$ and $x^i = y^j$ then every $z \in A^i$ satisfies $z^i \leq x^j$ and \{u: u \in A^i, u^i = x^j\} \subseteq A^i$. P.S.O.

Lemma 1.11. Let $S_k = \{i, j\}$. If $\{A^i, A^j\}$ intersects maximally and $x^i$ and $x^j$ take their maxima in $F^k$ at the points $\alpha$ and $\beta$ respectively then $F^k = [\alpha \beta]$ and one of the following possibilities holds:

(a) $\alpha = \beta, \quad [\alpha \beta] \subseteq A^i \cap A^j$.
(b) $\alpha \neq \beta, \quad [\alpha \beta] \subseteq A^i \cap A^j, \quad \alpha \in A^i \cap A^j,$
(c) $\alpha \neq \beta, \quad [\alpha \beta] \subseteq A^i \cap A^j, \quad \beta \in A^i \cap A^j.$

Proof. W.L.G. $i = 2$ and $j = 3$. We saw in the proof of I.8 that $\alpha \in F^1 \cap a^3$ and $\beta \in F^1 \cap a^2$. If $\alpha = \beta$ then (a) holds. If $\alpha \neq \beta$ we have the following possibilities for the relative positions of $a^3$ and $F^1$:

1. There is no $x \neq \alpha$ in $a^3 \cap F^1$, i.e., $[\alpha \beta] \subseteq A^3$.
2. There is an $x \neq \alpha$ in $a^3 \cap F^1$, and, therefore, by I.10, $F^1 \subseteq a^3$. And similarly for $a^2$ and $F^1$:

3. There is no $y \neq \beta$ in $a^2 \cap F^1$, i.e., $[\alpha \beta] \subseteq A^2$.
4. There is a $y \neq \beta$ in $a^2 \cap F^1$ and therefore $F^1 \subseteq a^2$.

Since $F^1 \subseteq a^2 \cup a^3$ (1) and (3) cannot hold together. If (2) and (4) hold together then we have (b). If (1) and (4) hold together, then we have (c).

We say that $F^k$ has a-shape if (a) holds; similarly for (b), (c), and (c).

For $x \in A$ the following sets are defined: $Q_i(x) = \{y: y \in A, y^i \leq x^i\}, T_i(x) = \{y: y \in A, y^i < x^i\}$ and $R_i(x) = A - T_i(x).$ We remark that:

(I.12) $\text{dom } Q_i(x) \cap R_i(x) = \text{dom } s_i Q_i(x) \cap R_i(x),$
(I.13) $x \notin A^i$ if and only if $x \in Q_i(x) - \text{dom } Q_i(x),$
(I.14) $x \notin A^i$ if and only if $Q_i(x) \cap A^i = \emptyset.$

Lemma 1.15. If $x \in A - A^i$, $y \neq x, y \in A^i \cap Q_i(x)$ then there is a $j \in S_i$ such that every $z \in A^i$ satisfies $z^i \leq x^j$. P.S.O.

Lemma 1.16. Let $S_k = \{i, j\}$. We denote the ends of $F^k$ by $\alpha$ and $\beta$ such that $\alpha^i \geq \beta^i.$ If $y \in A$ satisfies $\gamma^k = \alpha^k$ and $\gamma^l < \beta^l$ then $Q_k(y) \cap A^i = \emptyset$. P.S.O.

Lemma 1.17. Let $S_k = \{i, j\}$. If $y \in F^k \cap A^i \cap A^j$ and $x \in R_k(y) - F^k$ then dom $x \cap Q_k(y) = \emptyset$.

Proof. W.L.G. $i = 2$ and $j = 3$. We denote the ends of $F^1$ by $\alpha$ and $\beta$ such that $\alpha^2 \geq \beta^2.$ Let $x \in R_1(\gamma) - F^1$ and $y \in Q_1(\gamma) - x^2 \leq \gamma^2 + y^3 \leq y^2 + y^3$ so $x \not\prec s_i y$ is impossible. If $x^1 > \gamma^1$ and $x \not\prec s_3 y$ or $x \not\prec s_3 y$ then $x \not\prec s_2 y$ or $x \not\prec s_2 y$ respectively, which is impossible. If $x^1 = \gamma^1$ then either $x^2 < \beta^2$ or $x^3 < \alpha^3.$ If $x^3 < \alpha^3$ then, by I.16, $x \notin A^3.$ Since $y^3 \geq \gamma^3 \geq \alpha^3 > x^3,$ if $x \not\prec y$ then $x \not\prec s_3 y,$ but this is impossible. Similarly if $x^2 < \beta^2$ then $x \not\prec y$ is impossible.
Lemma 1.18. If a $B \subset A$ is convex and compact then $(N, v, B)$ is 3-P.C.G., $B' = A^1 \cap B$ and $\bar{B}' = \bar{A}^1 \cap B$. P.S.O.

If $B \subset A$ is convex and compact we say that $B$ is solvable or that $B$ has a solution if $(N, v, B)$ is solvable. We also write $m(B)$ instead of $m((N, v, B)).$

Lemma 1.19. If $x \in D^k$ and $I \in S_k$ then $R^I_k(x) = \bar{A}^I \cap R_k(x)$.

Proof. If $y \in \bar{A}^I \cap R_k(x)$ then $y_k \geq x_k$ and there is a $z \in A^I$ such that $z \succeq_{s^c} y$.

Since $z^k > y^k$, we have $z \in R_k(x) \cap A^I = R_k^I(x)$ and therefore $y \in \bar{R}^I_k(x)$. We have shown that $\bar{R}^I_k(x) \supset A^I \cap R_k(x)$. By 1.18 $\bar{R}^I_k(x) \subset \bar{A}^I \cap R_k(x)$, so $\bar{R}^I_k(x) = \bar{A}^I \cap R_k(x)$.

Lemma 1.20. If $x \in D^k - \bar{A}^I$, $I \in S_k$ and $\{A^k, A^I\}$ does not intersect maximally, then $\{R^I_k(x), R^I_k(x)\}$ does not intersect maximally.

Proof. Since $\{A^k, A^I\}$ does not intersect maximally, by 1.8 at least one of the following possibilities holds: $\bar{A}^k \cap \bar{A}^I = \emptyset$, $\bar{A}^I \supset A^k$, or $\bar{A}^k \supset A^I$. If $\bar{A}^k \cap \bar{A}^I = \emptyset$ then $\bar{R}^I_k(x) \cap \bar{R}^I_k(x) = \bar{A}^k \cap \bar{A}^I \cap R_k(x) = \emptyset$. If $\bar{A}^I \supset A^k$ then $\bar{R}^I_k(x) = \bar{A}^I \cap R_k(x)$.

