ON GENERATORS OF THE BANACH ALGEBRAS l_1 AND $L_1(0, \infty)$(1)

BY

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In Memory of the "Math Table" at City College 1945–1949.

Introduction. By l_1 we mean the Banach algebra of complex sequences $\{a_n\}$, $n = 0, 1, 2, \cdots$ for which $\sum_{n=0}^{\infty} |a_n| < \infty$, where multiplication is defined by convolution: the product of $\{a_n\}$ and $\{b_n\}$ is $\{c_n\}$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. The norm of $\{a_n\}$ is $\sum_{n=0}^{\infty} |a_n|$. The *Fourier transform* of $\{a_n\}$ is

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

the series converging for $|z| \leqq 1$. The algebra of functions (1) is denoted by A, and we write

$$\|f\|_{l_1} = \sum_{n=0}^{\infty} |a_n|, \tag{2}$$

and

$$\|f\|_{\infty} = \max_{|z| \leqq 1} |f(z)|, \tag{3}$$

the latter being the Gelfand norm in A.

The closure of A in the norm (3) is the algebra C of functions continuous in $|z| \leqq 1$ and analytic in $|z| < 1$, endowed with the norm (3).

The problem studied in the present paper is: given a sequence $\{a_n\} \in l_1$, when is the closed algebra which it generates all of l_1? Or equivalently, when are polynomials in $f(z)$, without constant term, dense in A in the l_1 norm?

Clearly, a necessary condition is that $f(z) \neq 0$ in $|z| \leqq 1$. In order to avoid trivial inconveniences arising from this condition, we therefore reword the problem: *For which $f(z)$ is the algebra generated by* 1 *and f dense in A, in the l_1 norm*? When this is the case, we say for brevity "f is a generator." A necessary condition is clearly that $f(z)$ separate points in $|z| \leqq 1$ since $\|f\|_{\infty} \leqq \|f\|_{l_1}$ and convergence in the l_1 norm implies uniform convergence in $|z| \leqq 1$. It is natural to inquire whether this condition is also sufficient. From a more abstract point of view the problem may be regarded as follows (as remarked to the authors by John Wermer): Certain Banach algebras have the property that every element

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which separates maximal ideals is a generator (in the above sense). In particular C is such an algebra (see §1). Is l_1 such an algebra?

We have not been able to decide this question, but obtain partial results in the present paper. Namely, if $f(z) \in A$ and separates points in $|z| \leqq 1$, then f maps $|z| < 1$ conformally on a certain Jordan domain Ω. *In the case that this domain has a rectifiable boundary*, we are able to give a necessary and sufficient condition that f be a generator. Although the condition obtained (see Theorem 1) is not easy to apply, we are able to deduce from it that when the domain Ω satisfies a certain condition first studied by Smirnov, f is a generator. Since, by virtue of a theorem of Hardy and Littlewood (see Zygmund [18, p. 293]), rectifiability of the boundary implies that the mapping function is in A, we are thus able to associate a generator of l_1 with every Jordan domain whose boundary is rectifiable and which satisfies Smirnov's condition.

In §1 certain preliminary results are gathered. In §2 the main theorem is stated and proved. §3 contains some remarks concerning the Smirnov condition. In §4 there is a brief discussion of the analogous problem for the algebra $L_1(0,\infty)$ of absolutely integrable functions. This problem had been studied earlier by Shapiro and Silverman [14], where it arose in connection with a sampling problem in the theory of random processes (see [14, Theorems 1 and 2]). In §5 some examples of generators are given; in particular, an apparently new completeness theorem involving Bessel functions is deduced from our general theory.

1. Preliminary material.

1.1. THEOREM A (CARATHÉODORY-WALSH). *Let Ω be a Jordan domain, and $F(w)$ continuous in the closure $\overline{\Omega}$ of Ω and analytic in Ω. Then $F(w)$ can be uniformly approximated in $\overline{\Omega}$ arbitrarily closely by polynomials.*

For a proof see Walsh [15].

COROLLARY. *If $f(z) \in C$ (see introduction) and is univalent (separates points) for $|z| \leqq 1$, f is a generator of C.*

Indeed, by the correspondence $w = f(z)$, $F(w) = F(f(z))$ Theorem A and the corollary say the same thing, in the w and in the z plane, respectively.

1.2. Following Beurling [3], we say that a function $f(z)$, analytic and of bounded characteristic in $|z| < 1$ (and so having radial boundary values $f(e^{i\theta})$ for almost all θ, with $\log |f(e^{i\theta})| \in L_1$) is *outer* if $\log |f(re^{i\theta})|$ is, for all $0 \leqq r < 1$, equal to the Poisson integral of $\log |f(e^{i\theta})|$; $f(z)$ is *inner* if $|f(e^{i\theta})| = 1$ a.e. and a *normalized* inner function if, moreover, the first nonvanishing Taylor coefficient is real and positive. Every function analytic in $|z| < 1$ and of bounded characteristic is uniquely representable as a product $I(z)O(z)$ of a normalized inner, and outer, function. If $f(z)$ is moreover of class H_p for some $0 < p \leqq \infty$, the same is true of I and O, and hence $|I(z)| \leqq 1$ for $|z| < 1$ in this case. Every

inner function may be written as a product λBS where $\lambda, |\lambda| = 1$, is a constant, B a Blaschke product, and

$$S(z) = \exp\left\{\frac{-1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)\right\}, \tag{1}$$

where $\mu(t)$ is a real measure singular with respect to Lebesgue measure. μ is called the *representing measure* of S. If $f \in H_p$ for some $0 < p \leqq \infty$, $d\mu$ is non-negative. For further details, the reader may consult Privalov [10] or Nevanlinna [8], where the terms "inner" and "outer" are not used. Beurling proved [3] that for $f \in H_2$ the multiples of $f(z)$ by polynomials are dense in H_2, in the H_2 norm, if and only if $f(z)$ is outer. More precisely, the norm closure of the functions $P(z)f(z)$, $P(z)$ polynomials, is the set IH_2 of all multiples of $I(z)$, the inner factor of $f(z)$, by functions of class H_2. An analogous theorem for the class H_1 was proved by de Leeuw and Rudin [4], which we state here for reference:

THEOREM B. *If $f(z) \in H_1$, the closure in H_1 norm of the multiples of $f(z)$ by polynomials is precisely the set IH_1 of all multiples of I by functions of class H_1.*

Given an inner function $I(z)$, IH_2 is a closed subspace of H_2, proper if $I \not\equiv$ constant. In what follows, the space $(IH_2)^{\perp}$, the orthogonal complement of IH_2 in H_2, plays an important role. Such subspaces are characterized by being closed, and invariant relative to the operator T: $Tf = (f(z) - f(0))/z$, adjoint to multiplication by z.

