

# BOUNDARY THEORY FOR RECURRENT MARKOV CHAINS<sup>(1)</sup>

BY

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1. **Introduction.** In [1] Doob introduced two boundaries, the Martin exit boundary and entrance boundary for denumerable Markov chains. Each boundary is obtained by completing the state space by means of a suitably chosen metric. Boundary points are the new points added by this completion. The completed space is a compact metric space. An alternate approach to boundaries was proposed by Feller in [2]. We use the theory as developed by Doob and extended by Hunt in [3].

Martin boundary theory applied directly to transient chains gives new and interesting information about these chains. But for recurrent chains the boundaries reduce to one point boundaries and give no new information. The purpose of this paper is to show that Martin boundary theory can be used to define a different boundary for recurrent chains, which gives useful information. Our recurrent boundary  $\mathbf{B}$  is defined as the Martin exit boundary obtained for a transient chain determined from the recurrent chain.

For transient chains the Martin boundaries are ideally suited to the representation of non-negative superregular functions and measures. Superregular measures are measures  $\nu$  such that  $\nu(I - P) = \mu$ , where  $\mu$  is non-negative. For, given  $\mu$ , there may be many such measures, but each is determined by a measure on a "minimal part" of the entrance boundary.

A recurrent chain has only one non-negative superregular measure. But we show that, for recurrent chains, non-negative solutions of  $\nu(I - P) = \mu$ , where  $\mu$  is assumed to be a signed measure of finite total measure, can be represented in a similar way using the recurrent boundary. This representation generalizes a result for sums of independent random variables, for subregular measures, obtained by Spitzer in [9]. The measure used in the representation is given a probabilistic interpretation in terms of an "associated transient chain."

Just as in the transient case we obtain a dual boundary  $\hat{\mathbf{B}}$  for recurrent chains. This boundary is obtained by forming the recurrent boundary for the reverse chain. It is used to represent all non-negative solutions of  $(I - P)h = f$  where  $f$  is integrable with respect to the stationary measure. As  $\mathbf{B}$  gives information

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about the long-range behavior of the chain  $P$ , so  $\hat{\mathbf{B}}$  yields long-range information about the reverse chain.

For transient chains the process tends with probability one to the exit boundary. Clearly, a recurrent chain converges almost nowhere, hence to obtain corresponding results for recurrent chains we replace convergence almost everywhere by weak convergence. We also restrict ourselves to a class of recurrent chains that we call *normal* chains. Let  $P$  be the transition matrix of an aperiodic irreducible recurrent chain. Let  $\alpha$  be the stationary measure. Then  $P$  is normal if the matrices

$$C_{ij} = \lim_{n \rightarrow \infty} \sum_{k=0}^n (P_{jj}^{(k)} - P_{ij}^{(k)}), \quad G_{ij} = \lim_{n \rightarrow \infty} \sum_{k=0}^n \left( \frac{\alpha_j}{\alpha_i} P_{ii}^{(k)} - P_{ij}^{(k)} \right)$$

are well defined. We showed in [4] that the matrices  $C$  and  $G$  can be used as dual potential operators for recurrent chains. It has been shown by Spitzer in [9] that all chains arising from sums of independent random variables are normal. The existence of non-normal chains has been shown by Orey in [8]. (See also [6].)

For a normal chain and  $E$  a finite set, there is a limiting probability  $\lambda_j^E$  that the set  $E$  is entered for the first time after time  $n$  at  $j$ . Let  $E_1, E_2, \dots$  be a sequence of finite sets increasing monotonically to the state space  $\mathbf{S}$ . Then we show that the measures  $\lambda^{E_k}$  converge weakly to a measure  $\beta$  on  $\mathbf{S}^* = \mathbf{S} \cup \mathbf{B}$ , where  $\mathbf{B}$  is a recurrent boundary. For ergodic chains this measure is the stationary measure  $\alpha$  and hence is all on  $\mathbf{S}$ . For null chains the measure is usually on the boundary but may be on both  $\mathbf{S}$  and  $\mathbf{B}$ . We show that  $G(I - P) = -I$  if and only if  $\beta$  assigns measure 0 to  $\mathbf{S}$ . In particular, this will be the case if all boundary points are minimal.

For ergodic chains it follows from standard results that  $\beta$  is also the weak limit of the  $i$ th row of the powers of the transition matrix. We show by an example that this is not true for general normal chains. We are led then to the study of the class of continuous functions for which  $P^n h \rightarrow \int h d\beta$ .

We obtain the following results. Let  $E$  be a finite set and  $B_{ij}^E$  be the probability, starting in  $i$ , that  $E$  is entered at  $j$ . As a function of the starting state  $i$ , this probability can be extended to the boundary in such a way that the resulting function  $B^{E(x,j)}$  is continuous on  $\mathbf{S}^*$ . A finite combination of such functions is called an elementary continuous function. We show that if  $h$  is in the strong closure of the elementary functions, then  $P^n h \rightarrow \int h d\beta$ . We give a method of characterizing this closure and show that if  $\mathbf{B}$  has no nonminimal boundary points, it contains all continuous functions. Hence for this case the rows of  $P^n$  tend weakly to  $\beta$ . As an application of these results we obtain a representation of several basic quantities, e.g.,

$$\lambda_j^E = \int_{\mathbf{S}^*} B^{E(x,j)} d\beta(x).$$

We obtain a class of stochastic processes, which start on  $S^*$  with an arbitrary distribution, and then proceed by  $P$  till they reach a designated state 0. This gives direct probabilistic interpretations for the representations developed. For example, above,  $B^E(x, j)$  is the probability of entering  $E$  at  $j$ , if the process starts at  $x$ —possibly on the boundary. Hence  $\lambda^E$  yields the entrance probabilities for the process started with distribution  $\beta$ .

Normal chains may then be characterized as chains for which there is a starting distribution, namely  $\beta$ , which represents the limiting behavior of the chain in the long run.

We apply recurrent boundary theory to extend and give new probabilistic interpretations to the results on recurrent potentials obtained by the authors in [4; 5]. In particular we obtain the following representation for the potential operator:  $G_{ij}$  may be interpreted as the mean number of entries to  $j$ , before hitting  $i$ , for a process started on  $S^*$  with the measure  $\beta$ .

2. **Notation.** For convenience we summarize here the basic notation and conventions that will be used in the paper.

We use  $P$  for the transition matrix of a recurrent chain and  $Q$  for the transition matrix of a transient chain. We use  $S$  for the state space. We consider both measures (row vectors) and functions (column vectors); the former are denoted by Greek letters, and the latter by ordinary lower case letters.

The product of two infinite matrices is well defined only if the series defining the components of the product converge absolutely.

When a quantity is written as  ${}^E A$  the set  $E$  is treated as a "taboo" set: the entries of  ${}^E A$  then refer to a quantity related to the process watched only as long as it avoids the set  $E$ . When we write  $A^E$  we are referring to a quantity relating to the process at a time when it hits the set  $E$ . When we write  $A_E$  we mean the matrix  $A$  restricted to the components which are in  $E$ . When  $E$  is a single state, say 0, we write  ${}^0 A$  for  ${}^E A$ .

- $\alpha$       A stationary measure for the recurrent chain  $P$ . That is,  $\alpha \geq 0$  and  $\alpha P = \alpha$ . (The choice of  $\alpha$  is unique up to a constant multiple.)
- $\hat{P}$       The reverse chain associated with  $P$ ,  $\hat{P}_{ij} = \alpha_j P_{ji} / \alpha_i$ .
- $P_{ij}^{(n)}$       The probability of moving from  $i$  to  $j$  in  $n$  steps. (We write  $Q_{ij}^{(n)}$  for transient chains.)
- $N_{ij}$       The mean number of visits to  $j$ , when the process is started in  $i$ ,  $N = \sum_n Q^n$ . (Finite for transient chains.)
- $N_{ij}^{(n)}$       The mean number of visits to  $j$  in  $n$  steps.  $N^{(n)} = \sum_{k=0}^n P^{(k)}$ .
- ${}^E N_{ij}$       The mean number of visits to  $j$  before hitting  $E$ , when the process is started in  $i$  (taken to be 0 if  $i$  or  $j$  is in  $E$ ).

- $H_{ij}$  The probability, starting at  $i$ , that  $j$  is ever reached. ( $H_{ii} = 1$ .)
- $H_{ij}^{(n)}$  The probability, starting at  $i$ , that  $j$  is reached by step  $n$ .
- ${}^E H_{ij}$  The probability, starting at  $i$ , that  $j$  is reached before  $E$ .
- $B_{ij}^E$  The probability starting in  $i$  that  $E$  is entered at  $j$ .
- $\bar{B}_{ij}^E$  The probability, starting at  $i$ , that  $E$  is exited at  $j$  (or, more precisely, that  $E$  is visited, that it is visited finitely often, and that  $j$  is the last state visited).
- $P_{ij}^E$  The probability starting at  $i$  in  $E$  that the next entry to  $E$  occurs at  $j$ . (I.e.  $P^E$  is the process  $P$  "watched in  $E$ ".)
- 1 The constant function  $h_i \equiv 1$ .

For normal chains:

$$G_{ij} = \lim_n [(\alpha_j/\alpha_i)N_{ii}^{(n)} - N_{ij}^{(n)}].$$

$$C_{ij} = \lim_n [N_{jj}^{(n)} - N_{ij}^{(n)}].$$

$$\begin{aligned} \lambda_j^E &= \lim_n \Pr[\text{first entry to } E \text{ after time } n \text{ is at } j] \\ &= \lim_n (P^n B^E)_{ij} \text{ (if this exists, it is independent of } i\text{).} \end{aligned}$$

We use the notation  $\delta_{ij}$  for the  $ij$ th component of the identity matrix  $I$ . And  $\delta^j$  will be the function (or measure) which is the  $j$ th column (or row) of  $I$ . That is,  $\delta^j(i) = \delta_{ij}$ .