Definition 1.21. Let $B_1, \ldots, B_l$ be convex compact subsets of $A$. $B_1, \ldots, B_l$ are called independent if there exist solutions $V_1, \ldots, V_l$, $V_i$ solution of $B_i$ respectively, such that $\text{dom} V_i \cap \bigcup_{j=1}^l V_j = \emptyset$ for $k = 1, \ldots, l$.

Lemma 1.22. If $B_1, \ldots, B_l$ are independent then there exist solutions $V_1, \ldots, V_l$, $V_i$ solution of $B_i$ for $i = 1, \ldots, l$, such that $\bigcup_{j=1}^l V_j$ is $U_{j=1}^l B_j$-stable. P.S.O.

In the following three subsections we shall prove:

Theorem. Every 3-P.C.G. $G$ is solvable.

The proof will be by induction on $m(G)$.

II. First part: $m(G) = 0$. In this subsection we show that every 3-P.C.G. $G$ for which $m(G) = 0$ is solvable. We also prove some additional auxiliary lemmas:

Lemma II.1. Let $G$ be 3-P.C.G. If $\bar{A}^1 \cap \bar{A}^2 = \bar{A}^2 \cap \bar{A}^3 = \bar{A}^3 \cap \bar{A}^1 = \emptyset$ then the $A$-core is the solution of $G$.

Proof. Denote $C = A - \text{dom} A$. If $x \in A - C$ then there is a $y \in A$ that dominates it. There is an $i$ such that $y \succeq_{s^c} x$, i.e., $x \in \bar{A}^I$. By 1.5 there is a $z \in a^I$ such that $z \succeq_{s^c} x$. If $z \notin C$ then $z \in \bar{A}^I$ where $I \neq i$. There is an $\varepsilon > 0$ such that $S(z, \varepsilon) \cap A \subset \bar{A}^I$. But $S(z, \varepsilon) \cap \bar{A}^I \neq \emptyset$; therefore $\bar{A}^I \cap \bar{A}^I \neq \emptyset$ which is impossible. We have shown that for every $x \in A - C$ there is a $z \in C$ such that $z \succeq_{s^c} x$. By 2.6 $C$ is the only $A$-stable set.

Lemma II.2. Let $G$ be 3-P.C.G. If $m(G) = 0$ then the $A$-core is the solution of $G$.

Proof. If $\bar{A}^1 \cap \bar{A}^2 = \bar{A}^2 \cap \bar{A}^3 = \bar{A}^3 \cap \bar{A}^1 = \emptyset$ then by II.1 the $A$-core is the solution of $G$. If it is not the case then, W.L.G., we assume that $\bar{A}^2 \cap \bar{A}^3 \neq \emptyset$. Since $\{A^2, A^3\}$ does not intersect maximally we have that either $\bar{A}^3 \supset A^2$ or $\bar{A}^2 \supset A^3$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
W.L.G. we suppose that $A^3 \supset A^2$. There are three possibilities for the relative position of $A^1$ and $A^3$: (a) $A^1 \cap A^3 = \emptyset$, (b) $A^3 \supset A^1$ or (c) $A^1 \supset A^3$. In each case we show that $C$, the $A$-core, is the solution of $G$. (a) $A^1 \cap A^3 = \emptyset$. If $x \in A - C$ then there is a $y \in A$ such that $y \prec_{A} x$ for some $i$. Since $A^3 \supset A^2$ we may assume that $i \in S_2$. There is a $z \in a^i$ such that $z \prec_{A} x$. If $i = 1$ then, since $A^1 \cap A^j = \emptyset$ for $j = 2, 3$, $z \in C$. If $i = 3$ then, since $a^3 \cap A^2 = \emptyset$ and $A^1 \cap A^3 = \emptyset$, $z \in C$. By 2.6 $C$ is the solution of $G$. (b) $A^3 \supset A^1$. If $x \in A - C$ then there is a $y \in A$ such that $y \prec_{A} x$. So there is a $z \in a^3$ such that $z \prec_{A} x$. Since $a^3 \cap (A^1 \cup A^3) = \emptyset$, $z \in C$. So $C$ is the solution of $G$. (c) $A^1 \supset A^3$. The proof in this case parallels that in case (b).

Let $G = (N, v, H)$ be 3-P.C.G.

**Lemma II.3.** If $\xi \in D^k - \hat{A}^k$, $\eta \in D^k \cap Q_k(\xi)$ and $U$ is a solution of $Q_k(\eta)$ then $V = U \cup [\eta]$ is a solution of $Q_k(\xi)$.

**Proof.** W.L.G. $k = 1$. $\eta_1 \leq \xi_1$, $\eta_2 \leq \xi_2$ and $\eta_3 \geq \xi_3$; therefore $(\text{dom}_{S_1}[\eta]) \cup (\text{dom}_{S_3}[\eta]) \cap [\xi] = \emptyset$. $\cdots \hat{A}^1$; therefore $Q_1(\xi) \cap \hat{A}^1 = \emptyset$. So $\text{dom}_{S_1}[\eta] \cap [\xi] = \emptyset$. Summing we have (1) $\text{dom}_{S_1}[\eta] \cap [\xi] = \emptyset$. Now we show (2) $Q_1(\xi) - [\xi] - Q_1(\eta) \subseteq \text{dom}_{S_1}[\eta]$. Let $x \in Q_1(\xi) - [\xi] - Q_1(\eta)$. If $x_1 \leq \eta_1$ then there is a $y \in [\xi]$ such that $y_1 = x_1$. $y \neq x$ so we may assume that $y_2 > x_2$. Under this assumption we can find a $z \in [\eta]$ with $z_{S_1} > x_{S_1}$, so $z \prec_{A} x$. If $x_1 > \eta_1$ then, since $x \notin Q_1(\eta)$, either $x_2 > \eta_2$ or $x_3 > \eta_3$ and therefore $x \in \text{dom}_{S_1}[\eta] \cup \text{dom}_{S_3}[\eta]$. We now prove (3) $\text{dom} U \cap [\xi] = \emptyset$ and $\text{dom} [\xi] \cap U = \emptyset$. Let $x \in [\xi]$ and $y \in U$. $y_1 \leq x_1$, $y_2 \geq x_2$ and $y_3 \geq x_3$; therefore $x \prec_{A} y$ is impossible and if $y \prec_{A} x$ then $y \prec_{A} x$, but, since $x \notin \hat{A}^1$, this is also impossible. Combining (1), (2) and (3) it follows that $V = U \cup [\xi]$ solves $Q_1(\xi)$.