1.3. Let $f(z)$ be regular and univalent in $|z| < 1$ and map $|z| < 1$ on a Jordan domain Ω with rectifiable boundary Γ. We shall say Ω is a *Smirnov domain* if $f'(z)$ is outer (notice that the rectifiability implies $f' \in H_1$, see e.g. Zygmund [18, p. 285], so that the designation outer is meaningful). As is shown in Privalov [10, p. 160], this is a property only of the domain Ω and does not depend on the choice of the particular mapping function. Keldysch and Lavrentiev have shown how to construct domains for which $f'(z)$ is a nonconstant inner function. This construction is given in Privalov [10, pp. 166 ff.]. It is a very difficult construction, and so far as is known to the authors no other example of a non-Smirnov domain has been given. A number of sufficient conditions that a domain be Smirnov are given in Privalov [10, pp. 181 ff.], but understanding of the Smirnov condition, is at present far from complete. A new sufficient condition is given in §3 of the present paper.

1.4. In this paragraph we prove a lemma of D. J. Newman, submitted at one time as a problem to the *American Mathematical Monthly*.

LEMMA. *Let $\{C_n\}$, $n = 0, 1, \cdots$ be a sequence of complex numbers, and $C_n = O(1/n)$. Then there is a bounded measurable function $F(\theta)$ on $[0, 2\pi]$ with*

$$\frac{1}{2\pi}\int_0^{2\pi} F(\theta)e^{in\theta}d\theta = C_n, \qquad n = 0, 1, 2 \cdots.$$

Proof. By a theorem of F. Riesz, the necessary and sufficient condition that the moment problem in question be solvable, with ess sup $|F(\theta)| \leqq M$, is that for every finite sequence of complex numbers $\{\lambda_n\}$, $n = 0, 1, \cdots, r$ we have $|\sum_0^r \lambda_n C_n| \leqq M \| \sum_0^r \lambda_n e^{in\theta} \|_1$, the subscript denoting L_1 norm. (See e.g. Banach [1, p. 75]. The extension to complex-valued functions is evident.) But, by an inequality of Hardy and Littlewood (see Zygmund [18, p. 286]), $\| \sum_0^r \lambda_n e^{in\theta} \|_1 \geqq (1/\pi) \sum_0^r |\lambda_n|/(n+1)$ and the result follows.

2. The main theorem.

THEOREM 1. *Let $f(z) \in A$ and be univalent in $|z| \leqq 1$, and suppose that the Jordan domain Ω on which f maps $|z| < 1$ has rectifiable boundary. Let $I(z)$ denote the normalized inner factor of $f'(z)$. Then the algebra generated by 1 and f is dense in A (in l_1 norm) if, and only if, $(IH_2)^\perp$ contains no non-null function whose Taylor coefficients are $O(1/n)$.*

COROLLARY. *If $I(z) = 1$ (i.e., Ω is a Smirnov domain), f is a generator of A.*

Proof. Let $f(z) = \sum_{n=0}^\infty a_n z^n$, $[f(z)]^k = \sum_{n=0}^\infty a_n^{(k)} z^n$ $(a_n^{(1)} = a_n)$. Then, the algebra generated by $1, f$ is nondense if and only if there exist a non-null bounded sequence $\{c_n\}$, $n = 0, 1, \cdots$ with

$$c_0 = 0, \tag{1}$$

$$\sum_{n=1}^\infty c_n a_n^{(k)} = 0, \qquad k = 1, 2, \cdots. \tag{2}$$

This last equation may be written

$$\sum_{n=1}^\infty \frac{c_n}{n} \cdot n a_n^{(k)} = 0, \qquad k = 1, 2, \cdots. \tag{3}$$

By the lemma of 1.4, there is a bounded measurable function $F(\theta)$, whose Fourier series is

$$F(\theta) \sim \sum_{-\infty}^{\infty} C_n e^{in\theta}, \tag{4}$$

where

$$C_{-n} = \frac{c_n}{n}, \qquad n = 1, 2, \cdots. \tag{5}$$

Moreover, $ke^{i\theta}[f(e^{i\theta})]^{k-1} f'(e^{i\theta})$ is of class L_1 and has the Fourier series $\sum_1^\infty n a_n^{(k)} e^{in\theta}$. Hence by Parseval's theorem (for the version needed here see Zygmund [18, p.158]), we get from (3):

$$\int_0^{2\pi} F(\theta)[f(e^{i\theta})]^k f'(e^{i\theta}) e^{i\theta} d\theta = 0, \qquad k = 0, 1, 2, \cdots. \tag{6}$$

Hence, for any polynomial $P(w)$:

$$\int_0^{2\pi} F(\theta)P[f(e^{i\theta})]f'(e^{i\theta})e^{i\theta}d\theta = 0. \tag{7}$$

By Theorem A, the $P[f(z)]$ are dense in C, hence

$$\int_0^{2\pi} F(\theta)g(e^{i\theta})f'(e^{i\theta})e^{i\theta}d\theta = 0, \qquad \text{all } g \in C. \tag{8}$$

By Theorem B, the functions $f'(e^{i\theta})g(e^{i\theta})$ span in L_1 norm all $I(e^{i\theta})h(e^{i\theta})$; here I is the inner factor of f', and $h \in H_1$. Passing to the limit in (8) gives

$$\int_0^{2\pi} F(\theta)I(e^{i\theta})h(e^{i\theta})e^{i\theta}d\theta = 0, \qquad \text{all } h \in H_1. \tag{9}$$

We have thus shown that if f fails to be a generator of A, a bounded measurable function $F(\theta)$ exists satisfying (9), and

$$C_{-n} = O\left(\frac{1}{n}\right), \qquad n = 1, 2, \cdots, \tag{10}$$

$$C_{-n} \text{ are not all } 0, \qquad n = 1, 2, \cdots. \tag{11}$$

Conversely, the existence of such an F implies that f is not a generator, since in (9) we may choose $h = f^k O$, where O is the outer factor of $f'(z)$, giving (6), which implies (2). We can now easily complete the proof of Theorem 1. Suppose first f is a nongenerator, and let

$$F(\theta) = e^{-i\theta}\,\overline{F_1(e^{i\theta})} + F_2(e^{i\theta}), \tag{12}$$

where $F_1(z) = \sum_{n=0}^{\infty} \bar{C}_{-n-1}z^n$ and $F_2(z) = \sum_{n=0}^{\infty} C_n z^n$ are in H_2. Since (9) is true *a fortiori* for $h \in H_2$, we obtain, on substituting (12) for $F(\theta)$, that F_1 is orthogonal to IH_2, and has Taylor coefficients $O(1/n)$, not all zero. On the other hand, suppose there is a $G(z) = \sum_{n=0}^{\infty} b_n z^n$, $G \in (IH_2)^{\perp}$, $G \not\equiv 0$, $b_n = O(1/n)$. Then

$$\int_0^{2\pi} \overline{G(e^{i\theta})}\, I(e^{i\theta})h(e^{i\theta})d\theta = 0, \qquad \text{all } h \in H_2. \tag{13}$$

