DIFFERENTIABLE OPEN MAPS ON MANIFOLDS

BY

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Introduction. This paper contains a detailed discussion with proofs of results announced in [4].

Let $M^n$ and $N^n$ be $n$-manifolds without boundary, and let $f: M^n \rightarrow N^n$ be continuous. The map $f$ is open if, whenever $U$ is open in $M^n$, $f(U)$ is open in $N^n$; it is light if, for every $y \in N^n$, $\dim (f^{-1}(y)) \leq 0$. For $n \geq 2$ there is a canonical light open map $F_{n,d}: E^n \rightarrow E^n$ given by $F_{n,d}(x_1, x_2, \ldots, x_n) = (u_1, u_2, x_3, \ldots, x_n)$, where

$$u_1 + iu_2 = (x_1 + ix_2)^d \quad (i = \sqrt{-1}; \ d = 1, 2, \ldots).$$

For $n = 2$ it is well known that a nonconstant complex analytic function is open and light. Conversely, Stoilow [12] proved that every light open map is locally topologically equivalent to an analytic map, and thus to some $F_{2,d}$ ($d = 1, 2, \ldots$). In fact (1.10), if $M^2$ is compact and $f$ is $C^2$ and open, then $f$ has this canonical structure. The main object of this paper is to prove (2.1) that the corresponding conclusion holds for arbitrary $n$ ($n \geq 2$), if we first remove an exceptional set of dimension at most $n - 3$. Examples are given, especially in §3, showing that the exceptional set and some of the hypotheses used are necessary.

Definition. As in [5] the branch set $B_f$ is the set of points in $M^n$ at which $f$ fails to be a local homeomorphism.

Notation. If $f: E^n \rightarrow E^p$ is $C^r$, then $f_i$ will be the $i$th component real-valued function, and $D_j f_i$ will be the first partial derivative of $f_i$ with respect to its $j$th coordinate. If $y$ is a point in $E^n$, then $y_i$ will be its $i$th coordinate. The symbols $M^n$ and $N^p$ will refer to manifolds of dimensions $n$ and $p$, respectively. The statement that $f: M^n \rightarrow N^p$ is $C^m$ will imply that the manifolds are also $C^m$. The set of points in $M^n$ at which the Jacobian matrix of $f$ has rank at most $q$ will be denoted by $R_q$.

The closure of a set $X$ is denoted by $\text{Cl}[X]$ or $\bar{X}$, its interior by $\text{int} X$, and the restriction of $f$ to $X$ by $f|X$. A map is a continuous function, the distance between the points $x$ and $y$ is $d(x, y)$, and $S(x, \varepsilon) = \{y: d(x, y) < \varepsilon\}$.

1. General results.

1.1. Lemma. Let $h: E^n \rightarrow E^p$, $h \in C^m$ ($m = 1, 2, \ldots$), and let the rank of the Jacobian matrix of $h$ at $\bar{x}$ be at least $q$ ($q = 1, 2, \ldots, n - 1$). Then there exist open neighborhoods $U$ of $\bar{x}$ and $V$ of $h(\bar{x})$, and $C^m$ diffeomorphisms (onto)
$k^1: E \to U$ and $k^2: V \to E^p$ such that $k^2 h k^1$, call it $g$, has the following properties:

1. For each $(p - q)$-plane $\alpha$ given by $g_i$ constant ($i = 1, 2, \cdots, q$), $g^{-1}(\alpha)$ is a (single) $(n - q)$-plane given by $x_j$ constant ($j = 1, 2, \cdots, q$).

2. For each $x$ in $g^{-1}(\alpha)$, the rank of the Jacobian matrix of $g$ at $x$ is $s$ if and only if the rank of the Jacobian of $g | g^{-1}(\alpha)$ at $x$ is $s - q$ ($s = q, q + 1, \cdots, \min(n, p)$).

Proof. By reordering the variables, both dependent and independent, we may suppose that the determinant $\det[D_j h_i(x)] \neq 0$ ($i, j = 1, 2, \cdots, q$). Let $W$ be a neighborhood of $x$ such that $\det[D_j h_i(x)] \neq 0$ on all of $W$, and let $h: W \to E^q$ be defined by $h_i = h_i (i = 1, 2, \cdots, q)$. Since $h$ has maximal rank at every point of $W$, we may apply the rank theorem [7, pp. 273-274]. Thus, there exists an open $n$-cell $U$ in $W$ about $x$ and $C^\infty$ diffeomorphisms $k^1: E^n \to U$ and $k: h(U) \to E^q$ such that $(k^2 h k^1)(x_1, x_2, \cdots, x_n) = x_i$ ($i = 1, 2, \cdots, q$). Using $V = h(U) \times E^{p-q}$, $k^2(x) = k_1(x_1, x_2, \cdots, x_q)$ ($i = 1, 2, \cdots, q$), and $k^2(x) = x_i$ ($i = q + 1, q + 2, \cdots, p$), conclusion (1) follows.