**Lemma II.4.** If $\xi \in D^k - \hat{A}^k$ then $Q_k(\xi)$ is solvable and if $V$ solves it then $\text{dom} V \supset T_k(\xi) - V$.

**Proof.** W.L.G. $k = 1$. Denote $J = Q_1(\xi) \cap D^1$. Let $\eta$ be a point where $x^1$ receives its minimum in $J$. We show that $Q_1(\eta) \cap Q_1(\eta) = \emptyset$ for all $i \neq j$. First, since $\xi \notin \hat{A}^1$, $\hat{A}^1 \cap Q_1(\xi) = \emptyset$. So we have $Q_1(\eta) \subset \hat{A}^1 \cap Q_1(\eta) = \emptyset$. Next, since $x^1$ receives its minimum in $J$ at $\eta$ we have $Q_1(\eta) \cap D^1 = \{\eta\}$. $\eta \notin \hat{A}^1$ therefore $\eta \in Q_1(\eta)$-core. We have $Q_1(\eta) \cap Q_1(\eta) \subseteq Q_1(\eta) \cap Q_1(\eta) = D^1 \cap Q_1(\eta) = \{\eta\}$, so $Q_1(\eta) \cap Q_1(\eta) = \emptyset$. By II.1 $Q_1(\eta)$ is solvable. Let $U$ be a solution of $Q_1(\eta)$; by II.3 $U \cup [\eta]$ solves $Q_1(\xi)$. Now let $V$ be a solution of $Q_1(\xi)$. $\xi \in Q_1(\xi)$-core so $\xi \in V$. $\text{dom} \xi \supset T_1(\xi) - Q_1(\xi)$ therefore $\text{dom} V \supset (Q_1(\xi) - V) \cup (T_1(\xi) - Q_1(\xi)) = T_1(\xi) - V$.

**Lemma II.5.** Let $S_k = \{i, j\}$. If $F_k$ has $c_2$-shape and $\mu \in F_k \cap \hat{A}^k$ then there is a solution $V$ of $Q_k(\mu)$ such that $V \cap F_k = \{\mu\}$.

**Proof.** W.L.G. $i = 2$ and $j = 3$. We denote the ends of $F^1$ by $\alpha$ and $\beta$ such that $\alpha^2 > \beta^2$. Since $F^1$ has $c_2$-shape we have (1) $\hat{A}^2 \cap R_1(\alpha) = \emptyset$ and (2) $[\beta \alpha] \subset \hat{A}^3$. From (1) it follows that (3) $Q_2(\mu) = \emptyset$. We show (4) $[\beta \mu] \subset Q_2(\mu)$. Let $x \in [\beta \mu]$. 

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\( x^1 = \mu^1, x^2 < \mu^2 \) and \( x^3 > \mu^3 \). By (2) \( x \in \hat{A}^3 \), therefore there is a \( y \in A^3 \) such that \( y \preceq_s x \). For small \( t > 0 \) \( z = ty + (1-t)x \) satisfy \( z^1 > x^1, z^2 > x^2 \) and \( z^3 > \mu^3 \), so \( z \in Q_2^2(\mu) \) and \( x \in \hat{Q}_2^2(\mu) \). For the relative position of \( Q_2^2(\mu) \) and \( Q_3^2(\mu) \) we have the following possibilities: (a) \( Q_2^2(\mu) \cap \hat{Q}_2^2(\mu) = \emptyset \) or (b) \( \hat{Q}_2^2(\mu) \cap \hat{Q}_3^2(\mu) \neq \emptyset \). If (a) holds then by (3) \( \hat{Q}_2^2(\mu) \cap \hat{Q}_3^2(\mu) = \emptyset \) for all \( i \neq j \). By II.1 \( Q_2(\mu) = \text{dom } Q_2(\mu) \) is a solution of \( Q_2(\mu) \) and since \( \mu \in Q_2(\mu) \)-core, \( (Q_2(\mu) = \text{dom } Q_2(\mu)) \cap F^1 = \{\mu\} \). If (b) then by I.2 there is an interior point \( \zeta \) of \( \hat{Q}_2^2(\mu) \cap \hat{Q}_3^2(\mu) \). \( \zeta^1 > \mu^1, \zeta^2 < \mu^2 \) and \( \zeta^3 > \mu^3 \). \( \zeta \in D^2 - A^2 \), therefore by II.4 \( Q_2(\zeta) \) is solvable. If \( U \) is a solution of \( Q_2(\zeta) \) then, by II.3, \( [\zeta, \mu] \cup U \) solves \( Q_2(\mu) \). Since \( ([\mu] \cup U) \cap F^1 = \{\mu\} \) this completes the proof.

**Definition II.6.** The pair \( \{A^i, A^j\} \) satisfies condition M if:

1. \( \{A^i, A^j\} \) intersects maximally,
2. \( F^k \cap A^i \cap A^j = \emptyset \) where \( \{k\} = N - \{i, j\} \).

We now formulate the induction hypothesis:

**II.7.** every 3-P.C.G. \( G \) for which \( m(G) \leq 1 \) is solvable. Let \( G \) be 3-P.C.G. for which \( m(G) = l \). We have to prove that \( G \) is solvable. We distinguish between the following possibilities:

**II.8.** there is at least one pair that satisfies condition M.

**II.9.** there is no pair that satisfies condition M.

**III. Second part: case II.8.** W.L.G. \( \{A^2, A^3\} \) satisfies condition M. The ends of \( F^1 \) will be denoted by \( a \) and \( \beta \) such that \( a^2 \preceq \beta^2 \). \( F^1 \cap a^2 \cap a^3 \notin \hat{A}^1 \) therefore at least one of the ends is in \( a^2 \cap a^3 - A^1 \). We shall prove that \( G \) is solvable when:

**III.1.** \( a \in a^2 \cap a^3 - A^1 \). The proof when \( \beta \in a^2 \cap a^3 - A^1 \) is similar to that in case (III.1). We shall distinguish three cases according to the three possible shapes of \( F^1 \) in case (III.1).