Again, by the lemma of (1.4) there is a bounded measurable $F(\theta) \sim \sum_{-\infty}^{\infty} C_n e^{in\theta}$ with $C_{-n} = \bar{b}_{n-1}$, $n = 1, 2, \cdots$. Hence (13) holds when $\overline{G(e^{i\theta})}$ is replaced by $e^{i\theta}F(\theta)$. This gives (9), initially only for $h \in H_2$ but, since F is bounded, also for $h \in H_1$. Theorem 1 is completely proved.

REMARK. If one seeks only the corollary, it can be obtained more quickly by means of the inequality $\|f\|_{l_1} < \pi \|f'\|_1$. Compare §4.

3. More on the Smirnov condition.

3.1. Theorem 1 is far from giving a satisfactory solution to the problem of generators of A, because of the restriction to rectifiable boundaries, and the difficulty even in that case of applying the criterion of Theorem 1. The authors know no example, on the one hand of any $f \in A$ which is univalent in $|z| \leqq 1$ and not a generator, nor on the other hand of a generator whose derivative has a nonconstant inner factor. To understand the rectifiable case more closely, requires deeper knowledge of the inner factors of derivatives of schlicht functions, and of subspaces of the type $(IH_2)^{\perp}$. In this section we prove a theorem concerning these inner factors, which has consequences for the theory of Smirnov domains.

LEMMA 1. *Let $f(z)$ map $|z| < 1$ one-to-one and conformally on a Jordan domain with rectifiable boundary. Let $I(z)$ denote the normalized inner factor of $f'(z)$ and let $g(z) \in (IH_2)^{\perp}$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$. If there exists a function $F(\theta)$ of period 2π and bounded variation, $F(\theta) \sim \sum_{-\infty}^{\infty} C_n e^{in\theta}$, with $C_n = b_n$ for all $n \geqq 0$, then $g \equiv 0$.*

Proof. $\int_0^{2\pi} \overline{g(e^{i\theta})} I(e^{i\theta}) h(e^{i\theta}) d\theta = 0$ for all $h \in H_2$. In this equation we may replace $g(e^{i\theta})$ by $F(\theta)$, since $F(\theta) - g(e^{i\theta}) = \overline{G(e^{i\theta})}$, where $G(z) \in H_2$, $G(0) = 0$. Hence

$$\int_0^{2\pi} \overline{F(\theta)} I(e^{i\theta}) h(e^{i\theta}) d\theta = 0, \qquad \text{all } h \in H_2. \tag{1}$$

Since F is bounded (1) holds for all $h \in H_1$, hence

$$\int_0^{2\pi} \overline{F(\theta) e^{i\theta}} f'(e^{i\theta}) [f(e^{i\theta})]^{k-1} e^{i\theta} d\theta = 0, \qquad k = 1, 2, \cdots. \tag{2}$$

Let $F_1(\theta) = \overline{F(\theta)} e^{-i\theta}$ and integrate by parts. We may assume without loss of generality that $0 \equiv 2\pi$ is a point of continuity of $F(\theta)$, since otherwise we could replace the interval of integration by $(\theta_0, \theta_0 + 2\pi)$ where $\theta_0 \equiv \theta_0 + 2\pi$ is a point of continuity.

$$\int_0^{2\pi} [f(e^{i\theta})]^k dF_1(\theta) = 0, \qquad k = 1, 2, \cdots. \tag{3}$$

Since this holds also for $k = 0$, we have by Theorem A:

$$\int_0^{2\pi} e^{ik\theta} dF_1(\theta) = 0, \qquad k = 0, 1, \cdots; \tag{4}$$

hence

$$b_n = \int_0^{2\pi} F(\theta) e^{-ik\theta} d\theta = 0, \qquad k = 0, 1, \cdots \tag{5}$$

as asserted.

In passing we observe that great delicacy would be needed in constructing a counter-example (a nongenerator) by the criterion of Theorem 1: one must produce in $(IH_2)^{\perp}$ a non-null function whose Taylor coefficients are $O(1/n)$, yet do not coincide with the positively-indexed Fourier coefficients of any function of bounded variation (hence in particular they cannot be $O(1/n^p)$ with $p > 3/2$), by virtue of Lemma 1(2).

THEOREM 2. *Let $f(z)$ map $|z| < 1$ one-to-one and conformally on a Jordan domain with rectifiable boundary. Let $I(z)$ denote the normalized inner factor of $f'(z)$, and ρ its representing measure. Then $\rho(E) = 0$ when E consists of a single point, i.e., ρ has no "mass points"*(3).

Proof. Suppose the contrary. Then (taking the mass point at $z = 1$):

$$I(z) = I_0(z) \exp\left[\frac{-a(1+z)}{1-z}\right] \tag{6}$$

where $a > 0$, and $I_0(z)$ is inner and bounded. Writing

$$I_1(z) = \exp\left[\frac{-a(1+z)}{1-z}\right],$$

and

$$\frac{1}{z}\left(1 - \frac{1}{z}\right)^2 I_1(z) = g(z) + h\left(\frac{1}{z}\right) \tag{7}$$

where $g(z) \in H_2$, $h(z) \in H_2$ and $h(0) = 0$ (h is a polynomial of degree 3), we obtain from (6)

$$\int_0^{2\pi} g(e^{i\theta})\overline{I(e^{i\theta})}\,\overline{f(e^{i\theta})}\,d\theta = 0, \qquad \text{all } f \in H_2. \tag{8}$$

Hence $g(e^{i\theta}) \in (IH_2)^{\perp}$. Moreover, since $I_1(e^{i\theta}) = \exp[-ia\cot(\theta/2)]$, the function $F(\theta) = e^{-i\theta}(1 - e^{-i\theta})^2 I_1(e^{i\theta})$ is of bounded variation, implying $g \equiv 0$ by Lemma 1. This is a contradiction, and Theorem 2 is proved.

