RECURSIVE FUNCTIONALS AND QUANTIFIERS OF FINITE TYPES II[18]

BY

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In 1957 we put about half of the material we then had on this subject into Part I [37] of this paper and [18]. (Not all of the topics intended for inclusion in Part II were mentioned in Part I.)

Now in 1961, with this Part II and [39; 40; 41 and 42] written, our project seems to us still as far from completed as it seemed in 1957, and we are planning a Part III. (Not all of the topics which we said in Part I would be in Part II are treated in the present series of papers.)

Early in §9 of this Part II we give the Kreisel result cited in Footnote 17 at the end of Part I. The remainder of §9 continues the analysis of substitution of functions begun in IV, XXII and XXIII (cf. 3.10). This analysis is fundamental for the derivation in §10 of a version of the first recursion theorem (extending IM Theorem XXVI p. 348 to higher types of variables under a restriction on the functional). That in turn underlies our proofs in [39; 41 and 42] that only functions general (or partial) recursive in the present sense (§3) are comprised by appropriate extensions to higher types of Turing-machine computability, of λ-definability and of general (or partial) recursiveness in the Herbrand-Gödel style. By [40], and the latter portions of [41 and 42], conversely all general (or partial) recursive functions in the present sense are comprised. (Cf. end 3.2.) In §11 of this Part II, a start is made toward the discussion of the extensions to higher types of the degree concept of [25 and 19] (cf. the introduction to Part I) and of transfinite hierarchies (e.g. [14, §§6–9] or [16, §§4–7]).

9. **Substitution of functions. 9.1.** We consider the "stages" in the computation, or attempted computation, of $\{z\}(\mathfrak{a})$ for a given $\mathfrak{a}$, where z is an index for $\mathfrak{a}$ 4.1, arranged on a "tree", as at the beginning of 5.3. However, we consider the vertices or "positions" γ in the tree as occupied not by expressions "$\{z_\gamma\}(\mathfrak{a}_\gamma)$" but by $(n_\gamma+1)$-tuples $(z_\gamma, \mathfrak{a}_\gamma)$ where $\mathfrak{a}_\gamma$ is an n_γ-tuple and z_γ is an index for $\mathfrak{a}_\gamma$. We call this tree the *computation tree for* $\{z\}(\mathfrak{a})$. Although, at each position $\gamma, (z_\gamma, \mathfrak{a}_\gamma)$ is always defined as an $n_\gamma+1$-tuple, $\{z_\gamma\}(\mathfrak{a}_\gamma)$ as a number may or may not be; when it is, we speak of this number as the *value of* $(z_\gamma, \mathfrak{a}_\gamma)$ (or *of* $\{z_\gamma\}(\mathfrak{a}_\gamma)$).

The computation tree for $\{z\}(\mathfrak{a})$ is similar to that for $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ considered later in 5.3, with the following differences. Here the number $n_\gamma+1$ of objects in the tuples $(z_\gamma, \mathfrak{a}_\gamma)$ in general varies within a tree, while there the number $r+2$ of objects in $(z_\gamma, a_\gamma, \alpha^1_\gamma, \cdots, \alpha^r_\gamma)$ was fixed throughout. Here we are

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constructing the tree only as far as we can with the quantities that are available; specifically, at a node we fill the upper next position when and only when the computation of the tuple at the lower next position can be completed with the determination of its value, while there we used a function η of position γ giving a true or fictitious value to permit the upper next position to be filled simultaneously with the lower one. Finally, here at an application of S1–S3, S7, or of S9 with the a not an index for the $\mathfrak{b}$, we terminate the branch in question (without carrying out the application), while there we repeated the $r+2$-tuple before the application ad infinitum.

REMARK 8. The computation tree for $\{z\}(a,\mathfrak{d},\sigma'',\mathfrak{b})$ where z is an index for $(a,\mathfrak{d},\sigma'',\mathfrak{b})$ and $(z)_0 = 5$ begins as follows:

$$(z,a,\mathfrak{d},\sigma,\mathfrak{b}) \;\_\; ((z)_3, a-1, b_{a-1}, \mathfrak{d},\sigma,\mathfrak{b}) \qquad (\simeq\ b_a).$$
$$\diagdown (z,a-1,\mathfrak{d},\sigma,\mathfrak{b}) \;\_\; ((z)_3, a-2, b_{a-2}, \mathfrak{d},\sigma,\mathfrak{b}) \qquad (\simeq\ b_{a-1}).$$
$$\diagdown \quad \cdots$$
$$(z,2,\mathfrak{d},\sigma,\mathfrak{b}) \;\_\; ((z)_3, 1, b_1, \mathfrak{d},\sigma,\mathfrak{b}) \qquad (\simeq\ b_2).$$
$$\diagdown (z,1,\mathfrak{d},\sigma,\mathfrak{b}) \;\_\; ((z)_3, 0, b_0, \mathfrak{d},\sigma,\mathfrak{b}) \qquad (\simeq\ b_1).$$
$$\diagdown (z,0,\mathfrak{d},\sigma,\mathfrak{b}) - ((z)_2, \mathfrak{d},\sigma,\mathfrak{b}) \qquad (\simeq\ b_0).$$

We write "$\simeq\ b$" to mean "with value b if defined". A branch containing the lower next position at each of the first a nodes can in any case be constructed as far as its $a+1$-position occupied by $((z)_2,\mathfrak{d},\sigma,\mathfrak{b})$. Only if the computation of that tuple can be completed with the determination of the value $b_0 = \{(z)_2\}(\mathfrak{d},\sigma,\mathfrak{b})$ can the tuple $((z)_3, 0, b_0, \mathfrak{d},\sigma,\mathfrak{b})$ be supplied for a branch which instead includes the upper next position at the ath node. Similarly only if the value $b_1 = \{(z)_3\}(0, b_0, \mathfrak{d},\sigma,\mathfrak{b})$ exists can the tuple $((z)_3, 1, b_1, \mathfrak{d},\sigma,\mathfrak{b})$ be supplied for a branch which instead includes the upper next position at the $a-1$st node, etc. The corresponding tree for the computation of $\{\gamma_n(z,w,p)\}(a,\mathfrak{d},\mathfrak{b},\mathfrak{c})$ begins as follows (cf. Case 5 for XXII in 4.2):

$$(\gamma_n(z,w,p),a,\mathfrak{d},\mathfrak{b},\mathfrak{c}) \;\_\; (\mathrm{F},a,a,\mathfrak{d},\mathfrak{b},\mathfrak{c}) \; -$$
$$\diagdown (\mathrm{G},a,\mathfrak{d},\mathfrak{b},\mathfrak{c}) \qquad \text{(with value } a\text{)}.$$
$$(\mathrm{E},a,\mathfrak{d},\mathfrak{b},\mathfrak{c},a) \;\_\; (\gamma_n((z)_3,\mathrm{B},p), a-1, b_{a-1}, \mathfrak{d},\mathfrak{b},\mathfrak{c},a) \qquad (\simeq\ b_a).$$
$$\diagdown (\mathrm{E},a-1,\mathfrak{d},\mathfrak{b},\mathfrak{c},a) \;\_\; (\gamma_n((z)_3,\mathrm{B},p,),a-2, b_{a-2}, \mathfrak{d},\mathfrak{b},\mathfrak{c},a) \qquad (\simeq\ b_{a-1}).$$
$$\diagdown \quad \cdots$$
$$(\mathrm{E},2,\mathfrak{d},\mathfrak{b},\mathfrak{c},a) \;\_\; (\gamma_n((z)_3,\mathrm{B},p), 1, b_1, \mathfrak{d},\mathfrak{b},\mathfrak{c},a) \qquad (\simeq b_2).$$
$$\diagdown (\mathrm{E},1,\mathfrak{d},\mathfrak{b},\mathfrak{c},a) \;\_\; (\gamma_n((z)_3,\mathrm{B},p), 0, b_0, \mathfrak{d},\mathfrak{b},\mathfrak{c},a) \qquad (\simeq b_1).$$
$$\diagdown (\mathrm{E},0,\mathfrak{d},\mathfrak{b},\mathfrak{c},a) - (\gamma_n((z)_2,\mathrm{A},p), \mathfrak{d},\mathfrak{b},\mathfrak{c},a) \qquad (\simeq\ b_0).$$

For (completely defined) $\sigma = \lambda\tau\{w\}(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau)$, the $b_0, b_1, \cdots, b_a$, so far as they exist in the first figure, exist and have the same values in the second, by the following reasoning. The b_0 in the first exists and $= \{(z)_2\}(\mathfrak{d}, \sigma, \mathfrak{b})$ only if the latter is defined; but then by XXII and the construction of A, $\{(z)_2\}(\mathfrak{d}, \sigma, \mathfrak{b}) = \{\gamma_n((z)_2, \mathrm{A}, p)\}(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, a)$, which is the b_0 for the second figure. Similarly for $b_1, b_2, \cdots, b_{a-1}, b_a$, successively.

LIII. *Let z be an index for $\mathfrak{a}$. Then $\{z\}(\mathfrak{a})$ is defined, or undefined, according as each branch in the computation tree for $\{z\}(\mathfrak{a})$ terminates at an application of* S1–S3 *or* S7, *or there is a branch either infinite or terminating at an application of* S9 *with the a not an index for the $\mathfrak{b}$. In the first case, each tuple in the tree has a value; in the second case, no tuple along a branch of the kind described has a value, each tuple at a position below such a branch* 5.11 *has a value, and such a branch is uppermost (i.e. at each node it contains the upper next position when both next positions are in the tree).*

Proof. PART I. Assume $\{z\}(\mathfrak{a})$ is defined. We show by induction on $\{z\}(—)$ (end 3.8) that each branch terminates at an application of S1–S3 or S7, and each tuple in the tree has a value. CASE 1. $(z)_0 = 1$. Then there is only one branch, this terminates at its 0-position at an application of S1, and the tuple $(z, \mathfrak{a})$ $(= (z, a, \mathfrak{b}))$ there has a value (namely $a + 1$). CASE 4. $(z)_0 = 4$. Because $\{z\}(\mathfrak{a})$ $(= \{(z)_2\}(\{(z)_3\}(\mathfrak{a}), \mathfrak{a}))$ is defined, so are both $\{(z)_3\}(\mathfrak{a})$, say with value b, and $\{(z)_2\}(b, \mathfrak{a})$. Application of the hyp. ind. to $\{(z)_3\}(\mathfrak{a})$ and $\{(z)_2\}(b, \mathfrak{a})$ gives the required conclusions. CASE 9. $(z)_0 = 9$. Since $\{z\}(\mathfrak{a})$ $(= \{z\}(a, \mathfrak{b}, \mathfrak{c}) = \{a\}(\mathfrak{b}))$ is defined, the hyp. ind. applies to $\{a\}(\mathfrak{b})$.

PART II. Assume $\{z\}(\mathfrak{a})$ undefined. Then, starting with $(z, \mathfrak{a})$ at the 0-position, we can choose successively a next position at which the tuple has no value, either ad infinitum or until the resulting branch terminates at an application of S9 as described. The resulting infinite or finite branch has the stated properties; for at a node, if the upper next position exists in the tree, the tuple at the lower has a value so we must choose the upper, while at all positions below that upper next position the tuple has a value, by Part I applied to the tuple at the lower next position.

9.2. We now prove the result announced at the end of Part I in the last sentence of Footnote 17. Our lemma is LIV, and (b) is answered using an argument of Kreisel by LV. A completely defined extension 8.3 is a *completion*.

LIV. *Let γ_2 be the primitive recursive function γ_n constructed in the proof of* XXII *for $n = 2$. Suppose z, w are indices of $\phi(\mathfrak{a}, \sigma^2, \mathfrak{b})$, $\theta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau^1)$, respectively, where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ consist of type-*0 *variables only. For values of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ such that $\theta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau^1)$ is defined for every general recursive τ^1: If $\phi(\mathfrak{a}, \sigma_1^2, \mathfrak{b})$ is defined for some completion σ_1^2 of $\lambda\tau^1\ \theta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau^1)$, then, for every completion σ_2^2 of $\lambda\tau^1\ \theta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau^1)$, $\phi(\mathfrak{a}, \sigma_2^2, \mathfrak{b})$ is defined and*

$$\phi(\mathfrak{a}, \sigma_2^2, \mathfrak{b}) = \{\gamma_2(z, w, 0)\}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}). \tag{90}$$

(The right side of (90), and hence the left, is independent of the choice of the completion σ_2^2.)

Proof. Since by hyp. $\phi(\mathfrak{a},\sigma_1,\mathfrak{b})$, i.e. $\{z\}(\mathfrak{a},\sigma_1,\mathfrak{b})$, is defined, we can use induction on $\{z\}(—)$ (in the course-of-values version). For the notation in the cases, cf. the proof of XXII.

Case 5. $(z)_0 = 5$. Since $\{z\}(a,\mathfrak{d},\sigma_1,\mathfrak{b})$ is defined, by Remark 8 so are $\{(z)_2\}(\mathfrak{d},\sigma_1,\mathfrak{b})$, $\{(z)_3\}(0,b_0,\mathfrak{d},\sigma_1,\mathfrak{b}),\cdots,\{(z)_3\}(a-1,b_{a-1},\mathfrak{d},\sigma_1,\mathfrak{b})$ with respective values $b_0,b_1,\cdots,b_a$. By the hyp. of the ind. on $\{z\}(—)$ and the construction of A and B,

$$b_0 = \{(z)_2\}(\mathfrak{d},\sigma_2,\mathfrak{b}) = \{\gamma_2((z)_2,\mathrm{A},0)\}(\mathfrak{d},\mathfrak{b},\mathfrak{c},a),$$

$$b_1 = \{(z)_3\}(0,b_0,\mathfrak{d},\sigma_2,\mathfrak{b}) = \{\gamma_2((z)_3,\mathrm{B},0)\}(0,b_0,\mathfrak{d},\mathfrak{b},\mathfrak{c},a),$$

$$\cdots$$

$$b_a = \{(z)_3\}(a-1,b_{a-1},\mathfrak{d},\sigma_2,\mathfrak{b}) = \{\gamma_2((z)_3,\mathrm{B},0)\}(a-1,b_{a-1},\mathfrak{d},\mathfrak{b},\mathfrak{c},a).$$

Hence by Remark 8,

$$b_a = \{z\}(a,\mathfrak{d},\sigma_2,\mathfrak{b}) = \{\gamma_2(z,w,0)\}(a,\mathfrak{d},\mathfrak{b},\mathfrak{c}).$$

Case 8. $(z)_0 = 8$. Subcase 1. $n=(z)_2$ & $p=0$, so σ is α^j. (We have $n=j=2$, $p=0$.) Since $\{z\}(\sigma_1,\mathfrak{b})$ ($\simeq$ $\sigma_1(\lambda\alpha^0\{(z)_3\}(\sigma_1,\alpha^0,\mathfrak{b}))$) is defined, $\lambda\alpha^0\{(z)_3\}(\sigma_1,\alpha^0,\mathfrak{b})$ is completely defined, and by the hyp. ind. on $\{z\}(—)$ and the construction of A, for each α^0, $\{(z)_3\}(\sigma_2,\alpha^0,\mathfrak{b})$ is defined and $=$ $\{\gamma_2((z)_3,\mathrm{A},0)\}(\alpha^0,\mathfrak{b},\mathfrak{c})$. Thus, since $\mathfrak{b},\mathfrak{c}$ are numbers (not higher-type objects), $\lambda\alpha^0\ \{(z)_3\}(\sigma_2,\alpha^0,\mathfrak{b})$ is a general recursive function. Hence by our hypothesis on θ, $\theta(\mathfrak{b},\mathfrak{c},\lambda\alpha^0\ \{(z)_3\}(\sigma_2,\alpha^0,\mathfrak{b}))$, i.e. $\{z\}(\sigma_2,\mathfrak{b})$, is defined. But then by XXII and the construction of C, $\theta(\mathfrak{b},\mathfrak{c},\lambda\alpha^0\{(z)_3\}(\sigma_2,\alpha^0,\mathfrak{b})) = \{\gamma_1(w,\mathrm{C},0)\}(\mathfrak{b},\mathfrak{c})$, i.e. $\{z\}(\sigma_2,\mathfrak{b}) = \{\gamma_2(z,w,0)\}(\mathfrak{b},\mathfrak{c})$.

Subcase 2. $n\neq(z)_2 \vee p\neq 0$, so σ is not α^j. This subcase cannot occur, since σ is the only function variable among $\mathfrak{a},\sigma,\mathfrak{b}$.

Case 9. $(z)_0 = 9$. Subcase 2. $p < (z)_{2,n}$. By hyp., $\{z\}(a,\mathfrak{d},\sigma_1,\mathfrak{b})$ ($=\{a\}(\mathfrak{d},\sigma_1,\mathfrak{e})$) is defined. Hence by the hyp. ind. on $\{z\}(-)$, $\{a\}(\mathfrak{d},\sigma_2,\mathfrak{e})$ $=\{\gamma_2(a,S^1(w,a),0)\}(\mathfrak{d},\mathfrak{b},\mathfrak{c})$. So $\{z\}(a,\mathfrak{d},\sigma_2,\mathfrak{b})=\{\gamma_2(z,w,0)\}(a,\mathfrak{d},\mathfrak{b},\mathfrak{c})$. —

We say $(x)(Ey)(z)R(x,y,z)$ is *recursively fulfillable*, if, for some general recursive β^1, $(x)(z)R(x,\beta^1(x),z)$ is true (Kleene [10, p. 69]).

LV. *Suppose* (A) $R(x,y,z)$ *is general recursive,* (B) $(x)(Ey)(z)R(x,y,z)$ *is true, but* (C) $(x)(Ey)(z)R(x,y,z)$ *is not recursively fulfillable. Let* $R(\beta^1,x)$ $\equiv R((x)_0,\beta^1((x)_0),(x)_1)$, $R(\alpha^2,\beta^1)\equiv R(\beta^1,\alpha^2(\beta^1))$. *Then* $(\alpha^2)(E\beta^1)R(\alpha^2,\beta^1)$ *is true, but* $(\alpha^2)R(\alpha^2,\lambda x\,\chi(\alpha^2,x))$ *is false for every general recursive* $\chi(\alpha^2,x)$.

Proof. First, from (B) by the transformation $(x)(Ey)(z)R(x,y,z)\equiv(E\beta^1)(x)(z)R(x,\beta^1(x),z)\equiv(E\beta^1)(x)R((x)_0,\beta^1((x)_0),(x)_1)\equiv(E\beta^1)(x)R(\beta^1,x)$ $\equiv(\alpha^2)(E\beta^1)R(\beta^1,\alpha^2(\beta^1))\equiv(\alpha^2)(E\beta^1)R(\alpha^2,\beta^1)$, $(\alpha^2)(E\beta^1)R(\alpha^2,\beta^1)$ is true.

Second, by (C), for every general recursive β^1, $(Ex)\bar{R}(\beta^1,x)$. So the function $\theta(\beta^1) = \mu x\bar{R}(\beta^1,x)$, which by XVI and (A) is partial recursive, is defined for every general recursive β^1, and

(91) $$(\beta^1)_{\beta^1 \text{ gen. rec.}}\bar{R}(\beta^1,\theta(\beta^1)).$$

Consider any general recursive $\chi(\alpha^2,x)$, say with index z. Let w be an index of $\lambda x\beta^1\,\theta(\beta^1)$. Let θ_2 be any completion of θ. By LIV, for every x, $\chi(\theta_2,x) = \{\gamma_2(z,w,0)\}(x)$. So $\lambda x\,\chi(\theta_2,x)$ is general recursive. Now were $(\alpha^2)R(\alpha^2,\lambda x\,\chi(\alpha^2,x))$ true, we would have $R(\theta_2,\lambda x\,\chi(\theta_2,x))$, i.e. $R(\lambda x\,\chi(\theta_2,x),\theta_2(\lambda x\,\chi(\theta_2,x)))$, i.e., since $\lambda x\,\chi(\theta_2,x)$ is general recursive and $\theta(\beta^1)$ is defined for every general recursive β^1, $R(\lambda x\,\chi(\theta_2,x),\theta(\lambda x\,\chi(\theta_2,x)))$, contradicting (91).

REMARK 9. Advancing quantifiers in

(92) $$(x)\{(Ey)T_1(x,x,y) \vee \overline{(Ey)}T_1(x,x,y)\}$$

gives $R(x,y,z) \equiv T_1(x,x,y) \vee \bar{T}_1(x,x,z)$ as an example of a predicate $R(x,y,z)$ satisfying (A)–(C) of LV. This simplifies slightly the second of the two original examples in Kleene [10, pp. 69–71].

9.3. As our theory stands, we cannot strengthen XXII to have

(a) $$\phi(\mathfrak{a},\lambda\tau^{n-1}\;\theta(\mathfrak{a},\mathfrak{b},\mathfrak{c},\tau^{n-1}),\mathfrak{b}) \simeq \{\gamma_n(z,w,p)\}(\mathfrak{a},\mathfrak{b},\mathfrak{c})$$

(for the γ_n constructed there or any other completely defined γ_n) under merely the self-evident condition that $\lambda\tau^{n-1}\;\theta(\mathfrak{a},\mathfrak{b},\mathfrak{c},\tau^{n-1})$ be completely defined. Thus (with $n=2$, $\mathfrak{a}$ empty, $p=0$ and $\mathfrak{b}=a$):

LVI. *There is a partial recursive function $\phi(\sigma^2,a)$ such that $\lambda a\mathfrak{c}\;\phi(\lambda\tau^1\;\theta(a,\mathfrak{c},\tau^1),a)$ is partial recursive for no completely defined $\theta(a,\mathfrak{c},\tau^1)$ with $\mathfrak{c}$ (empty or) consisting of variables of types $\leqq 1$.*

Proof. Let

$$\chi(\sigma^2,x,a) \simeq \chi(x,a) \simeq \begin{cases} 0 \text{ if } \bar{T}_1(a,a,x), \\ \mu y(y \neq y) \text{ otherwise,} \end{cases}$$

$$\phi(\sigma^2,a) \simeq \sigma^2(\lambda x\;\chi(\sigma^2,x,a)).$$

Then $\lambda x\;\chi(\sigma^2,x,a)$ is completely defined if and only if $(x)\bar{T}_1(a,a,x)$; and hence, under our condition for S8 in 3.7, $\phi(\sigma^2,a)$ is defined exactly in this case. Let $\phi(a,\mathfrak{c}) \simeq \phi(\lambda\tau^1\;\theta(a,\mathfrak{c},\tau^1),a)$ for a θ as stated, and let $\phi(a) \simeq \phi(a,\mathfrak{c}_0)$ where $\mathfrak{c}_0$ comes from $\mathfrak{c}$ by substituting 0 for the number variables and $\lambda y\;0$ for the function variables. Then $\phi(a)$ is defined exactly if $(x)\bar{T}_1(a,a,x)$, and hence by XXXI and IM p. 331 Example 3 is not partial recursive. Neither is $\phi(a,\mathfrak{c})$, since the class of the partial recursive functions of variables of types 0 and 1 is closed under substitution (using XXXI, XVII and IM Lemma VI p. 344 with Theorem XVII(a) p. 329; or by LXI (95) below).