**III.a.** \( F^1 \) has a-shape. By (III.1) and II.4 \( Q_1(\alpha) \) is solvable and if \( V \) solves it then (1) \( \text{dom } V \supseteq T_1(\alpha) - V \). Since \( \alpha \in F^1 \hat{A}^2 \cap \hat{A}^3 \cap R_1(\alpha) = \emptyset \). By I.19 \( \hat{R}_2^1(\alpha) \cap \hat{R}_3^1(\alpha) = \emptyset \), so \( \{\hat{R}_2^1(\alpha), \hat{R}_3^1(\alpha)\} \) does not intersect maximally. From I.20 it follows now that \( m(R_1(\alpha)) \leq l - 1 \). By II.7 \( R_1(\alpha) \) is solvable. If \( Q_1(\alpha) \) and \( R_1(\alpha) \) are independent then from I.22 and (1) it follows that \( A \) has a solution. If \( Q_1(\alpha) \) and \( R_1(\alpha) \) are not independent then if \( V \) solves \( Q_1(\alpha) \) and \( W \) solves \( R_1(\alpha) \) either (2) \( \text{dom } V \cap W \neq \emptyset \) or (3) \( \text{dom } W \cap V \neq \emptyset \). From III.a and I.17 it follows that (4) \( \text{dom } R_1(\alpha) \cap Q_1(\alpha) = \emptyset \). By (4) we have that (3) is impossible. By (III.1) \( a \in A - \text{dom } A \text{ therefore } (5) \alpha \in W \cap V \). From (2), (5) and I.12 it follows that there is a \( \alpha \notin \alpha \in V \cap A \). By I.15 and due to III.a, we may assume that every \( y \in A^1 \) satisfies \( y^2 \leq \alpha^2 \). Let \( \zeta \) be a point where \( x^3 \) receives its maximum in \( V \cap A \). If \( u \in V \cap A \) then \( \zeta^2 = u^2 \) and \( \zeta^3 \geq u^3 \) and therefore (6) \( \text{dom } s, \zeta \cap \text{dom } s, u \). \( \alpha \in D^3 - A^3 \) therefore by II.4, \( Q_3(\alpha) \) is solvable. If \( U \) solves \( Q_3(\alpha) \) then by (4) we have that (7) \( \text{dom } U \cap V = \emptyset \). We remark that (8) \( Q_3(\alpha) \cap A^3 \subseteq \{x: x^2 = \alpha^2 \} \). Let \( x^1 \) receive its maximum in \( U \cap A^3 \) at the point \( \eta \). We define: \( v = (\eta^1, e - \eta^1 - \zeta^3, \zeta^3) \). By I.12 and (6) we have that (9) \( R_1(\alpha) = \text{dom } V = R_1(\alpha) \).

(2) Observe that a solution of a compact set is compact. see [3, Theorem 3].
- \text{dom} s_1 \zeta = Q_3(\alpha) \cup \{x : x \in R_1(\alpha), x^3 \geq \zeta^3\}. \text{By (8) we have that (10) } Q_2(\alpha) - \text{dom } U = Q_2(\alpha) - \text{dom } s_1 \eta = \{x : x \in Q_2(\alpha), x^3 \geq \eta^3\}. \text{Combining (1), (9) and (10) we have (11) } A - \text{dom } (U \cup V) = \{x : x \in A, x^3 \geq v^3\}. \text{If } v \notin A \text{ then from I.4, (11), (9) and (7) it follows that } U \cup V \text{ solves } A. \text{Suppose now that } v \in A. \text{We define } v_1 = (\alpha^1, e - \alpha^1 - \zeta^3, \zeta^3). \text{By I.16 we have that (12) } \emptyset = A^2 \cap Q_2(v_1) \supset A^2 \cap Q_2(v). \text{Q}_1(\alpha) \cup Q_3(\alpha) \subset R_2(v) \text{ therefore by (12) and I.12 we have (13) } \text{dom } Q_2(v) \cap (Q_1(\alpha) \cup Q_3(\alpha)) = \emptyset. v \in D^2 - A^2 \text{ therefore } Q_2(v) \text{ is solvable. If } U_1 \text{ solves } Q_2(v) \text{ then by (11) and (13) } U \cup U \cup U_1 \text{ is a solution of } A. \text{III.b. } F^1 \text{ has } b\text{-shape. Due to III.b, we have that } R_1(\alpha) \cap (A^2 \cup A^3) = \emptyset \text{ and therefore (1) } \mathcal{R}_2^2(\alpha) = \mathcal{R}_2^3(\alpha) = \emptyset. \text{From (1) it follows that (2) } \mathcal{R}_2^2(\alpha) \cap \mathcal{R}_3^1(\alpha) = \emptyset \text{ for all } i \neq j. \text{If } x \in R_1(\alpha) \text{ then } x^2 + x^3 \leq \alpha^2 + \alpha^3 \text{ therefore (3) } [\alpha^2] \cap \text{dom } s_1 R_1(\alpha) = \emptyset. \text{From (1) and (3) we have that (4) } R_1(\alpha) - \text{dom } R_1(\alpha) = [\alpha^2]. \text{We also have that (5) } T_1(\alpha) = \text{dom } R_1(\alpha) = T_1(\alpha) - \text{dom } \{x, \beta \} = \{x : x \in T_1(\alpha), x^2 \leq \alpha^2, x^3 \geq \beta^3\}. \text{Define } \mu = (e - \alpha^2 - \beta^3, \alpha^2, \beta^3). \text{III.b.1. } \mu \notin A. \text{By I.4 we have that (6) } \{x : x^{s_1} \geq \mu^{s_1}\} \cap A = \emptyset. \text{By (2) and II.1 the } R_1(\alpha)\text{-core is the solution of } R_1(\alpha). \text{By (4), (5) and (6) we have that } R_1(\alpha) - \text{dom } R_1(\alpha) \text{ solves } A. \text{III.b.2. } \mu \in A. \mu \in D^1 - A^1 \text{ therefore } Q_1(\mu) \text{ is solvable. We remark that (7) } Q_1(\mu) \cap A^1 \subset \{x : x^2 = \alpha^2\}. \text{We distinguish several subcases of III.b.2.} \text{III.b.2.1. There is a solution } V_1 \text{ of } Q_1(\mu) \text{ such that } V_1 \cap A^1 = \emptyset. \text{We have that (8) } \text{dom } V_1 \cap R_1(\alpha) = \emptyset. \text{From (5) and (8) it follows that } V_1 \cup (R_1(\alpha) - \text{dom } R_1(\alpha)) \text{ is a solution of } G. \text{III.b.2.2. There is a solution } V_2 \text{ of } Q_1(\mu) \text{ such that } V_2 \cap A^1 = \{\mu\}. \text{In this case: } R_1(\alpha) - \text{dom } V_2 = Q_3(\alpha) \cup Q_2(\beta). \alpha \in D^3 - A^3 \text{ and } \beta \in D^2 - A^2 \text{ so } Q_3(\alpha) \text{ and } Q_2(\beta) \text{ are solvable and if } U \text{ solves } Q_3(\alpha) \text{ and } W \text{ solves } Q_2(\beta) \text{ then, by I.13, } \alpha \in U \text{ and } \beta \in W. \text{Since } Q_3(\alpha) \subset R_2(\beta) \text{ and } Q_2(\beta) \subset R_3(\alpha) \text{ it follows from I.12 that } \text{dom } U \cap W = \text{dom } W \cap U = \emptyset. \text{From these results and (5) it follows that } U \cup U \cup V_2 \text{ is a solution of } G. \text{III.b.2.3. There is a solution } V_3 \text{ of } Q_1(\mu) \text{ such that } V_3 \cap A^1 - \{\mu\} \neq \emptyset. \text{Let } x^3 \text{ receive its maximum in } V_3 \cap A^1 \text{ at the point } \zeta \cdot \zeta^2 = \mu^2 \text{ and } \zeta^3 > \mu^3. \text{By II.3 } [\alpha x] \cup V_3 = V_3 \text{ solves } Q_1(\alpha). \text{We define } v = (\alpha^1, e - \alpha^1 - \zeta^3, \zeta^3). R_1(\alpha) - \text{dom } V_3 = Q_3(\alpha) \cup \{x : x \in A, x^{s_2} \geq v^{s_2}\}. \text{Let } U \text{ solve } Q_3(\alpha), \text{dom } U \cap (Q_1(\alpha) \cup \{x : x \in A, x^{s_2} \geq v^{s_2}\}) = \emptyset. \text{If } v \notin A \text{ then } V_3 \cup U \text{ solves } G. \text{If } v \in A \text{ then } v \in D^2 - A^2. \text{By I.16 } Q_2(\alpha) \cup A^2 = \emptyset, Q_1(\alpha) \cup Q_3(\alpha) \subset R_3(\alpha) \text{ therefore by I.12 } \text{dom } Q_2(\alpha) \cap (Q_1(\alpha) \cup Q_3(\alpha)) = \emptyset. \text{If } W \text{ solves } Q_2(\alpha) \text{ we have that } V_3 \cup U \cup V = \emptyset \text{ is a solution of } G. \text{III.c. } F^3 \text{ has } c_3\text{-shape. As in III.a, we have that } R_1(\alpha) \text{ is solvable and if } Q_1(\alpha) \text{ and } R_1(\alpha) \text{ are independent then } G \text{ is solvable. If } Q_1(\alpha) \text{ and } R_1(\alpha) \text{ are not independent and } V \text{ solves } Q_1(\alpha) \text{ and } W \text{ solves } R_1(\alpha) \text{ then either } \text{dom } V \cap W = \emptyset \text{ or } \text{dom } W \cap V = \emptyset. \text{III.c.1. There exist } V_0 \text{ and } W_0 \text{ such that } \text{dom } V_0 \cap W_0 = \emptyset. \text{If } \alpha \in W_0 \text{ therefore there must be a } z \neq \alpha \text{ in } V_0 \cap A^1. \text{We have that either } z^3 = \alpha^3 \text{ or } z^2 = \alpha^2.
III.c.1.1. $z^3 = \alpha^3$. In this case we have that (1) $y \in A^1$ implies $y^3 \leq z^3$. Let $x^2$ take its maximum in $V_0 \cap A^1$ at the point $\zeta$. Define $v = (x^1, z^2, e - x^1 - z^2)$. From (1) it follows that (2) $R_1(x) - \text{dom } V_0 = Q_2(x) \cup \{x : x \in A, x^5 \geq y^5\}$.