Given $x \in E^n$, let $s$ be the rank of the Jacobian matrix of $g$ at $x$. If $\alpha$ is the $(p - q)$-coordinate plane containing $g(x)$, then $J = (D_j g_i(x))$ ($i = q + 1, q + 2, \cdots, p$; $j = q + 1, q + 2, \cdots, n$) is the Jacobian matrix of $g | g^{-1}(\alpha)$ (as a map into $\alpha$) at $x$. Since (by (1)) $D_j g_i(x) = 0$ ($i = 1, 2, \cdots, q$; $j = q + 1, q + 2, \cdots, n$), and $\det(D_j g_i(x)) \neq 0$ ($i, j = 1, 2, \cdots, q$), $J$ has rank $s - q$, yielding (2). (Clearly, the same result holds if we interpret $g | g^{-1}(\alpha)$ as a map into $E^p$.)

Remark. If $p = n$ and if $f | U$ has Jacobian determinant non-negative or nonpositive, then, for each $(n - q)$-cell $\gamma$ given by conclusion (1), $f | \gamma$ (i.e., $g | \gamma$) has Jacobian determinant non-negative or nonpositive (not "respectively," in general). In particular, if $q = n - 1$, then each map $f | \gamma$ is monotone.

1.2. Remark. If $X$ is a compact set contained in $E^n = E^{n-r} \times E^r$, and if dim($X \cap (E^{n-r} \times \{x\}) \leq q - r$ for each $x$ in $E^r$, then dim $X \leq q$.

Let $g: E^{n-r} \times E^r \to E^p$ be the projection map, and let $f$ be the restriction of $g$ to $X$. The proof, pointed out to the author by E. Connell, follows from an application of [9, pp. 91-92].

The following result is related to Sard's theorem [11].

1.3. Proposition. If $f: M^n \to N^p$, $f$ and the manifolds are $C^\infty$, then dim($f(R_q)$) $\leq q$ (where $R_q$ is the set of points of $M^n$ at which the Jacobian matrix of $f$ has rank at most $q$). In particular, dim($f(M^n)$) $\leq n$. If $f$ is also light then \( \dim(R_q) \leq q \).

Proof. Clearly, it is sufficient to prove the theorem for $f: E^n \to E^p$. If $X_i$ is the set of critical points of $f_i$ (the points at which all first partials are zero), then the measure of $f_i(X)$ is zero [10, p. 68, (4.3)] ($i = 1, 2, \cdots, p$). Thus, $\dim(\bigcap_{i=1}^p f_i(X)) \leq 0$. Since $R_0 = \bigcap_{i=1}^p X_i$, it follows that $\dim(f(R_0)) \leq 0$.  

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(In fact, for each \( u \) in \( f(R_0) \), there exists a coordinate \( p \)-cube \( C \) containing \( u \) such that the sides of \( C \) are coordinate \((p - 1)\)-planes, \( \text{diam}(C) < \varepsilon \), and \( f(R_0) \cap \text{bdy} \ C = \emptyset \).)

The proposition follows for \( p = 1 \) and all \( n \) and \( q \); we proceed by induction on \( p \).

Since each \( R_q \) is closed, \( f(R_q) \) is the countable union of compact sets; thus it is sufficient to prove that \( \dim(f(R_q - R_0)) \leq q \), i.e., to prove the result in the case that the rank is at least one at each point. Furthermore, it suffices to prove the conclusion for \( f|U \), where \( U \) is the open set given by (1.1) for \( f, q = 1 \), and an arbitrary point \( \bar{x} \). For each \((n - 1)\)-cell \( y \) given in (1.1) and each point \( x \) of \( R_q \cap y \), \( f|y \) (as a map into the corresponding \((p - 1)\)-cell) has rank at most \( q - 1 \) at \( x \). From the inductive hypothesis, \( \dim(f(R_q \cap y)) \leq q - 1 \). Thus \( f(R_q \cap U) \) meets each \((p - 1)\)-cell of (1.1) in a set of dimension at most \( q - 1 \). Since \( R_q \cap U \) is the countable union of compact sets, it follows from (1.2) that \( \dim(f(R_q \cap U)) \leq q \); thus \( \dim(f(R_q)) \leq q \).

