TCHEBYCHEFF QUADRATURE
ON THE INFINITE INTERVAL(1)

BY

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1. Introduction. Theorem A, the principal theorem of this paper can be interpreted as a result on the zeros of Faber polynomials or as a result on the theory of Tchebycheff quadrature.

1.1. The proof of Theorem A is achieved by means of two auxiliary theorems. In §2 Theorem B is stated and it is shown that it implies Theorem A. In §3 Theorem C is stated and proved. In §4 it is shown that Theorem C implies Theorem B, thus completing the proof of Theorem A. In §5 Theorem A is related to a paper on the zeros of Faber polynomials by the author [1], and a paper on Tchebycheff quadrature by Wilf [2].

1.2. A unit mass distribution on (— ∞, ∞), possessing moments of all positive integer order will be said to belong to class D. If ψ, ψj, ψ* are in class D, we will denote the kth moments by mk, mkj, mk*, respectively.

Theorem A. There is an element ψ* of class D which has the properties:

(a) the equations

\[ m_k^* = \frac{1}{n} \sum_{i=1}^{n} x_{i,n}^k, \quad k = 1, \ldots, n, \]

have a real solution for infinitely many positive integers n,

(b) the mass set of ψ* does not lie on a finite interval.

An element of D which satisfies (a) and (b) will be called a T distribution. The set of integers for which (1.2.1) has real solutions will be called the T set of ψ*.

2. The first auxiliary theorem.

2.1. Let ψj be an element of class D. The equations

\[ m_k^j = \frac{1}{n} \sum_{i=1}^{n} x_{i,n}^k, \quad k = 1, \ldots, n, \]

have a unique solution t1,n, ⋯, tn,n, up to a permutation of the first subscripts. Let

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where \( a_0^i = 1 \). We adopt the convention that if \( \psi \) without a subscript is used, then the coefficients of the third term of (2.1.2) are written as \( a_i \). The quantity \( a_i \) is a polynomial with real coefficients in the quantities \( m_1^i, \ldots, m_n^i \), \( i = 1, \ldots, n \).

2.2. Let \( \psi, \psi_1 \) be two elements of class \( D \). Let

\[
\| \psi - \psi_1 \|_n = \max \{ |m_1^1 - m_1^1|, \ldots, |m_n^1 - m_n^1| \}.
\]

The defined quantity is called the \( n \)th order distance between the two mass distributions.

2.3. A unit mass distribution with \( n \) equal masses located at \( n \) distinct real points will be called a simple mass distribution of degree \( n \). If the mass points of a simple mass distribution, say \( \psi \), are \( s_1, \ldots, s_n \), then

\[
F_n(z) = \prod_{i=1}^{n} (z - s_i).
\]

2.4. Lemma A. Let \( \psi \) be a simple mass distribution of degree \( n \), and let \( \psi_1 \) be any element of class \( D \). There is a number \( \varepsilon, \varepsilon > 0 \), called a proximity number of \( \psi \), such that if

\[
\| \psi - \psi_1 \|_n < \varepsilon,
\]

then the polynomial \( F_n(z) \psi_1 \) has real zeros.

**Proof.** Let \( s_1, \ldots, s_n \) be the mass points of \( \psi \). Let \( \sigma_i \) be the circle \( |z - s_i| = \delta_i \), \( i = 1, \ldots, n \). The \( \delta_i \) are chosen to be positive, and such that \( s_i \) is the only point of the set \( s_1, \ldots, s_n \) inside or on \( \sigma_i \). Let \( E = \bigcup_{i=1}^{n} \sigma_i \), and let

\[
\min_{z \in E} |F_n(z) \psi_1| = \delta.
\]

The quantity \( \delta \) is positive. By Rouché's theorem, we know that if

\[
|F_n(z) \psi_1 - F_n(z) \psi| < \delta
\]

for \( z \) on \( E \), then \( F_n(z) \psi_1 \) has one root inside each of the circles \( \sigma_i \). By §2.1, the coefficients of \( F_n(z) \psi_1 \) are real, so these roots must be real.