LEMMA 2. *Let $I(z)$ be a nonconstant bounded inner function without zeroes, ρ its representing measure, and E the carrier of ρ (so E is a nonempty closed subset of $0 \leqq \theta < 2\pi$). In every open subinterval of $0 \leqq \theta < 2\pi$ whose intersection with E is not empty, there is a θ for which $\lim_{r\to 1-0} I(re^{i\theta}) = 0$.*

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(2) By another line of argument based on the Dirichlet integral, A. L. Shields has shown that this orthocomplement space contains no non-null function with $\Sigma_1^{\infty} n\,|\,b_n\,|^2 < \infty$; in particular $b_n = O(n \log n)^{-1}$ is impossible.

(3) A stronger theorem will be proved, by another method in §3.2 below.

REMARK. This lemma is a slight extension of a result of Seidel (see Noshiro [9]) and occurs also in Rudin [11].

Proof. It is convenient here to work with a function of bounded variation, rather than a measure. Let $u(\theta)$ be the ρ-measure of the closed interval $[0, \theta]$ $(0 \leqq \theta < 2\pi)$. Then $u(\theta)$ is nondecreasing and $u'(\theta) = 0$ outside of E. Hence, in every open interval intersecting E there is a point θ_0 with $u'(\theta_0) = +\infty$ (see Saks [12, p. 128]). Now, $-\log I(re^{i\theta})$ is the Poisson-Stieltjes integral of $du(\theta)$, hence (Bari [2, p. 163]) tends to $+\infty$ as $r \to 1-0$, for $\theta = \theta_0$. This proves Lemma 2.

LEMMA 3. *Let $f(z) \in H_p$, $p > 0$ be nonvanishing in $|z| < 1$, and $f = IO$ its canonical factorisation. Suppose that for $\theta_1 < \theta < \theta_2$:* (i) *$f(e^{i\theta})$ coincides a.e. with a continuous function* (ii) $\liminf_{r\to 1} |f(re^{i\theta})| \geqq a$ *where a is a positive constant. Then, if ρ is the representing measure of I, the ρ-measure of the interval $\theta_1 < \theta < \theta_2$ is zero.*

Proof. Since $|f(e^{i\theta})| \leqq M$ a.e. for $\theta_1 < \theta < \theta_2$ (M a positive constant), $\log |O(re^{i\theta})|$, which is the Poisson integral of $\log |f(e^{i\theta})|$, is bounded above uniformly for $\theta_1' \leqq \theta \leqq \theta_2'$, $r \geqq 1 - \varepsilon$ where $\theta_1 < \theta_1' < \theta_2 < \theta_2$ and $\varepsilon > 0$ is sufficiently small. Thus $|O(re^{i\theta})|$ is bounded for $\theta_1' \leqq \theta \leqq \theta_2$, $r \geqq 1 - \varepsilon$ and combining this with (ii) gives $\liminf_{r\to 1} |I(re^{i\theta})| \geqq a'$ for $\theta_1' \leqq \theta \leqq \theta_2'$. By Lemma 2, the result follows.

For the statement of the next theorem, we make a definition: a rectifiable Jordan arc represented parametrically by $w = w(s)$ where s is arc length and w complex, is *of class* $C^{1+\alpha}$ (where $\alpha > 0$) if $w'(s)$ exists and $\in \operatorname{Lip} \alpha$. Kellogg proved (see Warschawski [16]):

LEMMA 4. *If $f(z)$ maps $|z| < 1$ one-to-one and conformally on a Jordan domain, and an arc $\gamma: \theta_1 \leqq \theta \leqq \theta_2$ of $|z| = 1$ is mapped on an arc of class $C^{1+\alpha}$, then $f'(z)$ can be extended to be continuous and nonvanishing for $0 \leqq r \leqq 1$, $\theta_1' \leqq \theta \leqq \theta_2'$ for any $\theta_1 < \theta_1' < \theta_2' < \theta_2$.*

(Kellogg proved also that $f'(e^{i\theta}) \in \operatorname{Lip} \alpha$ on the interval $[\theta_1', \theta_2']$, but we do not require this fact.)

THEOREM 3. *Let Ω be a Jordan domain with rectifiable boundary Γ. Suppose that every point w of Γ, except for an enumerable set, is interior to an arc of Γ which is of class $C^{1+\alpha}$ for some $\alpha > 0$ (which may depend on the point w). Then Ω is a Smirnov domain.*

Proof. By Lemma 4 and Lemma 3, the representing measure of the inner factor of $f'(z)$ is concentrated on an at most enumerable set. Hence the measure contains a mass point, or else is the zero-measure; and the former alternative is excluded by Theorem 2, proving Theorem 3.

Theorem 3 generalizes several of the known sufficient conditions for a Smirnov domain. It enables one to construct a Smirnov domain whose boundary contains spirals.

3.2. Theorem 2 bis admits a generalization. To state this, let us make a definition: A subset E of $0 \leqq \theta < 2\pi$ is said to be *thin* if for every $\varepsilon > 0$, it can be covered with a sequence of intervals $\{J_n\}$, length $J_n = \delta_n$, such that $\sum \delta_n \log 1/\delta_n < \varepsilon$. Note that a thin set is of Lebesgue measure zero, and a set of Hausdorff dimension less than one is thin. We also mention by way of orientation that a symmetric perfect set with variable ratio of dissection $\{r_n\}$ is thin if and only if $\lim_{n\to\infty} n \prod_{i=1}^{n} (1 - r_i) = 0$.

THEOREM 2 BIS. *Under the hypotheses of Theorem* 2, $\rho(E)=0$ *when* E *is any thin set.*

We give a proof based on an idea suggested to the authors by A. L. Shields. By a well-known distortion theorem for schlicht functions (see, e.g., [8, p. 93]),

$$|f'(z)| \geqq |f'(0)|(1 - |z|), \qquad \text{for } |z| < 1. \tag{9}$$

Since moreover $f'(z)$ is of class H_1 so is its outer factor $O(z)$, hence $|O(z)| \leqq A/(1 - r)$, $r = |z| < 1$, for some constant A, hence $I(z)$, the normalized inner factor of $f'(z)$, satisfies

$$|I(re^{i\theta})| \geqq c(1 - r)^2, \qquad c > 0 \text{ constant} \tag{10}$$

$$\log I(re^{i\theta}) = \frac{-1}{2\pi} \int_0^{2\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} d\rho(t). \tag{11}$$

From (10) and (11) we have (A_1, A_2 denote positive constants):

$$\int_0^{2\pi} \frac{1 - r^2}{|e^{it} - re^{i\theta}|^2} d\rho(t) < 2 \log \frac{1}{1 - r} + A_1 . \tag{12}$$

Let F denote a closed t-interval of length $\delta < 1$ centered at $t = \theta$; choosing $r = 1 - \delta$ and replacing the integral in (12) by the integral over F we get by trivial estimates

$$\rho(F) < 4\delta \log \frac{1}{\delta} + A_2\delta. \tag{13}$$

Since θ is arbitrary, (13) holds for every interval F of length $\delta < 1$ and this evidently implies Theorem 2 bis (4).