DISCUSSION. This failure of the principle that substitution of a recursive function for a variable of a recursive function should produce a recursive function is connected with the special character of our type-2 and higher objects. As introduced in 1.2, and interpreted in 3.7 (where for the first time we had to consider them with partial functions as arguments), they are functionals which are defined exactly for completely defined functions as arguments. A general recursive functional $\lambda\tau^1\,\theta(\mathfrak{a},\tau^1)$ with $\mathfrak{a}$ of types $\leqq 1$ however depends on only finitely many values of its function argument τ^1, and so is most naturally to be regarded as defined for some incompletely defined τ^1. (For $\mathfrak{a}$ including objects of type $\geqq 2$, $\lambda\tau^1\,\theta(\mathfrak{a},\tau^1)$ may depend on all values of τ^1.)

In order to have an unqualified extension of IV to partial recursive functions, we should have to extend our types $2,3,4,\cdots$ to include functionals that may be defined for some incompletely defined arguments and need not be defined for all completely defined ones. Then in applying S8, only those values of $\lambda\alpha^{j-2}\chi(\alpha^j,\alpha^{j-2},\mathfrak{b})$ would be used which are required by the particular α^j. This is what is happening now (for $\sigma_0^n = \lambda\tau^{n-1}\,\theta(\mathfrak{a},\mathfrak{b},\mathfrak{c},\tau^{n-1})$ as the α^j) in computing the right member $\{\gamma_n(z,w,p)\}(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ of (a) with the γ_n of XXII, while for the left $\phi(\mathfrak{a},\sigma_0^n,\mathfrak{b})$ the convention of 3.7 is in force.

The theory thus extended may be of interest. However, after some prospecting in this direction[19], we have decided upon continuing for the present to restrict our "ultimate" variables (the $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ of (a), though not for the left side the σ_0^n; and the variables of the $\psi_1,\cdots,\psi_l$ in 3.14, though not the $\psi_1,\cdots,\psi_l$ themselves) to range over the (unextended) types introduced in 1.2 and interpreted for partial arguments by 3.7.

9.4. For $n=1$, (a) will hold (inessentially restated) for a new γ_1 (called γ_1'). For $n>1$, (a) will hold, for a γ_n', when the $\lambda\tau^{n-1}\,\theta(\mathfrak{a},\mathfrak{b},\mathfrak{c},\tau^{n-1})$ is taken from a special class, for each member σ_0^n of which the computation of $\sigma_0^n(\tau^{n-1})$ will use every value of τ^{n-1}. These results are given in LXI, toward which we first establish:

LVII. *There is a primitive recursive function γ' with the following property. For each $n\geqq 1$, write $\gamma_n'(z,w,p)=\gamma'(n,z,w,p)$. Suppose $\mathfrak{a}$ contains exactly p type-n variables, z is an index of $\phi(\mathfrak{a},\sigma^n,\mathfrak{b})$, and for $n=1$ (for $n>1$) w is an index of $\theta(t,\mathfrak{a},\mathfrak{b},\mathfrak{c})$ (of $\theta(b_1,\cdots,b_q,\mathfrak{a},\mathfrak{b},\mathfrak{c})$ where $(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ contains exactly $q>0$ type-n variables $\alpha_1^n,\cdots,\alpha_q^n$). Then for values of $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ such that $\lambda t\,\theta(t,\mathfrak{a},\mathfrak{b},\mathfrak{c})$ for $n=1$ ($\lambda\tau^{n-1}\,\theta(\alpha_1^n(\tau^{n-1}),\cdots,\alpha_q^n(\tau^{n-1}),\mathfrak{a},\mathfrak{b},\mathfrak{c})$ for $n>1$) is completely defined, $=\sigma_0^n$ say, and $\phi(\mathfrak{a},\sigma_0^n,\mathfrak{b})$ is defined, $\{\gamma_n'(z,w,p)\}(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ is defined and*

$$\phi(\mathfrak{a},\lambda t\ \theta(t,\mathfrak{a},\mathfrak{b},\mathfrak{c}),\mathfrak{b}) = \{\gamma_1'(z,w,p)\}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) \qquad (n=1), \tag{93}$$

$$\begin{aligned}\phi(\mathfrak{a},\lambda\tau^{n-1}\ \theta(\alpha_1^n(\tau^{n-1}),\cdots,\alpha_q^n(\tau^{n-1}),\ \mathfrak{a},\mathfrak{b},\mathfrak{c}),\mathfrak{b})&\\ = \{\gamma_n'(z,w,p)\}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) \qquad (n>1).&\end{aligned} \tag{94}$$

(19) An extension of the theory in this direction may be called for in connection with a simultaneous specialization or restriction in the other direction considered end 3.1 and in Footnote 9, which may make the extension easier to handle than in the most general setting.

Proof. We follow the same general plan as for XXII. But there is no induction on n. We take $\gamma'(0, z, w, p) = 0$.

For $n = 1$, the cases are treated as for XXII (Case 8 Subcase 1 not arising), except for the t (or τ^0) being taken now as the first variable of the θ, and except for a modification in the treatment of Case 9 Subcase 2 also made for $n > 1$, where we present it. This modification is not necessary for LVII, but is used for LIX and LX.

For $n > 1$, we take $\gamma_n'(z, w, p) = 0$ except for $\mathrm{Ix}(z) \,\&\, (z)_{1,n} > p \,\&\, \mathrm{Ix}(w) \,\&\, (w)_{1,n} > 0 \,\&\, (2\exp(w)_{1,n}) \cdot (z)_1 \mid p_n \cdot (w)_1$. The cases are treated generally similarly to those for XXII, except Case 8 Subcase 1, and except for the modification in Case 9 Subcase 2. In the case demonstrations of (94) by induction on $\{z\}(-)$, we tacitly assume we are dealing with values of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ such that

$$\lambda\tau\ \theta(\alpha_1^n(\tau), \cdots, \alpha_q^n(\tau),\ \mathfrak{a}, \mathfrak{b}, \mathfrak{c})$$

is completely defined. For abbreviation we write $q = (w)_{1,n}$.

Case 5. $(z)_0 = 5$. We express $\theta(b_1, \cdots, b_q, u, \mathfrak{d}, \mathfrak{b}, \mathfrak{c})$ as $\theta_1(b_1, \cdots, b_q, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u)$ with index $\mathrm{A} = \pi(0, (w)_{1,0} \dot{-} 1, \langle 6, (w)_1, 0, q, w \rangle)$, and further as $\theta(b_1, \cdots, b_q, a, b, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u)$ with index $\mathrm{B} = \pi(0, q+1, \pi(0, q+1, \langle 6, 4 \cdot (w)_1, 0, (w)_{1,0} + 1, \langle 6, 4 \cdot (w)_1, 0, (w)_{1,0} + 1, \iota(\mathrm{A}, 4) \rangle\rangle))$. Next we introduce the function $\psi(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u)$ having as index $C = \gamma_n'((z)_2, \mathrm{A}, p)$, which by the hyp. ind. on $\{z\}(-)$ will have the value $\psi(\mathfrak{d}, \lambda\tau\ \theta_1(\alpha_1^n(\tau), \cdots, \alpha_q^n(\tau), \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u), \mathfrak{b})$ when the latter is defined. Similarly we introduce $\chi(a, b, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u)$ with index $\mathrm{D} = \gamma_n'((z)_3, \mathrm{B}, p)$. The rest of the index construction, and the proof that $\gamma_n'(z, w, p) = \mathrm{H} = \langle 4, (\mathrm{C})_1, \mathrm{F}, \mathrm{G} \rangle$ has the desired property, are as for XXII, or using Remark 8 stated with γ_n' instead of γ_n.

Case 8. $(z)_0 = 8$. Write $j = (z)_2$. Subcase 1. $n = j \,\&\, p = 0$, so $\sigma = \alpha^j$. We have $\phi(\sigma, \mathfrak{b}) = \sigma(\lambda\alpha^{j-2}\ \chi(\sigma, \alpha^{j-2}, \mathfrak{b}))$. When this is defined for $\sigma = \lambda\tau\ \theta(\alpha_1^j(\tau), \cdots, \alpha_q^j(\tau), \mathfrak{b}, \mathfrak{c})$ (then $\chi(\lambda\tau\ \theta(\alpha_1^j(\tau), \cdots, \alpha_q^j(\tau), \mathfrak{b}, \mathfrak{c}), \alpha^{j-2}, \mathfrak{b})$ is defined for every α^{j-2}), its value is

$$\theta(\alpha_1^j(\lambda\alpha^{j-2}\ \chi(\lambda\tau\ \theta(\alpha_1^j(\tau), \cdots, \alpha_q^j(\tau), \mathfrak{b}, \mathfrak{c}), \alpha^{j-2}, \mathfrak{b})), \cdots,$$
$$\alpha_q^j(\lambda\alpha^{j-2}\ \chi(\lambda\tau\ \theta(\alpha_1^j(\tau), \cdots, \alpha_q^j(\tau), \mathfrak{b}, \mathfrak{c}), \alpha^{j-2}, \mathfrak{b})), \mathfrak{b}, \mathfrak{c}).$$

First we express $\theta(b_1, \cdots, b_q, \mathfrak{b}, \mathfrak{c})$ as $\theta(b_1, \cdots, b_q, \alpha^{j-2}, \mathfrak{b}, \mathfrak{c})$ with index $\mathrm{A} = \pi(0, (\overline{\mathrm{sg}}(j \dot{-} 2)) \cdot q, \langle 6, p_{j \dot{-} 2} \cdot (w)_1, j \dot{-} 2, (w)_{1, j \dot{-} 2}, \iota(w, p_{j \dot{-} 2}) \rangle)$. Next we introduce $\chi(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c})$ with index $\mathrm{B} = \gamma_n'((z)_3, \mathrm{A}, 0)$; by the hyp. ind. and the construction of A, this will have the value $\chi(\lambda\tau\ \theta(\alpha_1^j(\tau), \cdots, \alpha_q^j(\tau), \mathfrak{b}, \mathfrak{c}), \alpha^{j-2}, \mathfrak{b})$ when the latter is defined, which is the case when $\phi(\sigma, \mathfrak{b})$ is defined for $\sigma = \lambda\tau\,\theta(\alpha_1^j(\tau), \cdots, \alpha_q^j(\tau), \mathfrak{b}, \mathfrak{c})$. So it will suffice to make $\gamma_n'(z, w, p)$ an index of the function $\phi(\mathfrak{b}, \mathfrak{c}) \simeq \theta(\alpha_1^j(\lambda\alpha^{j-2}\chi(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c})), \cdots, \alpha_q^j(\lambda\alpha^{j-2}\chi(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c})), \mathfrak{b}, \mathfrak{c})$. To obtain this index, we first build $\chi_i(\mathfrak{b}, \mathfrak{c}) \simeq \alpha_{i+1}^j(\lambda\alpha^{j-2}\chi(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c}))$ with index

$$\mathrm{C}_i = \pi(j, i, \langle 8, (\mathrm{B})_1, j, \langle 6, (B)_1, j, i, \mathrm{B} \rangle\rangle) \qquad (i = 0, \cdots, q-1).$$

Now $\phi(\mathfrak{b},\mathfrak{c}) \simeq \theta(\chi_0(\mathfrak{b},\mathfrak{c}),\cdots,\chi_{q-1}(\mathfrak{b},\mathfrak{c}),\mathfrak{b},\mathfrak{c})$, which we must construct by S4, etc., as in the proof of V. We need

$$\begin{aligned}\chi_0(b_2,\cdots,b_q,\mathfrak{b},\mathfrak{c}) &\simeq \chi_0(\mathfrak{b},\mathfrak{c}),\\ \chi_1(b_3,\cdots,b_q,\mathfrak{b},\mathfrak{c}) &\simeq \chi_1(\mathfrak{b},\mathfrak{c}),\\ &\cdots\\ \chi_{q-2}(b_q,\mathfrak{b},\mathfrak{c}) &\simeq \chi_{q-2}(\mathfrak{b},\mathfrak{c}),\\ \chi_{q-1}(\mathfrak{b},\mathfrak{c}) &\simeq \chi_{q-1}(\mathfrak{b},\mathfrak{c}).\end{aligned}$$

For $i \leqq q-1$, let $\chi_{0,i}(\mathfrak{b},\mathfrak{c},b_{i+2},\cdots,b_q) \simeq \chi_i(\mathfrak{b},\mathfrak{c})$ with index $\mathrm{D}_{0\ i} = \iota(\mathrm{C}_i, 2^{q \dot- (i+1)})$; and for $t+1 \leqq q-(i+1)$, let

$$\chi_{t+1,i}(b_{(q+1)\dot-(t+1)},\cdots,b_q,\mathfrak{b},\mathfrak{c},b_{i+2},\cdots,b_{q\dot-(t+1)}) \simeq \chi_i(\mathfrak{b},\mathfrak{c})$$

with index $D_{t+1,i} = \langle 6,(\mathrm{D}_{0,i})_1,0,(\mathrm{D}_{0,i})_{1,0} \dot- 1,\mathrm{D}_{t,i}\rangle$. Now $\chi_i(b_{i+2},\cdots,b_q,\mathfrak{b},\mathfrak{c})$ has as index $\mathrm{D}_i = \mathrm{D}_{q\dot-(i+1),i}$. Finally we need

$$\begin{aligned}\phi_0(b_1,\cdots,b_q,\mathfrak{b},\mathfrak{c}) &= \theta(b_1,\cdots,b_q,\mathfrak{b},\mathfrak{c}),\\ \phi_1(b_2,\cdots,b_q,\mathfrak{b},\mathfrak{c}) &\simeq \phi_0(\chi_0(b_2,\cdots,b_q,\mathfrak{b},\mathfrak{c}),b_2,\cdots,b_q,\mathfrak{b},\mathfrak{c}),\\ \phi_2(b_3,\cdots,b_q,\mathfrak{b},\mathfrak{c}) &\simeq \phi_1(\chi_1(b_3,\cdots,b_q,\mathfrak{b},\mathfrak{c}),b_3,\cdots,b_q,\mathfrak{b},\mathfrak{c}),\\ &\cdots\\ \phi_{q-1}(b_q,\mathfrak{b},\mathfrak{c}) &\simeq \phi_{q-2}(\chi_{q-2}(b_q,\mathfrak{b},\mathfrak{c}),b_q,\mathfrak{b},\mathfrak{c}),\\ \phi(\mathfrak{b},\mathfrak{c}) \simeq \phi_q(\mathfrak{b},\mathfrak{c}) &\simeq \phi_{q-1}(\chi_{q-1}(\mathfrak{b},\mathfrak{c}),\mathfrak{b},\mathfrak{c}).\end{aligned}$$

Here $\phi_0(b_1,\cdots,b_q,\mathfrak{b},\mathfrak{c})$ has as index $\mathrm{E}_0 = w$; and for $i+1 \leqq q$, $\phi_{i+1}(b_{i+2},\cdots,b_q,\mathfrak{b},\mathfrak{c})$ has as index $\mathrm{E}_{i+1} = \langle 4,(\mathrm{D}_i)_1,\mathrm{E}_i,\mathrm{D}_i\rangle$. Thus $\phi(\mathfrak{b},\mathfrak{c})$ has as index $\gamma'_n(z,w,p) = \mathrm{F} = \mathrm{E}_q$.
SUBCASE 2. $n \neq j \vee p \neq 0$. If $n \neq j \dot- 2$, we express $\theta(b_1,\cdots,b_q,\alpha^j,\mathfrak{d},\mathfrak{b},\mathfrak{c})$ as $\theta(b_1,\cdots,b_q,\alpha^j,\alpha^{j-2},\mathfrak{d},\mathfrak{b},\mathfrak{c})$ with index A as in Subcase 1; while if $n = j \dot- 2$, we express it as $\theta(b_0,b_1,\cdots,b_q,\alpha^j,\alpha^{j-2},\mathfrak{d},\mathfrak{b},\mathfrak{c})$ with index $\mathrm{A} = \langle 6,2\cdot p_{j\dot-2}\cdot(w)_1,0,(w)_{1,0},\langle 6,2\cdot p_{j\dot-2}\cdot(w)_1,j\dot-2,q,\iota(w,2\cdot p_{j\dot-2})\rangle\rangle$. We continue as for XXII.

CASE 9. $(z)_0 = 9$. SUBCASE 1. $p \geqq (z)_{2,n}$. Let $\gamma'_n(z,w,p) = \langle 9,[(w)_1/2^q],(z)_2\rangle$.
SUBCASE 2. $p < (z)_{2,n}$. Now $\phi(a,\mathfrak{d},\sigma,\mathfrak{b}) \simeq \{a\}(\mathfrak{d},\sigma,\mathfrak{e})$ and $\theta(b_1,\cdots,b_q,a,\mathfrak{d},\mathfrak{b},\mathfrak{c}) = \{S^1(\langle 6,(w)_1,0,q,w\rangle,a)\}(b_1,\cdots,b_q,\mathfrak{d},\mathfrak{b},\mathfrak{c})$. When $\phi(a,\mathfrak{d},\sigma,\mathfrak{b})$ is defined, a is an index for $\mathfrak{d},\sigma,\mathfrak{e}$, whence $(a)_1 = (z)_2$. So, for values of $a,\mathfrak{d},\mathfrak{b},\mathfrak{c}$ which make $\sigma_0 = \lambda\tau\,\theta(\alpha^n_1(\tau),\cdots,\alpha^n_q(\tau),a,\mathfrak{d},\mathfrak{b},\mathfrak{c})$ completely defined and $\phi(a,\mathfrak{d},\sigma_0,\mathfrak{b})$ defined, the hyp. ind. on $\{z\}(-)$ gives that

$$\begin{aligned}\phi(a,\mathfrak{d},\sigma_0,\mathfrak{b}) &\simeq \{\gamma'_n(a,S^1(\langle 6,(w)_1,0,q,w\rangle,a),p)\}(\mathfrak{d},\mathfrak{b},\mathfrak{c})\\ &\simeq \{(\overline{\mathrm{sg}}\,|(a)_1-(z)_2|)\cdot\gamma'_n(a,S^1(\langle 6,(w)_1,0,q,w\rangle,a),p)\}(\mathfrak{d},\mathfrak{b},\mathfrak{c}).\end{aligned}$$

So it will suffice to take $\gamma'_n(z,w,p) = S^5(\mathrm{D},g,n,z,w,p)$ where D is an index of $\lambda gnzwpa\mathfrak{d}\mathfrak{b}\mathfrak{c}\,\{(\overline{\mathrm{sg}}\,|(a)_1-(z)_2|)\cdot\{g\}(n,a,S^1(\langle 6,(w)_1,0,q,w\rangle,a),p)\}(\mathfrak{d},\mathfrak{b},\mathfrak{c})$ and g is an index of γ'. To obtain D, we pick an index e of

$$\lambda gnzwpa\ (\overline{\mathrm{sg}}\,|(a)_1-(z)_2|)\cdot\{g\}(n,a,S^1(\langle 6,(w)_1,0,q,w\rangle,a),p),$$

and define thence A, B_0, B_{i+1}, C, D as before, except that for A and B_0 the p_{n-1} becomes 2^q and for B_0 and C the 4 becomes 6.

9.5. If in the computation tree for $\{z_0\}(\mathfrak{g}_0)$ (at the 0-position), $(z,\mathfrak{g})$ occurs at an $s+1$-position not the lower next position to a node by S5b, we write $(z_0,\mathfrak{g}_0) \succ (z,\mathfrak{g})$. For example, in the first tree of Remark 8, in any case $(z,a,\mathfrak{d},\sigma,\mathfrak{b}) \succ ((z)_2,\mathfrak{d},\sigma,\mathfrak{b})$; but (provided tuples shown do not occur also at positions not shown) not $(z,a,\mathfrak{d},\sigma,\mathfrak{b}) \succ (z,0,\mathfrak{d},\sigma,\mathfrak{b})$; while for $a>1$ (with the same proviso) $(z,a,\mathfrak{d},\sigma,\mathfrak{b}) \succ ((z)_3,0,b_0,\mathfrak{d},\sigma,\mathfrak{b})$ only if $\{(z)_2\}(\mathfrak{d},\sigma,\mathfrak{b})$ is defined (with value b_0), as otherwise the position shown as occupied by $((z)_3,0,b_0,\mathfrak{d},\sigma,\mathfrak{b})$ would be nonexistent. The relation $\succ$ is transitive.

If $(z,\mathfrak{g})$ occurs in the computation tree for $\{z_0\}(\mathfrak{g}_0)$ at an $s+1$-position not the lower next position to a node by S5b, but in the path from the 0-position to this $s+1$-position no intermediate position occurs not the lower next position to a node by S5b, we write $(z_0,\mathfrak{g}_0) \succ^\circ (z,\mathfrak{g})$. (Thus $(z_0,\mathfrak{g}_0) \succ^\circ (z,\mathfrak{g}) \rightarrow (z_0,\mathfrak{g}_0) \succ (z,\mathfrak{g})$.) If $(z_0,\mathfrak{g}_0) \succ (z,\mathfrak{g})$, we can find $(z_0,\mathfrak{g}_0),\cdots,(z_l,\mathfrak{g}_l)$ $(l \geqq 1)$, where $(z_l,\mathfrak{g}_l)$ is $(z,\mathfrak{g})$, such that $(z_0,\mathfrak{g}_0) \succ^\circ (z_1,\mathfrak{g}_1) \succ^\circ \cdots \succ^\circ (z_l,\mathfrak{g}_l)$; to do this, we pick a particular $s+1$-position as described for $(z_0,\mathfrak{g}_0) \succ (z,\mathfrak{g})$, and select in order along the path from the 0-position to this $s+1$-position each tuple not at the lower next position to a node by S5b.

If $(z_0,\mathfrak{g}_0) \succ (z,\mathfrak{g})$, then, relative to a particular choice of the $s+1$-position in question and thus of a path from $(z_0,\mathfrak{g}_0)$ to $(z,\mathfrak{g})$, a *correspondence* between some of the members of $\mathfrak{g}_0$ and the same objects occurring as members of $\mathfrak{g}$ is established through the identifications of certain of the objects at the individual schema applications and transitivity. Thus, in Remark 8, if $(z,a,\mathfrak{d},\sigma,\mathfrak{b}) \succ ((z)_3,0,b_0,\mathfrak{d},\sigma,\mathfrak{b})$ (for the shown occurrence of the latter), the $\mathfrak{d},\sigma,\mathfrak{b}$ of the former correspond 1-1 in the given order to the $\mathfrak{d},\sigma,\mathfrak{b}$ of the latter, while the a of the former, and the $0,b_0$ of the latter, have no mates.

Take $z,w,p,\mathfrak{a},\sigma^n,\mathfrak{b},\mathfrak{c}$ as in LVII with completely defined $\sigma^n = \lambda t\{w\}(t,\mathfrak{a},\mathfrak{b},\mathfrak{c})$ for $n=1$ $(= \lambda\tau^{n-1}\{w\}(\alpha_1^n(\tau^{n-1}),\cdots,\alpha_q^n(\tau^{n-1}),\mathfrak{a},\mathfrak{b},\mathfrak{c})$ for $n>1)$, not assuming $\{z\}(\mathfrak{a},\sigma^n,\mathfrak{b})$ defined; we call $(\gamma_n'(z,w,p),\mathfrak{a},\mathfrak{b},\mathfrak{c})$ a *γ'-transform of* $(z,\mathfrak{a},\sigma^n,\mathfrak{b})$ *on* (the indicated occurrence of) σ^n.