We shall have occasion to extend certain matrix quantities to be defined on the state space and the boundary. Since the boundary need not be a denumerable set we shall indicate the extension say of  $A_{ij}$ , as a function of  $i$ , by  $A(x, j)$ .

**3. Boundary theory for transient chains.** Let  $Q$  be the transition matrix of a denumerable, transient Markov chain, with state space  $S$ . A function  $h$  is said to be *regular* if  $h = Qh$ , *superregular* if  $h \geq Qh$ , and *subregular* if  $h \leq Qh$ . A non-negative regular function  $h$  is called *minimal* if for every non-negative regular function  $h'$ , which is not 0, and  $h' \leq h$ , we have  $h' = ch$ . If  $h$  can be written in the form  $h = Nf$ , then  $h$  is called a *potential*, and  $f = (I - Q)h$  is its *charge*.

We shall summarize a number of results from Martin boundary theory. If  $h$  is non-negative and superregular, then

$$h \geq Qh \geq Q^2h \geq \dots \geq 0,$$

hence  $Q^n h$  converges to a function  $r$ . We may represent  $h$  as

$$(3.1) \quad h = r + Nf,$$

where  $f = (I - Q)h \geq 0$ . This represents  $h$  (uniquely) as the sum of a regular function and a potential with  $f \geq 0$ , which corresponds to the Riesz representation of a superharmonic function as a harmonic function plus a potential with a positive charge.

For each non-negative superregular  $h$  we may define the modified Markov chain with transition matrix

$$(3.2) \quad Q_{ij}^h = Q_{ij}h_j/h_i.$$

This new chain is known as the  $h$ -process. The chain  $Q^h$  has as state space the set of all states for which  $h_i > 0$ . Quantities for an  $h$ -process may be computed in terms of the original chain, for example

$$(3.3) \quad N_{ij}^h = N_{ij}h_j/h_i.$$

For simplicity let us assume that all states in  $S$  may be reached from some designated state 0. We introduce the function

$$(3.4) \quad K(i, j) = N_{ij}/N_{0j}.$$

We metrize the space  $S$  by the distance

$$(3.5) \quad d(i, j) = \sum_k |K(k, i) - K(k, j)| H_{0k} w_k + |\varepsilon_i - \varepsilon_j|,$$

where the weights  $w_k$  are non-negative and have sum 1, while  $\varepsilon_i$  are distinct and tend to 0 on any infinite sequence of states. [The inclusion of the  $\varepsilon$ -terms was suggested by Hunt to assure that no state in  $S$  be "on the boundary". We adopt this convenient modification of Doob's metric.]

Let  $S^*$  be the completion of this metric space. The set  $B = S^* - S$  is the *exit boundary* of the Markov chain. Since the metric (3.5) was so chosen that all points in  $S$  are isolated, the boundary (which consists entirely of points added during the completion) contains all limit points.

A sequence of states  $\{j_n\}$  converges if and only if  $K(i, j_n)$  converges for all  $i$ . Hence if  $j_n \rightarrow x$ , we may define

$$(3.6) \quad K(i, x) = \lim_{j_n \rightarrow x} K(i, j_n).$$

Then the function  $K(i, \cdot)$  is continuous on  $S^*$ . For every  $x \in S^*$ ,  $K(\cdot, x)$  is a non-negative superregular function which is 1 at state 0. A boundary point is called *minimal* if this is a minimal regular function. The set of minimal boundary points is a Borel set.

We will now state two basic theorems from Martin boundary theory. They are proved in [1; 3].

**THEOREM 3.1. (REPRESENTATION THEOREM.)** *If  $h$  is a non-negative superregular function, with  $h_0 = 1$ , then  $h$  can be represented in the form*

$$(3.7) \quad h_i = \int_{S^*} K(i,x) d\beta^h(x),$$

where  $\beta^h$  is a probability measure on  $S^*$ . There is a unique such measure if we require that the measure of the nonminimal part of the boundary be 0. In this representation the integral over  $\mathbf{B}$  gives the regular function and the integral over  $\mathbf{S}$  the potential of the representation (3.1).

**THEOREM 3.2. (CONVERGENCE THEOREM.)** *If  $h$  is a non-negative superregular function, and  $h_0 > 0$ , and the  $h$ -process is started at 0, then almost every path either ends at a state of  $\mathbf{S}$  in finite time or converges to a minimal boundary point (in the sense of the metric topology). The measure  $\beta^h$  in the representation theorem can be interpreted as the "distribution of the final position" in the  $h$ -process. More precisely,  $\beta^h(i)$  is the probability that the  $h$ -process started at 0 ends at  $i$ , while for a Borel subset  $A$  of  $\mathbf{B}$ ,  $\beta^h(A)$  is the probability that the  $h$ -process converges to a point of  $A$ .*

If  $h$  is a regular function, then by Theorem 3.1,  $\beta^h$  puts all its measure on the boundary, hence the  $h$ -process does not terminate in finite time. A minimal function must actually be a multiple of a  $K(\cdot, x)$  for  $x$  a minimal boundary point, and in this case the  $h$ -process goes to  $x$  with probability 1. Since  $\mathbf{1}$  is always superregular, and  $Q^1$  is  $Q$ , the original chain itself converges, and the  $\beta$  so obtained is the measure needed in (3.7) to represent the constant function. This measure will be on the boundary if  $\mathbf{1}$  is regular, that is,  $Q$  has row-sums equal to 1, which is precisely the condition that the process should not stop.

**4. Boundary for a recurrent chain.** Let  $P$  be the transition matrix of a denumerable, aperiodic recurrent chain in which it is possible to go from any state to any other state. Let  $\mathbf{S}$  be its state space. Let  $\alpha$  be its stationary measure, i.e.,  $\alpha > 0$  and  $\alpha P = \alpha$ . (This measure is determined up to a constant multiple, and only ratios of components will be used.)

The reverse chain has the transition matrix

$$(4.1) \quad \hat{P}_{ij} = \alpha_j P_{ji} / \alpha_i,$$

and it is again a recurrent chain with stationary measure  $\alpha$ . We select one state, which we shall designate by 0, and define the transient chain associated with  $P$ , relative to 0. This chain proceeds according to  $\hat{P}$ , but it is stopped when  $\hat{P}$  steps into state 0. More precisely,

$$(4.2) \quad Q_{ij} = \begin{cases} \hat{P}_{ij} & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

This is clearly a transient chain, and we need to find its potential operator  $N$ . Since  $N_{ij}$  is the mean number of times in  $j$  if the transient chain starts at  $i$ , this quantity must be  ${}^0\hat{N}_{ij} = \alpha_j {}^0N_{ji}/\alpha_i$ , the mean number of times, in the reverse chain started at  $i$ , that  $j$  is visited before reaching 0. However, if  $i = 0$ , we want the mean number of entries into  $j$  before returning to 0, which is  $\alpha_j/\alpha_0$ . Hence

$$(4.3) \quad N_{ij} = (\alpha_j/\alpha_i)({}^0N_{ji} + \delta_{i0}).$$

Thus

$$(4.4) \quad K(i, j) = N_{ij}/N_{0j} = (\alpha_0/\alpha_i)({}^0N_{ji} + \delta_{i0}).$$

Since all states can be reached from 0 in the associated transient chain its state space is  $S$ . We now adjoin to this space its Martin boundary  $B$ , so that  $S^* = S \cup B$  is a compact metric space. A boundary point  $x$  corresponds to a sequence of states  $j_n$  along which  $K(i, j_n)$  converges for all  $i$ . From (4.4) we see that this means that  $\text{Lim}_n {}^0N_{j_n i}$  exists for all  $i$ . We may thus extend  ${}^0N$  to the boundary:

$$(4.5) \quad {}^0N(x, i) = \begin{cases} {}^0N_{ji} & \text{if } x = j \in S, \\ \text{Lim}_n {}^0N_{j_n i} & \text{if } x = \text{Lim}_n j_n. \end{cases}$$

Then

$$(4.6) \quad K(i, x) = (\alpha_0/\alpha_i)({}^0N(x, i) + \delta_{i0}).$$

We will next show that we obtain the same boundary if 0 is replaced by another state, and that this choice also does not affect the question of whether a boundary point is minimal.

By computing in two ways the mean number of visits to state  $i$  before the process reaches both 0 and 1, if it starts at state  $j$ , we obtain the identity:

$$(4.7) \quad {}^1N_{ji} + {}^0H_{j1} \cdot {}^0N_{1i} = {}^0N_{ji} + {}^1H_{j0} \cdot {}^1N_{0i}.$$

We can transform this by using the fact that  ${}^1H_{j0} = 1 - {}^0H_{j1}$ , and that  ${}^0N_{1i} + {}^1N_{0i} = (\alpha_i/\alpha_1) {}^0N_{11}$  (see [4, §3]), into

$$(4.8) \quad {}^1N_{ji} = {}^0N_{ji} + {}^1N_{0i} - (\alpha_i/\alpha_1) {}^0N_{j1}.$$

If  $j_n \rightarrow x$  in the transient chain defined relative to 0, then  ${}^0N_{j_n i}$  converges for all  $i$ . Hence, in (4.8) the right-hand side converges, and therefore  ${}^1N_{j_n i}$  also converges for all  $i$ . Thus  $x$  is also a boundary point of the process defined relative to 1. Since the roles of 0 and 1 are interchangeable, the two boundaries are the same. And we may extend (4.8) to the common boundary.

From (4.4) and (4.8) we obtain

$$(4.9) \quad K_1(i, x) = (\alpha_1/\alpha_0)[K_0(i, x) - K_0(1, x)] + (\alpha_1/\alpha_i)[{}^1N_{0i} - \delta_{i0} + \delta_{i1}],$$

where the kernels for the two processes are distinguished by subscripts. Suppose

that  $y$  is a nonminimal boundary point of the 0-process; then  $K_0(i,y)$  may be represented as

$$K_0(i,y) = \int K_0(i,x) d\pi(x),$$

where  $\pi$  is a non-negative measure of total measure 1, which puts weight on more than one point. By integrating (4.9) we obtain:

$$\int K_1(i,x) d\pi(x) = K_1(i,y),$$

and hence  $y$  is also nonminimal for the 1-process, and conversely, by symmetry.