The left side of (2.4.3) is less than

\[
\gamma \sum_{i=0}^{n} |a_i - a_i^1|,
\]

for \( z \in E \), where

\[
\gamma = \max_{z \in E} \{1, |z|, \ldots, |z|^{n-1} \} > 0.
\]

The quantity (2.4.4) is a continuous function of \( m_i^1, i = 1, \ldots, n \), and takes on the value zero when \( m_i^1 = m_i, i = 1, \ldots, n \). Hence there is a number \( \varepsilon > 0 \), such that
if (2.4.1) is satisfied, (2.4.4) is less than $\delta$, and therefore (2.4.3) is satisfied. This completes the proof.

2.5. Theorem B. There is an element $\psi^*$ of class $D$, and a denumerable sequence of simple mass distributions $\psi_k$ of degree $r_k$ and proximity numbers $\varepsilon_k$, $k = 1, 2, \ldots$, where $r_1 < r_2 < \ldots$ such that

\begin{equation}
\int_0^\infty d\psi^* > 0, \text{ for all } a > 0,
\end{equation}

and

\begin{equation}
\|\psi^* - \psi_k\|_{r_k} < \varepsilon_k, \quad k = 1, 2, \ldots.
\end{equation}

2.6. Theorem B implies Theorem A. We will show that if a function $\psi^*$ exists satisfying (2.5.1) and (2.5.2), then the same function satisfies conditions (a) and (b) of Theorem A. It is clear that (2.5.1) implies (b). Also (2.5.2) implies that (1.2.1) has real solutions for $n = r_k$, $k = 1, 2, \ldots$ because by Lemma A the zeros of $F_{r_k}(z | \psi^*)$ are real, and by (2.1.2) they form a solution to (1.2.1) for $n = r_k$.

3. The second auxiliary theorem.

3.1. The proof of Theorem C is based on a construction. In §5 methods for generalizing the construction to arrive at a wider class of $T_1$ distributions is discussed.

We first define a family of sets $O$ which will remain fixed throughout §3 and §4. The family $O$ consists of a denumerable number of nonoverlapping intervals on the positive real axis, say $\{O_j\}$, $j = 1, 2, \ldots$, having centers at $u_j$, $j = 1, 2, \ldots$ such that $0 < u_1 < u_2 < \ldots$ and $u_n$ tends to infinity.

3.2. Theorem C. Let the family of sets $O$ be given. There exists a denumerable sequence of simple mass distributions, $\psi_k$, $k = 1, 2, \ldots$, having degree $r_k$, $r_1 < r_2 < \ldots$, and proximity numbers $\varepsilon_k$ such that

\begin{equation}
\int_{O_j} d\psi_k = \gamma_j > 0, \quad j = 1, \ldots, k - 1, \quad k = 2, 3, \ldots,
\end{equation}

and

\begin{equation}
\|\psi_k - \psi_{k-1}\|_{r_{k-1}} < \min \left\{ \frac{\varepsilon_1}{2k-1}, \ldots, \frac{\varepsilon_{k-1}}{2} \right\}, \quad k = 2, 3, \ldots.
\end{equation}

3.3. An element $\psi$ of $D$ is said to be $(M, k)$ compatible if $\psi$ is a simple mass distribution, if the mass points of $\psi$ lie in the sets $O_1, \ldots, O_k$, and if there is a positive mass on $O_k$, all of which is located at $u_k$. The operation $M(k, n)$, where $n$ is an integer and automatically greater than one, can be applied to a $(M, k)$ compatible mass distribution, and yields a unique mass distribution $\psi = \psi M(k, n)$.