We shall consider γ'-transforms of tuples $(z_0,\mathfrak{g}_0) \succ (z_1,\mathfrak{g}_1) \succ (z_2,\mathfrak{g}_2) \succ \cdots$ on corresponding occurrences of σ^n throughout $\mathfrak{g}_0,\mathfrak{g}_1,\mathfrak{g}_2,\cdots$ or an initial segment of $\mathfrak{g}_0,\mathfrak{g}_1,\mathfrak{g}_2,\cdots$. Specifically, an occurrence of σ^n in $\mathfrak{g}_0$ may be specified, and the occurrences in question are this and the corresponding occurrence in each of $\mathfrak{g}_1,\mathfrak{g}_2,\cdots$ so long as there is one (relative to choices of the $s+1$-positions in question); or no occurrence of σ^n in $\mathfrak{g}_0$ may be specified, and then the initial segment is empty. By a *γ'-transform of* $(z_i,\mathfrak{g}_i)$ *on* σ^n when such an occurrence of σ^n is missing from $\mathfrak{g}_i$ we mean $(z_i,\mathfrak{g}_i)$ itself.

9.6. LVIII. *Let* $(z_0,\mathfrak{g}_0) \succ (z,\mathfrak{g})$, *where a specified one of* $\mathfrak{g}_0$, *or none, is* σ^n. *Consider a given γ'-transform* $(z_0,\mathfrak{g}_0)^\dagger$ *of* $(z_0,\mathfrak{g}_0)$ *on* σ^n. *Then there is a γ'-transform* $(z,\mathfrak{g})^\dagger$ *of* $(z,\mathfrak{g})$ *on* σ^n *such that* $(z_0,\mathfrak{g}_0)^\dagger \succ (z,\mathfrak{g})^\dagger$.

Proof. From $(z_0,\mathfrak{g}_0) \succ (z,\mathfrak{g})$ we obtain $(z_0,\mathfrak{g}_0) \succ^\circ (z_1,\mathfrak{g}_1) \succ^\circ \cdots \succ^\circ (z_l,\mathfrak{g}_l) = (z,\mathfrak{g})$ $(l \geqq 1)$. Using the transitivity of $\succ$, it will accordingly suffice to prove the theorem with $\succ$ replaced by $\succ^\circ$ in the hypothesis (but not in the conclusion). Then if $\mathfrak{g}_0$ does not contain σ^n, neither does $\mathfrak{g}$, and the conclusion follows trivially.

In the alternative case, we change the notation to write $(z_0,\mathfrak{g}_0)$, $(z,\mathfrak{g})$ as $(z,\mathfrak{a},\sigma^n,\mathfrak{b})$, $(z_1,\mathfrak{g})$, taking $z,w,p,\mathfrak{a},\sigma^n,\mathfrak{b},\mathfrak{c}$ as in LVII with completely defined $\sigma^n = \lambda t\{w\}(t,\mathfrak{a},\mathfrak{b},\mathfrak{c})$ for $n = 1$ $(= \lambda\tau^{n-1}\{w\}(\alpha_1^n(\tau^{n-1}),\cdots,\alpha_q^n(\tau^{n-1}),\mathfrak{a},\mathfrak{b},\mathfrak{c})$ for $n > 1)$, not assuming $\{z\}(\mathfrak{a},\sigma^n,\mathfrak{b})$ defined. It will suffice, given that $(z,\mathfrak{a},\sigma^n,\mathfrak{b}) \succ^\circ (z_1,\mathfrak{g})$, to find a γ'-transform $(z_1,\mathfrak{g})^\dagger$ of $(z_1,\mathfrak{g})$ on σ^n such that $(\gamma_n'(z,w,p),\mathfrak{a},\mathfrak{b},\mathfrak{c}) \succ (z_1,\mathfrak{g})^\dagger$. The same cases come under consideration as in the proof of LVII.

CASE 5. $(z)_0 = 5$. By the first figure in Remark 8 restated for γ_n', $(z_1,\mathfrak{g})$ has to be one of $((z)_2,\mathfrak{d},\sigma,\mathfrak{b})$, $((z)_3,0,b_0,\mathfrak{d},\sigma,\mathfrak{b}),\cdots,((z)_3,a-1,b_{a-1},\mathfrak{d},\sigma,\mathfrak{b})$. Then by the second figure, it suffices to take as $(z_1,\mathfrak{g})^\dagger$ the corresponding one of $(\gamma_n'((z)_2,\mathrm{A},p),\mathfrak{d},\mathfrak{b},\mathfrak{c},a)$, $(\gamma_n'((z)_3,\mathrm{B},p),0,b_0,\mathfrak{d},\mathfrak{b},\mathfrak{c},a),\cdots,(\gamma_n'((z)_3,\mathrm{B},p),a-1,b_{a-1},\mathfrak{d},\mathfrak{b},\mathfrak{c},a)$.

CASE 8. $(z)_0 = 8$. SUBCASE 1. $n = j$ & $p = 0$ (only for $n > 1$). Then $\sigma = \alpha^j$ and $(z_1,\mathfrak{g})$ is $((z)_3,\sigma,\alpha^{j-2},\mathfrak{b})$ for some α^{j-2}. By the construction of $\gamma_n'(z,w,p)$, we have in particular (since $q > 0$) $(\gamma_n'(z,w,p),\mathfrak{b},\mathfrak{c}) \succ (\mathrm{D}_{q-1},\mathfrak{b},\mathfrak{c}) = (\mathrm{D}_{0,q-1},\mathfrak{b},\mathfrak{c}) = (\iota(\mathrm{C}_{q-1},1),\mathfrak{b},\mathfrak{c}) = (\mathrm{C}_{q-1},\mathfrak{b},\mathfrak{c}) \succ (\langle 8,(\mathrm{B})_1,j,\langle 6,(\mathrm{B})_1,j,q-1,\mathrm{B}\rangle\rangle,\mathfrak{b}_1,\mathfrak{c}_1)$ [where $\mathfrak{b}_1,\mathfrak{c}_1$ is $\mathfrak{b},\mathfrak{c}$ with α_q^j advanced to the front] $\succ (\langle 6,(\mathrm{B})_1,j,q-1,\mathrm{B}\rangle,\alpha^{j-2},\mathfrak{b}_1,\mathfrak{c}_1) \succ (\mathrm{B},\alpha^{j-2},\mathfrak{b},\mathfrak{c}) = (\gamma_n'((z)_3,\mathrm{A},0),\alpha^{j-2},\mathfrak{b},\mathfrak{c})$. But the last is a γ'-transform of $((z)_3,\sigma,\alpha^{j-2},\mathfrak{b})$ on σ. SUBCASE 2. $n \neq j \vee p \neq 0$. Then $(z_1,\mathfrak{g})$ is $((z)_3,\alpha^j,\alpha^{j-2},\mathfrak{d},\sigma,\mathfrak{b})$ for some α^{j-2}. But

$$(\gamma_n'(z,w,p),\alpha^j,\mathfrak{d},\mathfrak{b},\mathfrak{c}) \succ (\gamma_n'((z)_3,\mathrm{A},p+\overline{\mathrm{sg}}\,|\,n-(j \dot{-} 2)\,|),\alpha^j,\alpha^{j-2},\mathfrak{d},\mathfrak{b},\mathfrak{c}),$$

which is a γ'-transform of $(z_1,\mathfrak{g})$ on σ.

CASE 9. $(z)_0 = 9$. SUBCASE 1. $p \geqq (z)_{2,n}$. Then $\mathfrak{g}$ does not contain σ, so $(z_1,\mathfrak{g})$ is its own γ'-transform on σ; and $(\gamma_n'(z,w,p),\mathfrak{a},\mathfrak{b},\mathfrak{c}) \succ (z_1,\mathfrak{g})$. SUBCASE 2. $p < (z)_{2,n}$. Then $\mathfrak{a}$ is $(a,\mathfrak{d})$, $(z_1,\mathfrak{g})$ is $(a,\mathfrak{d},\sigma,\mathfrak{e})$, and, e.g. for $n > 1$,

$$\begin{aligned}
&(\gamma_n'(z,w,p),a,\mathfrak{d},\mathfrak{b},\mathfrak{c}) = (S^5(\mathrm{D},g,n,z,w,p),a,\mathfrak{d},\mathfrak{b},\mathfrak{c}) \succ (\mathrm{D},g,n,z,w,p,a,\mathfrak{d},\mathfrak{b},\mathfrak{c})\\
&\succ (\mathrm{C},\{\mathrm{A}\}(g,n,z,w,p,a,\mathfrak{d},\mathfrak{b},\mathfrak{c}),g,n,z,w,p,a,\mathfrak{d},\mathfrak{b},\mathfrak{c})\\
&= (\mathrm{C},(\overline{\mathrm{sg}}\,|(a)_1-(z)_2|)\cdot\gamma_n'(a,S^1(\langle 6,(w)_1,0,q,w\rangle,a),p),g,n,z,w,p,a,\mathfrak{d},\mathfrak{b},\mathfrak{c})\\
&\succ (\mathrm{B}_0,(\overline{\mathrm{sg}}\,|(a)_1-(z)_2|)\cdot\gamma_n'(a,S^1(\langle 6,(w)_1,0,q,w\rangle,a),p),\mathfrak{d},\mathfrak{b},\mathfrak{c},g,n,z,w,p,a)\\
&= (\mathrm{B}_0,\gamma_n'(a,S^1(\langle 6,(w)_1,0,q,w\rangle,a),p),\mathfrak{d},\mathfrak{b},\mathfrak{c},g,n,z,w,p,a)
\end{aligned}$$

[for, $(a)_1 = (z)_2$, since otherwise $(z,\mathfrak{a},\sigma,\mathfrak{b})$ would come under S9 with the a not an index for the $\mathfrak{b}$, and so would be the only tuple into the tree, contradicting $(z,\mathfrak{a},\sigma,\mathfrak{b}) \succ^\circ (z_1,\mathfrak{g})$]

$$\succ (\gamma_n'(a,S^1(\langle 6,(w)_1,0,q,w\rangle,a),p),\mathfrak{d},\mathfrak{b},\mathfrak{c}),$$

which is a γ'-transform of $(a,\mathfrak{d},\sigma,\mathfrak{e})$ on σ.

CASES 1, 2, 3, 7. $(z)_0 = 1,2,3,7$. These cases cannot occur, since the computation tree for $\{z\}(\mathfrak{a},\sigma,\mathfrak{b})$ would then have no further tuple $(z_1,\mathfrak{g})$.

REMARK 10. Except in Case 8 Subcase 1 we have the like for the function γ_n of XXII.

LIX. *If* $(z,\mathfrak{g})$ *comes under* S9 *with the a not an index for the* $\mathfrak{b}$, *and* $(z,\mathfrak{g})^\dagger$ *is a* γ'*-transform of* $(z,\mathfrak{g})$ *on* σ^n, *then either* $(z,\mathfrak{g})^\dagger$ *itself, or a tuple* $(u,\mathfrak{h})$ *with* $(z,\mathfrak{g})^\dagger \succ (u,\mathfrak{h})$, *comes under* S9 *with the a not an index for the* $\mathfrak{b}$.

Proof. If $\mathfrak{g}$ does not contain (a specified occurrence of) σ (so $(z,\mathfrak{g})^\dagger$ is $(z,\mathfrak{g})$), or $\mathfrak{g}$ contains σ but Subcase 1 of Case 9 for LVII applies, $(z,\mathfrak{g})^\dagger$ itself comes under S9 with the a not an index for the $\mathfrak{b}$. If $\mathfrak{g}$ contains σ and Subcase 2 applies, then, e.g. for $n > 1$ (cf. the calculation in Case 9 for LVIII),

$$\begin{aligned}(z,\mathfrak{g})^\dagger &= (\gamma_n'(z,w,p),a,\mathfrak{d},\mathfrak{b},\mathfrak{c})\\ &\succ (B_0,(\overline{\mathrm{sg}}\,|(a)_1 - (z)_2|)\cdot\gamma_n'(a,S^1(\langle 6,(w)_1,0,q,w\rangle,a),p),\mathfrak{d},\mathfrak{b},\mathfrak{c},g,n,z,w,p,a)\\ &\qquad\qquad [\text{where } B_0 = \langle 9,[(w)_1/2^{q-6}],[(w)_1/2^{q+1}]\rangle]\\ &= (B_0,0,\mathfrak{d},\mathfrak{b},\mathfrak{c},g,n,z,w,p,a)\end{aligned}$$

[for, a is not an index for the $\mathfrak{b}$, i.e. for $(\mathfrak{d},\sigma,\mathfrak{e})$, so either $(a)_1 \neq (z)_2$, or $\overline{\mathrm{Ix}}(a)$ whence $\gamma_n'(a,S^1(\langle 6,(w)_1,0,q,w\rangle,a),p) = 0$], which comes under S9 with the a not an index for the $\mathfrak{b}$ (since 0 is not an index).

LX. *In* LVII, *for values of* $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ *such that* $\lambda t\ \theta(t,\mathfrak{a},\mathfrak{b},\mathfrak{c})$ *for* $n = 1$ $(\lambda\tau^{n-1}\ \theta(\alpha_1^n(\tau^{n-1}),\cdots,\alpha_q^n(\tau^{n-1}),\mathfrak{a},\mathfrak{b},\mathfrak{c})$ *for* $n > 1)$ *is completely defined*, $= \sigma_0^n$ *say*, $\{\gamma_n'(z,w,p)\}(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ *is defined only if* $\phi(\mathfrak{a},\sigma_0^n,\mathfrak{b})$ *is defined.*

Proof. Assume $\phi(\mathfrak{a},\sigma_0^n,\mathfrak{b})$ $(=\{z\}(\mathfrak{a},\sigma_0^n,\mathfrak{b}))$ undefined. We shall infer that $\{\gamma_n'(z,w,p)\}(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ is undefined. By LIII, in the computation tree for $\{z\}(\mathfrak{a},\sigma_0,\mathfrak{b})$ there is a branch either infinite or terminating at an application of S9 with the a not an index for the $\mathfrak{b}$. If along this branch we take in order all but the tuples from the lower next positions to nodes by S5b (only finitely many of which can occur consecutively, since the second member a of the tuple is decreased by one between such a node and its lower next position), we obtain a sequence $(z,\mathfrak{a},\sigma_0,\mathfrak{b}) = (z_0,\mathfrak{g}_0) \succ^\circ (z_1,\mathfrak{g}_1) \succ^\circ (z_2,\mathfrak{g}_2) \succ^\circ \cdots$, either infinite or terminating at an application of S9 with the a not an index for the $\mathfrak{b}$. In both cases, by LVIII there is a corresponding sequence $(\gamma_n'(z,w,p),\mathfrak{a},\mathfrak{b},\mathfrak{c}) = (z_0,\mathfrak{g}_0)^\dagger \succ (z_1,\mathfrak{g}_1)^\dagger \succ (z_2,\mathfrak{g}_2)^\dagger \succ \cdots$ of γ'-transforms on σ_0. This is infinite in the first case; and in the second case, by LIX it or an extension of it terminates at an application of S9 with the a not an index for the $\mathfrak{b}$. Hence by LIII, $\{\gamma_n'(z,w,p)\}(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ is undefined.

9.7. Combining LVII and LX:

LXI. *In* LVII, *for values of* $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ *such that* $\lambda t\ \theta(t,\mathfrak{a},\mathfrak{b},\mathfrak{c})$ *for* $n = 1$ $(\lambda\tau^{n-1}\,\theta(\alpha_1^n(\tau^{n-1}),\cdots,\alpha_q^n(\tau^{n-1}),\mathfrak{a},\mathfrak{b},\mathfrak{c})$ *for* $n > 1)$ *is completely defined*:

$$\phi(\mathfrak{a},\lambda t\,\theta(t,\mathfrak{a},\mathfrak{b},\mathfrak{c}),\mathfrak{b}) \simeq \{\gamma_1'(z,w,p)\}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) \quad (n = 1), \tag{95}$$

$$\phi(\mathfrak{a},\lambda\tau^{n-1}\ \theta(\alpha_1^n(\tau^{n-1}),\cdots,\alpha_q^n(\tau^{n-1}),\mathfrak{a},\mathfrak{b},\mathfrak{c}),\mathfrak{b}) \simeq \{\gamma_n'(z,w,p)\}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) \quad (n > 1). \tag{96}$$

9.8. We shall now provide for indefinitely many substitutions of functions of a suitable class. To do this we adjoin the substitution schema S4.j of Remark 1 in 1.6 for a function of this class as the $\lambda\alpha^{j-1}\chi(\cdots)$ of the schema. The class will be the subclass of the functions $\lambda\tau\,\theta(\cdots)$ considered in 9.4–9.7 obtained by making the θ primitive recursive. This assures that $\lambda\tau\,\theta(\cdots)$ will be completely defined whatever the values of $\mathfrak{a},\mathfrak{b},\mathfrak{c}$. This subclass will suffice for our applications, but we could employ instead any other such subclass the indices w of the members of which constitute a general recursive class of numbers.

We accordingly define the *partial′* (*general′*) *recursive* functions in the same manner as the partial (general) recursive functions were defined in 3.7 except admitting the additional schema

S4′.1 $\quad \phi(\mathfrak{a}) \simeq \psi(\lambda t\,\theta(t,\mathfrak{a}),\mathfrak{a}) \quad \langle 4,\langle n_0,\cdots,n_r\rangle,g,h,1\rangle,$

where θ is primitive recursive, and

S4′.j ($j > 1$) $\quad \phi(\mathfrak{a}) \simeq \psi(\lambda\tau^{j-1}\,\theta(\alpha_1^j(\tau^{j-1}),\cdots,\alpha_{n_j}^j(\tau^{j-1}),\mathfrak{a}),\mathfrak{a}) \quad \langle 4,\langle n_0,\cdots,n_r\rangle,g,h,j\rangle,$

where $\mathfrak{a}$ includes exactly the $n_j > 0$ type-j variables $\alpha_1^j,\cdots,\alpha_{n_j}^j$ and $\theta(b_1,\cdots,b_{n_j},\mathfrak{a})$ is primitive recursive. The presence of the new schema extends the indexing, and indices in the new sense we call *indices′*. A function introduced by an application of S4′ shall have the index′ shown at the right opposite the schema, where g is an index′ of ψ, and h is a primitive recursive index of θ (cf. 4.1). Schema S9 requires restatement as

S9′. $\quad \phi(a,\mathfrak{b},\mathfrak{c}) \simeq \{a\}'(\mathfrak{b}) \quad \langle 9,\langle n_0,\cdots,n_r\rangle,\langle m_0,\cdots,m_s\rangle\rangle,$

where $\{a\}'(\mathfrak{b})$ is the value for $\mathfrak{b}$ as arguments of the function with the index′ a, if a is an index′ for $\mathfrak{b}$ and that value is defined, and is undefined otherwise.

In a *computation tree for* $\{z\}'(\mathfrak{a})$ (cf. 9.1), an application of S4′.1 (S4′.j, $j > 1$) constitutes a new kind of "node", with infinitely many lower next positions, corresponding to the values of t (of τ^{j-1}) and giving rise collectively to a function rather than a number, namely the function $\lambda t\,\theta(t,\mathfrak{a})$ (the function $\lambda\tau^{j-1}\,\theta(\alpha_1^j(\tau^{j-1}),\cdots,\alpha_{n_j}^j(\tau^{j-1}),\mathfrak{a})$) for use as the first argument in the tuple for $\psi(\lambda t\,\theta(t,\mathfrak{a}),\mathfrak{a})$ (for $\psi(\lambda\tau^{j-1}\,\theta(\cdots),\mathfrak{a})$) at the upper next position. Since the θ is primitive recursive, the tuple at each lower next position has a value, and so the upper next position will always be in the tree.

9.9. In an irredundant partial′ recursive description of a function $\phi(\mathfrak{a})$ of variables $\mathfrak{a}$ of maximum type $r \geqq 1$, each function has the same maximum type r of its variables (cf. I); and hence the r does not increase (unless from 0 to 1) in computing $\{z\}'(\mathfrak{a})$ (cf. 5.1). We have versions II′, III′, XII′–XVI′, XVIII′–XXI′, LIII′ of II, III, XII–XVI, XVIII–XXI, LIII (in III′, applications of S6 may have to be added in placing new b's in the θ's of applications of S4′.j, $j > 1$).

9.10. LXII. *Each partial′ (general′) recursive function is partial (general) ecursive.*

Proof. (A) *There is a primitive recursive function* δ *such that, for each index'* z *for* $\mathfrak{a}$, *if* $\{z\}'(\mathfrak{a})$ *is defined, then* ($\{\delta(z)\}(\mathfrak{a})$ *is defined and*) $\{z\}'(\mathfrak{a}) = \{\delta(z)\}(\mathfrak{a})$.

We take $\delta(z) = 0$, except for $\mathrm{Ix}'(z)$ (using XIX′), when the appropriate following case shall apply. We use induction on $\{z\}'(-)$ (analogous to 3.8) in proving that, when $\{z\}'(\mathfrak{a})$ is defined, $\{z\}'(\mathfrak{a}) = \{\delta(z)\}(\mathfrak{a})$. That a primitive recursive δ exists satisfying all the specifications will follow using the recursion theorem (as for γ_n of XXII but more simply).

Cases 1, 2, 3, 7. $(z)_0 = 1, 2, 3, 7$. Let $\delta(z) = z$.

Case 4. $(z)_0 = 4 \,\&\, (z)_4 = 0$. Let $\delta(z) = \langle 4, (z)_1, \delta((z)_2), \delta((z)_3)\rangle$. By the case, $\{z\}'(\mathfrak{a}) \simeq \{(z)_2\}'(\{(z)_3\}'(\mathfrak{a}), \mathfrak{a})$. Assume $\{z\}'(\mathfrak{a})$ defined. Then $\{(z)_3\}'(\mathfrak{a})$ is defined, $= b$ say, and $\{(z)_2\}'(b,\mathfrak{a})$ is defined, $= \{z\}'(\mathfrak{a})$. By the hyp. ind., $b = \{\delta((z)_3)\}(\mathfrak{a})$ and $\{z\}'(\mathfrak{a}) = \{\delta((z)_2)\}(b,\mathfrak{a})$. Now $\{\delta(z)\}(\mathfrak{a}) = \{\delta((z)_2)\}(\{\delta((z)_3)\}(\mathfrak{a}), \mathfrak{a}) = \{z\}'(\mathfrak{a})$.

Case 4′. $(z)_0 = 4 \,\&\, (z)_4 > 0$. Write $j = (z)_4$. Let $\delta(z) = \gamma_j'(\delta((z)_2), (z)_3, 0)$. By the case, $\mathrm{PRI}((z)_3)$, and $\{z\}'(\mathfrak{a}) \simeq \{(z)_2\}'(\sigma^j, \mathfrak{a})$ where $\sigma^j = \lambda t\{(z)_3\}(t, \mathfrak{a})$ if $j = 1$ ($= \lambda\tau^{j-1}\{(z)_3\}(\alpha_1^j(\tau^{j-1}), \cdots, \alpha_{n_j}^j(\tau^{j-1}), \mathfrak{a})$ if $j > 1$). Assume $\{z\}'(\mathfrak{a})$ defined. Then $\{(z)_2\}'(\sigma^j, \mathfrak{a})$ is defined, and by hyp. ind. $= \{\delta((z)_2)\}(\sigma^j, \mathfrak{a})$. By LVII, $\{\delta((z)_2)\}(\sigma^j, \mathfrak{a}) = \{\gamma_j'(\delta((z)_2), (z)_3, 0)\}(\mathfrak{a})$. So $\{z\}'(\mathfrak{a}) = \{\delta(z)\}(\mathfrak{a})$.