Hence the boundary  $\mathbf{B}$  is determined by  $P$  alone, and so is the question of the minimality of a boundary point. Hence we call  $\mathbf{B}$  the *boundary of the recurrent chain  $P$* , and we can speak of the *minimal boundary points* of  $P$ . That this boundary has the desired properties will be seen later.

It is worth noting how the boundary  $\mathbf{B}$  could have been constructed directly in terms of  $P$ . First of all we note that  ${}^0N_{ij} = {}^0H_{ij} {}^0N_{jj}$ , hence  ${}^0N_{i,nj}$  converges as  $i_n \rightarrow x$  if and only if  ${}^0H_{i,nj}$  does. This shows that the metric

$$(4.10) \quad d(i,j) = \sum_k |{}^0H_{ik} - {}^0H_{jk}| w_k + |\varepsilon_i - \varepsilon_j|$$

is the same as the metric (3.5) yielded by the associated transient chain.

We shall use  $i, j, k$  as variables for states, while  $x$  and  $y$  will be variables over  $\mathbf{S}^*$ .

Since column vectors are thought of as functions on  $\mathbf{S}$ , it is often important to know whether this function can be extended to be continuous on  $\mathbf{S}^*$ . We introduce the following terminology:

DEFINITION. A column vector  $h$  is said to be *continuous* if there is a continuous function  $h(x)$  on  $\mathbf{S}^*$  such that  $h(i) = h_i$  for all states  $i$ . Then  $h(x)$  is the *extension* of  $h$ .

Clearly, a vector  $h$  is continuous if and only if for every sequence  $i_n$  tending to a fixed boundary point  $x$  the values  $h_{i_n}$  converge to the same value  $h(x)$ .

From the definition of  $\mathbf{B}$  we know that a column of  ${}^0N$  is continuous, and hence so is a finite combination of columns. A simple probabilistic argument shows that if  $0 \in E, j \in E,$

$$(4.11) \quad \sum_{k \in E} {}^0N_{ik} P_{kj}^E = {}^0N_{ij} + \delta_{j0} - B_{ij}^E,$$

hence a column of  $B^E$ , for a finite set  $E$ , is continuous. Therefore, a finite combination of such columns is continuous, which means that for any column vector  $h$  and finite set  $E, B^E h$  is continuous.

DEFINITION. For a finite set  $E$ , we write  $B^E(x,j)$  for the extension of the  $j$ th column of  $B^E$ . By an *elementary continuous function* we mean a finite linear combination of such functions. We will frequently consider a sequence of finite sets

$E_k$ , which are monotone increasing and  $U_k E_k = S$ , to be called a *fundamental sequence* of sets. For example, in the next section we will show what happens to  $B^{E_k} h$  if the  $\{E_k\}$  is a fundamental sequence.

**5. Associated transient chains.** In the last section we introduced the transient chain  $Q$  associated with a given recurrent chain. Various  $h$ -processes of this chain will be of interest to us.

Let us suppose that a measure  $\nu$  has the following three properties: (i)  $\nu \geq 0$  (ii)  $\nu_0 = 0$ , and (iii)  $\nu P \leq \nu + \delta^0$ . We may then define the function  $h$  by

$$(5.1) \quad h_i = (\alpha_0/\alpha_i)(\nu + \delta^0)_i.$$

Clearly  $h$  is non-negative and a direct computation shows that  $Qh \leq h$ . We may therefore introduce the  $h$ -process. Since this process is determined by the measure  $\nu$ , we will write  $Q^\nu$  instead of  $Q^h$ :

$$(5.2) \quad Q_{ij}^\nu = Q_{ij} h_j / h_i = \nu_j P_{ji} / (\nu_i + \delta_{i0}).$$

By Theorems 3.1 and 3.2 we know that this process determines a measure  $\beta^h$  on  $S^*$  (with the measure on  $B$  being concentrated on minimal boundary points), which represents the "final position" of the process, and we know that  $h$  can be represented by integrating the kernel  $K$  with respect to this measure. Thus (3.7) yields a representation for  $\nu$ , by means of (5.1). We again write  $\beta^\nu$  in place of  $\beta^h$

$$\nu_i = (\alpha_i/\alpha_0) \int_{S^*} K(i,x) d\beta^\nu(x) - \delta_{i0},$$

and using (4.6) and the fact that  $\beta^\nu$  has total measure 1, we obtain

$$(5.3) \quad \nu_i = \int_{S^*} {}^0N(x,i) d\beta^\nu(x),$$

and this form of representation is unique if the measure puts no weight on non-minimal boundary points.

The potential operator for the  $h$ -process is

$$(5.4) \quad N_{ij}^\nu = \begin{cases} \nu_j {}^0N_{ji}/\nu_i & \text{if } i \neq 0, \\ \nu_j + \delta_{j0} & \text{if } i = 0. \end{cases}$$

If we start the chain  $Q^\nu$  at 0, then  $\nu_j$  is the mean number of times the process is at  $j$ , not counting the original position.

**DEFINITION.**  $\theta_j^E = \bar{B}_{0j}^E$  is the probability in the  $Q^\nu$ -process, started at 0, that there is a last time that the process is in set  $E$  and that this is at state  $j$ . ( $\theta^E$  depends both on  $P$  and on  $\nu$ .)

Let us compute  $\theta^E$ . Clearly,

$$\begin{aligned}
 \theta_j^E &= N_{0j}^v \cdot \text{Pr}_j [\text{process leaves } E \text{ and never returns}] \\
 &= N_{0j}^v [1 - \sum_{k \in E} (Q_{jk}^v + \sum_{a,b \in \tilde{E}} Q_{ja}^v {}^E N_{ab}^v Q_{bk}^v)] \\
 &= N_{0j}^v [1 - \sum_{k \in E} (v_k/(v_j + \delta_{j0})) (P_{kj} + \sum_{a,b \in \tilde{E}} P_{kb} {}^E N_{ba} P_{aj})] \\
 &= (v_j + \delta_{j0}) [1 - (1/(v_j + \delta_{j0})) \sum_{k \in E} v_k P_{kj}^E] \\
 &= v_j + \delta_{j0} - \sum_{k \in E} v_k P_{kj}^E \qquad \text{for all } j \in E.
 \end{aligned}$$

Hence,

$$(5.5) \qquad v_E(I - P^E) = \theta^E - \delta^0.$$

In general  $\theta^E$  will have total measure less than 1, since the process may never enter  $E$  or even if it does it may come back to it infinitely often. However, if  $E$  is finite and  $0 \in E$ , then  $\theta^E$  has total measure 1.

In particular, if  $E = S$ , then the set can be left only if the process ends in a finite number of steps, and hence  $\theta_j^S = \beta^v(j)$ . Letting  $\bar{\beta}$  stand for the row-vector that represents  $\beta$  within  $S$ , we have

$$(5.6) \qquad v(I - P) = \bar{\beta}^v - \delta^0.$$

From the almost everywhere convergence of the  $Q^v$ -process (see Theorem 3.2), we infer that if  $h$  is continuous, then  $[(Q^v)^n h]_0$  converges to  $\int h(x) d\beta^v(x)$ . Let  $E_k$  be a fundamental sequence of sets, and let us observe the  $Q^v$ -process when it is last in the sets  $E_k$ . This observes the process at a sequence of random times, which tend to infinity (or the terminal time) almost everywhere. Hence, for any continuous vector  $h$ ,  $\theta^{E_k} h \rightarrow \int h(x) d\beta^v(x)$ . That is, the measures  $\theta^{E_k}$  converge weakly to  $\beta^v$  on  $S^*$ .

In (4.11), if  $E = S$ , then

$$(5.7) \qquad \sum_k {}^0 N_{ik} P_{kj} = {}^0 N_{ij} - \delta_{ij} + \delta_{0j}.$$

Hence

$$\sum_k {}^0 N_{ik} P_{kj} \leq {}^0 N_{ij} + \delta_{0j}.$$

By letting  $i$  tend to a boundary point  $x$  we obtain

$$(5.8) \qquad \sum_k {}^0 N(x,k) P_{kj} \leq {}^0 N(x,j) + \delta_{0j}, \qquad \text{for } x \in S^*.$$

Also,  ${}^0 N(x,0) = 0$  and  ${}^0 N(x,j) \geq 0$ . Hence the measure  ${}^0 N(x, \cdot)$  satisfies the three conditions for  $v$ . We shall refer to the corresponding  $h$ -process by the notation  $Q^x$ .

$$(5.9) \qquad Q_{ij}^x = {}^0 N(x,j) P_{ji} / ({}^0 N(x,i) + \delta_{i0}), \qquad \text{for } x \in S^*.$$

The corresponding superregular function is  $h = K(\cdot, x)$ , by (4.6), (5.1), and (5.2). Hence the measure  $\beta^x$  is the one needed to represent  $K(\cdot, x)$ . If  $x \in \mathbf{S}$  or  $x$  is a minimal boundary point, then this is the measure putting unit mass at  $x$ . In the latter case,  $\bar{\beta}^x = 0$ , and thus (5.6) states that for a minimal boundary point  $x$  we have equality in (5.8). Thus for a minimal boundary point  ${}^0N(x, \cdot)$  is regular at all states except 0. For  $x$  a nonminimal boundary point,  $\beta^x$  is some measure on states and minimal boundary points.

We also obtain a useful interpretation of  ${}^0N(x, j)$  for a minimal  $x$ : Start the process  $Q^x$  at 0. This chain goes to  $x$  almost everywhere. And  ${}^0N(x, j)$  is the mean number of times in state  $j$ , not counting the starting position.