It will be noted that the hypotheses of the theorem are only used in a rather weak way, which suggests that sharper statements concerning the measure ρ may

(4) Another way of stating inequality (13) is that $u(\theta)$, the ρ-measure of the closed interval $[0,\theta]$, has a modulus of continuity $\omega(\delta)$ which is $O(\delta \log 1/\delta)$.

be possible. Thus it is perhaps of some interest that the method used to prove Theorem 2 pursued to its ultimate generality leads also the concept of "thin" sets. The two methods use the schlichtness of $f(z)$ in very different ways, namely the property expressed in Theorem A (corollary), and the distortion theorem, respectively. We prove now a lemma which shows that a certain type of closed set studied earlier by Beurling and by Carleson in connection with sets of uniqueness for analytic functions (see [4, p. 326]) is "thin" in the above sense.

LEMMA 5. *Let E be a closed set of Lebesgue measure zero whose complement consists of open intervals $B_1, B_2, \cdots$ length $B_n = \beta_n$ and*

$$\sum \beta_n \log \frac{1}{\beta_n} < \infty; \tag{14}$$

then E is thin.

The converse statement is not true since there are enumerable sets E for which (14) does not hold.

Proof of Lemma 5. We may assume $\beta_1 \geqq \beta_2 \geqq \cdots$. The complement of the union of the intervals $B_1, \cdots, B_n$ consists (on the circle) of n closed intervals $J_1, \cdots, J_n$ (some of which may degenerate to points), of lengths $\delta_1, \cdots, \delta_n$ say, which cover E. Now,

$$\sum_{i=1}^{n} \delta_i = 2\pi - \sum_{i=1}^{n} \beta_i = \sum_{n+1}^{\infty} \beta_i \tag{15}$$

since E has measure zero. Hence

$$\sum_{i=1}^{n} \delta_i \leqq \frac{\varepsilon_n}{\log \dfrac{1}{\beta_{n+1}}}, \tag{16}$$

where

$$\varepsilon_n = \sum_{i=n+1}^{\infty} \beta_i \log \frac{1}{\beta_i}, \qquad \lim \varepsilon_n = 0. \tag{17}$$

Moreover, since β_n are decreasing and $\sum \beta_n < \infty$, $\beta_n = o(1/n)$, $\log(1/\beta_{n+1}) > c \log n$, and (16) gives

$$\sum_{i=1}^{n} \delta_i \leqq \frac{A\varepsilon_n}{\log n}. \tag{18}$$

(Here c, A are positive constants, as also B below.) Now, the function $\phi(t) = t \log(1/t)$ is concave, hence

$$\sum_{i=1}^{n} \phi(\delta_i) \leqq n\phi\left(\frac{\sum_{i=1}^{n} \delta_i}{n}\right) \leqq n\phi\left(\frac{A\varepsilon_n}{n \log n}\right) \tag{19}$$

for sufficiently large n since ϕ is increasing for small t. Hence

$$\sum_{i=1}^{n} \phi(\delta_i) \leqq \frac{A\varepsilon_n}{\log n} \log\left(\frac{n \log n}{A\varepsilon_n}\right)$$

$$< \frac{A\varepsilon_n}{\log n}\left(B \log n + \log\frac{1}{\varepsilon_n}\right) \to 0 \text{ as } n \to \infty,$$

and the lemma is proved.

We sketch a proof of Theorem 2 bis by the method of Theorem 2 (for the special class of thin sets described in Lemma 5). One introduces the function $\Delta(t)$ defined by: $\Delta(t) = 0$ for $t \in E$, and for $t \notin E$, $\Delta(t)$ is the product of the distances from t to the ends of the interval B_n containing it.

It follows from (14) that $\log \Delta(t)$ is integrable and one can construct a function $h(z)$ analytic and bounded in $|z| < 1$ with $|h(e^{it})| = \Delta(t)$, by use of a Poisson integral (see [4, p. 328] where this construction is used for another purpose). One can then show that, if the representing measure ρ of the normalized inner function $I(z)$ satisfies $\rho(E) > 0$, i.e., if $I(z)$ has a divisor $I_1(z)$ whose representing measure is concentrated on E, then for suitably large positive p the function $e^{-i\theta}\overline{h(e^{i\theta})}^{p} I_1(e^{i\theta})$ is of bounded variation, and since its projection into H_2 belongs to $(IH_2)^{\perp}$, one has a contradiction of Lemma 1. We shall not, however, carry out the details.

Finally, we remark that it might be possible to generalize Theorem 3 on the basis of Theorem 2 bis, i.e., replace "enumerable set" by one or another type of metrically thin set on Γ. For such an extension one must determine which sets on Γ correspond to thin sets (in the sense of Theorem 2 bis) on the circle, under conformal transformation.

3.3. We mention in passing that the Smirnov condition on Ω (which, as we have seen, *implies* that polynomials in f span A in l_1 norm) is *equivalent* to the statement: polynomials in f span $A \cap V$ in the norm $\|f\|_V = \text{Variation } f(e^{i\theta})$. (Here V denotes functions of finite total variation; clearly $A \cap V$ is a subalgebra of A.) For this is equivalent to saying that the functions $P[f(e^{i\theta})]f'(e^{i\theta})$ (P polynomial) span the boundary functions of H_1, in L_1 norm. By Theorem B, this occurs if and only if f' is outer.

4. Generators of $L_1(0, \infty)$.

4.1. Let us denote by L_1 the Banach algebra of complex-valued functions $f(t)$ of class $L_1(0, \infty)$, multiplication being defined by convolution: $f * g = h$, where

$$h(t) = \int_0^t f(t-u)g(u)du = \int_0^t f(u)g(t-u)du, \tag{1}$$

and

$$F(z) = \int_0^\infty f(t)e^{itz}dt, \qquad \operatorname{Im} z \geqq 0 \tag{2}$$

is the Fourier transform of $f(t)$. We will use the letters f, g, h to denote elements of L_1 and F, G, H for the corresponding Fourier transforms. We denote by π the half-plane $y > 0$, and by $\bar{\pi}$ the half-plane $y \geqq 0$, plus the boundary point ∞ (here $z = x + iy$).