Case 9′. $(z)_0 = 9$. Then $\{z\}'(a, \mathfrak{b}, \mathfrak{c}) \simeq \{a\}'(\mathfrak{b})$. Assume $\{z\}'(a, \mathfrak{b}, \mathfrak{c})$ defined. Then $\{a\}'(\mathfrak{b})$ is defined, so a is an index′ for $\mathfrak{b}$, so $(a)_1 = (z)_2$. By hyp. ind., $\{a\}'(\mathfrak{b}) = \{\delta(a)\}(\mathfrak{b})$. So $\{z\}'(a, \mathfrak{b}, \mathfrak{c}) = \{(\overline{\mathrm{sg}}\,|(a)_1 - (z)_2|) \cdot \delta(a)\}(\mathfrak{b})$. It will suffice to take $\delta(z) = S^2(\mathrm{D}, d, z)$ where D is an index of $\lambda dza\mathfrak{b}\mathfrak{c}\{(\overline{\mathrm{sg}}\,|(a)_1 - (z)_2|) \cdot \{d\}(a)\}(\mathfrak{b})$ and d is an index of δ. To obtain D, pick an index e of $\lambda dza(\overline{\mathrm{sg}}\,|(a)_1 - (z)_2|) \cdot \{d\}(a)$, let $\mathrm{A} = \iota(e, [(z)_1/2])$, $\mathrm{B}_0 = \langle 9, 2^3 \cdot (z)_1, (z)_2\rangle$, and define B_{i+1}, C, D as for XXII Case 9 Subcase 2 (except that in C, 4 becomes 3).

(B) *For the* $\delta, z, \mathfrak{a}$ *of* (A), *if* $\{z\}'(\mathfrak{a})$ *is undefined, so is* $\{\delta(z)\}(\mathfrak{a})$.

Assume $\{z\}'(\mathfrak{a})$ undefined. Then by LIII′, in the computation tree for $\{z\}'(\mathfrak{a})$ there is a branch, either infinite or terminating at an application of S9′ with the a not an index′ for the $\mathfrak{b}$, along which each tuple has no value. Define $\succ'$, $\succ^{\circ\prime}$ analogously to $\succ$, $\succ^{\circ}$ in 9.5. Omitting from this branch the tuples at lower next positions to nodes by S5b (only finitely many consecutively), we find a sequence $(z, \mathfrak{a}) = (z_0, \mathfrak{g}_0) \succ^{\circ\prime} (z_1, \mathfrak{g}_1) \succ^{\circ\prime} (z_2, \mathfrak{g}_2) \succ^{\circ\prime} \cdots$, either infinite or terminating with $(z_l, \mathfrak{g}_l)$ coming under S9′ with the a not an index′ for the $\mathfrak{b}$. To this sequence we shall correlate another sequence $(\delta(z), \mathfrak{a}) = (z_0^*, \mathfrak{g}_0^*), (z_1^*, \mathfrak{g}_1^*), (z_2^*, \mathfrak{g}_2^*), \cdots$ so that (i) $(z_i^*, \mathfrak{g}_i^*) = (z_{i+1}^*, \mathfrak{g}_{i+1}^*)$ ($(z_i^*, \mathfrak{g}_i^*) \succ (z_{i+1}^*, \mathfrak{g}_{i+1}^*)$) if $(z_{i+1}, \mathfrak{g}_{i+1})$ comes from $(z_i, \mathfrak{g}_i)$ by an application of S4′ (otherwise), and (ii) if $(z_l, \mathfrak{g}_l)$ comes under S9′ with the a not an index′ for the $\mathfrak{b}$, there is a $(u, \mathfrak{h})$ with $(z_l^*, \mathfrak{g}_l^*) \succ (u, \mathfrak{h})$ which comes under S9 with the a not an index for the $\mathfrak{b}$. The first case under (i) can occur consecutively only finitely many times, since in an application of S4′ the first member of the tuple is decreased. Hence when the original sequence is infinite, $(z_i^*, \mathfrak{g}_i^*) \succ (z_{i+1}^*, \mathfrak{g}_{i+1}^*)$ holds for infinitely many i. Using (ii) also, it will follow by LIII that $\{\delta(z)\}(\mathfrak{a})$ is undefined.

We define $(z_i^*,\mathfrak{g}_i^*)$ by recursion on i (simultaneously establishing (i)) so that it is a *k_i-fold γ'-transform* $(\delta(z_i),\mathfrak{g}_i)^\#$ of $(\delta(z_i),\mathfrak{g}_i)$, i.e. the result of performing γ'-transformations successively on k_i objects $\sigma_1^{n_1},\cdots,\sigma_{k_i}^{n_{k_i}}$ $(k_i \geqq 0)$. BASIS. $(z_0^*,\mathfrak{g}_0^*) = (\delta(z),\mathfrak{a})(k_0 = 0)$. IND. STEP. Given $(z_i^*,\mathfrak{g}_i^*) = (\delta(z_i),\mathfrak{g}_i)^\#$, and that $(z_{i+1},\mathfrak{g}_{i+1})$ exists, consider the case by which $(z_i,\mathfrak{g}_i) \succ^{\circ\prime} (z_{i+1},\mathfrak{g}_{i+1})$. In all but Case 4′ $(\delta(z_i),\mathfrak{g}_i) \succ (\delta(z_{i+1}),\mathfrak{g}_{i+1})$, as is obvious in Cases 4, 5, 6, 8 (1, 2, 3, 7 cannot occur), and in Case 9′ is confirmed by the following calculation:

$$
\begin{aligned}
(\delta(z_i),\mathfrak{g}_i) &= (S^2(\mathrm{D},d,z),a,\mathfrak{b},\mathfrak{c})\\
&\succ (\mathrm{D},d,z,a,\mathfrak{b},\mathfrak{c})\\
&\succ (\mathrm{C},\{\mathrm{A}\}(d,z,a,\mathfrak{b},\mathfrak{c}),d,z,a,\mathfrak{b},\mathfrak{c})\\
&= (\mathrm{C},(\overline{\mathrm{sg}}\,|(a)_1 - (z)_2|)\cdot\delta(a),d,z,a,\mathfrak{b},\mathfrak{c})\\
&\succ (\mathrm{B}_0,(\overline{\mathrm{sg}}\,|(a)_1 - (z)_2|)\cdot\delta(a),\mathfrak{b},\mathfrak{c},d,z,a) \ [\text{where } \mathrm{B}_0 = \langle 9,2^3\cdot(z)_1,(z)_2\rangle]\\
&= (\mathrm{B}_0,\delta(a),\mathfrak{b},\mathfrak{c},d,z,a) \ [\text{for, } a \text{ is an index}' \text{ for } \mathfrak{b}, \text{ since } (z_{i+1},\mathfrak{g}_{i+1}) \text{ exists,}\\
&\qquad\qquad \text{so } (a)_1 = (z)_2]\\
&\succ (\delta(a),\mathfrak{b}) = (\delta(z_{i+1}),\mathfrak{g}_{i+1}).
\end{aligned}
$$

Applying LVIII k_i times successively to $(\delta(z_i),\mathfrak{g}_i) \succ (\delta(z_{i+1}),\mathfrak{g}_{i+1})$, we obtain a k_i-fold γ'-transform $(\delta(z_{i+1}),\mathfrak{g}_{i+1})^\#$ of $(\delta(z_{i+1}),\mathfrak{g}_{i+1})$ such that $(\delta(z),\mathfrak{g}_i)^\# \succ (\delta(z_{i+1}),\mathfrak{g}_{i+1})^\#$. Letting $k_{i+1} = k_i$ and $(z_{i+1}^*,\mathfrak{g}_{i+1}^*) = (\delta(z_{i+1}),\mathfrak{g}_{i+1})^\#$, thus $(z_i^*,\mathfrak{g}_i^*) \succ (z_{i+1}^*,\mathfrak{g}_{i+1}^*)$. In Case 4′, $(z_{i+1},\mathfrak{g}_{i+1})$ is at the upper next position, since the tuples at the lower next positions all have values (end 9.8); so $(z_{i+1},\mathfrak{g}_{i+1}) = ((z_i)_2,\mathfrak{g}_{i+1})$. Taking

$$(z_{i+1}^*,\mathfrak{g}_{i+1}^*) = (z_i^*,\mathfrak{g}_i^*) = (\delta(z_i),\mathfrak{g}_i)^\# = (\gamma_j'(\delta((z_i)_2),(z_i)_3,0),\mathfrak{g}_i)^\#,$$

$(z_{i+1}^*,\mathfrak{g}_{i+1}^*)$ is a $k_i + 1$-fold γ'-transform of $(\delta(z_{i+1}),\mathfrak{g}_{i+1})$ $(k_{i+1} = k_i + 1)$.

Finally, (ii) will follow from the lemma: *If $(z,\mathfrak{g})$ comes under* S9′ *with the a not an index′ for the* $\mathfrak{b}$, *and $(z^*,\mathfrak{g}^*)$ is a k-fold γ'-transform of $(\delta(z),\mathfrak{g})$, then there is a $(u,\mathfrak{h})$ with $(z^*,\mathfrak{g}^*) \succ (u,\mathfrak{h})$ which comes under* S9 *with the a not an index for the* $\mathfrak{b}$. This we prove by induction on k. BASIS. $(z^*,\mathfrak{g}^*) = (\delta(z),\mathfrak{g})$. Now $(\delta(z),\mathfrak{g}) \succ (\mathrm{B}_0,(\overline{\mathrm{sg}}|(a)_1 - (z)_2|)\cdot\delta(a),\mathfrak{b},\mathfrak{c},d,z,a)$ [where $\mathrm{B}_0 = \langle 9,2^3\cdot(z)_1,(z)_2\rangle$] $= (\mathrm{B}_0,0,\mathfrak{b},\mathfrak{c},d,z,a)$ [for now a is not an index′ for $\mathfrak{b}$, so either $(a)_1 \neq (z)_2$, or $\overline{\mathrm{Ix}}'(a)$ whence $\delta(a) = 0$], which comes under S9 with the a not an index for the $\mathfrak{b}$ [since 0 is not an index]. IND. STEP. $(z^*,\mathfrak{g}^*)$ is a $k + 1$-fold γ'-transform of $(\delta(z),\mathfrak{g})$. Then there is a k-fold γ'-transform $(z_1^*,\mathfrak{g}_1^*)$ of $(\delta(z),\mathfrak{g})$ with $(z^*,\mathfrak{g}^*)$ a simple γ'-transform $(z_1^*,\mathfrak{g}_1^*)^\dagger$ of $(z_1^*,\mathfrak{g}_1^*)$. By hyp. ind., there is a $(u_1,\mathfrak{h}_1)$ with $(z_1^*,\mathfrak{g}_1^*) \succ (u_1\mathfrak{h}_1)$ which comes under S9 with the a not an index for the $\mathfrak{b}$. By LVIII, $(z_1^*,\mathfrak{g}_1^*)^\dagger \succ (u_1,\mathfrak{h}_1)^\dagger$ for a suitable simple γ'-transform $(u_1,\mathfrak{h}_1)^\dagger$ of $(u_1,\mathfrak{h}_1)$.

By LIX, either $(u_1,\mathfrak{h}_1)^\dagger$ itself (call it $(u,\mathfrak{h})$), or a $(u,\mathfrak{h})$ with $(u_1,\mathfrak{h}_1)^\dagger \succ (u,\mathfrak{h})$, comes under S9 with the a not an index for the $\mathfrak{b}$. Now $(z^*,\mathfrak{g}^*) = (z_1^*,\mathfrak{g}_1^*)^\dagger \succ (u,\mathfrak{h})$.

9.11. LXIII. *Each partial (general) recursive function is partial′ (general′) recursive.*

Proof. (A) *There is a primitive recursive function* δ' *such that, for each index* z *for* $\mathfrak{a}$, *if* $\{z\}(\mathfrak{a})$ *is defined, then* $\{z\}(\mathfrak{a}) = \{\delta'(z)\}'(\mathfrak{a})$.

Let $\delta'(z) = 0$ unless $\mathrm{Ix}(z)$.

CASE 4. $(z)_0 = 4$. Let $\delta'(z) = \langle 4,(z)_1,\delta'((z)_2),\delta'((z)_3)\rangle$.

CASE 9. $(z)_0 = 9$. We want $\{\delta'(z)\}'(a,\mathfrak{b},\mathfrak{c}) \simeq \{(\overline{\mathrm{sg}}\,|(a)_1 - (z)_2|)\cdot\delta'(a)\}'(\mathfrak{b})$, which we obtain by taking $\delta'(z) = S^2(\mathrm{D},d,z)$ etc. as for LXII.

(B) *For the* $\delta',z,\mathfrak{a}$ *of* (A), *if* $\{z\}(\mathfrak{a})$ *is undefined, so is* $\{\delta'(z)\}'(\mathfrak{a})$.

Given, in the computation tree for $\{z\}(\mathfrak{a})$, a branch $(z,\mathfrak{a}) = (z_0,\mathfrak{g}_0)$, $(z_1,\mathfrak{g}_1)$, $(z_2,\mathfrak{g}_2),\cdots$ either infinite or terminating in a tuple $(z_l,\mathfrak{g}_l)$ coming under S9 with the a not an index for the $\mathfrak{b}$, we obtain the like in the computation tree for $\{\delta'(z)\}(\mathfrak{a})$ by correlating $(\delta'(z_i),\mathfrak{g}_i)$ to $(z_i,\mathfrak{g}_i)$, and whenever $(z_i,\mathfrak{g}_i)$ comes under S9 supplying after $(\delta'(z_i),\mathfrak{g}_i)$ $(= (S^2(\mathrm{D},d,z),a,\mathfrak{b},\mathfrak{c}))$ the tuples which lead from it to $(\mathrm{B}_0,(\overline{\mathrm{sg}}\,|(a)_1 - (z)_2|)\cdot\delta'(a),\mathfrak{b},\mathfrak{c},d,z,a)$ where B_0 is $\langle 9,2^3\cdot(z)_1,(z)_2\rangle$ and, when a is an index for $\mathfrak{b}$, further to $(\delta'(a),\mathfrak{b})$.

10. **The first recursion theorem. 10.1.** We seek a version of the first recursion theorem IM p. 348 (also cf. pp. 234, 326). The equation $\zeta(x_1,\cdots,x_n) \simeq \mathbf{F}(\zeta;x_1,\cdots,x_n)$ of IM becomes

$$\text{(a)} \qquad \zeta(\mathfrak{a}) \simeq \mathbf{F}(\zeta;\mathfrak{a}),$$

where $\mathfrak{a}$ is a list of variables of our types 1.2, ζ ranges over partial functions of $\mathfrak{a}$, and $\mathbf{F}(\zeta;\mathfrak{a})$ belongs to a suitable class of functionals.

We now restrict $\mathbf{F}(\zeta;\mathfrak{a})$ to belong to the class of the 'normal recursive functionals', which will suffice for our applications. In 10.6 we shall see that some such restriction is necessary.

A *normal recursive functional* $\mathbf{F}(\zeta;\mathfrak{a})$ is one describable (i.e. $\lambda\mathfrak{a}\,\mathbf{F}(\zeta;\mathfrak{a})$ is derivable uniformly from ζ; cf. 1.8, 1.9) using the simple schema

$$\text{S0}' \qquad \phi(\mathfrak{b},\mathfrak{c}) \simeq \zeta(\mathfrak{b}) \qquad \langle 0,\langle n_0,\cdots,n_r\rangle,\langle m_0,\cdots,m_s\rangle\rangle$$

to introduce ζ, and further the schemata S1–S8, S4′ $(=$ S4′.j for $j \geqq 1$, 9.8), and

$$\text{S5}' \qquad \begin{cases} \phi(0,\mathfrak{b}) \simeq \psi(\mathfrak{b}), \\ \phi(a',\mathfrak{b}) \simeq \chi(\mathfrak{b}) \end{cases} \qquad \langle 5,\langle n_0,\cdots,n_r\rangle,g,h\rangle.$$

The ψ's and χ's for the applications of S4–S6, S8, S4′, S5′ are to be functions previously described using the same schemata S0′, S1–S8, S4′, S5′, but the θ's for S4′ are to be primitive recursive, i.e. describable using only S1–S8. The function variable ζ (introduced by S0′) ranges over partial functions (of $m_0,\cdots,m_s$ variables of types $0,\cdots,s$, respectively), like the $\psi_1,\cdots,\psi_l$ last sentence of 3.14.

S5′ gives a strong form of definition by cases; in an application of S5′, to compute $\phi(a,\mathfrak{b})$, when $a = 0$ we compute only $\psi(\mathfrak{b})$, and when $a \neq 0$ we compute only $\chi(\mathfrak{b})$ (cf. XV and IM Theorem XX(c) p. 337[20]), in contrast to IM #F p. 229 and Theorem XVII(b) p. 329).

10.2. By *the function* ϕ *computed by* the recursion (a), for a normal recursive $\mathbf{F}$, we mean the function $\phi(\mathfrak{a})$ such that, for each $\mathfrak{a}$, $\phi(\mathfrak{a})$ has the value obtained for $\zeta(\mathfrak{a})$ by the following computation procedure when this procedure leads to a value, and $\phi(\mathfrak{a})$ is undefined otherwise. The procedure consists in identifying $\zeta(\mathfrak{a})$ with $\mathbf{F}(\zeta;\mathfrak{a})$, computing (or attempting to compute) the latter, via the applications of S1–S8, S4′, S5′ used in its description, from $\zeta(\mathfrak{b}_\gamma)$ for certain $\mathfrak{b}_\gamma$'s as called for by the applications of S0′, while computing those $\zeta(\mathfrak{b}_\gamma)$'s (if they can be computed) by the same procedure. (Similarly for other sorts of functionals $\mathbf{F}(\zeta;\mathfrak{a})$ such that, in the computation of $\mathbf{F}(\zeta;\mathfrak{a})$ for given $\mathfrak{a}$, values of $\zeta(\mathfrak{b}_\gamma)$ are called for only for certain $\mathfrak{b}_\gamma$'s arising in the course of the computation. How the $\mathfrak{b}_\gamma$'s arise will be clear for each sort of $\mathbf{F}(\zeta;\mathfrak{a})$ considered.)

Indices (similar to the indices of 3.5, 3.7 and the indices′ of 9.8) can be used in formulating the computation procedure more explicitly. In general, indices serve as code numbers to say at each stage of a computation what schema applies and thus what step is to be performed next. In the present situation with $\mathbf{F}$ normal recursive, the indices are those $d_1, \cdots, d_k$ determined in the usual manner (cf. 3.5) from a description $\phi_1, \cdots, \phi_k$ of $\mathbf{F}(\zeta;\mathfrak{a})$ by the present schemata. Using (a), we identify $\zeta(\mathfrak{a})$ with the last function $\phi_k(\mathfrak{a})$ ($\simeq \mathbf{F}(\zeta;\mathfrak{a})$), having the index d_k. At an application of S0′, the computation does not terminate with a value of $\zeta(\mathfrak{b}_\gamma)$ being supplied by an "oracle" (with indefinition if $\zeta(\mathfrak{b}_\gamma)$ is undefined), as it would in simply computing $\mathbf{F}(\zeta;\mathfrak{a})$ for ζ an "ultimate" function argument; but instead, the recursion (a) is utilized again with the index at the next step becoming d_k.

As in 9.1 and end 9.8, the computation of $\phi(\mathfrak{a})$ by (a) can be arranged on a tree. The 0-position is occupied by the tuple $(d_k,\mathfrak{a})$. After a position γ occupied by a tuple $(d_\gamma, \mathfrak{b}_\gamma, \mathfrak{c}_\gamma)$ coming under S0′ (i.e. with $(d_\gamma)_0 = 0$), there is a single next position occupied by $(d_k,\mathfrak{b}_\gamma)$.

We have an analog of induction on $\{z\}(-)$ (end 3.8) for the present computation procedure and indices. This may be taken in the course-of-values version, so that in proving a property of $(d,\mathfrak{a})$, where d is any one of $d_1, \cdots, d_k$ and $(d,\mathfrak{a})$ has a value, we assume $(d_\gamma, \mathfrak{a}_\gamma)$ to have the property for any position γ after the 0-position of the computation tree for $(d,\mathfrak{a})$. We call this *induction over the computation* (*of* the value of $(d, \mathfrak{a})$, or, when d is d_k, *of* $\phi(\mathfrak{a})$) *by the recursion* (a).

10.3. LXIV. *If* $\mathbf{F}(\zeta;\mathfrak{a})$ *is normal recursive, the function* ϕ *computed by* (a) *is a solution of* (a) *for* ζ *such that any solution* ϕ' *for* ζ *is an extension of* ϕ, *and this solution* ϕ *is partial recursive.*

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(20) On IM p. 338, after "SECOND METHOD" insert ", for $Q_1, \cdots, Q_m$ simultaneously defined".

Proof. (i) *For each* $\mathfrak{a}$, $\phi(\mathfrak{a}) \simeq \mathbf{F}(\phi;\mathfrak{a})$. By the computation procedure for the function ϕ, we compute (or attempt to compute) $\phi(\mathfrak{a})$, for given $\mathfrak{a}$, by computing $\mathbf{F}(\zeta;\mathfrak{a})$ from $\zeta(\mathfrak{b}_y)$ for certain $\mathfrak{b}_y$'s, where $\zeta(\mathfrak{b}_y)$ is taken to be whatever we obtain (a value, or indefinition, making $\mathbf{F}(\zeta;\mathfrak{a})$ undefined, if we obtain no value) for $\mathbf{F}(\zeta;\mathfrak{b}_y)$ by the same procedure. So, for each of the $\mathfrak{b}_y$'s, $\zeta(\mathfrak{b}_y) \simeq \phi(\mathfrak{b}_y)$. Thus the computation procedure makes $\phi(\mathfrak{a}) \simeq \mathbf{F}(\phi;\mathfrak{a})$.

(ii) *If, for each* $\mathfrak{a}$, $\phi'(\mathfrak{a}) \simeq \mathbf{F}(\phi';\mathfrak{a})$, *then* ϕ' *is an extension of* ϕ. Consider any $\mathfrak{a}$ for which $\phi(\mathfrak{a})$ is defined; we must show that $\phi'(\mathfrak{a})$ (is defined and) $= \phi(\mathfrak{a})$. But $\phi(\mathfrak{a}) =$ the result $\mathbf{F}(\phi;\mathfrak{a})$ of computing $\mathbf{F}(\zeta;\mathfrak{a})$ using as value of the $\zeta(\mathfrak{b}_y)$ at each application of S0′ the number $\phi(\mathfrak{b}_y)$. By the hypothesis of an induction over the computation of $\phi(\mathfrak{a})$ by the recursion (a), for each such $\mathfrak{b}_y$, $\phi'(\mathfrak{b}_y) = \phi(\mathfrak{b}_y)$. So the computation makes $\phi(\mathfrak{a}) = \mathbf{F}(\phi;\mathfrak{a}) = \mathbf{F}(\phi';\mathfrak{a})$. This with $\phi'(\mathfrak{a}) \simeq \mathbf{F}(\phi';\mathfrak{a})$ gives $\phi'(\mathfrak{a}) = \phi(\mathfrak{a})$.

(iii) ϕ *is partial recursive.*

To prove this, first we pick a variable z not used in writing a given description of $\mathbf{F}(\zeta;\mathfrak{a})$ by S0′, S1–S8, S4′, S5′. Consider each application of S0′ in this description; say $\phi_i(\mathfrak{b}_i,\mathfrak{c}_i) \simeq \zeta(\mathfrak{b}_i)$ is introduced. Replace ζ by $\{z\}'$, so that instead $\phi_i'(z, \mathfrak{b}_i, \mathfrak{c}_i) \simeq \{z\}'(\mathfrak{b}_i)$ is introduced by an application of S9′ (cf. 9.8), and follow this by a series of applications of S6 to bring the variable z to the end (cf. II), so that altogether the application of S9′ and the applications of S6 introduce $\phi_i(\mathfrak{b}_i, \mathfrak{c}_i, z) \simeq \phi_i'(z, \mathfrak{b}_i, \mathfrak{c}_i) \simeq \{z\}'(\mathfrak{b}_i)$. This "spoils" the description, but as in the proof of III it becomes a description again when we further replace each other function in the original description by a function with z inserted as an additional variable at the end. To this description we suffix an application of S6 to bring the variable z to the front in the final function $\psi(z,\mathfrak{a})$.