By II.5 there is a solution $U$ of $Q_3(x)$ such that $U \cap F^1 = \{z\}$; it follows that (3) dom $U \cap Q_1(x) = \emptyset$. If $v \notin A$ then $U \cup V_0$ is a solution of $G$. If $v \in A$ then $v \in D^3 - A^3$, so $Q_3(v) \cap A^3 = \emptyset$. It follows that (4) dom $Q_3(v) \cap (Q_2(x) \cup Q_1(x)) = \emptyset$.

We also have that (5) dom $U \cap Q_3(v) = \emptyset$. Now if $U_1$ is a solution of $Q_3(v)$ then, combining (2), (3), (4) and (5), we have that $V_0 \cup U \cup U_1$ is a solution of $G$.

III.c.1.2. $z^2 = \alpha^2$. We now have that $y \in A^1$ implies $y^2 \leq \alpha^2$. We show that we may suppose: (**) there is no $u \in A^3$ such that $u^2 = \alpha^2$ and $u^1 > \alpha^1$. If (**) fails then $F^2$ has $b$-shape and $\{A^1, A^3\}$ satisfies condition $M$, so by III.b $G$ is solvable.

We also notice that: (***) if $\theta \in A, \theta^2 = \alpha^2, \theta^3 \leq \beta^3$ and $J$ solves $Q_3(\theta)$ then $J \cup [\alpha \theta]$ solves $Q_1(x)$. Now if (**) holds and $U$ is a solution of $Q_3(x)$ then dom $U \cap (Q_1(x) \cup Q_2(x)) = \emptyset$. Let $V$ solve $Q_1(x)$. We denote by $\mu(V)$ the point where $x^3$ takes its maximum in $A^1 \cap V$. The point where $x^3$ takes its maximum in $A^1 \cap \{x : x^1 = \alpha^1\}$ is denoted by $\zeta$. $U$ denotes a fixed solution of $Q_3(x)$. We remark that $x \in U$.

III.c.1.2.1. There is a solution $V_1$ of $Q_1(x)$ such that $\beta^3 < \mu^3(V_1)$. Define $v = (x^1, e - x^2 - \mu^3(V_1), \mu^3(V_1))$. We have that $R_1(x) - \text{dom } V_1 = Q_3(x) \cup \{x : x \in A, x^5 \geq \mu^5\}$. If $v \notin A$ then $U \cup V_1$ is a solution of $G$. If $v \in A$ then $v \in D^2 - A^2$. Let $U_1$ solve $Q_2(v)$. dom $U_1 \cap (Q_1(x) \cup Q_2(x)) = \emptyset$, so $U \cup U_1 \cup U_1$ is a solution of $G$.