If in (4.11) we choose a finite set  $E$ , such that  $0 \in E$ , then we may extend the equation to  $\mathbf{S}^*$ :

$$(5.10) \quad \sum_{k \in E} {}^0N(x, k)(I - P^E)_{kj} = B^E(x, j) - \delta_{0j}.$$

If we compare this with (5.5), we see that

$$(5.11) \quad B^E(x, j) = \theta_j^E, \text{ for } E \text{ finite and } 0 \in E,$$

where  $\theta^E$  gives the exit probabilities for the  $Q^x$ -process.

**DEFINITION.**  $T$  is the transformation that changes the continuous function  $h(x)$  into  $Th(x) = \int_{\mathbf{S}} h(y) d\beta^x(y)$ . A continuous function such that  $Th = h$  is said to be  $T$ -continuous.

Clearly,  $Th(x) = h(x)$  if  $x \in \mathbf{S}$  or  $x$  is a minimal boundary point. Hence if  $P$  has only minimal boundary points, then  $T$  is the identity mapping.  $T$  is a continuous transformation of norm 1.

**THEOREM 5.1.** *For any continuous function  $h$ , the extensions of  $B^{E_k}h$  converge to  $Th$  for any fundamental sequence.*

**Proof.** We are interested in the limit of  $B^{E_k}(x, \cdot)h$ . By (5.11) we know that for each  $x$  this is the limit of  $\theta^{E_k}h$ , which is  $Th(x)$  by the weak convergence of  $\theta^{E_k}$  to  $\beta^x$ .

**THEOREM 5.2.** *A continuous function is  $T$ -continuous if and only if the extensions of  $B^{E_k}h$  tend to  $h$ . And if  $P$  has only minimal boundary points, then every continuous function is  $T$ -continuous.*

**Proof.** The first part of the theorem follows from Theorem 5.1 and the definition of  $T$ -continuity. The second part follows from the fact that if all boundary points are minimal, then  $T$  is the identity transformation.

**THEOREM 5.3<sup>(3)</sup>.** *Let  $h$  be  $T$ -continuous. Let  $E_k$  be a fundamental sequence of sets. Then for every  $\varepsilon > 0$  there are sets  $E_1, \dots, E_r$  in the sequence and weights  $c_1, \dots, c_r$  (positive and with sum 1) such that*

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(<sup>3</sup>) The authors are indebted to W. A. Veech for this theorem.

$$|h(x) - \sum_{s=1}^r c_s B^{E_s}(x, \cdot)h| < \varepsilon \quad \text{for all } x \in S^*$$

**Proof.** By Theorem 5.2,  $h^k(x) = B^{E_k}(x, \cdot)h$  converge to  $h(x)$  pointwise. Hence for any finite measure  $\pi$  on  $S^*$ ,

$$\int h^k(x) d\pi(x) = \int h(x) d\pi(x)$$

by dominated convergence. Since on a compact space the finite measures are dual to the continuous functions, we see that the function  $h$  is in the weak closure of the set  $\{h^k\}$ .

Let  $C$  be the convex hull of the set  $\{h^k\}$ . Since  $h$  is in the weak closure of  $C$ ,  $h$  must be in the strong closure of  $C$ . Hence some element of  $C$  is within  $\varepsilon$  of  $h$  in norm, which completes the proof.

**LEMMA.** For any elementary continuous function  $h$ ,  $Th = h$ .

**Proof.** It suffices to show this for  $h_i = B_{ij}^E$ , for a finite set  $E$ . By Theorem 5.1,  $Th$  is the limit of a column of  $B^{E_k}B^E$ , for a fundamental sequence  $\{E_k\}$ . But once  $E \subseteq E_k$ ,  $B^{E_k}B^E = B^E$ , hence the limit is  $h$ .

**THEOREM 5.4.** The set of  $T$ -continuous functions is the strong closure of the set of elementary continuous functions.

**Proof.** We shall apply Theorem 5.3. Each function  $h^k$  is an elementary continuous function, and so is any convex combination. Hence the theorem states that every  $T$ -continuous function is in the strong closure of the set of elementary continuous functions. Conversely, if  $h$  is in the strong closure, then it is the limit (in norm) of a sequence of elementary continuous functions  $f^k$ , hence it is continuous. And  $Th = \text{Lim } Tf^k = \text{Lim } f^k = h$ , by the continuity of  $T$  and the lemma. Hence  $h$  is  $T$ -continuous.

**6. The dual boundary and the representation theorem.** In [4] we developed complete duality between functions of  $P$  and measures of  $\hat{P}$ . We shall now apply this duality to represent functions. Since  $\hat{P}$  is itself a recurrent chain of the type considered, we may construct its boundary  $\hat{B}$  by the methods of §4. All the results of §5 are then applicable.

**THEOREM 6.1.** If  $Ph = h + \delta^0$ ,  $h \geq 0$ , and  $h_0 = 0$ , then there is a probability measure  $\pi$  on  $\hat{B}$  such that

$$(6.1) \quad h_i = (\alpha_0/\alpha_i) \int_{\hat{B}} {}^0\hat{N}(x, i) d\pi(x),$$

and  $\pi$  is unique if it is restricted to the minimal points of  $\hat{B}$ .

**Proof.** Let  $v_i = \alpha_i h_i / \alpha_0$ . Then  $v$  satisfies conditions (i)–(iii) of §5 for  $\hat{P}$ . As a matter of fact, in (iii) we have equality. This means that  $Q^v$  has row sums equal

to 1, and hence the process does not stop in finite time. Hence  $\beta^v$  puts all its measure on the boundary, and the theorem follows from (5.3).

**DEFINITION.** If we let  $\mu = v(I - P)$ , then  $\mu$  is the *deviation* (from regularity) of the measure  $v$ . If  $\mu \mathbf{1}$  is finite, then we say that  $v$  is a measure of *finite deviation*. Dually,  $f = (I - P)h$  is the deviation of  $h$ , and  $h$  is of finite deviation if  $\alpha f$  is finite.

**THEOREM 6.2. (REPRESENTATION THEOREM.)** *If  $h$  is a non-negative function of finite deviation, then  $\alpha f \leq 0$  and there is a probability measure  $\pi$  on the minimal part of  $\hat{\mathbf{B}}$  such that*

$$(6.2) \quad h_i = h_0 + ({}^0Nf)_i - (\alpha f/\alpha_i) \int_{\hat{\mathbf{B}}} {}^0\hat{N}(x, i) d\pi(x),$$

and if  $\alpha f \neq 0$ ,  $\pi$  is uniquely determined.

**Proof.** Since  $\alpha f$  is finite, so is  ${}^0Nf$  (see [4, §3]).

Start the chain in state  $i$  and stop it the first time that it reaches 0. The expected value of  $h$  after  $n$  steps in this stopped process is

$$(6.3) \quad \sum_k {}^0P_{ik}^{(n)} h_k + H_{i0}^{(n)} h_0.$$

However, since  $Ph = h - f$ , we may also write this expected value as

$$(6.4) \quad h_i - \sum_k {}^0N_{ik}^{(n)} f_k.$$

Equating (6.3) and (6.4), and recalling that  $h \geq 0$ , we obtain

$$(6.5) \quad h_i \geq H_{i0}^{(n)} h_0 + \sum_k {}^0N_{ik}^{(n)} f_k.$$

If we let  $n$  tend to infinity,  $H_{i0}^{(n)}$  tends to 1, and since  ${}^0Nf$  is finite, we have dominated convergence in the second term on the right. Therefore,

$$(6.6) \quad h_i \geq h_0 + ({}^0Nf)_i.$$

Let  $h_i^* = h_i - h_0 - ({}^0Nf)_i$ . Then  $h_0^* = 0$ , and by (6.6)  $h^* \geq 0$ . From a direct probability computation we find that

$$(6.7) \quad (I - P) {}^0N = \{\delta_{ij} - \delta_{i0}(\alpha_j/\alpha_0)\}.$$

Hence  $(I - P)h^* = (\alpha f/\alpha_0)\delta^0$ . We see that  $\alpha f > 0$  is impossible, since otherwise  $h^*$  is superregular without being constant. If  $\alpha f = 0$ , then  $h^*$  is regular and non-negative, hence a constant, hence 0. Then (6.2) follows from the definition of  $h^*$ .

On the other hand, if  $\alpha f < 0$ , then Theorem 6.1 is applicable to  $-(\alpha_0/\alpha f)h^*$ , and hence (6.2) follows from (6.1) and the definition of  $h^*$ . This completes the proof.

The dual of this theorem is:

**THEOREM 6.2'.** *If  $\nu$  is a non-negative measure of finite deviation, then  $\mu \mathbf{1} \leq 0$  and there is a probability measure  $\pi$  on the minimal part of  $\mathbf{B}$  such that*

$$(6.2') \quad \nu_j = \nu_0(\alpha_j/\alpha_0) + (\mu \mathbf{0}N)_j - \mu \mathbf{1} \int_{\mathbf{B}} \mathbf{0}N(x,j) d\pi(x),$$

and if  $\mu \mathbf{1} \neq 0$ , then  $\pi$  is uniquely determined.

**NOTE.** The assumption that  $h$  is non-negative can be weakened to the condition that  $h$  is bounded from below, since in that case the theorem is applicable to  $h + \text{const.}$  Similarly in the dual it suffices to assume that  $\nu$  is bounded from below by a multiple of  $\alpha$ .