Next we eliminate successively the applications of S5′, each time operating on the earliest one remaining in the description. This application introduces say $\phi(a,\mathfrak{b},z)$, where $\phi(a,\mathfrak{b},z) \simeq \psi(\mathfrak{b},z)$ if $a = 0$, $\simeq \chi(\mathfrak{b},z)$ if $a \neq 0$. Adapting the proof of XV′ (cf. 9.9, 3.13), we insert an additional variable a (distinct from those already used) at the end in the parts of the description leading to $\psi(\mathfrak{b},z)$ and $\chi(\mathfrak{b},z)$, and bring the a to the front by S6. Thus we obtain descriptions of $\psi(a,\mathfrak{b},z) \simeq \psi(\mathfrak{b},z)$ and $\chi(a,\mathfrak{b},z) \simeq \chi(\mathfrak{b},z)$. In these, only S1–S8, S4′, S9′ are used, so $\psi(a,\mathfrak{b},z)$ and $\chi(a,\mathfrak{b},z)$ have indices′ e_0 and e_1, respectively. We now introduce

$$\pi(a, \mathfrak{b}, z) = \begin{cases} e_0 \text{ if } a = 0, \\ e_1 \text{ if } a \neq 0 \end{cases}$$

by S1–S6, $\rho(c,a,\mathfrak{b},z) \simeq \{c\}'(a,\mathfrak{b},z)$ by S9′, and finally $\phi(a,\mathfrak{b},z) \simeq \rho(\pi(a,\mathfrak{b},z),a,\mathfrak{b},z)$ $\simeq \{\pi(a,\mathfrak{b},z)\}'(a,\mathfrak{b},z)$ by S4.

Now the given description of the normal recursive functional $\mathbf{F}(\zeta;\mathfrak{a})$ by S0′, S1–S8, S4′, S5′ has been transformed into a description of a partial′ recursive function $\psi(z,\mathfrak{a})$ by S1–S8, S4′, S9′. By the recursion theorem XIV′ (cf. 9.9, 3.12), there is an index′ e of $\psi(e,\mathfrak{a})$, i.e. a number e such that

$$(97) \qquad \{e\}'(\mathfrak{a}) \simeq \psi(e,\mathfrak{a}).$$

For (B) below, we shall assume a particular construction of this e using the method of proof of XIV′. We shall show ((A) and (B)) that $\phi(\mathfrak{a}) \simeq \{e\}'(\mathfrak{a})$. The partial recursiveness of $\phi(\mathfrak{a})$ will follow by LXII, since $\{e\}'(\mathfrak{a})$ is partial′ recursive (by XII′).

(A) *For each* $\mathfrak{a}$, *if* $\phi(\mathfrak{a})$ *is defined, then* $\{e\}'(\mathfrak{a})$ (*is defined and*) $= \phi(\mathfrak{a})$. Consider the part of the computation of $\phi(\mathfrak{a})$ by (a) which consists in computing $\mathbf{F}(\zeta;\mathfrak{a})$ from the $\zeta(\mathfrak{b}_y)$'s introduced by S0′ and evaluated as $\phi(\mathfrak{b}_y)$. By the hypothesis of an induction over the computation of $\phi(\mathfrak{a})$ by (a), for each such $\mathfrak{b}_y$, $\{e\}'(\mathfrak{b}_y) = \phi(\mathfrak{b}_y)$. Via the foregoing transformation of the given description of $\mathbf{F}(\zeta;\mathfrak{a})$ into one of $\psi(z,\mathfrak{a})$, there corresponds to the said part of the computation of $\phi(\mathfrak{a})$ a computation of $\psi(e,\mathfrak{a})$ from the $\{e\}'(\mathfrak{b}_y)$'s introduced by the corresponding applications of S9′ which gives $\psi(e,\mathfrak{a}) = \mathbf{F}(\phi;\mathfrak{a}) = \phi(\mathfrak{a})$. Using (97), $\{e\}'(\mathfrak{a}) = \phi(\mathfrak{a})$.

(B) *For each* $\mathfrak{a}$, *if* $\{e\}'(\mathfrak{a})$ *is defined, so is* $\phi(\mathfrak{a})$. We assume $e = S^1(f,f)$ where $S^1(u,y) = \langle 4,[(u)_1/2],u,\langle 2,[(u)_1/2],y\rangle\rangle$, and f is the index′ of $\lambda y\mathfrak{a}\ \psi(S^1(y,y),\mathfrak{a})$ constructed as follows. Let g be the index′ of $\lambda z\mathfrak{a}y\ \psi(z,\mathfrak{a})$ which results by introducing the new variable y at the end into each of the functions in the above description of $\psi = \lambda z\mathfrak{a}\ \psi(z,\mathfrak{a})$. Let $g_0 = \pi(0,1,\langle 6,(g)_1,0,(g)_{1,0} \dot{-} 1,g\rangle)$, which is an index′ of $\lambda zy\mathfrak{a}\,\psi(z,\mathfrak{a})$. Let h be an index′ of $\lambda y\mathfrak{a}\,S^1(y,y)$. Let $f = \langle 4,(h)_1,g_0,h\rangle$. Now along a branch of the computation tree for $\{e\}'(\mathfrak{a})$ including the upper next position at each node, there occur in order (via only S4 and S6) the tuples $(e,\mathfrak{a})$, $(f,f,\mathfrak{a})$, $(g_0,e,f,\mathfrak{a})$, $(g,e,\mathfrak{a},f)$. Thus the tuple for $\{g\}'(e,\mathfrak{a},f)$ occurs in the computation of $\{e\}'(\mathfrak{a})$. Now consider the part of the computation (or attempted computation) of $\phi(\mathfrak{a})$ which consists in computing $\mathbf{F}(\zeta;\mathfrak{a})$ from the $\zeta(\mathfrak{b}_y)$'s called for by applications of S0′ for certain $\mathfrak{b}_y$'s and evaluated (if possible) as $\phi(\mathfrak{b}_y)$. By the relationship between the descriptions of $\mathbf{F}(\zeta;\mathfrak{a})$ and $\psi(z,\mathfrak{a})$, corresponding applications of S9′ in the computation (or attempted computation) of $\psi(e,\mathfrak{a})$ introduce $\{e\}'(\mathfrak{b}_y)$ for the same $\mathfrak{b}_y$'s. By the construction of g, there correspond in turn applications of S9′ in the computation of $\{g\}'(e,\mathfrak{a},f)$ which introduce the same $\{e\}'(\mathfrak{b}_y)$'s. So far each of the $\mathfrak{b}_y$'s in question, $(e,\mathfrak{b}_y)$ occurs (before the 0-position) in the computation of $\{g\}'(e,\mathfrak{a},f)$, and hence of $\{e\}'(\mathfrak{a})$; and by the hypothesis of an induction on $\{e\}'(-)$ in the course-of-values version, $\phi(\mathfrak{b}_y)$ is defined. Thus the values $\phi(\mathfrak{b}_y)$ of the $\zeta(\mathfrak{b}_y)$'s called for in the said part of the computation of $\phi(\mathfrak{a})$ exist; and $\phi(\mathfrak{a})$ is defined.

10.4. We give also a relativized version LXIV* of the first recursion theorem LXIV (with LXIV included as the $l = 0$ case). Consider assumed functions $\psi_1,\cdots,\psi_l$ or briefly Ψ, where ψ_i is a partial function of $m_{i0},\cdots,m_{is_i}$ variables of types $0,\cdots,s_i$, respectively.

We define the functions *partial′* (*general′*) *recursive in* Ψ in the same manner as the partial′ (general′) recursive functions were defined in 9.8 except adding the simple schema

S0′.i $\qquad \phi(\mathfrak{b},\mathfrak{c}) \simeq \psi_i(\mathfrak{b}) \qquad \langle 0, \langle n_0, \cdots, n_r \rangle, \langle m_{i0}, \cdots, m_{is_i} \rangle, i \rangle$

for introducing each one ψ_i of the functions Ψ. The θ's for S4′ are still to be primitive recursive absolutely. The indices′ become *indices′ from* Ψ and S9′ becomes

S9′$^\Psi$. $\qquad \phi(a,\mathfrak{b},\mathfrak{c}) \simeq \{a\}'^{\Psi}(\mathfrak{b}) \qquad \langle 9, \langle n_0, \cdots, n_r \rangle, \langle m_0, \cdots, m_s \rangle \rangle.$

Relativized versions II′*, III′* etc. of II′, III′ etc. (cf. 9.9) hold (in the second alternative of LIII′*, the branch may terminate at an application of S0′.i with the $\psi_i(\mathfrak{b})$ undefined).

A *normal recursive functional* $\mathbf{F}(\zeta,\Psi;\mathfrak{a})$ is as in 10.1 except adding S0′.i to introduce ψ_i $(i = 1, \cdots, l)$.

LXIV*. *If* $\mathbf{F}(\zeta,\Psi;\mathfrak{a})$ *is normal recursive, the function* ϕ *computed from* Ψ *by*

(a*) $$\zeta(\mathfrak{a}) \simeq \mathbf{F}(\zeta,\Psi;\mathfrak{a})$$

is a solution of (a*) *for* ζ *such that any solution* ϕ' *for* ζ *is an extension of* ϕ, *and this solution* ϕ *is partial′ recursive in* Ψ.

10.5. The relation 'ϕ is partial′ recursive in Ψ' is transitive, i.e. if the Ψ are in turn partial′ recursive in (zero or more) partial functions Θ, then ϕ is partial′ recursive in Θ. For, say Θ are $\theta_1, \cdots, \theta_p$ $(p \geqq 0)$, where θ_j is a partial function of $q_{j0}, \cdots, q_{jt_j}$ variables of types $0, \cdots, t_j$, respectively. Write $m_i = \langle m_{i0}, \cdots, m_{is_i} \rangle$ $(i = 1, \cdots, l)$ and $q_j = \langle q_{j0}, \cdots, q_{jt_j} \rangle$ $(j = 1, \cdots, p)$. Similarly to Kleene [17, §4]:

LXV. *There is a primitive recursive function* $\mathrm{tr}^{m_1,\dots,m_l;q_1,\dots,q_p}$ *such that: If* ϕ *is partial′ recursive in* Ψ *with index′* z, *and* Ψ *are partial′ recursive in* Θ *with respective indices′* $y_1, \cdots, y_l$, *then* ϕ *is partial′ recursive in* Θ *with index′* $\mathrm{tr}^{m_1,\dots,m_l;q_1,\dots,q_p}(z,y_1, \cdots, y_l)$.

Proof. We write $\mathrm{tr}^{m_1,\dots,m_l;q_1,\dots,q_p}$ as tr simply. Let $\mathrm{tr}(z,y_1,\cdots,y_l) = 0$ unless $\mathrm{Ix}'^{m_1,\dots,m_l}(z)$ (using XIX′*). (Cf. 9.11 and earlier.)

CASE 0′. i. $(z)_0 = 0$ & $(z)_3 = i$ $(i = 1, \cdots, l)$. Let $\mathrm{tr}(z,y_1,\cdots,y_l) = \iota'^{q_1,\dots q_p}(y_i,[(z)_1/(z)_2])$ (using XXI′*).

CASE 9′. $(z)_0 = 9$. Let $\mathrm{tr}(z,y_1,\cdots,y_l) = S^{l+2}(\mathrm{D},t,z,y_1,\cdots, y_l)$ where D is an index′ from Θ of $\lambda tzy_1\cdots y_l a\mathfrak{b}\mathfrak{c}\ \{(\overline{\mathrm{sg}}|(a)_1 - (z)_2|)\cdot\{t\}'^{\Theta}(a,y_1,\cdots,y_l)\}'^{\Theta}(\mathfrak{b})$ and t is an index′ from Θ of tr.

10.6. A functional $\mathbf{F}(\zeta;\mathfrak{a})$ is *primitive* [*partial*] *recursive* if $\lambda\mathfrak{a}\,\mathbf{F}(\zeta;\mathfrak{a})$ is primitive [partial] recursive uniformly in the partial function $\zeta = \lambda\mathfrak{a}\ \zeta(\mathfrak{a})$ in the sense of 1.9 [of 3.14]. By the convention stated in 3.14, the function ϕ described by an application of S0.1 with $\psi_1 = \zeta$ shall be undefined for arguments which make any of the $\lambda\alpha^{j-2}\ \chi_k(\alpha^{j-2},\mathfrak{b})$ (the $\lambda t\ \chi_2(t,\mathfrak{b})$, $\lambda\sigma\ \chi_3(\sigma,\mathfrak{b})$ in the illustration of S0.i in 1.8) incompletely defined.

For the normal recursive functionals 10.1, ζ is introduced by a more restricted schema S0′ than the S0.1 allowed for the primitive recursive functionals; but also S5′ is admitted (and S4′ retained).

For an unrestricted generalization of the first recursion theorem IM p. 348, the $\mathbf{F}(\zeta;\mathfrak{a})$ for (a) would be any partial recursive functional. However the generalization does not hold (under the convention of 3.14 for the functionals) even for the subclass of the partial recursive functionals obtained from the normal recursive functionals by liberalizing S0′ to S0.1 (S4′ becoming redundant).

LXVI. *If* $\mathbf{F}(\zeta;\mathfrak{a})$ *is a partial recursive functional, then the function* ϕ *computed by* (a) *is a solution of* (a) *for* ζ *such that any solution* ϕ' *for* ζ *is an extension of* ϕ. *But a partial recursive* $\mathbf{F}$ *can be chosen so that this function* ϕ *is not partial recursive.*

Proof. Cf. LXIV. (iii) Adapting the example for LVI, let

$$\mathbf{F}(\zeta;b,\tau^1,a) \simeq \begin{cases} \zeta(1,\lambda x\ \zeta(2,\tau^1,\langle a,x\rangle),a) & \text{if } b=0, \\ 0 \quad \text{if } b=1 \vee [b=2\ \&\bar{T}_1((a)_0,(a)_0,(a)_1)], \\ \zeta(2,\tau^1,a) & \text{otherwise.} \end{cases}$$

Suppose the $\phi(b,\tau^1,a)$ computed by (a) with this $\mathbf{F}$ were partial recursive. Then using LXI (95) with $\theta(t,\mathfrak{a},\mathfrak{b},\mathfrak{c})=0$, so would be $\phi(a) \simeq \phi(0,\lambda t\ 0,a)$. But from the computation procedure for $\phi(b,\tau^1,a)$ by (a), we see that $\phi(a)$ is defined if and only if $(x)\bar{T}_1(a,a,x)$.

10.7. In 5.1 we introduced the function $\lambda za\alpha^1\cdots\alpha^r\{z\}[a,\alpha^1,\cdots,\alpha^r]$ for studying the partial recursive functions without showing it to be itself partial recursive, though there is a calculation procedure for it similar to that for the functions introduced by S9. There, with only XXII–XXIII to handle substitution for function variables, we were only in a position to show, by a direct but meticulous construction of indices (similar to that in the proof of XXV but more complicated), that an extension of it is partial recursive. If it were not itself partial recursive, there would be a gap in the defense of Church's thesis for our partial recursive functions (with variables as in 1.2, 3.7) in the form corresponding to IM Thesis I†(a) p. 332 (rather than I†(b)). In showing that it is itself partial recursive, we could employ a direct index construction with LXI replacing XXII, but we prefer to use the first recursion theorem LXIV.

LXVII. *For each* $r \geqq 0$, $\lambda za\alpha^1\cdots\alpha^r\ \{z\}[a,\alpha^1,\cdots,\alpha^r]$ *is partial recursive.*

Proof. A computation procedure for $\{z\}[a,\alpha^1,\cdots,\alpha^r]$ is determined from that for $\{z\}(\mathfrak{a})$ (3.9, beginning 5.3, 9.1) by the definition of $\{z\}[a,\alpha^1,\cdots,\alpha^r]$ in terms of $\{z\}(\mathfrak{a})$ in 5.1.

Consider the following recursion, where $k=(z)_3$ (cf. 5.4).

$$
\begin{aligned}
\zeta(z,a,\alpha^1,\cdots,\alpha^r) &\simeq (a)_0+1 \text{ if } \mathrm{Ix}(z) \ \&\, \mathrm{tp}(z) \leqq r \ \&\ (z)_0 = 1 \quad (\text{CASE } 1),\\
&\simeq (z)_2 \text{ if } \mathrm{Ix}(z) \ \&\, \mathrm{tp}(z) \leqq r \ \&\ (z)_0 = 2 \quad (\text{CASE } 2),\\
&\simeq (a)_0 \text{ if } \mathrm{Ix}(z) \ \&\, \mathrm{tp}(z) \leqq r \ \&\ (z)_0 = 3 \quad (\text{CASE } 3),\\
&\simeq \zeta((z)_2, 2^{\zeta((z)_3,a,\alpha^1,\ldots,\alpha^r)} \cdot \prod_{i<a} p_{i+1}^{(a)_i}, \alpha^1,\cdots,\alpha^r)\\
&\qquad \text{if } \mathrm{Ix}(z) \ \&\ \mathrm{tp}(z) \leqq r \ \&(z)_0 = 4 \quad (\text{CASE } 4),\\
&\simeq \zeta((z)_2, \prod_{i<a} p_i^{(a)_{i+1}}, \alpha^1,\cdots,\alpha^r) \text{ if } \mathrm{Ix}(z) \ \&\ \mathrm{tp}(z) \leqq r \ \&\ (z)_0 = 5 \ \&\\
&\qquad (a)_0 = 0 \quad (\text{CASE } 5\text{a}),\\
(98) \qquad &\simeq \zeta((z)_3, 2^{(a)_0 \dot- 1} \cdot 3^{\zeta(z,[a/2],\alpha^1,\ldots,\alpha^r)} \cdot \prod_{i<a} p_{i+2}^{(a)_{i+1}}, \alpha^1,\cdots,\alpha^r)\\
&\qquad \text{if } \mathrm{Ix}(z) \ \&\ \mathrm{tp}(z) \leqq r \ \&\ (z)_0 = 5 \ \&\ (a)_0 > 0 \quad (\text{CASE } 5\text{b}),\\
&\simeq \zeta((z)_4, (\prod_{i<k} p_i^{(a)_{i+1}}) \cdot p_k^{(a)_0} \cdot (\prod_{k<i<a} p_i^{(a)_i}), \alpha^1,\cdots,\alpha^r)\\
&\qquad \text{if } \mathrm{Ix}(z) \ \&\, \mathrm{tp}(z) \leqq r \ \&\ (z)_0 = 6 \ \&\ (z)_2 = 0 \quad (\text{CASE } 6.0),\\
&\simeq \zeta((z)_4, a, \alpha^1,\cdots,\alpha^{j-1}, \lambda v\, (\prod_{i<k} p_i^{(\alpha^j(v))_{i+1}}) \cdot p_k^{(\alpha^j(v))_0} \cdot\\
&\qquad (\prod_{k<i<\alpha^j(v)} p_i^{(\alpha^j(v))_i}), \alpha^{j+1},\cdots,\alpha^r) \text{ if } \mathrm{Ix}(z) \ \&\ \mathrm{tp}(z) \leqq r\\
&\qquad \&\ (z)_0 = 6 \ \&\ (z)_2 = j \quad (\text{CASE } 6.j \text{ for } j = 1,\cdots,r),\\
&\simeq (\alpha^1((a)_0))_0 \text{ if } \mathrm{Ix}(z) \ \&\, \mathrm{tp}(z) \leqq r \ \&\ (z)_0 = 7 \quad (\text{CASE } 7),\\
&\simeq (\alpha^2(\lambda x\, \zeta((z)_3, 2^x \cdot \prod_{i<a} p_{i+1}^{(a)_i}, \alpha^1,\cdots,\alpha^r)))_0\\
&\qquad \text{if } \mathrm{Ix}(z) \ \&\ \mathrm{tp}(z) \leqq r \ \&\ (z)_0 = 8 \ \&\ (z)_2 = 2 \quad (\text{CASE } 8.2),\\
&\simeq (\alpha^j(\lambda\sigma^{j-2}\, \zeta((z)_3, a, \alpha^1,\cdots,\alpha^{j-3}, \lambda v\, 2^{\sigma^{j-2}(v)} \cdot \prod_{i<\alpha^{j-2}(v)} p_{i+1}^{(\alpha^{j-2}(v))_i},\\
&\qquad \alpha^{j-1},\cdots,\alpha^r)))_0 \text{ if } \mathrm{Ix}(z) \ \&\ \mathrm{tp}(z) \leqq r \ \&\ (z)_0 = 8 \ \&\ (z)_2 = j\\
&\qquad (\text{CASE } 8.j \text{ for } j = 3,\cdots,r),\\
&\simeq \zeta((a)_0, \prod_{i<a} p_i^{(a)_{i+1}}, \alpha^1,\cdots,\alpha^r) \text{ if } \mathrm{Ix}(z) \ \&\ \mathrm{tp}(z) \leqq r \ \&\ (z)_0 = 9\\
&\qquad \&\ \mathrm{Ix}((a)_0) \ \&\ (a)_{0,1} = (z)_2 \quad (\text{CASE } 9),\\
&\simeq \zeta(z,a,\alpha^1,\cdots,\alpha^r) \text{ otherwise} \quad (\text{CASE } 10).
\end{aligned}
$$

The right side of (98) is a functional $\mathbf{F}(\zeta; z,a,\alpha^1,\cdots,\alpha^r)$ defined by exhaustive mutually-exclusive cases. The functions $\lambda v \cdots$ substituted in Cases $6.j$ ($j = 1,\cdots,r$) and $8.j$ ($j = 3,\cdots,r$) are of the form of the $\lambda\tau\ \theta(\cdots)$ for S4′, while the substitutions of $\lambda x \cdots$ in Case 8.2 and $\lambda\sigma^{j-2} \cdots$ in Cases $8.j$ ($j = 3,\cdots,r$) are directly as arguments of the function variables and thus come under S8. The other operations in building the expressions given for the values in the cases can be effected using only S1–S7, after introductions of ζ by S0′. Finally, the case hypotheses are primitive recursive (with number variables only), so the definition by cases can be handled by iterated use of S5′ with S1–S6. Thus $\mathbf{F}(\zeta;z,a,\alpha^1,\cdots,\alpha^r)$ is normal recursive, and by LXIV: *The function* ϕ *computed by* (98) *is partial recursive.*

Next we show by induction over the computation of ϕ by (98) that: *For any* $z,a,\alpha^1,\cdots,\alpha^r$, *if* $\phi(z,a,\alpha^1,\cdots,\alpha^r)$ *is defined, then* $\{z\}[a,\alpha^1,\cdots,\alpha^r]$ (*is defined and*) $=\phi(z,a,\alpha^1,\cdots,\alpha^r)$. When $\phi(z,a,\alpha^1,\cdots,\alpha^r)$ is defined, one of Cases 1–9 of (98) must apply, since in Case 10 there is an infinite branch in the computation tree (cf. LIII). Consider that part of the computation of $\phi(z,a,\alpha^1,\cdots,\alpha^r)$ which consists in computing $\mathbf{F}(\zeta;z,a,\alpha^1,\cdots,\alpha^r)$ from the $\zeta(z_y,a_y,\alpha^1_y,\cdots,\alpha^r_y)$ introduced by S0′ for zero, one or more $r+2$-tuples $(z_y,a_y,\alpha^1_y,\cdots,\alpha^r_y)$ and evaluated as $\phi(z_y,a_y,\alpha^1_y,\cdots,\alpha^r_y)$. In each of Cases 1–3, 7, there are no such $r+2$-tuples, and the same value is given outright to $\mathbf{F}(\zeta;z,a,\alpha^1,\cdots,\alpha^r)$ as $\{z\}[a,\alpha^1,\cdots,\alpha^r]$ has in the case; so $\{z\}[a,\alpha^1,\cdots,\alpha^r]=\phi(a,\alpha^1,\cdots,\alpha^r)$. In each of Cases 4–6, 8, 9, the value of $\mathbf{F}(\zeta;z,a,\alpha^1,\cdots,\alpha^r)$ is made to depend on that of ζ for the same $r+2$-tuples $(z_y,a_y,\alpha^1_y,\cdots,\alpha^r_y)$ as in the first step in the computation of $\{z\}[a,\alpha^1,\cdots,\alpha^r]$, by the same formula. By the hyp. ind., for each such $r+2$-tuple,

$$\{z_y\}[a_y,\alpha^1_y,\cdots,\alpha^r_y]=\phi(z_y,a_y,\alpha^1_y,\cdots,\alpha^r_y);$$

and hence

$$\{z\}[a,\alpha^1,\cdots,\alpha^r]=\mathbf{F}(\phi;z,a,\alpha^1,\cdots,\alpha^r)=\phi(z,a,\alpha^1,\cdots,\alpha^r).$$

Finally, by a similar induction over the computation of $\{z\}[a,\alpha^1,\cdots,\alpha^r]$: *For any* $z,a,\alpha^1,\cdots,\alpha^r$, *if* $\{z\}[a,\alpha^1,\cdots,\alpha^r]$ *is defined, so is* $\phi(z,a,\alpha^1,\cdots,\alpha^r)$.