III.c.1.2.2. Every solution $V$ of $Q_1(x)$ satisfies $\mu^3(V) \leq \beta^3$ and there is a solution $V_1$ of $Q_1(x)$ such that $\mu^3(V_1) = \beta^3$. $\beta \in A^2 \cap D^2$ therefore $Q_3(\beta)$ is solvable. If $U_1$ solves $Q_2(\beta)$ then $U_1 \cap A^2 = \{\beta\}$ and dom $U_1 \cap (Q_1(\mu(V_1)) \cup Q_3(x)) = \emptyset$. Let $U_2$ be a solution of $Q_3(\mu(V_1))$. From (**) it follows that $U_2 \cap A^1 = \{\mu(V_1)\}$. So we have that $U \cup U_1 \cup U_2$ is a solution of $G$.

III.c.1.2.3. If $V$ solves $Q_1(x)$ then $\mu^3(V) < \beta^3$.

III.c.1.2.3.1. $\zeta^3 \geq \beta^3$. Define $v = (x - x^2 - \beta^2, \alpha^2, \beta^3)$. By I.17 $Q_1(x) \cap \text{dom } R_1(x) = \text{dom } [\alpha \beta] \cap Q_1(x)$, so dom $R_1(x) \cap \{x : x \in A, x^5 \geq \mu^5\} = \emptyset$. If $v \notin A$ and $U_1$ solves $Q_2(\beta)$ then $U \cup [\alpha \beta] \cup U_1$ solves $G$. If $v \in A$ let $U_2$ be a solution of $Q_1(v)$. By (**) and the definition of $U_2$ we have $U_1 \cup U_2 \cup U_1 \cup U_2 \cup U_1$ is a solution of $G$.

III.c.1.2.3.2. $\zeta^3 < \beta^3$. Define $v = (x - x^2 - \zeta^3, \alpha^2, \zeta^3)$. By II.5 there is a solution $U_1$ of $Q_3(\zeta)$ such that $U_1 \cap F^1 = \{\zeta\}$.

If $v \notin A$ then $U \cup U_1 \cup [\alpha \zeta]$ is a solution of $G$. If $v \in A - A^1$ and $U_2$ solves $Q_1(v)$ then $U \cup U_1 \cup U_2 \cup [\alpha \zeta]$ is a solution of $G$. If $v \in A^1$ then $U_1 \cup U_2 \cup U$ is a solution of $G$.

III.c.2. If $V$ solves $Q_1(x)$ and $W$ solves $R_1(x)$ then dom $W \cap V \neq \emptyset$ and dom $V \cap W = \emptyset$. By I.17 we have that dom $(R_1(x) - [\beta x]) \cap Q_1(x) = \emptyset$. By I.19 $\hat{R}_2(x) = A^3 \cap R_1(x)$. $[\beta x] \subset A^3$ therefore $[\beta x] \subset R_3(x)$. We conclude that

(1) \[ \text{dom } (R_1(x) - \text{dom } R_1(x)) \cap Q_1(x) = \emptyset. \]
Since $F^1$ has $c_3$-shape we have also that $A^2 \cap R_i(a) = \emptyset$. If $\{A^1, A^3\}$ does not intersect maximally then, by I.20, $\{R^1_i(a), R^2_i(a)\}$ does not intersect maximally and we have that $m(R_i(a)) = 0$. By II.2 the $R_i(a)$-core is the solution of $R_i(a)$ which, by (1), contradicts III.c.2. So $\{A^1, A^3\}$ intersects maximally. The $x^2$ coordinate of $F^2$ will be denoted by $d$.

III.c.2.1. $d < a^2$. We denote the ends of $F^2$ by $\gamma$ and $\delta$ such that $\delta^1 \geq \gamma^1$. We remark that $\delta \in D^2 - A^2$. So $Q_2(\delta)$ is solvable and if $U$ solves it then $dom U \supset T_2(\delta) - U$ and $dom U \cap R_2(\delta) = \emptyset$. Denote $v = (x^1, d, \varepsilon - d - \alpha^1)$ and $P = \{x: x \in A, x^2 \geq \gamma^1\}$. We remark that $\hat{B}_i \cap \hat{B}_j = \emptyset$ for all $i \neq j$ and that $(P - dom P) \cap ((\alpha \beta] \cup [\gamma \delta]) = \emptyset$. So $P - dom P$ solves $P$ and $dom (P - dom P) \cap (Q_2(\delta) \cup Q_1(a)) = \emptyset$.

Summing we have that $U_1 = U \cup (P - dom P)$ solves $R_i(a)$ and that $dom U_1 \cap Q_i(a) = \emptyset$. Since this result contradicts III.c.2 $d < a^2$ is impossible.

III.c.2.2. $d = a^2$. If $F^2$ has $a$-shape or $b$-shape then $\{A^1, A^3\}$ satisfies condition $M$ and by III.a or III.b $G$ is solvable. It remains only to complete the proof when $F^2$ has $c_3$ or $c_1$-shape.

III.c.2.2.1. $\gamma < a^1$. In this case $F^2$ has $c_3$-shape. Let $\eta$ satisfy $\eta^2 = a^2$ and $\alpha^1 > \eta^1 > \gamma^1$. Let $U_2 = [\alpha \eta] \cup U_1$ solves $Q_1(a)$. Let $\zeta \in (\alpha \beta] \cap A^1$, $U_3$ be a solution of $Q_2(\zeta)$ and $U_4$ a solution of $Q_3(\alpha)$. $U_5 = U_3 \cup U_4 \cup [\alpha \zeta]$ is a solution of $R_i(a)$. But $dom U_2 \cap U_5 \neq \emptyset$ contradicting III.c.2, so III.c.2.2.1 is impossible.

III.c.2.2.2. $\gamma = a^1$. In this case $F^2$ has $c_1$-shape. By II.5 there exist solutions $U_1$ of $Q_3(\alpha)$ and $U_2$ of $Q_2(\alpha)$ such that $U_1 \cap F^2 = \{a\} = U_2 \cap F^1$. $U = U_1 \cup U_2$ is a solution of $R_i(a)$ but $dom \beta$ contradicting III.c.2.

III.c.2.2.3. $\gamma > a^1$. In this case $F^2$ has $c_3$-shape and $y \in a^1 \cap a^3 - A^2$. Let $U$ solve $Q_2(\gamma)$. $U_1 = U \cup (Q_3(\alpha) - dom Q_3(\alpha))$ solves $R_i(a)$ but $dom U_1 \cap Q_i(a) = \emptyset$ which is impossible.