Since  $\mathbf{0}\hat{N}_{ij} = (\alpha_j/\alpha_i)\mathbf{0}N_{ji}$ , we see from (4.4) that

$$(6.8) \quad K(i,j) = (\alpha_0/\alpha_j)\mathbf{0}\hat{N}_{ij} + \delta_{i0}.$$

Hence  $\hat{\mathbf{B}}$  is characterized by sequences  $i_n \rightarrow x$  for which  $K(i_n, j)$  converges for all  $j$ . Thus  $K$  serves to characterize both  $\mathbf{B}$  and  $\hat{\mathbf{B}}$ . And in the representation theorems  $K$  could have been used for both functions and measures. This yields the alternate representations:

$$(6.9) \quad h_i = h_0 + (\mathbf{0}Nf)_i - (\alpha f/\alpha_0) \int_{\hat{\mathbf{B}}} K(x,i) d\pi(x),$$

and

$$(6.9') \quad \nu_j = \nu_0(\alpha_j/\alpha_0) + (\mu \mathbf{0}N)_j - (\mu \mathbf{1})(\alpha_j/\alpha_0) \int_{\mathbf{B}} [K(j,x) - \delta_{j0}] d\pi(x).$$

**THEOREM 6.3.** *If  $h$  is an elementary continuous function, then*

- (a)  $(I - P)h = f$  has finite support,
- (b)  $h = \text{const.} + \mathbf{0}Nf$ ,
- (c)  $\alpha f = 0$ , and
- (d)  $B^E h = h$  for any set  $E$  such that  $0 \in E$  and the support of  $f$  is in  $E$ .

And conversely, if (a) and (b) hold, then  $h$  is an elementary continuous function.

**Proof.**  $(I - P)B^E$  is equal to  $(I - P^E)$  for states in  $E$ , and 0 otherwise. Hence  $\alpha \cdot (I - P)B^E = 0$ . Since an elementary continuous function is a finite linear combination of columns of  $B^E$  (with  $E$  finite), (a) and (c) follow. Since a continuous function is bounded and since  $\alpha f$  is finite, Theorem 6.2 is applicable, which establishes (b). We shall complete the proof by showing that (a) and (b) imply both (d) and that  $h$  is an elementary continuous function.

Suppose (a) and (b) hold. If  $0 \in E$ ,  $\mathbf{0}N = B^{E0}N + {}^E N$ . If  $E$  also contains the support of  $f$ , then  ${}^E Nf = 0$ , hence (d) follows from (b). And since in  $B^E h$  we may by (a) choose a finite set  $E$ ,  $B^E h$  is an elementary continuous function. This completes the proof.

**7. Application to normal chains.** In [4] we developed a potential theory for recurrent chains in terms of the dual non-negative potential operators:

$$(7.1) \quad G_{ij} = \lim_n [N_{ii}^{(n)}(\alpha_j/\alpha_i) - N_{ij}^{(n)}],$$

$$(7.2) \quad C_{ij} = \lim_n [N_{jj}^{(n)} - N_{ij}^{(n)}].$$

A chain for which both operators exist is called a *normal chain*. We shall assume from now on that our chain is normal. The operators are dual since

$$(7.3) \quad C_{ij} = \alpha_j \hat{G}_{ji} / \alpha_i.$$

Let  $v_j = G_{0j}$ . From (7.1),

$$(7.4) \quad (vP)_j \leq (v + \delta^0)_j - \lim_n P_{0j}^{(n+1)}.$$

Hence  $v$  satisfies condition (iii) of §5, and it clearly satisfies (i) and (ii) as well.

DEFINITION. For any normal chain, the measure  $\beta^v$  for  $v_j = G_{0j}$  is called the *basic measure* of the chain  $P$ .

From now on, if we write  $\beta$  without a superscript, we shall always mean this basic measure. Applying the results of §5 we have:

$$(7.5) \quad G_{0j} = \int_{S^*} {}^0N(x, j) d\beta(x),$$

$$(7.6) \quad [G_E(I - P^E)]_0 = \theta^E - \delta^0,$$

$$(7.7) \quad [G(I - P)]_0 = \bar{\beta} - \delta^0$$

from (5.3), (5.5), and (5.6), respectively.

In [4] we showed that for any finite set  $E$ ,

$$(7.8) \quad \lim_n \sum_k P_{ik}^{(n)} B_{kj}^E = \lambda_j^E$$

exists and is independent of  $i$ , and  $\lambda^E$  is a probability measure. It measures the probability "in the long run" that  $E$  is entered at various states. In [5] we showed that

$$(7.9) \quad G_E(I - P^E) = 1\lambda^E - I.$$

Comparing the 0th row of this with (7.6), we see that if  $0 \in E$ , then  $\theta^E$  is  $\lambda^E$ . Suppose now that we had chosen 1 as the designated state, and  $v_j = G_{1j}$ . For  $1 \in E$  we would obtain  $\theta^E = \lambda^E$ , hence for  $E$  sufficiently large to contain both 0 and 1 we obtain the same  $\theta^E$ . And since  $\beta$  is the weak limit of  $\theta^{E_k}$  for a fundamental sequence  $\{E_k\}$ , we obtain the same  $\beta$  for both processes. Hence  $\beta$  depends only on  $P$  and not on the state 0. This justifies calling it the "basic measure of  $P$ " without mentioning 0.

We may now rewrite (7.5) and (7.7) for state  $i$  in place of 0. We summarize the results in matrix form:

$$(7.10) \quad G_{ij} = \int_{S^*} {}^iN(x,j)d\beta(x),$$

$$(7.11) \quad G(I - P) = \mathbf{1}\bar{\beta} - I.$$

We may also translate the weak convergence of  $\theta^E$  to  $\lambda^E$ . That is, for any continuous function  $h$  and fundamental sequence  $\{E_k\}$ ,

$$(7.12) \quad \lambda^{E_k}h \rightarrow \int_{S^*} h(x)d\beta(x).$$

If we apply Theorem 6.2' to  $G_{0j}$ , with  $\mu$  given by (7.7), we obtain

$$(7.13) \quad G_{0j} = (\bar{\beta} {}^0N)_j + (1 - \bar{\beta}\mathbf{1}) \int_{\mathbf{B}} {}^0N(x,j) d\pi(x).$$

Since  $\pi$  is to be a probability measure on the boundary, we may choose  $\beta$  restricted to the boundary as  $(1 - \bar{\beta}\mathbf{1})\pi$ . Hence (7.13) agrees with (7.5), the first term giving the integral over  $S$ .

**THEOREM 7.1.** *In a normal chain, if  $h$  is  $T$ -continuous, then  $P^n h \rightarrow \int_{S^*} h(x)d\beta(x)$  for any starting state.*

**Proof.** Let  $\pi^n$  be the row of  $P^n$  corresponding to the starting state chosen. Let  $b = \int_{S^*} h(x) d\beta(x)$ .

$$|b - \pi^n h| \leq |b - \sum_s c_s \lambda^{E_s} h| + |\sum_s c_s (\lambda^{E_s} - \pi^n B^{E_s}) h| + |\pi^n (\sum_s c_s B^{E_s} h - h)|.$$

We choose a fundamental sequence  $\{E_k\}$ . Then  $\lambda^{E_k} h \rightarrow b$ , by (7.12), and hence for sufficiently large  $k$ ,  $|b - \lambda^{E_k} h| < \varepsilon$ . We truncate the sequence so that it contains only sets  $E_k$  sufficiently large for this. We then form a convex combination in accordance with Theorem 5.3; this determines our choice of the  $E_s$  and the  $c_s$ . Since  $\pi$  has total measure 1, the quoted theorem guarantees that the last term is less than  $\varepsilon$ , and the first term is less than  $\varepsilon$  by the above argument and the fact that  $\sum c_s = 1$ . For the second term we note that only a finite number of differences  $(\lambda^{E_s} - \pi^n B^{E_s})_j$  are involved. Hence by (7.8) we make the second term less than  $\varepsilon$  by choosing  $n$  large enough. Thus  $|b - \pi^n h| < 3\varepsilon$  for  $n$  sufficiently large.

**THEOREM 7.2.** *If  $P$  has only minimal boundary points, then each row of  $P^n$  converges weakly to  $\beta$ .*

**Proof.** By Theorems 5.2 and 7.1.

We will show later, by an example, that the condition  $Th = h$  is really necessary in the theorem.

Let us see what our results say for an ergodic (positive recurrent) chain  $P$ . All ergodic chains are normal. For such a chain  $\alpha$  may be chosen with total mea-

sure 1, in which case the rows of  $P^n$  tend to  $\alpha$  componentwise. (We assume as usual that the chain is aperiodic.) In [4] we showed that for an ergodic chain

$$(7.14) \quad G(I - P) = \mathbf{1}\alpha - I.$$

Hence, from (7.11),  $\alpha = \bar{\beta}$ . Thus in an ergodic chain the entire basic measure is on  $\mathbf{S}$ , and the measure  $\beta$  is a generalization of the role played by  $\alpha$  for ergodic chains. Our results in this section generalize known results for ergodic chains to all normal chains. For example, (7.10) is a generalization of  $G_{ij} = \sum_k \alpha_k {}^iN_{kj}$ , which holds for ergodic chains.

Most important of all, for an ergodic chain  $P^n h \rightarrow \alpha h$  for any bounded function, for any starting state. This is the result generalized in Theorem 7.1; but in this theorem we had to make a stronger assumption about  $h$ . While for an ergodic chain  $\alpha_j$  is the limiting probability of being in state  $j$ , for null chains we can only say that  $\beta$  indicates what the chain is “near” in the long run—an intuitive statement made precise by the theorem. We can further strengthen this interpretation for null chains as follows.

**LEMMA.** *If  $\beta(j) = a > 0$  in a null chain, then there is a boundary point  $x$  such that  $\beta^{(x)}(j) \geq a$ .*

**Proof.**  $\beta$  is the weak limit of  $\lambda^{E_k}$ , for a fundamental sequence  $\{E_k\}$ . The characteristic function of  $\{j\}$  is continuous (since all states in  $\mathbf{S}$  are isolated), hence  $a = \beta(j) = \lim_k \lambda_j^{E_k}$ . Let us select a subsequence of the  $\{E_k\}$  so that in the new sequence  $\lambda_j^{E_k} \geq a - (1/k)$  for all  $k$ . Since the chain is null, for each  $k$  there must be infinitely many states  $i$  such that  $B_{ij}^{E_k} \geq a - (2/k)$ . Let us select one such  $i$  for each  $k$ , always selecting a state not previously used. Then  $B_{i_m j}^{E_k} \geq a - (2/k)$  for all  $m \geq k$ . Let  $x$  be a limit point of  $\{i_m\}$ , then  $B^{E_k}(x, j) \geq a - (2/k)$  for each  $k$ . Therefore  $\beta^{(x)}(j) = \lim_k B^{E_k}(x, j) \geq a$ .