10.8. LXVIII. *The partial recursive functions constitute the least class of functions which to each list* Ψ *of* $l\geqq 0$ *functions in the class and each normal recursive functional* $\mathbf{F}(\zeta,\Psi;\mathfrak{a})$ *contains the function* ϕ *computed from* Ψ *by the recursion* (a*) *of* 10.4 *and which is closed under the schema*

S4′.0 $$\phi(\mathfrak{a})\simeq\psi(\theta(\mathfrak{a}),\mathfrak{a}) \qquad \langle 4,\langle n_0,\cdots,n_r\rangle,g,h\rangle,$$

where $\theta(\mathfrak{a})$ *is primitive recursive, and the schemata* S4′.j $(j=1,2,3,\cdots)$ *of* 9.8, *as operations generating* ϕ *from* ψ.

Proof. (i) By LXIV*, the ϕ computed (from Ψ) by (a*) for such an $\mathbf{F}$ is partial′ recursive in Ψ, and so by LXV for Θ empty with LXIII and LXII is partial recursive when Ψ are partial recursive. By S4 (cf. 3.7) and LXI, the class of the partial recursive functions is closed under S4′.j $(j=0,1,2,\cdots)$.

(ii) Conversely, consider any partial recursive function ϕ_1 of variables $(a_1,\cdots,a_{n_0},\alpha^1_1,\cdots,\alpha^1_{n_1},\cdots,\alpha^r_1,\cdots,\alpha^r_{n_r})$ $(=\mathfrak{b})$ with $n_r>0$ if $r>0$. Introducing $\mathfrak{b}$ as parameters into (98), we compute by (a) (i.e. (a*) with the Ψ empty) a function ϕ such that $\{z\}[a,\alpha^1,\cdots,\alpha^r]\simeq\phi(z,a,\alpha^1,\cdots,\alpha^r,\mathfrak{b})$. Using (13) in 5.1, we thence obtain ϕ_1 by S4′.0–S4′.r (cf. the proof of V in 1.6), provided that $n_j>0$ for $2\leqq j\leqq$

$r-1$ (otherwise the $\langle \alpha_1^j, \cdots, \alpha_n^j, \rangle$ of 2.1 which we would substitute for α^j does not satisfy the requirement $n_j > 0$ for S4$'.j$).

(iii) To deal with the case that not all of $n_2, \cdots, n_{r-1} > 0$, we use recursions (a*) to introduce successively sets of functions like $\{z\}[a,\alpha^1,\cdots,\alpha^r]$ but with $1,2,\cdots,r-2$ of the variables $\alpha^2,\cdots,\alpha^{r-1}$ missing. Then we will need to substitute $\langle \alpha_1^j,\cdots,\alpha_{n_j}^j\rangle$ for α^j only for those j's for which $n_j > 0$. The parameters $\mathfrak{b}$, which we will not write, can be carried in each of the recursions. For example, suppose $r=6$ and we have already introduced the functions with 2 of $\alpha^2,\alpha^3,\alpha^4,\alpha^5$ missing; and consider $\{z\}[a,\alpha^1,\alpha^4,\alpha^6]$, which has 3 missing. The recursion for this is like (98) for $r=6$, except that Cases 6.2, 6.3, 6.5, 8.2, 8.3, 8.5 are missing, and in Case 8.4 the expression for the value is

$$(\alpha^4(\lambda\sigma^2\ \{(z)_3\}[a,\alpha^1,\langle\sigma^2\rangle,\alpha^4,\alpha^6]))_0,$$

using the previously introduced $\lambda z a\alpha^1\alpha^2\alpha^4\alpha^6\ \{z\}[a,\alpha^1,\alpha^2,\alpha^4,\alpha^6]$, call it ψ. The right side is a normal recursive functional $\mathbf{F}(\zeta,\psi;z,a,\alpha^1,\alpha^4,\alpha^6)$. So, since ψ is already in the class, the function $\phi = \lambda z a\alpha^1\alpha^4\alpha^6\ \{z\}[a,\alpha^1,\alpha^4,\alpha^6]$ computed by (a*) for this $\mathbf{F}$ is in the class.

Discussion. Thus (a*) for normal recursive $\mathbf{F}$ suitably understood can replace S9. Indeed, a finite number of particular applications of (a*) suffices for all partial recursive functions ϕ_1 of variables of types $\leqq$ a given r, if we allow the subsequent substitutions to have the more general forms $\psi(\theta(\mathfrak{b},\mathfrak{c}),\mathfrak{b})$, $\psi(\lambda t\,\theta(t,\mathfrak{b},\mathfrak{c}),\mathfrak{b})$, $\psi(\lambda\tau^{j-1}\,\theta(\alpha_1^j(\tau^{j-1}),\cdots,\alpha_{n_j}^j(\tau^{j-1}),\mathfrak{b},\mathfrak{c}),\mathfrak{b})$ ($j>1$; $n_j>0$). This avoids the use of indices, if one accepts without indices the notion 10.2 of the function computed by a recursion (a) or more generally computed from Ψ by (a*). We did use indices in a supplementary explanation of that; but there is the following difference. In computing $\{z\}(\mathfrak{a})$ for a given z and $\mathfrak{a}$ (3.7–3.9), infinitely many different indices may occur. In describing via indices the computation by (a*) for a given $\mathbf{F}$ but any $\mathfrak{a}$, only the finitely many indices $d_1,\cdots,d_k$ are used; we can say that only finitely many kinds of computation situations occur.

Remark 11. A definition of 'partial' and 'general recursive function' by using (a) for a normal recursive $\mathbf{F}$ instead of S9 has been available for the type-0 and type-1 cases in IM Example 4 pp. 350–351 and, for type 1, its uniformly relativized version. (The subsequent substitutions in Theorems IX, IX*, XIX pp. 288, 292, 330 use S4 and not simply S4$'$.)

11. **Degrees and sections. 11.1.** One of our principal objectives in developing the notions of partial and general recursiveness for functions of variables of types >1 was to make available relative general recursiveness for functions of variables of types >0 (cf. the introduction to Part I)([21]).

([21]) Thus in [14] we were forced to forgo a relativized version of the theory except for the case all independent variables are of type 0 (cf. [14, Footnote 4]). Our first work in the present area, specifically on the notions formulated in [42] and [39], was in 1952 in connection with the first work on [14] (presented to the Association for Symbolic Logic, December 29, 1952).

11.2. The objects whose relations of relative recursiveness we shall be considering are the total (i.e. completely defined) functions $\lambda\mathfrak{a}\,\phi(\mathfrak{a})$, where $\mathfrak{a}$ is a nonempty list of variables of the types $0,1,2,\cdots$ called *pure* in [18, 5.2], with values of type 0; these functions $\lambda\mathfrak{a}\,\phi(\mathfrak{a})$ themselves (and numbers) are of *special* types in the terminology of [18, 5.2]. Predicates (and sets) are included in the treatment via their representing functions(22).

11.3. The relations 'ϕ is partial recursive in ψ' and 'ϕ is general recursive in ψ' are defined in 3.14(23).

These relations are reflexive; e.g. if ψ is $\psi(c,\gamma_2^1,\gamma_3^2)$, we obtain

$$\psi(c,\gamma_2^1,\gamma_3^2) \simeq \psi(c,\lambda t\;\chi_2(t,\gamma_2^1,\gamma_3^2),\lambda\sigma\,\chi_3(\sigma,\gamma_2^1,\gamma_3^2))$$

by S0.1 in 1.8 after using VI to obtain

$$\chi_2(t,\gamma_2^1,\gamma_3^2) = \gamma_2^1(t), \qquad \chi_3(\sigma,\gamma_2^1,\gamma_3^2) = \gamma_3^2(\sigma).$$

11.4. As in 10.4 and 10.5, let Ψ be $\psi_1,\cdots,\psi_l$ $(l \geqq 0)$ where ψ_i is a partial function of $m_{i0},\cdots,m_{is_i}$ variables of types $0,\cdots,s_i$, respectively; Θ be $\theta_1,\cdots,\theta_p$ $(p \geqq 0)$ where θ_j is a partial function of $q_{j0},\cdots,q_{jt_j}$ variables of types $0,\cdots,t_j$, respectively; and write $m_i = \langle m_{i0},\cdots,m_{is_i}\rangle$, $q_j = \langle q_{j0},\cdots,q_{jt_j}\rangle$.

LXIX. *There is a primitive recursive function* $\mathrm{tre}^{m_1,\cdots,m_l;\,q_1,\cdots,q_p}$ *such that: If ϕ is partial recursive in Ψ with index z, and Ψ are partial recursive in Θ with respective indices $y_1,\cdots,y_l$, then an extension of ϕ is partial recursive in Θ with index* $\mathrm{tre}^{m_1,\cdots,m_l:q_1,\cdots,q_p}(z,y_1,\cdots,y_l)$. *For $l=0$,* $\mathrm{tre}^{q_1,\cdots,q_p}(z)$ *$= z$ simply.*

Proof for $l>0$. For illustration, say $l=1$, $m=m_1=\langle 1,1,1\rangle$. We write $\mathrm{tre}^{\langle 1,1,1\rangle;q_1,\cdots,q_p}$ as tre simply. We take $\mathrm{tre}(z,y)=0$ except for $\mathrm{Ix}^{\langle 1,1,1\rangle}(z)$ (using the relativized version XIX* of XIX), when the appropriate one of the following cases shall apply. That a primitive recursive function tre exists satisfying the specifications in all of the cases will follow using the recursion theorem (as for XXII, LXV). We use induction on $\{z\}^{\psi}(-)$ (3.14, 3.8) in proving that, for each $\mathfrak{a}$ for which $\phi(\mathfrak{a})$ $(=\{z\}^{\psi}(\mathfrak{a}))$ is defined, $\{z\}^{\psi}(\mathfrak{a}) = \{\mathrm{tre}(z,y)\}^{\Theta}(\mathfrak{a})$. This proof is to be given in each of the following cases, with the simple case hypothesis for the definition of $\mathrm{tre}(z,y)$ supplemented by the hypothesis of the theorem.

Case 0. $(z)_0 = 0$. Suppose $\phi(\mathfrak{a})$ is partial recursive in ψ with index z, and ψ is partial recursive in Θ with index y. Then $\phi(\mathfrak{a})$ is introduced by S0.1 in 1.8; writing $\mathfrak{a}=(c,\mathfrak{b})$, $\phi(\mathfrak{a}) \simeq \psi(c,\lambda t\;\chi_2(t,\mathfrak{b}),\;\lambda\sigma\,\chi_3(\sigma,\mathfrak{b}))$ where χ_2 and χ_3 are partial recursive in ψ with indices $(z)_3$ and $(z)_4$, respectively. Consider an $\mathfrak{a}$ for which

(22) The basis is available for the treatment of objects of all finite types [18, 5.2], via 1.5, [18, 5.3 and 5.4].

(23) Indices from Ψ (required in 3.14) are assigned to functions introduced by applications of S0.i in 1.8 in the manner illustrated there with the variables of ψ_i written in an order of nondecreasing type; thus in the example, h_2 and h_3 are indices from Ψ of χ_2 and χ_3, respectively. (We can, continuing the convention following S1–S8 in 1.3, write the variables of ψ_i in other orders, but these shall not give rise to additional indices from Ψ.)

$\phi(\mathfrak{a})$ is defined; by the convention of 3.14, $\lambda t\ \chi_2(t,\mathfrak{b})$ and $\lambda\sigma\ \chi_3(\sigma,\mathfrak{b})$ are completely defined, so by hyp. ind., for each t, $\chi_2(t,\mathfrak{b}) = \{\text{tre}((z)_3,y)\}^{\Theta}(t,\mathfrak{b})$, and, for each σ, $\chi_3(\sigma,\mathfrak{b}) = \{\text{tre}((z)_4,y)\}^{\Theta}(\sigma,\mathfrak{b})$. Thus the functions $\chi_2'(t,\mathfrak{b})$, $\chi_3'(\sigma,\mathfrak{b})$ with the following indices from Θ are extensions of $\chi_2(t,\mathfrak{b})$, $\chi_3(\sigma,\mathfrak{b})$, respectively.

$$\chi_2'\colon \mathrm{A} = \text{tre}((z)_3,y).$$

$$\chi_3'\colon \mathrm{B} = \text{tre}((z)_4,y).$$

The functions $\chi_2''(c,\sigma^2,\mathfrak{b},t) \simeq \chi_2'(t,\mathfrak{b})$, $\chi_3''(c,\mathfrak{b},\sigma) \simeq \chi_3'(\sigma,\mathfrak{b})$ have the following indices from Θ (using the relativized versions XX*, XXI* of XX, XXI).

$$\chi_2''\colon \mathrm{C} = \langle 6, \langle 1,0,1\rangle \cdot (\mathrm{A})_1, 0, (\mathrm{A})_{1,0}, \langle 6, \langle 1,0,1\rangle \cdot (\mathrm{A})_1, 2, (\mathrm{A})_{1,2}, \iota^{q_1,\dots,q_p}(\pi(0,(\mathrm{A})_{1,0} \dot{-} 1, \mathrm{A}), \langle 1,0,1\rangle)\rangle\rangle.$$

$$\chi_3''\colon \mathrm{D} = \langle 6, \langle 1\rangle \cdot (\mathrm{B})_1, 0, (\mathrm{B})_{1,0}, \iota^{q_1,\dots,q_p}(\pi(1,(\mathrm{B})_{1,1} \dot{-} 1, \mathrm{B}), \langle 1\rangle)\rangle.$$

Next consider the function $\psi_1(c,\sigma^2,\mathfrak{b})$ with the following index from Θ (using the relativized version XXII* of XXII; cf. 4.4).

$$\psi_1\colon \mathrm{E} = \gamma_1^{q_1,\dots,q_p}(y,\mathrm{C},0).$$

For values of $\mathfrak{a}$ $(= (c,\mathfrak{b}))$ for which $\phi(\mathfrak{a})$ is defined, so $\sigma^1 = \lambda t\,\chi_2(t,\mathfrak{b})$ is completely defined, and also of σ^2 for which $\psi(c,\sigma^1,\sigma^2)$ is defined, by XXII* $\psi_1(c,\sigma^2,\mathfrak{b}) = \{\mathrm{E}\}^{\Theta}(c,\sigma^2,\mathfrak{b}) = \psi(c,\lambda t\,\chi_2''(c,\sigma^2,\mathfrak{b},t),\sigma^2) = \psi(c,\lambda t\,\chi_2(t,\mathfrak{b}),\sigma^2)$. Finally consider the function $\psi_2(c,\mathfrak{b})$ with the following index from Θ.

$$\psi_2\colon \mathrm{F} = \gamma_2^{q_1,\dots,q_p}(\mathrm{E},\mathrm{D},0).$$

For values of $\mathfrak{a}(=(c,\mathfrak{b}))$ for which $\phi(\mathfrak{a})$ is defined, so $\sigma^2 = \lambda\sigma\ \chi_3(\sigma,\mathfrak{b})$ is completely defined and $\psi_1(c,\sigma^2,\mathfrak{b})$ is defined, by XXII*

$$\psi_2(c,\mathfrak{b}) = \{\mathrm{F}\}^{\Theta}(c,\mathfrak{b}) = \psi_1(c,\lambda\sigma\ \chi_3''(c,\mathfrak{b},\sigma),\mathfrak{b}) = \psi(c,\lambda t\ \chi_2(t,\mathfrak{b}), \lambda\sigma\,\chi_3(\sigma,\mathfrak{b})) = \phi(\mathfrak{a}).$$

Thus $\psi_2(c,\mathfrak{b})$ is an extension of $\phi(\mathfrak{a})$. Let $\text{tre}(z,y) = \mathrm{F}$.

CASE 9. $(z)_0 = 9$. $\phi(a,\mathfrak{b},\mathfrak{c}) \simeq \{a\}^{\psi}(\mathfrak{b})$. Let $\text{tre}(z,y) = S^2(\mathrm{D},t,y)$ where D is an index from Θ of $\lambda t y a \mathfrak{b}\mathfrak{c}\{\{t\}^{\Theta}(a,y)\}^{\Theta}(\mathfrak{b})$ and t is an index from Θ of tre.

Proof for $l = 0$. Consider any $\mathfrak{a}$ for which $\phi(\mathfrak{a})$ $(= \{z\}(\mathfrak{a}))$ is defined. By LIII, each tuple in the computation tree for $\{z\}(\mathfrak{a})$ is defined. This computation tree is already one for $\{z\}^{\Theta}(\mathfrak{a})$; the extra cases for S0.i in which a tuple would be defined in the tree for $\{z\}^{\Theta}(\mathfrak{a})$ but not in that for $\{z\}(\mathfrak{a})$ do not arise. Thus $\phi(\mathfrak{a}) = \{z\}^{\Theta}(\mathfrak{a})$.

REMARK 12. For $l = 0$, by using a more complicated index function $\text{tre}_0^{q_1,\dots,q_p}$ (briefly, tre_0), ϕ itself is partial recursive in Θ. Let $\text{tre}_0(z) = 0$ except when $\text{Ix}(z)$. In Case 9, let $\text{tre}_0(z) = S^2(\mathrm{D},t,z)$ where D is an index from Θ of $\lambda t z a \mathfrak{b}\mathfrak{c}\{(\overline{\text{sg}}\,|(a)_1 - (z)_2|) \cdot \{t\}^{\Theta}(a)\}^{\Theta}(\mathfrak{b})$ and t is an index from Θ of tre_0. (Cf. 9.11 and earlier.)

11.5. Applying LXIX with Ψ and Θ single functions ψ and θ ($l = p = 1$) and all the functions total: The relation 'ϕ is general recursive in ψ' is transitive.

REMARK 13. We stated LXIX for partial functions, even though we are primarily interested in total functions, since e.g. in Case 4 $\phi(\mathfrak{a}) = \chi_1(\chi(\mathfrak{a}), \mathfrak{a})$ with $\phi(\mathfrak{a})$ total, $\chi_1(c, \mathfrak{a})$ may not be total.

11.6. Applying LXIX [Remark 12] with Ψ empty ($l = 0$) and Θ a single function θ ($p = 1$): If ϕ is general [partial] recursive, ϕ is general [partial] recursive in any function θ.

Applying LXIX with Ψ a single function ψ and Θ empty: If ϕ is general recursive in a general (or partial) recursive function ψ, ϕ is general recursive.

REMARK 14. Under the convention of 3.14, a function ϕ may be partial recursive in a partial recursive function ψ without being partial recursive; e.g. $\phi(a) \simeq \psi(\lambda x\, \chi(x, a))$ for the $\chi(x, a)$ of the proof of LVI with $\psi(\alpha) = 0$.

11.7. By 2.1–2.4 for each $k, l \geqq 1$, $\langle \alpha^k, \beta^l \rangle = \langle \gamma^i, \gamma^j \rangle$ (where $\gamma^i = \alpha^k$ and $\gamma^j = \beta^l$ if $k \leqq l$, $\gamma^i = \beta^l$ and $\gamma^j = \alpha^k$ if $k > l$) $= \langle \gamma^i, \gamma^j \rangle^j = \langle \mathrm{mp}_i^j(\gamma^i), \mathrm{mp}_j^j(\gamma^j) \rangle$ $= \langle \lambda\tau^{j-1} \mathrm{mp}_i^j(\gamma^i, \tau^{j-1}), \lambda\tau^{j-1} \mathrm{mp}_j^j(\gamma^j, \tau^{j-1}) \rangle = \lambda\tau^{j-1} \langle \mathrm{mp}_i^j(\gamma^i, \tau^{j-1}), \mathrm{mp}_j^j(\gamma^j, \tau^{j-1}) \rangle$, which is of the form $\lambda\tau^{j-1}\, \phi(\tau^{j-1}, \gamma^k, \gamma^l)$ where $\phi(\tau^{j-1}, \gamma^k, \gamma^l)$ is primitive recursive. So by 1.9, for each $k, l \geqq 1$, $\langle \alpha^k, \beta^l \rangle$ is a function of type $j = \max(k, l)$ primitive recursive (uniformly) in α^k, β^l; a fortiori, by 3.14, $\langle \alpha^k, \beta^l \rangle$ is general recursive (uniformly) in α^k, β^l.

By 2.4, $(\langle \alpha^k, \beta^l \rangle)_0^k = \alpha^k$ for $k \leqq l$ $[(\langle \alpha^k, \beta^l \rangle)_1^k = \alpha^k$ for $k > l]$, whence (for $k, l \geqq 1$) via 2.1–2.4, 1.9 and 3.14, α^k is general recursive (uniformly) in $\langle \alpha^k, \beta^l \rangle$. Similarly, β^l is general recursive (uniformly) in $\langle \alpha^k, \beta^l \rangle$.

Applying LXIX with $\langle \alpha^k, \beta^l \rangle$, (α^k, β^l), γ^m as the ϕ, Ψ, Θ: If α^k and β^l are each general recursive in γ^m, then $\langle \alpha^k, \beta^l \rangle$ is general recursive in γ^m.

11.8. For pure-type objects, relative partial and general recursiveness can be expressed most simply via 1.9 extended to include S9, as remarked in 3.15 and used in 7.3, 7.4 Part (b) and 7.9; in particular: α^k is general recursive in β^l if and only if there is a partial recursive function $\phi(\tau^{k-1}, \gamma^l)$ such that, for each $\tau^{k-1}, \alpha^k(\tau^{k-1}) = \phi(\tau^{k-1}, \beta^l)$. – The index constructions required for this extension of 1.9 to include S9 were passed over in silence in Part I. We give them now for importing (exporting) a single function ψ_k into (out of) the list Ψ of arguments. By repetitions, the general form of the result will follow.