If \( f \) is also light, then by [9, pp. 91-92] \( \dim(R_q) \leq \dim(f(R_q)) \). The condition that \( f \in C^n \) is necessary [17] in the above result for \( p = 1 \).

The following result is, for open maps, an extension of the inverse function theorem.

1.4. Theorem. Let \( f : E^n \rightarrow E^n \) be open and \( C \). If the rank of the Jacobian matrix of \( f \) at \( \bar{x} \) is at least \( n - 1 \), then \( f \) is locally a homeomorphism at \( \bar{x} \). In other words, \( B_f \subset R_{n-2} \).

Proof. For \( n = 1 \), the openness alone implies that \( f \) is a homeomorphism into. For \( n > 1 \), let \( U \) be the neighborhood of \( \bar{x} \) given by (1.1) for \( q = n - 1 \), and let \( y \) be one of the \( 1 \)-cells. Since \( f(y) \) is contained in a \( 1 \)-cell, and since the restriction \( f|y \) is open [20, p. 147, (7.2)], \( f|y \) is a homeomorphism (into). Thus, \( f|U \) is one-to-one and open, so that \( \bar{x} \notin B_f \).

1.5. Corollary. If \( f : M^n \rightarrow N^n, f \) open and \( C^n \), then \( \dim(f(B_f)) \leq n - 2 \). If \( f \) is also light, then \( \dim(B_f) \leq n - 2 \).

The proof follows from (1.2) and (1.4).

1.6. Corollary. Let \( f : M^n \rightarrow N^n, f \) light and \( C^n \). Then \( f \) is open if and only if \( B_f \subset R_{n-2} \).

Proof. If \( B_f \subset R_{n-2} \), then \( \dim(f(B_f)) \leq n - 2 \) by (1.2); thus \( f \) is open [5, p.531, (2.4)].

A sufficient condition for openness was given in [14] by Titus and Young. We observe that (for \( f \in C^n \)) the condition is necessary, and give an independent proof of the sufficiency.

1.7. Corollary. Let \( f : E^n \rightarrow E^n \) be \( C^n \) and light. Then \( f \) is open if and only if the Jacobian determinant \( J \) is non-negative or nonpositive everywhere.
Proof. If \( J \) is open, then, by (1.3) and (1.6), \( \dim(B_J) \leq n - 2 \); thus \( J \) does not change sign.

Suppose that \( J \geq 0 \) (or \( J \leq 0 \)). If the rank of the Jacobian matrix of \( f \) at \( \bar{x} \) is (at least) \( n - 1 \), then (from the remark after (1)) \( f|U \) is one-to-one. For each closed \( n \)-cell \( C \) in \( U \), \( f|C \) is a homeomorphism onto its image; thus \( f|\text{int } C \) is a homeomorphism onto its image, which is open by the theorem on invariance of domain. It follows that \( x \notin B_J \). Since \( B_J \subset R_{n-2} \), the conclusion follows from (1.6).

1.8. Theorem. If \( f: M^n \to N^n \), \( M^n \) compact, \( f \) open and \( C^n \), then \( f \) is light.

In fact, \( f \) is a pseudo-covering map \([5, \text{pp. 529 and 531, (2.4)}]\).

Proof. By (1.5) \( \dim(f(B_f)) \leq n - 2 \). By the second paragraph of the proof of \([5, \text{p. 531, (2.4)}]\) the restriction of \( f \) to \( M^n - f^{-1}(f(B_f)) \) is a \( k \)-to-1 covering map for some \( k \).

Suppose that for some \( y \) in \( f(B_f) \), \( f^{-1}(y) \) contains at least \( k + 1 \) distinct points \( y^i \) \((i = 1, 2, \ldots, k + 1)\). Then there exist disjoint open neighborhoods \( U^i \) of these points and \( \bigcap_{i=1}^{k+1} f(U^i) \) is an open set; thus it meets \( N^n - f(B_f) \), yielding a contradiction.

Remark. This result contrasts with the examples by R.D. Anderson \([1; 2]\) of monotone open (not \( C^n \)) maps. The compactness of the domain is necessary, as we see in (3.6).

1.9. Corollary. If \( f: E^n \to E^n \) is a \( C^n \) light open map, then point inverses are isolated. Moreover, if \( f: S^n \to S^n \) is \( C^n \) open with (Brouwer) degree \( d \), then, for each \( p \in S^n \), \( f^{-1}(p) \) has at most \( |d| \) points and \( |d| \) is the least such number.