**THEOREM 7.3.** *If all boundary points in a null chain are minimal, then  $\beta(\mathbf{S}) = 0$ ; i.e., the measure is entirely on the boundary.*

**Proof.** By the lemma and the fact that for a minimal boundary point  $\beta^{(x)}(\mathbf{S}) = 0$ .

We thus see that for a normal null chain without nonminimal boundary points  $\beta$  is a measure on the boundary. Here  $\beta(A)$  for a Borel subset  $A$  of  $\mathbf{B}$  may be interpreted as the probability that in the long run our chain  $P$  is near this part of the boundary. For example, if  $x$  is at a positive distance from other boundary points, then any sufficiently small neighborhood  $A$  of  $x$  will have a continuous characteristic function. Then, by Theorem 7.2,  $\sum_{j \in A} P_{ij}^n \rightarrow \beta(A)$ , independently of  $i$  and of the choice of  $A$ . If there are only a finite number of points in  $\mathbf{B}$ , and all are minimal, we may partition  $\mathbf{S}^*$  into arbitrarily small neighborhoods of the boundary points plus a finite set of states; and  $\beta(x)$  gives the probability for each  $x \in \mathbf{B}$  that the process will be in its neighborhood in the long run.

Let us briefly look at the duals of our results. Equations (7.10) and (7.11) become:

$$(7.15) \quad C_{ij} = (\alpha_j/\alpha_i) \int_{S^*} {}^j\hat{N}(x, i) d\hat{\beta}_i(x)$$

and

$$(7.16) \quad (I - P)C = m\alpha - I, \text{ where } m_i = \hat{\beta}_i/\alpha_i.$$

One can also use duality to obtain sufficient conditions on measures  $\nu$  such that  $\nu P^n$  converges.

We can also connect these results to potentials for recurrent chains, as introduced in [4]. We say that  $g$  is a *potential function* with *charge*  $f$  if  $\alpha f$  is finite and  $g = \text{Lim}_n N^{(n)}f$ . Dually,  $\nu$  is a *potential measure* with *charge*  $\mu$  if  $\mu \mathbf{1}$  is finite and  $\nu = \text{Lim}_n \mu N^{(n)}$ . The set on which the charge is nonzero is its *support*. We showed in [4] that if  $f$  has finite support and  $\alpha f = 0$ , then  $g = -Gf$  is a potential. We also showed there that  $g = \text{const.} + {}^0Nf$ . Hence any elementary continuous function differs from a potential only by a constant, by Theorem 6.3. And since all potentials of finite support (i.e., whose charges have finite support) satisfy (a) and (b) of the theorem, they are all elementary continuous functions. We have thus shown:

**THEOREM 7.4.** *For a normal chain the elementary continuous functions are the functions that may be written as a potential of finite support plus a constant.*

**THEOREM 7.5.** *If  $g$  is a  $T$ -continuous potential, then*

$$\int_{S^*} g(x) d\beta(x) = 0.$$

**Proof.** By Theorem 7.1,  $P^n g \rightarrow \int g(x) d\beta(x)$ . But it was shown in [4] that for any potential  $P^n g \rightarrow 0$ .

**THEOREM 7.6.** *A function  $h$  of finite deviation is  $T$ -continuous if it can be written as*

$$h = c\mathbf{1} + g,$$

where  $g$  is a  $T$ -continuous potential and

$$c = \int_{S^*} h(x) d\beta(x).$$

**Proof.** If  $(I - P)h = f$ , and  $h$  is  $T$ -continuous, then

$$(I + P + \dots + P^n)f = (I - P^{n+1})h \rightarrow h - c\mathbf{1},$$

by Theorem 7.1. If  $h$  is of finite deviation, then  $f$  is a charge and the limit is a potential. Hence  $h$  differs from the potential by  $c\mathbf{1}$ . The converse is trivial.

**COROLLARY.** *If  $h$  is of finite deviation and  $T$ -continuous, then  $\alpha f = 0$  and  $h = \text{const.} + {}^0Nf$ .*

**THEOREM 7.7.**  *$g$  is a  $T$ -continuous potential if and only if (1)  ${}^0Nf$  is continuous, and (2) for any fundamental sequence  $E_k$ , the extensions of  ${}^{E_k}Nf$  tend to 0. If (3)  $\text{Lim}_{i \rightarrow x} ({}^E Nf)_i = (\text{Lim}_{i \rightarrow x} {}^E N_i) f$  for all finite sets  $E$ , then these two conditions are satisfied, and if  $Gf$  is finite then  $g = -Gf$ .*

**Proof.**  $B^E({}^0Nf) = {}^0Nf - {}^E Nf$ . Since a potential  $g = g_0 \mathbf{1} + {}^0Nf$ , conditions (1) and (2) are precisely the conditions that  $g$  be continuous and that  $Tg = g$ , respectively. If (3) holds, then (1) follows by letting  $E = \{0\}$ . Also  $(\text{Lim}_{i \rightarrow x} {}^E N_i) |f|$  is monotone decreasing in  $E$  to 0, hence (2) follows. We may also infer from (3) and from the finiteness of  $Gf$ , that  $\int {}^0Nf d\beta = \int {}^0Nd\beta \cdot f = (Gf)_0$ . But  $\int g d\beta = 0$ , hence  $g_0 = -(Gf)_0$ , and 0 may be replaced by any state.

**THEOREM 7.8.** *If  $E_k$  is a sequence increasing to  $E$  and  $\theta^{E_k} \mathbf{1} = 1$ , then  $\text{Lim } \theta^{E_k} = \theta^E$ .*

**Proof.** Since  $\theta_j^{E_k}$  is an exit probability, it is monotone decreasing in  $k$ . Hence it tends to a limit, say  $\theta_j$ . Since  $\theta^{E_k} \mathbf{1} = 1$ , therefore  $\theta \mathbf{1} \leq 1$ . From the interpretation,  $\theta_j^E \leq \theta_j^{E_k}$ , hence  $\theta_j^E \leq \theta_j$ , and  $1 = \theta^E \mathbf{1} \leq \theta \mathbf{1}$ . This can only happen if  $\theta^E = \theta$ .

This theorem enables us to give an interpretation for  $\theta^E$  for infinite sets, if we have an interpretation for it for finite sets. For example, if we choose  $v_j = G_{0j}$ , then  $\theta^E = \lambda^E$  for finite sets, and  $\theta^E$  is a limit of  $\lambda^E$ -measures in any case, whether or not  $\lambda^E$  exists for the infinite set  $E$ . (We have shown in [6] that  $\lambda^E$  need not exist.) The necessary condition is that  $\theta^E$  have total measure 1. If  $0 \in E$ , this means that the transient chain is sure to leave  $E$ ; such a set is known as an *equilibrium set*, since it has an equilibrium potential. (See [4, §2].)

For an equilibrium set  $E$  and any  $i \in E$  we have  $[G_E(I - P^E)]_i = \theta^E - \delta^i$ . Since for two different states  $i$  we have  $\theta^E$  as the limit of the same  $\lambda^{E_k}$ , by Theorem 7.8 we may write:  $G_E(I - P^E) = \mathbf{1}\theta^E - I$ ; and hence  $\theta^E G_E(I - P^E) = 0$ . But then  $\theta^E G_E = k\alpha$ . The dual of this result is that if we let  $f_i = \hat{\theta}_i^E / \alpha_i$ , then  $Cf$  is constant on  $E$ , and  $\alpha f = 1$ . This is the criterion for  $f$  to be the "equilibrium charge,"  $Cf$  the "equilibrium potential," and the constant the "capacity" of  $E$ , in a sense developed by Spitzer. (See [9].) The existence of such an "equilibrium potential" was proved in [5], but only for finite sets  $E$ , since only for these could we be sure of the existence of  $\lambda^E$ . By means of boundary theory and  $\theta^E$  we have now extended these results to all sets  $E$  which are equilibrium sets of the associated chain.

Let us consider the special case where  $P$  has only a single boundary point  $x$ . Then a neighborhood of  $x$  is simply the complement of a finite set. Hence the probability of being within this neighborhood tends to 1 for any null chain. Thus  $P^n h$  tends to  $h(x)$  for any null chain and any continuous function  $h$ .

If  $P$  and  $\hat{P}$  both have only a single boundary point, then the chain must be normal. For a null chain  $G_{ij} = {}^i N(x, j)$ , and  $C_{ij} = (\alpha_j / \alpha_i) {}^j \hat{N}(x, i)$ . The representation theorem takes on a much simpler form in this special case:

$$h_i = h_0 + ({}^0Nf)_i - (\alpha f/\alpha_0)C_{i0}.$$

Furthermore, from [4] we know that  ${}^0N_{ij} - (\alpha_j/\alpha_0)C_{i0} = C_{0j} - C_{ij}$ , hence

$$(7.17) \quad h = h_0\mathbf{1} + (C_0 - C_i).f.$$

Hence, if  $Cf$  is finite, then  $h = \text{const.} - Cf$ . Similarly, if  $\mu G$  is finite, then  $v = c\alpha - \mu G$ .

**8. Reinterpretation of the associated processes.** Let us consider the associated chain  $Q^v$ , started at 0, and reverse time for the chain: We denote by  $X_n$  the position of the  $Q^v$ -chain at time  $n$ . Then we consider the stochastic process  $Y_n$ , for  $n \leq 0$ , for which  $Y_{-n} = X_n$ . This process either starts at some random time  $\tau \leq 0$  (if  $X_n$  terminates in finite time), or it may "come from the boundary" at time  $-\infty$ .