(a) *For Ψ, m_i as in* 11.4 *with $l \geqq k \geqq 1$ and ψ_k total of pure type $j > 0$, let $\Psi_k = (\psi_1, \cdots, \psi_{k-1}, \psi_{k+1}, \cdots, \psi_l)$. There is a primitive recursive function $\mathrm{im}^{m_1, \cdots, m_l, k}(z)$ with the following property. If $\phi(\mathfrak{a})$ is partial recursive in Ψ with index z, then $\mathrm{im}^{m_1, \cdots, m_l, k}(z)$ is an index from Ψ_k of a function $\phi(\mathfrak{a}, \alpha^j)$ such that $\phi(\mathfrak{a}) \simeq \phi(\mathfrak{a}, \psi_k)$. (When ψ_k varies, if the index z is uniform in $\psi_k, \phi(\mathfrak{a}) \simeq \phi(\mathfrak{a}, \psi_k)$ for all ψ_k.)*

Proof. Let $\mathrm{im}^{m_1, \cdots, m_l, k}(z)$ (write it $\mathrm{im}(z)$ simply) $= 0$ unless $\mathrm{Ix}^{m_1, \cdots, m_l}(z)$ (using XIX*). Note that $j = 1 + \mathrm{tp}(m_k)$ (cf. 4.1).

CASE 0. $(z)_0 = 0$. SUBCASE 1. $(z)_2 \neq k$. Let $\mathrm{im}(z) = 3^{p_j \cdot (z)_1} 5^{(z)_2 \dot{-} \mathrm{sg}((z)_2 \dot{-} k)} \cdot \Pi_{i < \mathrm{lh}(z) \dot{-} 2} p_{i+3}^{\mathrm{im}((z)_{i+3}}$. SUBCASE 2. $(z)_2 = k$ & $j = 1$. Let $\mathrm{im}(z) = \pi(1,(z)_{1,1}, \langle 7, p_1 \cdot (z)_1 \rangle)$. SUBCASE 3. $(z)_2 = k$ & $j > 1$. Let $\mathrm{im}(z) = \pi(j,(z)_{1,\ j}, \langle 8, p_j \cdot (z)_1 j \langle 6, p_j \cdot p_{j-2} \cdot (z)_1, j, (z)_{1,j}, \mathrm{im}((z)_3) \rangle \rangle)$.

CASE 9. $(z)_0 = 9$. Write $\mathfrak{a} = (a, \mathfrak{b}, \mathfrak{c})$. Let $\mathrm{im}(z) = S^2(\mathrm{D}, i, z)$ where D is an index from Ψ_k of $\lambda iza\mathfrak{b}\mathfrak{c}\alpha^j \{(\overline{\mathrm{sg}}|(a)_1 - (z)_2|) \cdot \{i\}^{\Psi_k}(a)\}^{\Psi_k}(\mathfrak{b}, \alpha^j)$ and i is an index from Ψ_k of im.

(b) *For Ψ etc. as in* (a), *there is a primitive recursive function* $\mathrm{ex}^{m_1, \ldots, m_l, k}(z,q)$ *with the following property. Let* $\mathfrak{a}$ *be a list of variables including at least q of type j, and let* $\mathfrak{d}$ *be* $\mathfrak{a}$ *with the qth type-j variable* ψ_k *removed. If* $\phi(\mathfrak{a})$ *is partial recursive in* Ψ_k *with index z, then* $\lambda\mathfrak{d}\ \phi(\mathfrak{a})$ *is partial recursive in* Ψ *uniformly in* ψ_k *with index* $\mathrm{ex}^{m_1, \ldots, m_l, k}(z,q)$.

Proof. Let $\mathrm{ex}(z,q) = 0$ except when $\mathrm{Ix}^{m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_l}(z)$.

CASE 0. $(z)_0 = 0$. Let

$$\mathrm{ex}(z,q) = 3^{[(z)_1/p_j]} 5^{(z)_2 + \mathrm{sg}((z)_2' \dot{-} k)} \Pi_{i < \mathrm{lh}(z) \dot{-} 2} p_{i+3}^{\mathrm{ex}((z)_{i+3}, q + ((z)_{i+3,1,j} \dot{-} (z)_{1,j}))}.$$

CASE 6. $(z)_0 = 6$. SUBCASE 1. $(z)_2 \neq j$. Let $\mathrm{ex}(z,q) = \langle 6, [(z)_1/p_j], (z)_2, (z)_3, \mathrm{ex}((z)_4, q) \rangle$. SUBCASE 2. $(z)_2 = j$. SUB²CASE 1. $q \neq 1$. Let $\mathrm{ex}(z,q) = \langle 6, [(z)_1/p_j], j, (z)_3 \dot{-} \mathrm{sg}((z)_3'' \dot{-} q), \mathrm{ex}((z)_4, q \dot{-} \mathrm{sg}((z)_3'' \dot{-} q)) \rangle$. SUB²CASE 2. $q = 1$. Let $\mathrm{ex}(z,q) = \mathrm{ex}((z)_4, (z)_3')$.

CASE 7. $(z)_0 = 7$. SUBCASE 1. $j \neq 1 \vee q \neq 1$. Let $\mathrm{ex}(z,q) = \langle 7, [(z)_1/p_j] \rangle$. SUBCASE 2. $j = q = 1$. Let $\mathrm{ex}(z,q) = \langle 0, [(z)_1/2], k \rangle$.

CASE 8. $(z)_0 = 8$. SUBCASE 1. $(z)_2 \neq j \vee q \neq 1$. Let

$$\mathrm{ex}(z,q) = \langle 8, [(z)_1/p_j], j, \mathrm{ex}((z)_3, q + \overline{\mathrm{sg}}|(z)_2 - j''|) \rangle.$$

SUBCASE 2. $(z)_2 = j$ & $q = 1$. Let $\mathrm{ex}(z,q) = \langle 0, [(z)_1/p_j], k, \mathrm{ex}((z)_3, 1) \rangle$.

11.9. Since 'ϕ is general recursive in ψ' is reflexive and transitive (11.3, 11.5), the functions of special types 11.2 are partitioned into equivalence classes by the relation 'ϕ is general recursive in ψ, and vice versa' as in Kleene-Post [19, 1.2]. Everything in [19, 1.2 and 1.3] goes through now with no essential change.

In particular (but extending [19, 1.2] to include raising types), the proof of XI (with S0.1, and 2.1–2.4 as in 11.7) correlates to each function $\alpha(\mathfrak{a})$, where $\mathfrak{a}$ is a nonempty list of variables the highest of whose types is m, a type-$m+1$ function $\lambda\alpha^m \alpha^{m+1}(\alpha^m)$ of the same degree ($= \alpha$ itself, if α is already of a pure type). We call this the *pure-type function correlated to* α.

By passing first from $\alpha(\mathfrak{a})$ for $\mathfrak{a}$ as above to the predicate $\alpha(\mathfrak{a}) = w$ (which is of the same degree, since $\alpha(\mathfrak{a}) = \mu w[\alpha(\mathfrak{a}) = w]$ and XVI holds in a relativized version XVI*), and then applying XI to the representing function of this predicate, we are led from $\alpha(\mathfrak{a})$ to a *correlated set* of type m objects (i.e. a function of the pure type $m+1$ taking only 0 and 1 as values) of the same degree.

Corresponding to [19, end 1.2], the cardinality of the set of the degrees into which (the set of the functions of) a given special type is partitioned is the cardinality of the type ($2^{\aleph_0}$ when the independent variables are all of type 0).

Using 11.6, the recursive functions again constitute the least degree **0**. We define $\mathbf{a} \cup \mathbf{b}$ as the degree of $\langle \alpha^k, \beta^l \rangle$ where α^k and β^l are pure-type objects of the degrees **a** and **b**, respectively (cf. 11.7); it depends only on **a** and **b**. Formulas (1)–(9) of [19] hold.

11.10. We now understand by a *section* $\bar{\mathbf{a}}$ the set-theoretic union of all the degrees $\leqq$ a given degree **a**; thus $\bar{\mathbf{a}}$ consists of all the functions (predicates and sets) $\beta(\mathfrak{b})$ general recursive in a given function $\alpha(\mathfrak{a})$ of the degree **a**.

A degree **a** is determined by any one $\alpha(\mathfrak{a})$ of its members, a section $\bar{\mathbf{a}}$ by the highest degree **a** it includes as subset or by any one $\alpha(\mathfrak{a})$ of its members of that degree; and we may write $\mathbf{a} = \mathrm{dg}(\alpha)$, $\bar{\mathbf{a}} = \mathrm{sc}(\mathbf{a}) = \mathrm{sc}(\alpha)$.

11.11. We say a degree **a** (section $\bar{\mathbf{a}}$) is a *^{k}degree* (*^{k}section*), or is *generated by a type-k object*, if it contains a member (a member of its highest degree) of the pure type k; in this case, we may write **a** as ${}^k\mathbf{a}$ ($\bar{\mathbf{a}}$ as ${}^k\bar{\mathbf{a}}$).

11.12. Each ^{k}degree (^{k}section) is also an ^{l}degree (^{l}section) for each $l \geqq k$. For, $\mathrm{mp}_k^l(\alpha^k)$ is primitive, a fortiori general, recursive in α^k (cf. 2.3, 11.7); and by (7) α^k is likewise general recursive in $\mathrm{mp}_k^l(\alpha^k)$. Thus $\mathrm{mp}_k^l(\alpha^k)$ is a type-l object of the same degree as α^k.

11.13. We say a type-k object α^k is *reducible* (*to type-l*) if there is a type-l object α^l ($l < k$) of the same degree as α^k.

In this case, we may also speak of the ^{k}degree ${}^k\mathbf{a} = \mathrm{dg}(\alpha^k)$ [the ^{k}section ${}^k\bar{\mathbf{a}} = \mathrm{sc}(\alpha^k)$] as being *reducible* (*from type k to type l*), though it is not the degree [section] itself that is being reduced but its type description (the prefixed superior index).

11.14. By an ${}_{m+1}$*degree* ${}_{m+1}\mathbf{a}$ [${}_{m+1}$*section* ${}_{m+1}\bar{\mathbf{a}}$], or the ${}_{m+1}$*part of* **a** [*of* $\bar{\mathbf{a}}$], we mean the subset of **a** [of $\bar{\mathbf{a}}$] consisting of those functions (predicates and sets) whose independent variables are of types $\leqq m$, i.e. whose correlated pure-type objects are of types $\leqq m + 1$.

This notation may be combined with that of 11.11; thus ${}^k_{m+1}\bar{\mathbf{a}}$ is the set of the functions of variables of types $\leqq m$ which are general recursive in a type-k function α^k of degree **a**.

If ${}^k\mathbf{a}$ is irreducible, then ${}^k_l\mathbf{a}$ is empty for $l < k$.

11.15. By [19, 1.4] $(Ex)T_1^\alpha(a,a,x)$ is a predicate of degree $\mathbf{a}'$, depending only on the degree **a** of α, with $\mathbf{a}' > \mathbf{a}$. ($'$ was called a "jump" operation on degrees.) We state an analogous result for higher types.

Using the T of XXXVIII, for each $r \geqq 1$ let $N(\alpha^{r+1}, \gamma^r) \equiv (\eta^r)(E\xi^{r-1})T((\gamma^r)_0^0, \gamma^r, \alpha^{r+1}, \eta^r, \xi^{r-1})$ and $J[\alpha^{r+1}] = \lambda\gamma^r N(\alpha^{r+1}, \gamma^r)$ ("J" for "jump").

LXX. *For each* $r \geqq 1$:

$$\mathrm{dg}(J[\alpha^{r+1}]) \textit{ depends only on } \mathrm{dg}(\alpha^{r+1}). \tag{99}$$

$$\mathrm{dg}(\alpha^{r+1}) \leqq \mathrm{dg}(\beta^{r+1}) \rightarrow \mathrm{dg}(J[\alpha^{r+1}]) \leqq \mathrm{dg}(J[\beta^{r+1}]). \tag{100}$$

$$\mathrm{dg}(\alpha^{r+1}) < \mathrm{dg}(J[\alpha^{r+1}]). \tag{101}$$

Proof. (99) and (100). As before [19, p. 384], it will suffice to show that, if α^{r+1} is recursive in β^{r+1}, then $J[\alpha^{r+1}]$ is recursive in $J[\beta^{r+1}]$. But, slightly modifying the proof of XLIV, $N(\alpha^{r+1},(\gamma^r)_1) \equiv (\eta^r)(E\xi^{r-1})T(f,\gamma^r,\beta^{r+1},\eta^r,\xi^{r-1})$ for some number f (by (73) with XXIII and 11.8), so $N(\alpha^{r+1},\gamma^r) \equiv N(\alpha^{r+1},(\langle f,\gamma^r\rangle)_1) \equiv N(\beta^{r+1},\langle f,\gamma^r\rangle)$.

(101). By XVI* (cf. 11.9), α^{r+1} is recursive in $\lambda\alpha^r w\, \alpha^{r+1}(\alpha^r) = w$. Slightly modifying the proof of XLIV, $\alpha^{r+1}((\alpha^r)_2) = (\alpha^r)_1^0 \equiv (\eta^r)(E\xi^{r-1})T(f,\alpha^r,\alpha^{r+1},\eta^r,\xi^{r-1})$ by (73) etc., so $\alpha^{r+1}(\alpha^r) = w \equiv N(\alpha^{r+1},\langle f,w,\alpha^r\rangle)$. Thus using S8, etc.' $\lambda\alpha^r w\, \alpha^{r+1}(\alpha^r) = w$, and hence α^{r+1}, is recursive in $J[\alpha^{r+1}]$.

Suppose $J[\alpha^{r+1}]$, with representing function $\lambda\gamma^r \phi(\gamma^r)$, were recursive in α^{r+1}. Then by XXXV $\phi(\gamma^r) = w$, and (by substituting 0 for w) $N(\alpha^{r+1},\gamma^r)$, would be $r+1$-expressible in $=, +, \cdot, \alpha^{r+1}$ with a prenex $r+1$-expression in which all type-r quantifiers are existential. Applying the proof of XXXVIIa, we would then obtain $N(\alpha^{r+1},\gamma^r)$ in the form $(E\eta^r)(\xi^{r-1})R(\gamma^r,\alpha^{r+1},\eta^r,\xi^{r-1})$ with a (primitive) recursive R. But by the proof of XL with α^{r+1} as the $\mathfrak{b}$ (observing that the $\mathfrak{b}$ is held constant in that proof), this is absurd.

11.16. The type-2 object $\mathbf{E}$ of 8.2 we now write also ${}^2\mathbf{E}$; and generally, for each $k = 2,3,4,\cdots$, we define a type-k object ${}^k\mathbf{E}$ by

$$ {}^k\mathbf{E}(\alpha^{k-1}) = \begin{cases} 0 \text{ if } (E\alpha^{k-2})[\alpha^{k-1}(\alpha^{k-2}) = 0], \\ 1 \text{ otherwise.} \end{cases} $$

(Raising the types of all variables by the same amount $k-2$, the formulas in 8.2 for the representing function of $\alpha^1 = \beta^1$ in terms of ${}^2\mathbf{E}$ and vice versa generalize from $k = 2$ to any $k \geqq 2$.)

XLVIII can now be stated as: ${}_1\mathrm{sc}({}^2\mathbf{E})$ = the hyperarithmetical number-theoretic functions.

11.17. LXXI. (a) *If* α *is general recursive in* ${}^2\mathbf{E}$, *so is* $(Ex)T_1^\alpha(a,a,x)$. (b) *For each* $k \geqq 3$, *if* α^{k-1} *is general recursive in* ${}^k\mathbf{E}$, *so is* $J[\alpha^{k-1}]$.

Proof. (a) By (79), 11.8 and XXIII. (b) $N(\alpha^{k-1},\gamma^{k-2}) \equiv (\eta^{k-2})(E\xi^{k-3})$ $T((\gamma^{k-2})_0^0, \gamma^{k-2}, \alpha^{k-1}, \eta^{k-2}, \xi^{k-3}) \equiv (\eta^{k-2})(E\xi^{k-2})T((\gamma^{k-2})_0^0, \gamma^{k-2}, \alpha^{k-1}, \eta^{k-2}, \mathrm{pm}_{k-3}(\xi^{k-2}))$ (by (6)). Let $\psi(\varepsilon^k, \xi^{k-2}, \eta^{k-2}, \gamma^{k-2}, \alpha^{k-1})$ be the representing function of $T((\gamma^{k-2})_0^0, \gamma^{k-2}, \alpha^{k-1}, \eta^{k-2}, \mathrm{pm}_{k-3}(\xi^{k-2}))$ (constant in ε^k). Using S8 etc., let $\theta(\varepsilon^k, \eta^{k-2}, \gamma^{k-2}, \alpha^{k-1}) = \varepsilon^k(\lambda\xi^{k-2}\, \psi(\varepsilon^k, \xi^{k-2}, \eta^{k-2}, \gamma^{k-2}, \alpha^{k-1}))$ and $\phi(\varepsilon^k, \gamma^{k-2}, \alpha^{k-1}) = \overline{\mathrm{sg}}(\varepsilon^k(\lambda\eta^{k-2}\, \overline{\mathrm{sg}}(\theta(\varepsilon^k,\eta^{k-2},\gamma^{k-2},\alpha^{k-1}))))$. Then $\phi(\varepsilon^k,\gamma^{k-2},\alpha^{k-1})$ is primitive recursive, and $\phi({}^k\mathbf{E},\gamma^{k-2},\alpha^{k-1})$ is the representing function of $N(\alpha^{k-1},\gamma^{k-2})$; so for a particular α^{k-1} general recursive in ${}^k\mathbf{E}$, using 11.8 $J[\alpha^{k-1}]$ is general recursive in ${}^k\mathbf{E}$.

11.18. LXXII. *For* $k = 2,3,\cdots$: $\mathrm{dg}({}^k\mathbf{E})$ $(\mathrm{sc}({}^k\mathbf{E}))$ *is an irreducible* ^{k}*degree* (^{k}*section*).

Proof for $k = 2$. By XLVIII, the contrary would imply that there is a hyperarithmetic number-theoretic function α of highest degree, which is known not to be the case. A more direct proof is similar to the:

Proof for $k > 2$. Suppose $\mathrm{sc}({}^k\mathbf{E})$ were reducible. Then by 11.12 $\mathrm{sc}({}^k\mathbf{E}) = \mathrm{sc}(\alpha^{k-1})$ for some α^{k-1} (i.e. the type k can be reduced by 1, if it can be reduced at all). This is absurd, since by (101) $J[\alpha^{k-1}]$ would be of higher degree than α^{k-1}, while by LXXI (b) $J[\alpha^{k-1}]$ would also be recursive in ${}^k\mathbf{E}$.

11.19. Substituting relative μ-recursiveness 8.1 for relative general recursiveness (S10 of 8.1 replacing S9 in 3.14) in the definitions of degree, section, etc., we get corresponding notions with the prefix "μ-". The transitivity of relative μ-recursiveness, and the extension of 1.9 to include S10, are immediate, index constructions like those in 11.4 and 11.8 not being required.

XLVII can now be stated as: ${}_1\mu\text{-sc}({}^2\mathbf{E})$ = the arithmetical number-theoretic functions [${}_2\mu\text{-sc}({}^2\mathbf{E})$ = the arithmetical functions of type-0 and type-1 variables].

11.20. It is an open problem whether the arithmetical number-theoretic functions are the ${}_1$section of any type-2 object α^2. If they are, the α^2 and their generation from it are so devious that no arithmetical function χ exists such that, for each k, $\chi(k)$ is an index from α^2 of $L_k(a)$ (or, putting $\lambda(k,a) = \{\chi(k)\}^{\alpha^2}(a)$, λ would be the representing function of $\lambda ka\ L_k(a)$ and arithmetical, which is absurd; cf. [16, p. 198], 8.3, 8.4). Relative general recursiveness seems a more fundamental relation than relative μ-recursiveness (when they differ, §8); cf. §3 (Church's thesis extended), [39; 41; 42]. From this standpoint, the arithmetical functions are a less natural class of functions than the hyperarithmetical, which by XLVIII are exactly those general recursive in the simplest irreducible type-2 object ${}^2\mathbf{E}$ which comes to mind(24).

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(24) Another irreducible type-2 object is $\mathbf{E}_1'$ where

$$\mathbf{E}_1'(\alpha) = \begin{cases} 0 \text{ if } (\beta)\,(Ex)T_1{}^{(\alpha)_1,\beta}((a)_0^0,(a)_0^0,x), \\ 1 \text{ otherwise,} \end{cases}$$

with the T-predicate of IM p. 292, or the equivalent obtainable as in 5.24. In our notes, March 6, 1957, we conjectured that the number-theoretic predicates recursive in $\mathbf{E}_2'$ might exhaust the set of the predicates expressible in both 2-function-quantifier forms ($\Pi_2^1 \cap \Sigma_2^1$, in the notation of Addison [31, p. 127]), to which they all belong (similarly to XLVIII, the first part). We planned to show that the predicates recursive in $\mathbf{E}_1'$ include all predicates recursive in predicates of the Addison-Kleene $\mathfrak{H}_y$-hierarchy for $y \in O_{2_O}$ [33] as well as those obtainable by extending that hierarchy through the constructive higher number classes as we then proposed to define them (adapting [34, §6] in the direction of [36, §5]). However we felt that publication in this area should await a closer scrutiny of the constructive higher number classes than we then had time to make. Soon thereafter the constructive higher number classes were discussed by Wang [46] and then more thoroughly by Kreider and Rogers [43]. Other type-2 objects with the same degree, and hence the same ${}_1$section, as $\mathbf{E}_1'$ are $\mathbf{E}_1''$ and $\mathbf{E}_1$ where

$$\mathbf{E}_1''(\alpha) = \begin{cases} 0 \text{ if } (\beta)\,(Ex)T_0^{\alpha,\beta}(\alpha(0),\ x), \\ 1 \text{ otherwise,} \end{cases} \qquad \mathbf{E}_1(\alpha) = \begin{cases} 0 \text{ if } (\beta)\,(Ex)\,[\alpha(\bar{\beta}(x)) = 0], \\ 1 \text{ otherwise;} \end{cases}$$

these come from $\mathbf{E}_1'$ via [14, p. 320, Proof for $n = 0$ and Footnote 14] with [16, XIX* (cf. XXVII)]. Using $\mathbf{E}_1$, Togué in [45] and in an unpublished manuscript seen by us in September 1960 covered some of this ground. Subsequently Shoenfield [44], using a result of Addison [32], disproved the above conjecture.

11.21. The reducible ^{2}degrees are isomorphic with the ^{1}degrees, with all the complexities of fine structure found in [19], Spector [27], etc. A priori, one would expect a similar complexity of fine structure for the irreducible ^{2}degrees. But, at the moment, it is not even known whether there is an irreducible ^{2}degree $< \mathrm{dg}({}^2\mathbf{E})$, or whether there is an irreducible type-2 object α^2 such that ${}_1\mathrm{sc}(\alpha^2)$ is a proper subset of the hyperarithmetical functions. One approach would be to try to generalize some one of the operations which have been used in [19], Spector [27], etc. to construct a host of degrees between two levels of the arithmetical hierarchy [10; 16, §2] into a uniform operation, which will constitute a type-2 object α^2, and then to investigate $\mathrm{dg}(\alpha^2)$ and ${}_1\mathrm{sc}(\alpha^2)$. Another open question is whether, for some type-2 objects α^2 and β^2, ${}_1\mathrm{sc}(\alpha^2) = {}_1\mathrm{sc}(\beta^2)$ but $\mathrm{sc}(\alpha^2) \neq \mathrm{sc}(\beta^2)$.