Proof. The first conclusion follows from (1.3), (1.6), and \([5, \text{p. 530, (2.2)}]\).

For the second, since the Jacobian determinant is non-negative or nonpositive \([17]\), \( f^{-1}(p) \) has precisely \( |d| \) points, for each \( p \in S^n - f(B_f) \). In fact, \( f \) is a \( |d| \)-to-1 pseudo-covering map, and the rest of the conclusion follows from the proof of (1.8).

The corollary is related to \([16, \text{p. 329, Theorem A and p. 335, (6a)}]\).

1.10. Stoilow \([12]; \text{cf. [20, p. 198, (5.1)]}\) proved that a light open map \( f: M^2 \to N^2 \) is locally at each point topologically equivalent to the complex analytic map \( g(z) = z^d \) \((d = 1, 2, \ldots)\). (Manifolds are assumed to be without boundary.) For completeness we give now an independent proof in the case that \( f \in C^2 \). In particular, from (1.7) follows the apparently new result that: If \( f: M^2 \to N^2 \), \( M^2 \) compact, \( f \) open and \( C^2 \), then \( f \) has that local structure.

By (1.5) \( \dim(f(B_f)) \leq 0 \). Given any \( x \in B_f \), by restriction \([5, \text{p. 529, (1.4) and its proof}] \) there exists a pseudo-covering map \( g \) such that its domain \( V \) is a compact connected neighborhood of \( x \) in \( E^2 \), \( g(V) \) is a closed topological disk \( D \), and \( (\text{bdy } D) \cap g(B_f) = \emptyset \). We may also suppose \([5, \text{p. 530, (2.2), conclusion (1)}] \) that \( g^{-1}(g(x)) = x \). Since each component of \( f^{-1}(\text{bdy } D) \) \((= \text{bdy } V) \) is a simple closed curve, \( V \) is a disk-with-holes.
Let $U$ be an open 2-cell about $x$ in int $V$. Let $h$ be a pseudo-covering map given, as above, for $g | U$ and $x$; call its domain $E$ and its range disk $D'$. If $\partial E$ had two or more components (simple closed curves), then $U$ would contain a disk whose image under $g$ contained $D - \text{int}(D')$, contradicting the fact that $U \subseteq \text{int} V$. Thus $E$ is a topological closed disk itself.

If $B_h$ contains a point $y \neq x$, let $\gamma$ be an arc in $D' - h(B_h)$, separating int$(D')$ into two components $X$ and $Y$ such that $h(x) \in X$ and $h(y) \in Y$. Then $h^{-1}(\gamma)$ consists of $k$ mutually disjoint arcs, where $k$ is the degree of $h$, and thus it separates int $E$ into $k + 1$ components. Precisely one of these components has image $X$ (since $h^{-1}(h(x)) = x$), so that the other $k$ have image $Y$, contradicting the fact that $Y$ meets $h(B_h)$.

Thus $\{x\} = B_h$, and the conclusion is evident.

2. The structure theorem. In this section we give a structure theorem for differentiable open maps defined on compact manifolds, or (more generally) differentiable light open maps defined on arbitrary manifolds, comparing them with the maps $F_{n,d}$ defined in the introduction.

2.1. Theorem. Let $f : M^n \to N^n$ be $C^1$ and open ($n \geq 2$); let $M^n$ be compact, or let $f$ be light. Then there exists a closed set $E$, $\dim E \leq n - 3$, such that for each $x$ in $M^n - E$ there exists a neighborhood $U$ of $x$ on which $f$ is topologically equivalent to one of the canonical maps $F_{n,d}$ ($d = 1, 2, \cdots$). Moreover, $E$ is nowhere dense in $B_f$ unless $f$ is a local homeomorphism.

Trivial examples show that "topologically equivalent" cannot be replaced by "diffeomorphically equivalent." The hypothesis that $f$ is $C^1$ results from the use of (1.3).

Proof. Since $f$ is light (1.8), $\dim(R_{n-3}) \leq n - 3$ (1.3); thus the set $E$ may as well include $R_{n-3}$. To prove the first part of the theorem we may suppose that $n \geq 3$ and that the rank of the Jacobian matrix is at least $n - 2$ at every point. For each $\bar{x}$ in $B_f$ the restriction $f | U$ of $f$ to some neighborhood $U$ of $\bar{x}$ has the structure of (1.1), where $q = n - 2$ and $p = n$. (We may as well suppose that $f | U$ is the $g$ of (1.1).) Thus the domain and range of $f | U$ are $E^n = E^{n-2} \times C$, where $C$ is the complex plane; for each $v \in E^{n-2}$ the restriction of $f$ to the plane $\{v\} \times C$ is light and open [20, p. 147, (7.2)]. By (1.6) $B_f \subseteq R_{n-2}$, and by (1.1) $R_{n-2} \cap (\{v\} \times C)$ is the set of points at which $f | (\{v\} \times C)$ has rank 0; thus (1.3)

$$\dim(f(B_f \cap (\{v\} \times C))) \leq 0$$

and $\dim(f(B_f)) \leq n - 2$.