The distribution $\psi_1$ is characterized by the following properties:

\begin{equation}
\psi_1(E) = \psi(E)
\end{equation}
for all sets to the left of $O_k$,

$$\psi_1(u_k) = \psi(u_k) \left( \frac{n - 1}{n} \right),$$

and

$$\psi_1(u_{k+1}) = \frac{\psi(u_k)}{n}.$$

3.4. **Lemma B.** Let $\varepsilon$ be an arbitrary positive number, $p$ an arbitrary integer, and let $\psi$ be $(M,k)$ compatible. There exists an integer $n_1$ such that

$$\| \psi_1 - \psi \|_p < \varepsilon$$

where

$$\psi_1 = \psi M(k,n)$$

and $n \geq n_1$.

**Proof.** By considering the explicit expressions for $m_r, m_r^1$, for any positive integer $r$, we find that

$$m_r^1 - m_r = \frac{\psi(u_k)}{n}(u_{k+1}^r - u_k^r).$$

This tends to zero as $n$ tends to infinity, so that the proof of the lemma is readily completed.

3.5. An element $\psi$ of $D$ is said to be $(S, k)$ compatible if all the mass is located at a finite number of points, say masses $b_i, b_i > 0, i = 1, \ldots, q$ at the points $v_1, \ldots, v_q$, $v_1 < v_{i+1}, i = 1, \ldots, q - 1$, if all the mass lies on the sets $O_1, \ldots, O_k$, if there is one mass point on $O_k$, namely the point $u_k$, and if the equations

$$b_i = \alpha_i b_q, \quad i = 1, \ldots, q - 1,$$

are satisfied by integer values for $\alpha_i$. The operation $S(k, \delta), \delta$ a positive number, can be applied to an $(S, k)$ compatible mass distribution, and will yield a unique mass distribution we denote by

$$\psi_1 = \psi S(k, \delta).$$

The distribution $\psi_1$ is defined as follows. The mass point $v_i, 1 \leq i \leq q - 1$ is replaced by $\alpha_i$ mass points of mass $b_q$, say at $v_{i,1}, \ldots, v_{i,\alpha_i}$, according to some fixed law which satisfies the condition

$$| v_i - v_{i,j} | \leq \delta, \quad j = 1, \ldots, \alpha_i.$$

We can say for definiteness that

$$v_{i,j} = v_i + \frac{j}{\alpha_i} \delta, \quad j = 1, \ldots, \alpha_i.$$
The mass $b_q$ and the mass point $v_q$ remains unaffected.

3.6. Lemma C. Let $\varepsilon$ be an arbitrary positive number, $p$ an arbitrary positive integer, and let $\psi$ be $(S, k)$ compatible. There exists a number $\delta_1 > 0$ such that

\[
\| \psi_1 - \psi \|_p < \varepsilon,
\]

where

\[
\psi_1 = \psi S(k, \delta)
\]

and $\delta \leq \delta_1$.

Proof. Using the notations of §3.5 we find that

\[
m_1 - m_r = \sum_{i=1}^{q-1} \left( b_q \sum_{j=1}^{a_1} v_{i,j} - b_i v_i' \right),
\]

where $r$ is an arbitrary positive integer.

Because of (3.5.3), as $\delta$ tends to zero, $v_{i,j}$ tends to $v_i$. Using (3.5.1) we then see that (3.6.3) tends to zero, so the proof of the lemma is readily completed.

3.7. An element $\psi$ of $D$ followed by a finite sequence of operations of the type being considered is said to be well defined when the following conditions are satisfied. $\psi$ must have the type of compatibility required to perform the first operation, and after any number of operations have been performed, the resulting mass distribution must have the proper compatibility condition for the next operation, when it exists.

We note that if $\psi$ is $(M, k)$ compatible, then $\psi M(k, n)$ is $(S, k + 1)$ compatible, so that

\[
\psi M(k, n) S(k + 1, \delta)
\]

is well defined.