As is to be expected, Theorem 1 goes over, with obvious modifications, to the algebra L_1. (The function $F(z)$ now separates points on $\bar{\pi}$, which implies that $F(z)$ has no finite zero in $\bar{\pi}$ since it must vanish at ∞.) To prove the L_1 version of Theorem 1 one has only to adapt the old proof step by step for the present context. The necessary theory of functions of bounded characteristic, and class H_p, for the half-plane may be taken from the paper of V. I. Krylov [7], the F. and M. Riesz theorem from Essen [6], and the modification of Theorem B for the half-plane is routine. We do not propose, however, to carry out these details here, as they lead to nothing new from the Banach algebra standpoint.

On the other hand, from the point of view of classical analysis the L_1 case has somewhat greater interest than the circle case. We have already mentioned the application to sampling theory; moreover L_1 convolutions may be calculated explicitly in many cases and the present theory then yields concrete completeness theorems.

We therefore prove in this section the analog of the *corollary* to Theorem 1, which suffices for applications, and requires a minimum of preliminary material. This is Theorem 4 below. In Theorem 5 we give a result for functions of class $L_1 \cup L_2$. The results of this section, and of the remainder of the paper, were given in a more extended form by Schwartz and Shapiro in the report [13].

THEOREM 4. *Let Ω be a Jordan domain in the w-plane with rectifiable boundary Γ, such that $w = 0$ lies on Γ. Suppose further that Ω is a Smirnov domain. Let $F(z)$ map π one-to-one and conformally on Ω so that $z = \infty$ goes into $w = 0$. Then*

(i) *$F(z)$ is the Fourier transform of some $f(t) \in L_1$,*

(ii) *The convolution powers $\{f^{*n}\}$ are complete in L_1.*

The proof will be preceded by a lemma. It should be noted that the Smirnov condition is required only for (ii).

In the present discussion we mean by H_p the class of functions $F(z)$ analytic for $y > 0$ and such that

$$\| F \|_p = \sup_{y>0} \left(\int_{-\infty}^{\infty} | F(x + iy) |^p dx \right)^{1/p} < \infty . \tag{3}$$

LEMMA. *Let $G \in H_1$. Then there exist functions G_1, G_2, of class H_2 such that*

$$G = G_1 G_2 . \tag{4}$$

$$\| G \|_1 = \| G_1 \|_2 = \| G_2 \|_2 . \tag{5}$$

Proof. The assumption $G \in H_1$ implies that the Blaschke product B formed with the zeroes of G converges (see, e.g., Krylov [7]), and that $G = BG_0$ where $G_0 \in H_1$, $\| G_0 \|_1 = \| G \|_1$. Now take $G_1 = B\sqrt{G_0}$, $G_2 = \sqrt{G_0}$.

Proof of Theorem 4. Let $\operatorname{Im} z_0 > 0$; then $\lim_{a\to\infty} \int_{-a}^{a} (F(x)/(x - z_0))dx$ exists, and equals $2\pi i F(z_0)$, as we see upon applying Cauchy's theorem to $\int_C (F(z)/(z - z_0))dz$, where C is the boundary, described counter-clockwise, of the square in the upper half-plane with base $[-a, a]$. The assumption that $F(z) \to 0$ as $|z| \to \infty$ guarantees that the integrals along the three sides of C other than $[-a, a]$ tend to zero as $a \to \infty$. In like manner,

$$\lim_{a\to\infty} \int_{-a}^{a} \frac{F(x)}{x - z_0} dx = 0,$$

and subtracting:

$$2\pi i F(z_0) = \lim_{z\to\infty} \int_{-a}^{a} \frac{F(x)(z_0 - \bar{z}_0)}{(x - z_0)(x - \bar{z}_0)} dx,$$

and we may now rewrite the last formula as a Lebesgue integral (wherein we also, for convenience of notation, replace z_0 by $z = x + iy$, and x by u):

$$F(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(u - x)^2 + y^2} \cdot F(u)du, \qquad y > 0. \tag{6}$$

Let us denote by

$$P = P(u; x, y) = \frac{1}{\pi} \frac{y}{(u - x)^2 + y^2} \tag{7}$$

the Poisson kernel appearing in (6). For fixed (x, y), $y > 0$, $P$is integrable in u, and $\int_{-\infty}^{\infty} P du = 1$. Differentiating (6) with respect to x gives

$$F'(z) = \frac{\partial F}{\partial x} = \int_{-\infty}^{\infty} \frac{\partial P}{\partial x} F(u)du = -\int_{-\infty}^{\infty} \frac{\partial P}{\partial u} F(u)du,$$

whence integrating by parts (the hypothesis on Γ implies that $F(x)$ is of bounded variation):

$$F'(z) = \int_{-\infty}^{\infty} P(u; x, y)dF(u), \qquad y > 0. \tag{8}$$

Hence, $|F'(x + iy)| \leqq \int_{-\infty}^{\infty} P(u; x, y) |dF(u)|$. Integrating, and interchanging the orders of integration, gives

$$\int_{-\infty}^{\infty} |F'(x + iy)| dx \leqq V(F) \tag{9}$$

where V denotes total variation. Thus $F' \in H_1$, and by the lemma:

$$F'(z) = F_1(z)F_2(z) \tag{10}$$

where F_1 and F_2 are of class H_2, and

$$\| F_1 \|_2 = \| F_2 \|_2 = \| F' \|_1 . \tag{11}$$

By a theorem of Paley and Wiener

$$F_j(z) = \int_0^\infty f_j(t) e^{itz}, \qquad y > 0; \qquad j = 1, 2 \tag{12}$$

where the $f_j \in L_2(0, \infty)$. Let $f_3(t) = \int_0^t f_1(s) f_2(t-s) ds$. Then $f_3(t)$ is bounded for $t > 0$. Finally, let

$$f(t) = \frac{f_3(t)}{it}, \qquad t > 0. \tag{13}$$

Then $f(t) \in L_1(0, \infty)$, and fulfills the requirement (i) of Theorem 4. For,

$$\begin{aligned} \int_0^\infty |f(t)|\, dt &\leqq \int_0^\infty \frac{dt}{t} \int_0^t |f_1(s)|\, |f_2(t-s)|\, ds \\ &= \int_0^\infty |f_1(s)|\, ds \int_0^\infty \frac{|f_2(t-s)|}{t} dt \\ &= \int_0^\infty \int_0^\infty \frac{|f_1(s)|\, |f_2(t)|}{s+t} \\ &< \pi \left\{ \int_0^\infty |f_1(s)|^2 ds \right\}^{1/2} \left\{ \int_0^\infty |f_2(s)|^2 ds \right\}^{1/2} = \pi \| F' \|_1 . \end{aligned} \tag{14}$$

(We have used the integral form of Hilbert's inequality.)