Let  $E$  be any finite set containing 0. Then there is a first time  $-t$  that  $Y_n$  enters  $E$ . We know that  $E$  is entered at  $j$  with probability  $\theta_j^E$ . Let  $Z_n = Y_{-t+n}$ , for  $n = 0, 1, \dots$ , and let us compute the transition probabilities for  $Z$ :

$$\begin{aligned} \Pr[Z_{n+1} = j | Z_n = i] &= \frac{\Pr[Y_{-t+n+1} = j \text{ and } Y_{-t+n} = i]}{\Pr[Y_{-t+n} = i]} \\ &= \frac{\sum_{m=1}^{\infty} \Pr[X_{m-1} = j \text{ and } x_m = i \text{ and } E \text{ is left at } m+n]}{\sum_{m=0}^{\infty} \Pr[X_m = i \text{ and } E \text{ is left at } m+n]} \\ &= \frac{\sum_{m=1}^{\infty} \Pr[X_{m-1} = j] Q_{ji}^v \Pr_i[E \text{ is left in } n \text{ steps}]}{\sum_{n=0}^{\infty} \Pr[X_m = i] \Pr_i[E \text{ is left in } n \text{ steps}]} \\ &= \frac{(v_j + \delta_{0j})Q_{ji}^v}{v_i} = P_{ij}. \end{aligned}$$

Hence the  $Z$ -process is a Markov chain which starts in  $E$  with starting distribution  $\theta^E$ , proceeds according to  $P$  and stops at 0. Thus the  $Y$ -process, observed when it first enters a set  $E$ , may be thought of as our original Markov chain (started with  $\theta^E$  and stopped at 0), but with a time-distortion. That is, in the  $Y$ -process, time is counted backwards from the first time 0 is reached instead of forwards from the time  $E$  is entered<sup>(4)</sup>.

If  $f$  is any function on the sample paths which depends only on the path after a set  $E$  (finite and containing 0) is entered, and does not depend on time, then

(4) This shows that the  $Y$ -process is a strong approximate  $P$ -chain, in the sense of Hunt [3].

it has the same mean in the  $Y$ -process as in the chain  $P$  started on  $E$  with  $\theta^E$ . For example,  $v_j$  for  $j \in E$  represents the mean number of times this chain is in  $j$  before 0 is reached. This gives us a direct probabilistic interpretation for (5.5):  $v_E$  gives the mean number of times in each state before 0, while  $v_E P^E$  is the mean number of the "next position in  $E$ ," hence the former contains the starting position in  $E$  while the latter contains the terminal visit to 0, which yield the two terms  $\theta^E$  and  $\delta^0$ , respectively. A similar interpretation can be given for (5.10).

It is natural to think of the entire stochastic process  $\{Y_n\}$  as follows: We start our chain on  $S^*$  with starting distribution  $\beta^v$ , then we proceed according to  $P$ , and stop when 0 is reached; however, time is computed backwards from 0. Again the mean of any quantity which depends only on paths and not on time may be computed in the  $Y$ -process but interpreted as if the  $P$ -chain had been started with  $\beta^v$ . This gives direct insight into several of our identities.

In particular, if we choose  $v_j = {}^0N(x, j)$ , then the resulting  $Y$ -process may be thought of as starting  $P$  at the point  $x$  (and counting time backwards from 0). Thus  $B^E(x, j)$  is the probability that the chain started at  $x$  will enter the set  $E$  at  $j$ .

More generally, if  $\pi$  is any probability measure on  $S^*$ , we may use it as a starting distribution for  $P$  by choosing

$$(8.1) \quad v_j = \int_{S^*} {}^0N(x, j) d\pi(x),$$

and forming the  $Y$ -process. It should be noted that the resulting  $\beta^v$  will be the same as  $\pi$  only if  $\pi$  puts 0 measure on nonminimal boundary points. Hence we note that a starting distribution on  $S^*$  may always be reinterpreted as starting with  $\beta^v$ , i.e., only on  $S$  and at minimal boundary points. In any case  $\beta^v$  is the limiting distribution of  $Y_n$  as  $n$  approaches its starting time (i.e.,  $\tau$  or  $-\infty$ ).

Let us again return to the function  $f$  which depends only on the paths after  $E$  is entered. Let  $g_i = M_i[f]$ . Then

$$(8.2) \quad g_i = \sum_{k \in E} B_{ik}^E M_k[f].$$

This is clearly a continuous function, and its extension is

$$(8.3) \quad g(x) = \sum_{k \in E} B^E(x, k) \cdot M_k[f],$$

for any finite set  $E$  containing 0. On the other hand, using our new interpretation of  $B^E(x, k)$ , we see that  $g(x)$  is the mean of the function  $f$  if the  $P$ -chain is started at  $x$ . Hence  $g(x) = M_x[f]$  for all  $x \in S^*$ . And more generally, for any probability measure  $\pi$  on  $S^*$ ,

$$(8.4) \quad M_\pi[f] = \sum_{k \in E} \theta_k^E M_k[f] = \sum_{k \in E} \int_{S^*} B^E(x, k) d\pi(x) \cdot M_k[f] = \int_{S^*} g(x) d\pi(x).$$

For example, let  $f$  be the number of times in  $j$  before a finite set  $F$  (containing 0) is reached. Let  $E = F \cup \{j\}$ . Then our above considerations apply;  $g_i = {}^F N_{ij}$  and  $g(x) = {}^F N(x, j)$  is continuous. We also obtain a formula for  $M_\pi[f]$  from (8.4).

We can use our new interpretation to characterize normal chains. Let us attempt to start a  $P$ -chain with a distribution  $\pi$  which represents the position of the Markov chain "in the long run." That is, we let  $\pi^{(n)}$  be a row of  $P^n$ , and we desire  $\pi$  to be the limit of  $\pi^{(n)}$  in some sense. We know that  $\pi$  is determined if we know the entrance probabilities  $\theta^E$  for all finite sets  $E$ . These should be the limits of  $\pi^{(n)} B^E$ . These limits exist for both  $P$  and  $\hat{P}$  if and only if the chain is normal (in which case  $\pi^{(n)} B^E \rightarrow \lambda^E$ ). Hence normal chains are those recurrent chains for which we have a basic measure  $\beta$  which enables us to start the chain the way it is "in the long run," and a measure  $\hat{\beta}$ , with the same property, for  $\hat{P}$ .

If a normal chain is started with the basic measure  $\beta$ , then  $G_{0j}$  is the mean number of visits to  $j$  before 0 is reached, and  $\lambda^E$  is the entrance distribution for any finite set  $E$  containing 0. This enables us to give interpretations for various results in [4]. For example,  ${}^i \lambda_j = \lambda_j^{[i, j]}$  is the probability in this process of reaching  $j$  before  $i$ , and hence the relation  $G_{0j} = {}^0 \lambda_j {}^0 N_{jj}$  is obvious.

For a normal chain we also considered

$${}^E v_j = \lim P^n {}^E N_j.$$

This limit exists because a column of  ${}^E N$  is an elementary continuous function. Then

$${}^E v_j = \int_{S^*} {}^E N(x, j) d\beta(x)$$

by Theorem 7.1.

From our interpretation of  ${}^E N(x, j)$  we see that  ${}^E v_j$  is the mean number of visits to  $j$ , before hitting  $E$ , for the  $P$ -chain started with  $\beta$ . This gives a simple proof of the useful identity

$$(8.5) \quad G_{0j} = \sum_{k \in E} G_{0k} {}^E \bar{N}_{kj} + {}^E v_j,$$

where  ${}^E \bar{N}_{kj}$  is the number of visits to  $j$  before returning to  $E$  from  $k \in E$ . The first term represents the mean number of visits to  $j$  after  $E$  is reached and the second term the mean number of times before  $E$  is reached.

We can also give a simple proof for the important identity

$$(8.6) \quad G_{i0} \frac{\alpha_j}{\alpha_0} + G_{0j} = G_{ij} + {}^0 N_{ij}.$$

We showed in [4] that

$${}^i N_{m0} \frac{\alpha_j}{\alpha_0} + {}^0 N_{mj} = {}^i N_{mj} + {}^0 N_{ij},$$

by observing that the two sides compute the same quantity in two different ways.

Each is the mean starting in  $m$ , of the number of times in  $j$  as we reach 0 via  $i$ . The same argument applied to the  $P$ -chain started with  $\beta$  gives (8.6).

Finally we mention that the identity just proved gives us an interpretation for  $\hat{C}_{0j}$ . In fact from the identity, taking  $i = j$ , we obtain

$$(8.7) \quad \hat{C}_{0j} = {}^0N_{jj} - G_{0j}.$$

Hence  $\hat{C}_{0j}$  is the difference between the mean number of visits to  $j$  before 0 for the process started at  $j$  and the process started with  $\beta$ .

**9. Sums of independent random variables.** We may illustrate our results in terms of results for sums of independent random variables, found by Spitzer in [9]. If we consider a null Markov chain whose  $n$ th position is the sum of  $n$  independent and identically distributed random variables, then Spitzer showed that the chain is always normal and there are only minimal boundary points (in our terminology). In two dimensions there is always a unique boundary point, while in one dimension there are one or two, depending on whether the random variables have infinite or finite variance.

In the case of a single boundary point the space  $S^*$  is the one-point compactification of  $S$ , hence a continuous function  $h$  is one having a limit at infinity, and we have already noted that the convergence of  $P^n h$  is trivial. In one dimension, with finite variance,  $+\infty$  and  $-\infty$  are two distinct boundary points, and  $h$  must have limits (not necessarily the same) in both directions. Then Theorem 7.1 states that  $P^n h$  converges to the average of these two limits. This result also follows from the Central Limit Theorem.

There are such chains for which  $P^n h$  fails to converge for so nice a function as the characteristic function of the positive integers. However, this can happen only if the variance is infinite; then there is only one boundary point, and  $h$  is not continuous.