Lemma D. Let $\psi$ be $(M, k)$ compatible and of degree $r$. Let $\varepsilon$ be an arbitrary positive number. Let $\psi_1 = \psi M(k, n)$, and $\psi_2 = \psi_1 S(k + 1, \delta)$. There exist numbers $\delta_1 > 0$ and $n_1 \geq 2$ such that for any $\delta \leq \delta_1$, and any integer $n \geq n_1$

\[
\psi_2 \text{ is } (M, k + 1) \text{ compatible}
\]

and

\[
\| \psi_2 - \psi_1 \|_r < \varepsilon.
\]

Proof. Consider the set consisting of the mass points of $\psi_1$ and the end points of $O_1, \ldots, O_k$. Let $\delta_2$ be the smallest distance between any pair of these points. If $\delta < \delta_2$, then

\[
\psi_2(O_j) = \psi_1(O_j), \quad j = 1, \ldots, k,
\]
This means that all the mass of \( \psi_2 \) is on the sets \( O_1, \ldots, O_{k+1} \). Further checking shows that it is simple and that all of its mass on \( O_{k+1} \) is concentrated at \( u_{k+1} \) and is positive. Therefore \( \psi_2 \) is \((M, k + 1)\) compatible.

We next make use of the inequality

\[
\| \psi_2 - \psi \|_r \leq \| \psi_1 - \psi \|_r + \| \psi_2 - \psi_1 \|_r,
\]

which indeed holds true for any three elements of \( D \). By Lemma B, we can choose \( n_1 \) so that the first term on the right is less than \( \epsilon/2 \) for \( n \geq n_1 \). By Lemma C we can choose \( \delta_3 \) so that the second term on the right is less than \( \epsilon/2 \) for \( \delta < \delta_3 \). Then Lemma D is true for \( \delta_1 = \min(\delta_2, \delta_3) \), and the above choice of \( n_1 \).

3.8. We note the following properties that hold when \( n \geq n_1, \delta \leq \delta_1 \):

\[
\psi_2(0_j) = \psi(0_j), \quad j = 1, \ldots, k - 1,
\]

\[
\psi_2(0_k) = \psi(u_k) - \frac{1}{p} n,
\]

and

\[
\psi_2(u_{k+1}) = \frac{1}{pn}.
\]

The mass distribution \( \psi_2 \) is simple and of degree \( pn \).

3.9. Lemma E. Let \( \psi_1 \) be the unit mass located at \( u_1 \). There exist integers \( n_1, n_2, \ldots, n_k \geq 2 \), and positive numbers \( \delta_1, \delta_2, \ldots \) such that:

(a)

\[
\psi_1 M(1, n_1) S(2, \delta_1) \cdots M(k - 1, n_{k-1}) S(k, \delta_{k-1})
\]

is a well-defined sequence for \( k \geq 2 \), (b) the mass distribution \( \psi_k \) defined by

\[
(3.9.1)
\]

is \((M, k)\) compatible, and (c)

\[
\| \psi_k - \psi_{k-1} \|_{r_{k-1}} = \min \left\{ \frac{\varepsilon_1}{2k-1}, \ldots, \frac{\varepsilon_{k-1}}{2} \right\},
\]

for \( k \geq 2 \), where \( \varepsilon_j \) is the proximity number of \( \psi_j \) and \( r_j \) is the degree of \( \psi_j \), \( j = 1, \ldots, k - 1 \).

Proof. We divide the proof into two cases.

Case I. \( k = 2 \). Let \( \varepsilon_1 \) be a proximity number of \( \psi_1 \). We have observed that \((3.9.1)\) is well defined for this case in §3.7. By Lemma D, \( n^*, \delta^* \) exist such that \( \psi_2 \) is \((M, 2)\) compatible and such that \((3.9.2)\) is satisfied for \( p = 2 \), providing \( n_1 \geq n^* \), \( \delta_1 \leq \delta^* \). Choose \( n_1 = n^* \) and \( \delta_1 = \delta^* \).