To complete the proof of Theorem 4, it is enough to show that linear combinations of the f^{*n} span all $g \in L_1$ whose Fourier transforms have derivatives of class H_1, and for this it suffices to show that $V[P(F) - G]$ can be made arbitrarily small for a suitable polynomial P without constant term. Let $\zeta = \zeta(z)$ map $y > 0$ on $|\zeta| < 1$, $F(z) = \phi(\zeta)$, $G(z) = \gamma(\zeta)$. Here $\gamma(e^{i\theta})$, like $G(x)$ is of bounded variation. The problem is thus reduced to making the H_1 (circle) norm of $Q[\phi(\zeta)]\phi'(\zeta) - \gamma'(\zeta)$ small, Q polynomial; and this can be done if $\phi'(\zeta)$ is outer, i.e., if Ω is Smirnov. This completes the proof.

In the next theorem we consider the class $L_1 \cap L_2$ on $(0, \infty)$. This is an ideal in L_1, so for $f \in L_1 \cap L_2$ it is meaningful to ask when the $\{f^{*n}\}$ are complete in L_2. Here we obtain a sufficient condition that is very simple: mere separation of points is enough. (The corresponding theorem in the circle is trivial, since there the max norm on $f(z)$ majorizes the l_2 norm on $\{a_n\}$.) In the present case we actually can obtain a result for arbitrary sub-algebras. By analogy with §2 we denote by A the set of transforms of L_1, and by C the uniform closure of A: functions continuous on $\bar{\pi}$, analytic in π, and vanishing at ∞.

THEOREM 5. *Let S be a sub-algebra of $L_1 \cap L_2$ and $B \subset A$ the set of Fourier transforms of $f \in S$. If B is dense in C, then S is dense in $L_2(0, \infty)$ in the L_2 norm.*

Proof. Let $a(t) \in L_2(0, \infty)$ and suppose $\int_0^\infty f(t)\overline{a(t)}dt = 0$ for $f \in S$. It suffices to show that this implies a is a null-function. Let us write $a_1(t) = a(t)$ for $t > 0$, $a_1(t) = \overline{a(-t)}$ for $t \leqq 0$, and $f_1(t)$ for the function which coincides with $f(t)$ in $(0, \infty)$ and vanishes for $t < 0$. Then we have

$$\int_{-\infty}^{\infty} f_1(t)\overline{a_1(t)}dt = 0, \qquad f \in S.$$

We now apply Parseval's theorem, noting that the L_2 transform of f_1 is $F \in A$, and get

$$\int_{-\infty}^{\infty} F(x)A_1(x)dx = 0, \qquad F \in B \tag{15}$$

where $A_1(x)$ denotes the (L_2) Fourier transform of $a_1(t)$, and is *real*.

The proof could now be completed very simply if we could extend (15) to $F \in C$, but since we do not know $A_1(x) \in L_1(-\infty, \infty)$ we cannot immediately replace the F in (15) by limits (in the sense of uniform convergence) of such F. Therefore we proceed as follows. For any F, $G \in B$ we have (since $FG \in B$)

$$\int_{-\infty}^{\infty} F(x)G(x)A_1(x)dx = 0. \tag{16}$$

Since all functions in B are of class $L_2(-\infty,\infty)$, $GA_1 \in L_1(-\infty,\infty)$ and thus (16) remains true if we replace F by any function in the uniform closure of B, i.e., any $F \in C$. In particular, we may choose $F(z) = 1/(z - u + i)$, where u is a real constant. Hence

$$\int_{-\infty}^{\infty} G(x)\,\frac{A_1(x)}{x - u + i}\,dx = 0, \qquad G \in B \tag{17}$$

and since $A_1(x)/(x - u + i) \in L_1(-\infty, \infty)$, we may now replace G by any function of C. Choosing $G(z) = 1/(1 - i\delta z)$, where δ is a positive constant

$$\int_{-\infty}^{\infty} \frac{1}{1 - i\delta x}\,\frac{A_1(x)}{x - u + i}\,dx = 0.$$

Now, as $\delta \to +0$, $1/(1 - i\delta x)$ tends to 1 *boundedly*, since $|1 - i\delta x| \geqq 1$. Hence, another passage to the limit gives

$$\int_{-\infty}^{\infty} \frac{A_1(x)}{x - u + i}\,dx = 0,$$

and finally, recalling that A_1 is real and setting the imaginary part of the last integral equal to zero:

$$\int_{-\infty}^{\infty} \frac{A_1(x)}{1+(u-x)^2}\, dx = 0, \qquad \text{for all real } u.$$

In other words, the convolution of the real functions $A_1(x) \in L_2(-\infty, \infty)$ and $1/(1+x^2)$ vanishes identically, hence $a_1(t)e^{-|t|}$, its Fourier transform, is a null function, whence $a_1 \equiv 0, a \equiv 0$. Q.E.D.

COROLLARY. *Let* $f(t) \in L_1 \cap L_2(0, \infty)$, *and suppose* $F(z) = \int_0^\infty f(t)e^{itz}dt$ *is univalent and* $\neq 0$ *for* $\operatorname{Im} z \geqq 0$. *Then the functions* $\{f^{*n}(t)\}$ *are complete in* $L_2(0, \infty)$.

For, the hypothesis is precisely that F separates points on $\bar{\pi}$, and by Theorem A of §1, polynomials in F are dense in C. (In the present case, a trivial modification is necessary: only polynomials without constant term are permitted, and only functions vanishing at ∞ need be approximated.)

Under rather strong restrictions, we can deduce a pure L_1 result from Theorem 5: Namely, if the functions of B are analytic in $y > -\delta$ for some $\delta > 0$, and dense in the class C for the larger half-plane, then S is dense in L_1, in the L_1 norm. The proof, based on the transformation $e^{\sigma t}f(t) = g(t)$ $(0 < \sigma < \delta)$ is simple, and left to the reader.

5. Examples.

5.1. The preceding investigation has focused attention on functions $f \in R$ whose transforms are univalent in $\bar{\pi}$ (as one sees easily by the argument principle, it is enough to verify that $F(x)$ is nonvanishing and separates points on the real axis to assure this). The following theorem provides us with a class of such functions.