Spitzer also proved a representation theorem for subregular functions for sums of independent random variables. He obtained the form

$$(9.1) \quad h = -Cf + \text{const.}$$

for the case of one boundary point, and

$$(9.2) \quad h_i = -(Cf)_i + ai + b$$

for the two boundary point one-dimensional case. We have already shown that the former formula holds for any chain with a single boundary point. And (9.2) follows from our representation theorem together with special knowledge Spitzer obtains of the form of  ${}^0N$  and  $C$  in the case of sums of independent random variables.

Spitzer proves his results for the case of deviation with finite support, but he remarks that his results could be extended to the case of finite deviation, as

we have done. Thus our representation theorem extends his results from these special chains to all recurrent chains. It should also be noted that while Spitzer requires  $h$  to be subregular, i.e.,  $f$  to be all of the same sign, we do not impose this restriction.

10. **A class of summability methods.** Let  $P$  be the transition matrix of either a transient or null recurrent Markov chain, and let  $W^{(i)}$  be the matrix whose  $n$ th row is the  $i$ th row of  $P^n$ . We require that  $P$  have row sums 1, even if it is transient. Then it follows from the fact  $P^n \rightarrow 0$  that  $W^{(i)}$  is a regular summation matrix. The question of whether  $W^{(i)}$  can sum the sequence  $h_i$  is the same as the question of whether the  $i$ th component of  $P^n h$  converges. Thus we obtain a summation method for each choice of  $i$ .

By the *field* of a summation method one means the set of bounded sequences which are summable by the method. Thus the functions  $h$  for which  $P^n h$  converges are precisely the functions which lie in the field of  $W^{(i)}$  for all  $i$ .

It is easy to show that for recurrent  $P$  all these methods are equivalent, i.e., they sum precisely the same sequences and to the same limit. Thus we are really considering a single summability method. By a well-known result on summability, there is a sequence of 0's and 1's for any method which is not summable by that method. This means that there is a function  $h$ , which is the characteristic function of a set of states, for which  $P^n h$  fails to converge. This was shown by Lamperti in [7] by other methods. Lamperti then raised the question which in the present terminology asks for the fields of the methods  $W$ . Our results provide at least a partial answer: The field must at least contain the  $T$ -continuous functions.

For transient chains the methods  $W^{(i)}$  need not be equivalent. Indeed, if  $h$  is a nonconstant regular function, then  $P^n h \rightarrow h$ , and hence  $W^{(i)}$  is inconsistent with  $W^{(j)}$  if  $h_i \neq h_j$ . But it follows from Martin boundary theory that if  $h$  is continuous on  $S^*$ , then  $P^n h$  converges. Hence any continuous function will lie in the field of all of these summability methods.

It is interesting to note that several of the best known summability methods may be obtained in the indicated manner. It should also be noted that if  $P$  is a non-negative summability matrix, which may be taken as a transition matrix without loss of generality, then  $P^{n+1}$  is always a stronger method than  $P^n$ , and  $W^{(i)}$  is always stronger than any of these methods.

11. **Examples.** We have already illustrated some of our results in terms of sums of independent random variables in §9. At present we wish to furnish some counter-examples.

Let us first observe that for any null chain, if we choose for  $f$  the 0th column of  $I - P$ , then  $\alpha f = 0$ , and  $N^{(n)} f = (I - P^{n+1})_{\cdot 0}$ , which tends to  $\delta^0$ . Therefore  $f$  is a potential charge and its potential is  $g = \delta^0$ , which is clearly continuous.

We introduced in [4] the very useful class of examples in which the states are the non-negative integers and from  $i - 1$  the process steps to  $i$  with probability  $p_i$  or to 0 with probability  $q_i = 1 - p_i$ . If we let  $\alpha_0 = 1$ ,  $\alpha_i = \prod_{k=1}^i p_k$ , then  $\alpha$  is a superregular measure. The chain is recurrent if  $\alpha_n \rightarrow 0$ , and null if in addition  $\sum_n \alpha_n = \infty$ .

This chain has

$${}^0N_{ij} = \begin{cases} \alpha_j/\alpha_i & \text{if } j \geq i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

There is one boundary point at  $+\infty$ , and  ${}^0N(+\infty, j) \equiv 0$ . (Hence  $+\infty$  is a nonminimal boundary point.) Therefore, the chain is normal and  $G_{0j} \equiv 0$ . Since for a finite set  $E$ , with  $0 \in E$ , from any sufficiently large  $i$  the set must be entered at 0,  $\lambda_0^E = 1$ . Hence  $\beta$  puts a unit mass on 0. This can also be seen by observing that  $Q^v$  for  $v_j = G_{0j}$  starts at 0 and immediately stops.

If  $g$  is the potential  $\delta^0$ , then  $Tg$  is the limit of the extensions of  $B^{E_k}g$  for a fundamental sequence  $\{E_k\}$ . We need only consider what happens at  $+\infty$ :  $\text{Lim}_{i \rightarrow +\infty} (B^E g)_i = 1$  for any finite set  $E$ . Hence  $Tg(+\infty) = 1$ . Thus  $Tg \neq g$ . Also  $\int g(x)d\beta(x) = g_0 = 1$  is not 0. And  $-(Gf)_0 = 0 \neq g_0$ . Thus we see that  $g$  being continuous is not sufficient to assure that  $Tg = g$  or other properties which would hold in that case.

The reverse of this example is even more instructive. The reverse chain moves deterministically to the left until it reaches 0, and then  $P_{0i} = \alpha_i - \alpha_{i+1}$ .

$$Q_{ij} = \begin{cases} p_{i+1} & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $N_{0j} = \alpha_j$  and  $N_{ij} = \alpha_j/\alpha_i$  if  $j \geq i$  or 0 otherwise. Hence  $K(i, j) = 1/\alpha_i$  for all  $j \geq i$ , and hence  $K(i, j) \rightarrow 1/\alpha_i$  as  $j \rightarrow +\infty$ . We have a single boundary point, and since  $1/\alpha_i$  is the unique regular function of  $Q$ , the boundary point is minimal.

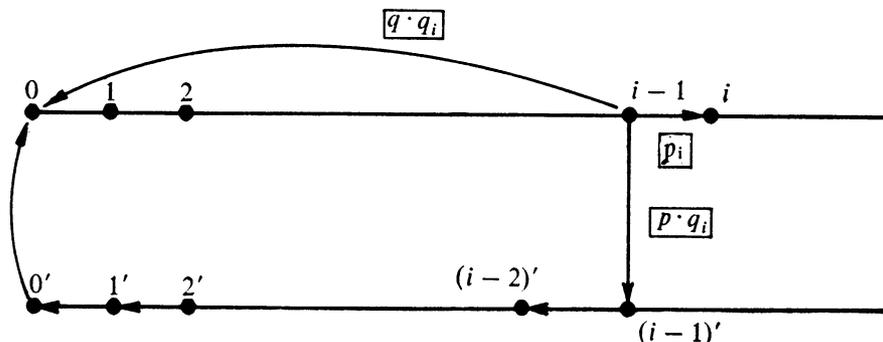
If  $v_j = G_{0j} = 1 - \delta_{0j}$ , then  $Q_{ij}^v = \delta_{j, i+1}$ ; hence the associated chain started at 0 marches out to  $+\infty$ . From this it is clear that the exit probabilities are  $\theta_j^E = 1$  if  $j$  is the last state in  $E$  and 0 otherwise. And hence,  $\beta(+\infty) = 1$ . We thus have convergence to the minimal boundary point. It is interesting to see that even in so simple an example the limiting behavior near the two boundaries can be quite different.

Let us modify the original example as follows (see figure):

On the upper branch at  $i - 1$  the chain goes to  $i$  with probability  $p_i$ , to 0 with probability  $q \cdot q_i$ , or to position  $(i - 1)'$  with probability  $p \cdot q_i$ . On the lower branch it moves a step at a time to the left, and from  $0'$  to 0. ( $p + q = 1$  are arbitrary but fixed positive numbers.) There are two boundary points,  $\infty$  and  $\infty'$ .

If  $E$  is any finite set and  $i'$  its last state on the lower branch, then  $\lambda_0^E = q$ ,  $\lambda_{i'}^E = p$ .

Hence  $\beta(0) = q$ ,  $\beta(\infty) = 0$ ,  $\beta(\infty') = p$ , and  $\infty'$  is minimal, while  $\infty$  is not.  $G_{0j} = pN(\infty', j)$  which is 0 on the upper branch and  $p$  on the lower. The asso-



ciated chain corresponding to  $v_j = G_{0j}$  starting at 0 stops immediately with probability  $q$  or moves to  $0'$  and out to  $\infty'$  along the lower branch.

Let  $u_n = P_{00}^{(n)}$ ,  $f_n = \text{Pr}_0[\text{first return to } 0 \text{ is at time } n]$ .

Let  $A(t) = \sum_n \alpha_n t^n$ ,  $F(t) = \sum_n f_n t^n$ , and  $U(t) = \sum_n u_n t^n$ . Then  $U(t) = 1/[1 - F(t)]$ , and  $F(t) = 1 - q(1-t)A(t) - p(1-t^2)A(t^2)$ . The probability of being on the upper branch after  $n$  steps is  $\sum_{k=0}^n u_k \alpha_{n-k}$ . This sequence of probabilities is Abel-summable if and only if

$$\lim_{t \rightarrow 1} (1-t)U(t)A(t) = \lim_{t \rightarrow 1} \frac{(1-t)A(t)}{q(1-t)A(t) + p(1-t^2)A(t^2)}$$

exist. But we need only choose the  $\alpha_n$  so that  $\lim_{t \rightarrow 1} A(t)/A(t^2)$  should fail to exist, and then the probabilities will not be Abel-summable.

Let  $h_i$  be 1 if  $i$  is on the upper branch and 0 if  $i$  is on the lower branch. Then  $h$  is continuous, but  $Th = qh$ . By the above argument  $P^n h$  does not converge, and is not even Abel-summable. This shows that continuity is not a sufficient condition for Theorem 7.1.

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