11.22. We next prepare to extend XLVII to higher types (in LXXV).

LXXIII (an extension of XXXV). *For each* $r \geqq 1$: *If a function* $\phi(\mathfrak{a})$ *of variables* $\mathfrak{a}$ *of types* $\leqq r+1$ *is partial recursive in* (*total*) *functions* Ψ *of variables of types* $\leqq r$, *then a completion* $(\phi(\mathfrak{a}) = w)'$ *of* $\phi(\mathfrak{a}) = w$ (cf. 9.2) *is* $r+1$-*expressible in* $=, +, \cdot, \Psi$, *with a prenex* $r+1$-*expression in which all the type-r quantifiers are universal, and also one in which all are existential.*

Proof. As before, except that "$(\phi(\mathfrak{a}) = w)'$" replaces the last "$\phi(\mathfrak{a}) = w$" in Part (b).

11.23. Analogously to **F**-height in 8.3, we can define the β^{r+2}-*height* of a description of $\phi(\beta^{r+2}, a_1, \cdots, a_{n_0}, \alpha_1^1, \cdots, \alpha_{n_1}^1, \cdots, \alpha_1^{r-1}, \cdots, \alpha_{n_{r-1}}^{r-1}, \beta_1^r, \cdots, \beta_{n_r}^r, \beta^{r+1})$ by S1–S5, S6.0–S6.r, S7, S8.2–S8.$(r+2)$, S10.

LXXIV. *For each* $r \geqq 1$: *Suppose* $\phi(\beta^{r+2}, a_1, \cdots, \alpha_{n_{r-1}}^{r-1}, \beta_1^r, \cdots, \beta_{n_r}^r, \beta^{r+1})$ *is partial* μ-*recursive with a description of* β^{r+2}-*height h. Then there are a function* $\phi(a_1, \cdots, \alpha_{n_{r-1}}^{r-1}, \beta_1^r, \cdots, \beta_{n_r}^r, \beta^{r+1})$ *and a predicate* $R_h(\beta^{r-1}, \beta_1^r, \cdots, \beta_{n_r}^r, \beta^{r+1})$ *expressible in the h-*(*type-r*)*-quantifier form with primitive recursive scope with* (*say*) *existence first* (*for* $h > 0$) *such that* (a) $\phi(a_1, \cdots, \alpha_{n_{r-1}}^{r-1}, \beta_1^r, \cdots, \beta_n^r, \beta^{r+1})$ *is an extension of*

$$\phi({}^{r+2}\mathbf{E}, a_1, \cdots, \alpha_{n_{r-1}}^{r-1}, \beta_1^r, \cdots, \beta_{n_r}^r, \beta^{r+1}),$$

and (b) $\lambda a_1 \cdots \alpha_{n_{r-1}}^{r-1}\ \phi(a_1, \cdots, \alpha_{n_{r-1}}^{r-1}, \beta_1^r, \cdots, \beta_{n_r}^r, \beta^{r+1})$ *is partial* μ-*recursive in* $\beta^r, \cdots, \beta_{n_r}^r, \beta^{r+1}, \lambda\beta^{r-1}\ R_h(\beta^{r-1}, \beta_1^r, \cdots, \beta_{n_r}^r, \beta^{r+1})$ *uniformly in* $\beta_1^r, \cdots, \beta_{n_r}^r, \beta^{r+1}$. *Similarly without the* β^{r+1}.

Proof (with the β^{r+1}), by induction on the length of the given description of $\phi(\beta^{r+2}, a_1, \cdots)$ by S1–S8, S10. For $h = 0$, $R_h(\beta^{r-1}, \beta_1^r, \cdots, \beta_{n_r}^r, \beta^{r+1}) \equiv 0 = 0$ simply.

Cases 1, 2, 3, 7. $\phi(\beta^{r+2}, a_1, \cdots)$ is introduced by S1, S2, S3 or S7. Then the β^{r+2}-height $h = 0$, and $\phi(\beta^{r+2}, a_1, \cdots)$ is independent of β^{r+2}. Defining $\phi(a_1, \cdots)$ by the same respective schema, $\phi(a_1, \cdots) = \phi({}^{r+2}\mathbf{E}, a_1, \cdots)$ and $\lambda a_1 \cdots \alpha_{n_{r-1}}^{r-1} \phi(a_1, \cdots)$ is primitive, a fortiori partial μ-, recursive absolutely or (for S7 with $r = 1$) uniformly in β_1^r, and hence in $\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1},\lambda\beta^{r-1}R_h(\beta^{r-1},\cdots)$ uniformly in $\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1}$, for $R_h(\beta^{r-1},\cdots)\equiv 0=0$.

CASE 4. $\phi(\beta^{r+2},a_1,\cdots)\simeq\psi(\beta^{r+2},\chi(\beta^{r+2},a_1,\cdots),a_1,\cdots)$ by S4. By hyp. ind. there are $\psi(c,a_1,\cdots)$ and R_{1,h_1} ($\chi(a_1,\cdots)$ and R_{2,h_2}) with the properties (a) and (b) with respect to $\psi(\beta^{r+2},c,a_1,\cdots)$ (to $\chi(\beta^{r+2},a_1,\cdots)$); and $h=\max(h_1,h_2)$. Letting $\phi(a_1,\cdots)\simeq\psi(\chi(a_1,\cdots),a_1,\cdots)$, clearly (a) is satisfied with respect to $\phi(\beta^{r+2},a_1,\cdots)$. For $h>0$, since $h\geqq h_i$ $(i=1,2)$, we can consider the existential h_i-(type-r)-quantifier expression for R_{i,h_i} as of the existential h-(type-r)-quantifier form, so (again adapting the proof of XLIV) by XXXVIII, for some f_i, $R_{i,h_i}(\beta^{r-1},\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1})\equiv(Q\mathfrak{x})S(f_i,\beta^{r-1},\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1},\mathfrak{x})$ where $(Q\mathfrak{x})$ is the prefix in question and S is T or $\bar{T}$ according as h is even or odd, whence $R_{i,h_i}(\beta^{r-1},\cdots)\equiv R_h(\langle f_i,\beta^{r-1}\rangle,\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1})$ for

$$R_h(\beta^{r-1},\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1})\equiv(Q\mathfrak{x})S((\beta^{r-1})_0^0,(\beta^{r-1})_1,\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1},\mathfrak{x}).$$

CASE 8.j $(2\leqq j<r+2)$. $\phi(\beta^{r+2},a_1,\cdots)\simeq\beta^j(\lambda\xi^{j-2}\chi(\beta^{r+2},\xi^{j-2},a_1,\cdots))$ where β^j is one of $\alpha_1^2,\alpha_1^3,\cdots,\alpha_1^{r-1},\beta_1^r,\beta^{r+1}$, by S8.$j$. By hyp. ind., there are $\chi(\xi^{j-2},a_1,\cdots)$ and $R_h(\beta^{r-1},\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1})$ with the properties (a) and (b) in relation to $\chi(\beta^{r+2},\xi^{j-2},a_1,\cdots)$. This R_h suffices as the R_h for this case, and we take $\phi(a_1,\cdots)\simeq\beta^j(\lambda\xi^{j-2}\chi(\xi^{j-2},a_1,\cdots))$.

CASE 8.$(r+2)$. $\phi(\beta^{r+2},a_1,\cdots)\simeq\beta^{r+2}(\lambda\xi^r\chi(\beta^{r+2},\xi^r,a_1,\cdots))$ by S8.$(r+2)$. By hyp. ind. there are $\chi(\xi^r,a_1,\cdots)$ and $R_{h-1}(\beta^{r-1},\xi^r,\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1})$ with the properties (a) and (b) for χ. Now let

$$\phi_0(a_1,\cdots,\alpha_{n_{r-1}}^{r-1},\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1})\simeq{}^{r+2}\mathbf{E}(\lambda\xi^r\chi(\xi^r,a_1,\cdots))$$

$$\simeq\begin{cases}0 \text{ if } (E\xi^r)[\chi(\xi^r,a_1,\cdots)=0],\\ 1 \text{ otherwise,}\end{cases}\tag{102}$$

this being undefined if $\lambda\xi^r\chi(\xi^r,a_1,\cdots)$ is incompletely defined. Since $\chi(\xi^r,a_1,\cdots)$ is an extension of $\chi({}^{r+2}\mathbf{E},\xi^r,a_1,\cdots)$, $\phi_0(a_1,\cdots)$ is an extension of $\phi({}^{r+2}\mathbf{E},a_1,\cdots)$. By (b) of the hyp. ind., $\lambda a_1\cdots\alpha_{n_{r-1}}^{r-1}\chi(\xi^r,a_1,\cdots)$ is partial μ-recursive, a fortiori (by XVI*) partial recursive, in $\xi^r,\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1},\lambda\beta^{r-1}R_{h-1}(\beta^{r-1},\xi^r,\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1})$ uniformly in $\xi^r,\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1}$. Hence by LXXIII, a completion $\lambda a_1\cdots\alpha_{n_{r-1}}^{r-1}(\chi(\xi^r,a_1,\cdots)=0)'$ of $\lambda a_1\cdots\alpha_{n_{r-1}}^{r-1}\chi(\xi^r,a_1,\cdots)=0$ is expressible (uniformly in $\xi^r,\beta_1^r,\cdots,\beta_{n_r}^r,\beta^{r+1}$) in a prenex form, consisting of a prefix of type-r and lower-type quantifiers only, with all those of type r existential, applied to a scope formed by propositional calculus from prime formulas built out of $a_1,\cdots,\alpha_{n_{r-1}}^{r-1}$, $=,+,\cdot$, ξ^r, $\beta_1^r,\cdots,\beta_{n_r}^r$, β^{r+1} and the representing function of the predicate $\lambda\beta^{r-1}R_{h-1}(\beta^{r-1},\cdots)$. Using 7.7(b), by introducing only some type-0 quantifiers, this representing function can be replaced by the predicate itself. Considering each occurrence of the latter (expressible in an $(h-1)$-(type-r)-quantifier form) as of the appropriate h-(type-r)-quantifier form, we can as in [14, the proof of Theorem 5] or 7.13 advance and contract quantifiers to transform the original scope to an expression in the h-(type-r)-quantifier form with existence first. Using this expression in place of the original scope, prefixing to it the original prefix and to that $(E\xi^r)$, the resulting expression comes to an h-(type-r)-quantifier expression $P(a_1, \cdots, \alpha^{r-1}_{n_{r-1}}, \beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1})$ with existence first. Then $\lambda a_1 \cdots \alpha^r_{n_{r-1}} P(a_1, \cdots)$ is primitive, a fortiori partial μ-, recursive in $\beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1}, \lambda\beta^{r-1} R_h(\beta^{r-1}, \beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1})$ uniformly in $\beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1}$ when we take $R_h(\beta^{r-1}, \beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1}) \equiv P((\beta^{r-1})^0_0, \cdots, (\beta^{r-1})^0_{n_0-1}, (\beta^{r-1})^1_{n_0}, \cdots, (\beta^{r-1})^1_{n_0+n_1-1}, \cdots, (\beta^{r-1})_{n_0+\cdots+n_{r-2}}, \cdots, (\beta^{r-1})_{n_0+\cdots+n_{r-1}-1}, \beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1})$. Now let

$$\phi(a_1, \cdots, \alpha^{r-1}_{n_{r-1}}, \beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1}) = \begin{cases} 0 \text{ if } P(a_1, \cdots), \\ 1 \text{ otherwise.} \end{cases} \tag{103}$$

Clearly (b) is satisfied; and (a) is satisfied because this makes $\phi(a_1, \cdots)$ an extension of $\phi_0(a_1, \cdots)$ and thence of $\phi({}^{r+2}\mathbf{E}, a_1, \cdots)$.

Case 10. $\phi(\beta^{r+2}, a_1, \cdots) \simeq \mu y[\chi(\beta^{r+2}, a_1, \cdots, y) = 0]$ by S10. Use the hyp. ind., with the R_h for the χ as the R_h.

11.24. LXXV (an extension of XLVII). *For each* $r \geqq 0$ *and each* k $(0 < k \leqq r+2)$: ${}_k\mu$-sc$({}^{r+2}\mathbf{E})$ *is exactly the set of the functions of order* $r+1$ *with variables of types* $< k$.

Proof for $r > 0$. (a) Suppose $\phi(\mathfrak{a})$ with variables $\mathfrak{a}$ of types $< k \leqq r+2$ is μ-recursive in ${}^{r+2}\mathbf{E}$, so by 1.9 extended (cf. 11.19) $\phi(\mathfrak{a}) = \phi({}^{r+2}\mathbf{E}, \mathfrak{a})$ where $\phi(\beta^{r+2}, \mathfrak{a})$ is partial μ-recursive. We must show that $\phi(\mathfrak{a}) = w$ is $r+1$-expressible in general recursive predicates. There will be no loss of generality in supposing the variables $\mathfrak{a}$ include either none or just one of type $r+1$ (e.g. were there two $\beta^{r+1}_1, \beta^{r+1}_2$, we could pass to a function ϕ_1 with one β^{r+1} by substituting $(\beta^{r+1})_0, (\beta^{r+1})_1$, draw our conclusion for ϕ_1, and then substitute $\langle \beta^{r+1}_1, \beta^{r+1}_2 \rangle$ for β^{r+1}). Consider the case of one type-$r+1$ variable β^{r+1}, so that the variables $\mathfrak{a}$ are the $a_1, \cdots, \alpha^{r-1}_{n_{r-1}}, \beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1}$ of the first case of LXXIV (for zero, we would instead apply the second case of LXXIV). Since $\lambda\mathfrak{a}\,\phi(\mathfrak{a}) = \lambda\mathfrak{a}\,\phi({}^{r+2}\mathbf{E}, \mathfrak{a})$ is completely defined, by LXXIV $\lambda a_1 \cdots a_{n_0} \alpha^1_1 \cdots \alpha^{r-1}_{n_{r-1}} \phi(\mathfrak{a})$ is μ-, a fortiori general, recursive in $\beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1}, \lambda\beta^{r-1} R_h(\beta^{r-1}, \beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1})$ uniformly in $\beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1}$. Hence by XXXV with 7.7, $\lambda a_1 \cdots a_{n_0} w \alpha^1_1 \cdots \alpha^{r-1}_{n_{r-1}} \phi(\mathfrak{a}) = w$ is $r+1$-expressible in $=, +, \cdot, \lambda\alpha^{r-1} w\, \beta^r_1(\alpha^{r-1}) = w, \cdots, \lambda\alpha^{r-1} w\, \beta^r_{n_r}(\alpha^{r-1}) = w$, $\lambda\alpha^r w\, \beta^{r+1}(\alpha^r) = w, \lambda\beta^{r-1} R_h(\beta^{r-1}, \cdots)$ uniformly in $\beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1}$ (with a prenex form in which all type-r quantifiers are universal, and one in which all are existential). But $R_h(\beta^{r-1}, \cdots)$ is expressible in an h-(type-r)-quantifier form with recursive scope. By advancing quantifiers suitably (as in the proof of XLV), we can obtain either $h+1$-(type-r)-quantifier form for $\phi(\mathfrak{a}) = w$ with $\mathfrak{a}, w$ as the variables (i.e. $\beta^r_1, \cdots, \beta^r_{n_r}, \beta^{r+1}$ are now included).

(b) Let $\mathfrak{a}$ be variables of types $\leqq r+1$, and consider any function $\phi(\mathfrak{a})$ of order $r+1$, i.e. $\phi(\mathfrak{a}) = w$ is of order $r+1$. By XXXVIIa, then $\phi(\mathfrak{a}) = w$ is expressible in one of the forms (c_1) (for $\mathfrak{a}, w$ as the $\mathfrak{a}$ of (c_1)), say with $h > 0$ type-r

quantifiers; and by XXXIX the R can be chosen to be primitive recursive. Then as in the proof of LXXI(b), using $h+1$ applications of S8.$(r+2)$, the representing function of $\phi(\mathfrak{a}) = w$ is primitive, and using S10 $\phi(\mathfrak{a})$ itself is μ-, recursive in ${}^{r+2}\mathbf{E}$.

11.25. Now we establish an extension to higher types of part of XLVIII.

LXXVI. *For each $r \geqq 0$: Each predicate of variables of types $< r+2$ general recursive in ${}^{r+2}\mathbf{E}$ is expressible in both one-(type-$r+1$)-quantifier forms.*

Proof. Say $P(\mathfrak{a})$, with variables $\mathfrak{a}$ of types $< r+2$, is general recursive in ${}^{r+2}\mathbf{E}$. Let $\phi(\mathfrak{a})$ be the representing function of $P(\mathfrak{a})$, so $P(\mathfrak{a}) \equiv \phi(\mathfrak{a}) = 0$. By 1.9 extended to include S9 (cf. 11.8), there is a partial recursive function $\phi(\mathfrak{a}, \varepsilon^{r+2})$ such that, for each $\mathfrak{a}, \phi(\mathfrak{a}) = \phi(\mathfrak{a}, {}^{r+2}\mathbf{E})$. By XXVI (for z an index of $\phi(\mathfrak{a}, \varepsilon^{r+2})$, and $w = 0$) and (13), there are primitive (a fortiori, μ-) recursive predicates $R(\mathfrak{a}, \varepsilon^{r+2}, \beta^{r+1}, \xi^r)$ and $S(\mathfrak{a}, \varepsilon^{r+2}, \beta^{r+1}, \xi^r)$ such that, for each $\mathfrak{a}$,

$$(104) \qquad \begin{aligned} P(\alpha) \;&\equiv\; (\beta^{r+1})(E\xi^r)R(\mathfrak{a}, {}^{r+2}\mathbf{E}, \beta^{r+1}, \xi^r) \\ &\equiv\; (E\beta^{r+1})(\xi^r)S(\mathfrak{a}, {}^{r+2}\mathbf{E}, \beta^{r+1}, \xi^r). \end{aligned}$$

Say the representing functions of R and S are ρ and σ. By LXXV (and 1.9 extended to include S10; cf. 11.19), $\rho(\mathfrak{a}, {}^{r+2}\mathbf{E}, \beta^{r+1}, \xi^r)$ is of order $r+1$, so using also XXXVIIa $R(\mathfrak{a}, {}^{r+2}\mathbf{E}, \beta^{r+1}, \xi^r)$ $(\equiv \rho(\mathfrak{a}, {}^{r+2}\mathbf{E}, \beta^{r+1}, \xi^r) = 0)$ is expressible in one of the forms (c_1); likewise $S(\mathfrak{a}, {}^{r+2}\mathbf{E}, \beta^{r+1}, \xi^r)$. Now an application of the method of proof of XXXVIIa with (104) brings $P(\mathfrak{a})$ to both one-(type-$r+1$)-quantifier forms.

11.26. We do not know whether the converse of LXXVI holds.

The hyperarithmetical number-theoretic predicates were characterized in a number of equivalent ways, among them by expressibility in both one-(type-1)-quantifier forms [16, p. 210], and by general recursiveness in ${}^{2}\mathbf{E}$ (using XLVIII). Of these characterizations, that by general recursiveness in ${}^{2}\mathbf{E}$ has the most appeal to us: the hyperarithmetical predicates are exactly those definable constructively except for using quantification over the natural numbers, embodied in ${}^{2}\mathbf{E}$ (cf. [38, p. 150]).

It cannot be presumed that all of the characterizations equivalent to one another in the case of number variables only will be equivalent at the higher types, i.e. that their natural extensions to predicates with variables of higher types are equivalent to one another. The facts will have to be investigated. (The situation was similar for general and partial recursiveness.)

We now propose (generalizing from hyperarithmetical = hyper-(order-1)) to define the *hyper-(order-r)* predicates to be the predicates general recursive in ${}^{r+1}\mathbf{E}$ $(r \geqq 1)$. In particular, the *hyperanalytic* predicates shall be those general recursive in ${}^{3}\mathbf{E}$. (Cf. 7.3.) We are primarily concerned with the case that the variables do not exceed in type the order in question, as the theory thus far available is limited to this case. The notion extends to functions via their representing predicates. (Cf. 7.1.)

11.27. Many questions arise in connection with the foregoing extension to higher types of the notion of hyperarithmeticity in the version general recursiveness in ${}^2\mathbf{E}$ and the extensions of its equivalents at type 0, on which we have work in progress which we hope later to report in print.

We shall close this discussion by proposing one method of obtaining for $r > 1$ a hierarchy within the hyper-(order-r) predicates of variables of type $r-1$ similar to that of the hyperarithmetic number-theoretic predicates given by the predicates H_y for $y \in O$ (cf. [16, p. 210] or [38, p. 149]). It will be an open question whether this new hierarchy exhausts the hyper-(order-r) predicates of type-$r-1$ variables.

To get a hierarchy of (some) such predicates, we could simply adapt the definition of H_y for $y \in O$ to the new type $r-1$ of the independent variables by substituting the new jump operation J of 11.15 for the old one $'$ of [19; 16], etc. However, it seems clear that this would not take us all the way through the hyper-(order-r) predicates, because the system O will not contain notations for all the ordinals which could be servicable now. For, as we use J (with ordinals already definable) to build new predicates, the possibility will arise of defining new ordinals recursively in the new predicates. This possibility did not arise in the case $r = 1$, because by Spector's [26, Theorem 6 Corollary 2 p. 161] no more ordinals are definably recursively in any hyperarithmetical predicate than recursively absolutely. What is called for now is a simultaneous generation of predicates of a hierarchy and of ordinal notations. Thus, at any stage, we should be able to construct a fundamental sequence of ordinal notations recursive in any predicate already defined, and to form an infinite join [19, 3.1] of predicates already defined indexed by the notations of that sequence. There is no loss of generality in taking the predicate in which the fundamental sequence is recursive as the first of the predicates to be joined. This leads us to the following definition, for $r \geqq 2$(25).

When γ^r is the representing function of a 1-place predicate $G(\alpha^{r-1})$, we may write $J[\gamma^r]$ as $J[G]$. In the following, we may write $H^r_y(\alpha^{r-1})$ as $H^r(y, \alpha^{r-1})$ for typographical simplification with complicated y.

O^r1. $1 \in O^r$ and $H^r_1(\alpha^{r-1}) \equiv 0 = 0$. O^r2. If $y \in O^r$, then $2^y \in O^r$ and $y <^r_O 2^y$ and $H^r_{2^y} = J[H^r_y]$. O^r3. If y defines y_n general recursively from H_u as a function of n_O, where $y_0 = u$ and $(n)[y_n \in O^r \,\&\, y_n <^r_O y_{n+1}]$, then $3 \cdot 5^y \cdot 7^u \in O^r$ and $(n)[y_n <^r_O 3 \cdot 5^y \cdot 7^u]$ and $H^r_{3 \cdot 5^y \cdot 7^u}(\alpha^{r-1}) \equiv H^r(y_{(\alpha^{r-1})^0_0}, (\alpha^{r-1})_1)$. O^r4. If $x \in O^r$, $y \in O^r$, $z \in O^r$, $x <^r_O y$ and $y <^r_O z$, then $x <^r_O z$. O^r5. $a \in O^r$ and $a <^r_O b$ only as required by O^r1–O^r4.

(25) The definition was proposed by the author in a seminar at the University of Wisconsin in the summer of 1960. An investigation of it is included in the Ph. D. thesis of D. A. Clarke [35].

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