THEOREM 6. *Let* $f(t) \in L_1$ *be real and strictly-decreasing in* $(0, \infty)$, *and suppose also its Fourier cosine transform* $F_c(x) = \sqrt{(2/\pi)} \int_0^\infty f(t)\cos tx dt$ *is strictly decreasing for* $0 \leqq x < \infty$. *Then* $F(x)$ *is nonvanishing and* $F(z)$ *is univalent in* $y \geqq 0$.

Proof. Clearly $F(x) = 0$ is excluded, since $\operatorname{Re} F(x) = \sqrt{(\pi/2)}F_c(x) > 0$. Now if $F(x_1) = F(x_2)$, then because of the assumption regarding $F_c(x), x_1 = -x_2$. Hence denoting by F_s the sine transform of f, $F_s(x_1) = F_s(x_2) = F_s(-x_1)$. But F_s is an *odd* function, so $F_s(x_1) = 0$. However, the sine transform of a strictly decreasing function is *positive* for $x > 0$.

REMARK. Closer inspection shows that when $f(t)$ is merely assumed nonincreasing, $F_s(x)$ still is positive for $x = 0$, except in the case that $f(t)$ coincides almost everywhere with a function which is constant in every interval $[2n\pi/x, 2(n+1)\pi/x]$, $n = 0, 1, \cdots$.

The following examples, quoted from [14], illustrate Theorem 6. (Note that, when $f(t)$ satisfies the hypotheses of Theorem 6, so does $g(t) = F_c(t)$, so we get examples in pairs.)

$$(1) \qquad f(t) = e^{-at}, \qquad F_c = \sqrt{\frac{2}{\pi}}\,\frac{a}{x^2 + a^2}, \qquad a > 0$$

$$(2) \qquad f(t) = \frac{1}{t^2 + a^2}, \qquad F_c(x) = \frac{\pi}{2a} e^{-at}, \qquad a > 0$$

$$(3) \qquad F(t) = e^{-(1/2)a^2t^2}, \qquad F_c(x) = e^{-(x^2/2a^2)}, \qquad a > 0.$$

We may use these examples to construct others by noting that any linear combination with positive weight factors, of functions $f(t)$ which satisfy the hypotheses of Theorem 6, again satisfies these hypotheses. Thus, for instance, the functions

$$(4) \qquad f(t) = \int_0^\infty e^{-tu}\, d\rho(u)$$

where $d\rho$ is any positive measure for which

$$(5) \qquad \int_0^\infty \frac{d\rho(u)}{u} < \infty$$

is a function of $L_1(0, \infty)$ whose transform separates points on $\bar{\pi}$. By a theorem of Bernstein (see, e.g., [17, p. 160]), every completely monotonic function can be represented in the form (4), i.e., every function of class C^∞ on $(0, \infty)$ whose nth derivative has the sign $(-1)^n$, for all $n \geqq 0$. Similar classes of functions can be constructed starting with the $F(t)$ of (2) or (3).

The class of functions characterized in Theorem 6 may be viewed as a limiting case of the following larger class: suppose for some x_0, $0 < x_0 < \infty$ we have (i) $F_c(x)$ strictly decreasing, $F_s(x) > 0$ for $x < x_0$ and (ii) $F_c(x) \leqq 0$, $F_s(x)$ strictly decreasing for $x > x_0$. Then $F(z)$ separates points on $\bar{\pi}$. (Theorem 6 is the limiting case $x_0 = \infty$.) We leave the simple proof to the reader. An example of such a function is $f(t) = te^{-t}$. Here

$$F_c(x) = \sqrt{\frac{2}{\pi}}\,\frac{1 - x^2}{(1 + x^2)^2}, \qquad F_s(x) = \sqrt{\frac{2}{\pi}}\,\frac{2x}{(1 + x^2)^2}.$$

$F_s(x) > 0$ for $x > 0$, and F_s decreases for $x > \sqrt{\frac{1}{3}}$, while $F_c(x)$ decreases for $x < \sqrt{3}$ and is negative for $x > 1$. Thus, conditions (i) and (ii) are satisfied with $x_0 = 1$. One notes independently, from the equation $F(z) = 1/(1 - iz)^2$, that in this case $F(z)$ maps $y > 0$ onto a cardioid, the image of the circle $|\zeta - \frac{1}{2}| < \frac{1}{2}$ under the map $w = \zeta^2$.

5.2. In a few cases the convolutions may be evaluated explicitly: from the equation

$$(e^{-t})^{*n} = \frac{t^{n-1}}{(n-1)!}e^{-t}$$

we have (since e^{-t} has a univalent transform $1/(1-iz)$): The Laguerre functions are complete in $L_1(0, \infty)$. Theorem 5 gives completeness also in L_2, but for the Laguerre functions completeness in either norm readily implies it in the other norm. Choosing $f(t) = te^{-t}$ gives the completeness of $\{t^{2n+1}e^{-t}\}$. On the other hand, since t^2e^{-t} has a nonschlicht transform, the $\{t^{3n+2}e^{-t}\}$ are not complete. These results are not new, following e.g. from the theory of weighted polynomial approximation (Bernstein problem).

We conclude with an example of a generator obtained by starting with a particular Smirnov domain, and deriving the associated element of L_1. Let Ω_ν denote the subdomain of $|w| < 1$ for which $|\arg w| < \pi\nu$, where $0 < \nu < 1$. I.e., Ω_ν is a circular sector of central angle $2\pi\nu$. A calculation shows that, for any $a > 0$, the function

$$F(z) = \left[-\left(\frac{z - \sqrt{(z^2 - a^2)}}{a} \right) \right]^{2\nu}$$

maps $y > 0$ onto Ω_ν. This mapping takes the positive y-axis onto the segment $(0, 1)$ of the real axis in the w-plane. We have, moreover, for $y > 0$

$$F(iy) = \left(\frac{\sqrt{(y^2 + a^2)} - y}{a} \right)^{2\nu}.$$

By consulting a table of Laplace transforms (e.g. [5, formula 34, p. 403]) we get $f(t) = 2\nu J_{2\nu}(at)/t$ where J_ν is Bessel's function of order ν. We thus have, applying Theorem 4: The functions $\{J_{n\nu}(t)/t\}$, $n = 1, 2, \cdots$ are complete in $L_1(0, \infty)$ for $0 < \nu < 2$. For $\nu = 2$ the completeness fails, since the mapping is onto a slit circle, i.e., does not separate points on the real axis.

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