Case II. The inductive step. Assume that the numbers \( n_1, \ldots, n_{p-1}, \delta_1, \ldots, \delta_{p-1} \) exist such that \((3.9.1)\) is well defined for \( k = 2, \ldots, p \), such that \( \psi_p \) is \((M, p)\) com-
compatible and such that (3.9.2) is satisfied for \( k = 2, \ldots, p \). These assumptions imply that \( \psi_p \) is simple. Let it have proximity number \( \varepsilon_p \) and degree \( r_p \). The proximity numbers \( \varepsilon_1, \ldots, \varepsilon_{p-1} \) used in (3.9.4) will be those introduced successively in previous steps. Since \( \psi_p \) is \((M, p)\) compatible, by Lemma D there are numbers \( n^*, \delta^* \) such that

\[
\psi^* = \psi_p M(p, n) S(p + 1, \delta)
\]

is \((M, p + 1)\) compatible, and

\[
\| \psi^* - \psi_p \|_{r_*} \leq \min \left\{ \frac{\varepsilon_1}{2^p}, \ldots, \frac{\varepsilon_p}{2} \right\}
\]

for \( n \geq n^*, \delta \leq \delta^* \). Let \( n_p = n^*, \delta_p = \delta^* \), and let \( \psi_{p+1} \) be the function defined by (3.9.3) by these values. Then for the values \( n_1, \ldots, n_p, \delta_1, \ldots, \delta_p \) (a), (b) and (c) are satisfied for \( k = 2, \ldots, p + 1 \). This completes the proof by induction.

3.10. We list properties of the functions \( \psi_k \) of this lemma:

\[
(3.10.1) \quad r_k = n_1, \ldots, n_{k-1}, \quad k \geq 2.
\]

\[
(3.10.2) \quad \psi_k(O_j) = \frac{n_j - 1}{n_1 \cdots n_j}, \quad j = 1, \ldots, k - 1, \quad k \geq 2.
\]

These follow from the fact that \( \psi_1 \) has degree 1, and the properties stated in §3.8 by induction.

3.11. To complete the proof of Theorem C, we use the mass distribution Lemma E, noting that (3.10.1) implies that \( r_1 < r_2 < \cdots \), that (3.10.2) implies (3.2.1), and that (3.9.2) corresponds to (3.2.2).

4. Proof of Theorem B.

4.1. Lemma F. Let \( \psi_i \) be a convergent sequence of unit mass distribution on \([0, \infty)\) which satisfies

\[
(4.1.1) \quad m^i_k \leq M_k,
\]

where \( M_k \) is a constant independent of \( i \). The limit distribution \( \psi \) will be of class \( D \) and

\[
(4.1.2) \quad \lim_{i \to \infty} m^i_k = m_k.
\]

Proof. We first show that for any \( \varepsilon > 0 \) there is a positive number \( A \) such that

\[
(4.1.3) \quad \int_R^S x^k d\psi_i \leq \varepsilon
\]

for any \( R, S \) which satisfy \( A < R < S \). This follows from the inequality
(4.1.4) \[ \int_R^S x^k d\psi_i \leq \frac{1}{A} \int_R^S x^{k+1} d\psi_i \leq \frac{M_{k+1}}{A}. \]

Hence
(4.1.5) \[ \int_R^S x^k d\psi < \varepsilon \text{ for } A < R < S, \]

and therefore \( \psi \) has moments of all orders. We next show that for any \( \varepsilon > 0 \) there is an \( i \), such that
(4.1.6) \[ \left| \int_0^\infty x^k d\psi_i - \int_0^\infty x^k d\psi \right| < \varepsilon \]

for \( i > i_1 \). Choose \( A \) so that
(4.1.7) \[ \int_0^\infty x^k d\psi_i \leq \frac{\varepsilon}{3}, \quad \int_A^\infty x^k d\psi \leq \frac{\varepsilon}{3}. \]

The left side of (4.1.6) is less than
(4.1.8) \[ \left| \int_0^A x^k d\psi_i - \int_0^A x^k d\psi \right| + \int_0^\infty x^k d\psi_i + \int_0^\infty x^k d\psi. \]

There is an \( i_1 \), such that for \( i > i_1 \), the first term is less than \( \varepsilon/3 \) and the proof of (4.1.6) is complete. In particular, the case \( k = 0 \) shows that \( \psi \) is a unit mass distribution, and since it has moments of all orders it is of class \( D \).