III.c.2.3. $d > a^2$. In this case we show that $A^1 \cap \hat{A}^3 \cap R_i(a) = \emptyset$. It follows that $\{R^1_3(a), R^2_3(a)\}$ does not intersect maximally which as we have already seen. Suppose that $A^1 \cap \hat{A}^3 \cap R_i(a) \neq \emptyset$. Let $x \in A^1 \cap \hat{A}^3 \cap R_i(a)$. $x^1 \geq a^1$. Since $x \in a^3 \cap x^2 < a^2$. Let $z \in a^3 \cap a^3 \cap F^2$ and $y$ be an interior point of $A^1 \cap \hat{A}^3$. There is a $u \in \{yz\} \cap A^1 \cap \hat{A}^3$ such that $u^2 > a^2$. So there is a $w \in \{ux\}$ for which $w^2 = a^2$. $w \in A^1 \cap \hat{A}^3$. If $w^1 \geq a^1$ then $x \in A^1 \hat{A}^3$ and if $w^3 \geq a^3$ then $x \in A^1 \hat{A}^3$. Since both cases are impossible we must have $\emptyset = A^1 \cap \hat{A}^3 \cap R_i(a)$.

IV. Third part: case II.9.

IV.1. $m(G) \leq 2$. W.L.G. $\{A^2, A^3\}$ does not intersect maximally.

IV.1.1. $A^2 \cap \hat{A}^3 = \emptyset$. We shall show that $m(G) = 0$. If $m(G) > 0$ then, W.L.G., $\{A^1, A^3\}$ intersects maximally. Let $z \in a^1 \cap a^3 \cap F^2$ and $y$ be an interior point of $A^1 \cap \hat{A}^3$. $\{yz\} \subset A^3$. If $z \in A^2$ then we have $A^2 \cap \hat{A}^3 = \emptyset$ which is impossible. If

(1) See I.11.
(4) If $x^2 \geq x^2$ then $\alpha \beta \geq \alpha \beta$ and $x \in \hat{A}^3$ imply $\alpha \in \hat{A}^3$ which is untrue.
then $\{A^1, A^3\}$ satisfies condition $M$, which is again impossible. Therefore $\{A^1, A^3\}$ does not intersect maximally.

IV.1.2. $A^3 \supseteq A^2$. If $\{A^1, A^3\}$ intersects maximally then there is a $z \in a^1 \cap a^3 \cap F^2$, $a^3 \cap A^2 = \emptyset$; therefore $z \notin A^2$. But since we have $a^1 \cap a^3 \cap F^2 \subseteq A^2$, $\{A^1, A^3\}$ cannot intersect maximally.

IV.1.2.1. $A^1 \cap A^3 = \emptyset$. $A^2 \cap A^1 \subseteq A^1 \cap A^2 \subseteq A^1 \cup A^3 = \emptyset$, so $m(G) = 0$.

IV.1.2.2. $A^3 \supseteq A^1$. We denote $C = A - \text{dom } A$. If $x \in A - C$ then there is a $y \in A$ such that $y <_{S_3} x$. So there is a $z \in a^3$ such that $z <_{S_3} x$. $a^2 \cap (A^1 \cup A^2) = \emptyset$ therefore $z \in C$. By 2.6 $C$ is a solution of $G$.

IV.1.2.3. $A^1 \supseteq A^3$. We have $A^1 \supseteq A^3 \supseteq A^2$, so $m(G) = 0$.

IV.1.3. $A^2 \supseteq A^3$. The proof in this case parallels that in IV.1.2.

IV.2. $m(G) = 3$. We denote $F^k = [a_k^2, a_k^3]$ and $D = \bigcap_{h=1}^3 A^h$.

**Lemma IV.2.1.** Under the assumptions of IV.2 we can find $i$ and $k$ such that $S_k = \{i, j\}$ and:

1. $\alpha^j_k = \beta^i_k$,
2. $\alpha^i_k \in a^1 \cap a^3$,
3. $x^i$ takes its maximum in $\{x : x^i = \alpha^i_k\} \cap D$ at a point $\theta \in a^k$ such that every $y \in A^i$ that satisfies $y^i = \theta^i$ and $y^i > \theta^i$ is in $A^i$, or
4. $\lambda^i$ takes its maximum in $\{x : x^i = \lambda^i_k\} \cap D$ at a point $\rho \in a^k \cap A^i$ and $F^k$ has $c_\rho$-shape.

**Proof.** $F^1 = [\alpha_1 \beta_1]$. W.L.G. $\alpha^j_k \geq \beta^j_k$. We also suppose that $\alpha \in a^2 \cap a^3$. If $\alpha \notin a^2 \cap a^3$ then $\beta \in a^2 \cap a^3$ and the proof is not altered much. We now consider $F^3$.

IV.2.1.1. Every $y \in F^3$ satisfies $y^2 < \alpha^2_2$. Let $\theta$ be the point where $x^3$ takes its maximum in $\{x : x^2 = \alpha^2_2\} \cap D$. $\theta \neq \alpha_1$. We show that $\theta \in A^2$. If $F^3 = [\alpha_2 \beta_2]$ and $\alpha^1_3 \geq \beta^1_3$ then $\alpha^3_3 > \theta^3$ since $\theta \notin F^3$. If $\alpha^1_3 > \theta^1$ then $\alpha^3 \geq \alpha^1_3$ then there is a $u \in [\alpha_2 \alpha_3]$ such that $u^3 \geq \theta^3$. Now if $\theta \in A^3$ then $F^2$ has $c_\rho$-shape and $\theta \in a^1$ and if $\theta \in A^3$ then it follows from I.3 that $\theta \in a^1$. So in this case we can choose $k = 1$ and $i = 2$.

IV.2.1.2. There is a $y \in F^3$ such that $y^2 = \alpha^2_2$. Let $\theta = y$. If $F^3$ has $a$-shape then $\theta \in a^1$ and there is no $u \in A^1$ that satisfies $u^3 = \theta^3$ and $u^2 > \theta^2$. So we can choose $k = 1$ and $i = 2$. If $F^3$ has not $a$-shape then $\beta^2_3 > \alpha^2_3$. If $F^3$ has $b$-shape and $y = \beta_3$ then there is no $u \in A^1$ such that $u^3 = \theta^3$, and $u^2 > \theta^2$ and we can choose $k = 1$ and $i = 2$. If $y \neq \beta_3$ then we have that $\beta^3_3 \in a^2 \cap a^3$ and every $x \in F^1$ satisfies $x^2 < \beta^2_3$. By IV.2.1.1. we may take $k = 3$ and $i = 2$. If $F^3$ has $c_1$-shape then $\theta \in a^1$ and if $u \in A$, $u^3 = \theta^3$ and $u^2 > \theta^2$ then $u \in A^3$. So we can take $k = 1$ and $i = 2$. If $F^3$ has $c_1$-shape and $y = \beta_3$ we choose $k = 1$ and $i = 2$. If $y \neq \beta_3$ then we have that $\beta^3 \in a^2 \cap a^3$ and every $x \in F^1$ satisfies $x^2 < \beta^2_3$. By IV.2.1.1. we can choose $k = 3$ and $i = 2$.