4.2. Proof of Theorem B. We now consider the sequence of mass distributions \( \psi_k \) of Theorem C. By (3.2.2), and (3.7.6) applied to arbitrary elements of \( D \), and the inequality \( \| \psi_1 - \psi_2 \|_r \leq \| \psi_1 - \psi_2 \|_{r+1} \), where \( r \) is an arbitrary integer and \( \psi_1, \psi_2 \) are arbitrary elements of \( D \), we find that
(4.2.1) \[ \| \psi_{k+p} - \psi_k \|_r \leq \frac{\varepsilon_k}{2} + \cdots + \frac{\varepsilon_k}{2^k} < \varepsilon_k. \]

In particular, this means that
(4.2.2) \[ m_q^{k+p} \leq m_q^k + \varepsilon_k, \quad q \leq r_k, \quad p \geq 1, \]

so that
(4.2.3) \[ m_q^i \leq \max\{m_q^1, \ldots, m_q^{k-1}, m_q^k + \varepsilon_k\}. \]

The right-hand side is a constant independent of \( i \), and there is an inequality for every value of \( q \) since \( r_k \) tends to infinity. Let \( \psi_k' \) be a convergent subsequence of the \( \psi_k \), which exists by Helly's theorem, and let \( \psi^* \) be the limit. Because of Lemma F we have
(4.2.4) \[ \| \psi^* - \psi_k \|_{r_k} < \varepsilon_k. \]

We now show that \( \psi^* \) satisfies the conditions of Theorem B. Indeed, (4.2.4) the same as (2.5.2), \( \psi^*(O_j) = y_j > 0 \) by (3.10.2), and since by §3.1, \( u_j \) tends to infinity, (2.5.1) is satisfied. Thus the proof of the theorem is complete.
5. Discussion of results.

5.1. Faber polynomials. The polynomials \( F_n(z|\psi) \) are the Faber polynomials of the series on the right-hand side of

\[
z \exp \left( - \sum_{k=1}^{\infty} \frac{m_k}{kz^k} \right) = z + a_0 + \frac{a_1}{z} + \cdots,
\]

which is obtained from the expression on the left-hand side by formal expansion. That is, they are the polynomial part of the formal \( n \)th power of the right-hand side of (5.1.1). In [1] a general theorem is given for the location of the zeros of Faber polynomials in the case

\[
\limsup |a_n|^{1/n} < \infty.
\]

If \( \psi \) is a \( T_1 \) distribution this condition is not satisfied, so that Theorem A can be interpreted as a result on the zeros of Faber polynomials for the case

\[
\limsup |a_n|^{1/n} = \infty.
\]

This example may suggest the proper formulation of a general theorem for zeros of Faber polynomials for the case (5.1.3).

5.2. Tchebycheff quadrature. Wilf in [2] raised the question whether \( T_1 \) distributions exist, which we have answered affirmatively. He has shown that if a \( T_1 \) distribution exists, then there must be large gaps in its \( T \) set. The question arises whether a gap condition can be devised which will discriminate between sequences of integers which are \( T \) sets of a \( T_1 \) distribution, and those sequences which are not.

5.3. The construction of §3 yields a \( T_1 \) distribution. There are several places where the construction can be generalized. The family of sets \( O \) can be on the entire real axis, the \( u_i \) need not be ordered, and we need only that \( \limsup |u_i| = \infty \).

The operation \( M(k,n) \) can be generalized. It takes mass from \( O_k \) to \( O_{k+1} \). Actually, quantities of mass can be taken from \( O_1, \cdots, O_k \) to \( O_{k+1}, \cdots, O_{k+p} \) but with a parameter which admits a convergence property similar to Lemma B. Likewise the operation \( S(k,\delta) \) can be generalized by modifying (3.5.4).

The carrying out of some such generalization would be justified if it could be shown that all \( T_1 \) distributions could be obtained by the new construction.

Bibliography


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