IV.2.1.3. Every $y \in F^3$ satisfies $y^2 > \alpha^2_3$. If $F^3$ has $a$, $b$ or $c_2$-shape then we have that $\beta^2_3 \geq \alpha^2_3$, $\beta^3_3 \in a^2 \cap a^1$ and every $x \in F^1$ satisfies $x^2 < \beta^2_3$. By IV.2.1.1. we can choose $k = 3$ and $i = 2$. Now suppose that $F^3$ has $c_2$-shape. Let $x^1$ take its maxi-
moment in $D \cap \{x: x^2 = \alpha_3^3\}$ at the point $\rho$. $\rho \neq \alpha_3$. We shall show that $\rho \in \hat{A}^1 \cap \hat{A}^2$. $\rho^2 = \alpha_3^3 < \beta_3^3$, $\rho^1 > \alpha_3^3$ therefore $\rho^3 < \alpha_3^3 = \beta_3^3$. So $\beta_3^3 \succ \alpha_3^3$. We have also that $\alpha_3^1 > \rho^1$ therefore there is a $u \in [\alpha_3^2, \alpha_3]$ such that $u \succ \alpha_3^2$. It follows from I.3 that $\rho \in a^3$. Summing we have: $\alpha_3^3 \geq \beta_3^3$, $\alpha_3 \in a^2 \cap a^1$ and $x^1$ takes its maximum in $D \cap \{x: x^2 = \alpha_3^3\}$ at a point $\rho \in a^3 \cap \hat{A}^2$. So we can take $k = 3$ and $i = 1$.

We now prove that $G$ is solvable in case IV.2. W.L.G. the results of IV.2.1 hold for $k = 1$ and $i = 2$.

IV.2.2. (3.a) holds in IV.2.1. We remark that if $z \in A^1 \cap Q_1(\theta)$ then $z^3 = \theta^3$. If there is a $z \in A^1$ such that $z^3 = \theta^3$ and $z^3 > \theta^2$ then $F^3$ has $c_3$-shape. By Lemma II.5 there is a solution $V$ of $Q_1(\theta)$ such that $V \cap F^3 = \{\theta\}$. So we can always find a solution $V_1$ of $Q_1(\theta)$ such that $V_1 \cap A^1 = \{\theta\}$. Similar reasoning shows that there is always a solution $V_2$ of $Q_3(\alpha_i)$ such that $V_2 \cap A^3 = \{\alpha_i\}$.

IV.2.2.1. $\theta^3 > \beta_3^3$. We define $v = (\alpha_3^1, e - \alpha_3^1 - \theta^3, \theta^3)$. Suppose $v \in A$. We have that $Q_2(v) \cap A^2 = \emptyset$. So if $U$ is a solution of $Q_2(v)$ then $V_1 \cup V_2 \cup U \cup [\theta \alpha_3]$ solves $G$. If $v \notin A$ then $V_1 \cup V_2 \cup [\theta \alpha_3]$ solves $G$.

IV.2.2.2. $\theta^3 = \beta_3^3$. If $U$ is a solution of $Q_2(\beta_3)$ then $V_1 \cup V_2 \cup U$ is a solution of $G$.

IV.2.2.3. $\theta^3 < \beta_3^3$. Let $x^3$ take its maximum in $A^1 \cap \{x: x^1 = \alpha_3^1\}$ at $\zeta$. We define $\mu = (e - \alpha_3^2 - \beta_3^3, \alpha_3^2, \beta_3^3)$. Suppose $\zeta^3 \geq \beta_3^3$ and $\mu \in A$. In this case if $W$ is a solution of $Q_2(\beta_3)$ and $W_1$ is a solution of $Q_1(\mu)$ then $V_2 \cup W \cup W_1 \cup F^1$ solves $G$. If $\mu \notin A$ then $V_2 \cup W \cup F^1$ is a solution of $G$. If $\zeta^3 < \beta_3^3$ then $F^1$ has $c_3$-shape. We define $\eta = (e - \zeta^3 - \alpha_3^3, \alpha_3^3, \zeta^3)$. By II.5 there is a solution $U$ of $Q_2(\zeta)$ such that $\{\zeta\} = U \cap F^1$. If $\eta \notin A$ then $U \cup V_2 \cup [\zeta \alpha_3]$ solves $G$. If $\eta \in A^1 \cup A^3$ and $U_1$ solves $Q_1(\eta)$ then $V_2 \cup U \cup U_1 \cup [\zeta \alpha_3]$ is a solution for $G$. If $\eta \in A^1$ then $\eta = \theta$ and $V_1 \cup V_2 \cup U$ is a solution of $G$.

IV.2.3.(3.b) holds in IV.2.1. So $F^1$ has $c_3$-shape. By II.5 there is a solution $V$ of $Q_2(\alpha_i)$ such that $V \cap F^1 = \{\alpha_i\}$. Next we show that there is a solution $V_1$ of $Q_1(\rho)$ such that $V_1 \cap A^1 = \{\rho\}$. If $Q_1(\rho) \cap A^1 = \{\rho\}$ this follows from the fact that $\rho$ belongs to every solution of $Q_1(\rho)$. If there is $x \in Q_1(\rho) \cap A^1$, $x \neq \rho$, then $x \in a^1$ and by $x^2 = \rho^2$. Using I.10 and observing that $\rho \in A^3$ we see that $F^2$ has $C^3$-shape and $\rho \in F^2$. II.5 yields a desired $V_1$. Now define $v = (\alpha_3^1, \rho^2, e - \alpha_3^1 - \rho^2)$. Observe that $Q_3(v) \cap A^3 = \emptyset$. If $v \notin A$ and $V_2$ is a solution of $Q_3(v)$ then $V_1 \cup V_2 \cup [\rho \alpha_3]$ is a solution of $G$. If $v \notin A$ then $V_1 \cup V_2 \cup [\rho \alpha_3]$ solves $G$.

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(5) Suppose there is $z \in A^1 \cap Q_1(\theta)$ with $z^3 > \theta^3$. $\alpha_3 \in A^1$ so there is $u \in A^1$ with $u^2 > \alpha_3^1 = \theta^2$.

For small $t > 0$ $y = tu + (1-t)z$ satisfy $x^3 > \theta^3$ and $y \in A^1$ which is impossible since $\theta \notin a^1$.

(6) By an argument similar to that in footnote (5).