The rest of the proof of the first conclusion uses only the topological properties of $f | U$ found above, and not the differentiability of $f | U$.

Let $A$ be a closed $n$-cell such that $\bar{x} \in \text{int} A$ and $A \subseteq U$. Since $\dim(f(B_f)) \leq n - 2$, there exists [5, p. 529, (1.4)] a connected open neighborhood $V$ of $\bar{x}$ such that
the restriction of \( f \) to \( V \) is a pseudo-covering map \( g \), and \( V \subseteq \text{int } A \). Choose \( \gamma \in E^{n-2} \) so that \( f(\bar{x}) \) is in the plane \( \{w\} \times C \) of (1.1). Since \( \text{Cl}[g(B_y)] \subseteq \text{cl}(B_f \cap A) \), \( \text{Cl}[g(B_y)] \) meets \( \{w\} \times C \) in a compact set of dimension 0. Let \( G \) be an open disk with center \( g(\bar{x}) \),

\[
G \subseteq g(V) \cap (\{w\} \times C).
\]

Let \( L \) be any straight line in \( \{w\} \times C \) through \( g(\bar{x}) \), and [9, p. 22, (D)] let \( a \) and \( b \) be points on opposite sides of \( L \cap G \) from \( g(\bar{x}) \), \( a \) and \( b \) disjoint from \( (\{w\} \times C) \cap \text{Cl}[g(B_y)] \). It follows from [9, p. 48, Corollary 1] that there exist arcs \( \Gamma_i \) joining \( a \) to \( b \), \( \Gamma_i \) disjoint from \( (\{w\} \times C) \cap \text{Cl}[g(B_y)] \) \((i = 1, 2) \), \( \Gamma_1 \setminus \{a, b\} \) contained in one component of \( G - L \), and \( \Gamma_2 \setminus \{a, b\} \) in the other.

Then \( \Gamma_1 \cup \Gamma_2 \) bounds a topological closed disk \( D \subseteq C \) such that \( g(\bar{x}) \in \{w\} \times (\text{int } D) \), \( \{w\} \times D \subseteq g(V) \), and \( \{w\} \times (\text{bdy } D) \) is disjoint from the 0-dimensional set \( (\{w\} \times C) \cap \text{Cl}[g(B_y)] \). Thus, for all \( w \) sufficiently near \( \bar{w} \), the corresponding-disks \( \{w\} \times D \) will also be disjoint from \( \text{Cl}[g(B_y)] \). Let \( T^{n-2} \) be such a small closed \((n-2)\)-cell in \( E^{n-2} \) for which \( \bar{w} \in \text{int}(T^{n-2}) \) and \( T^{n-2} \times D \subseteq g(V) \). The restriction of \( g \) to the component of \( g^{-1}(T^{n-2} \times D) \) containing \( \bar{x} \) is also a pseudo-covering map; for convenience we now call this map \( g \) and its domain \( V \).

Each set \( g^{-1}(\{w\} \times D) \) is the closure of a region in the plane, each boundary component a simple closed curve. Thus, each \( g^{-1}(\{w\} \times D) \) is homeomorphic to the same disk-with-holes \( H \), and we will denote \( g^{-1}(\{w\} \times D) \) by \( H_w \).

For each \( w \) in \( T^{n-2} \), let \( g \mid H^w \) be denoted by \( g^w \); and let its branch set be denoted by \( B(g^w) \). Clearly, \( \bigcup B(g^w) \subseteq B_g \). Suppose that \( x \in \text{int } H^w \) but \( x \notin B(g^w) \). Choose an open neighborhood \( N \) of \( x \) in \( \text{int } V \) such that \( g^w \mid (N \cap H^w) \) is a homeomorphism. Let \( h \) be a pseudo-covering map whose domain contains \( x \) and is contained in \( N \). Then the degree of \( h \) is one, and \( h \) is a homeomorphism; therefore, \( x \notin B_g \). As a result, \( \bigcup_w B(g^w) = B_g \).

For each \( w \) in \( T^{n-2} \), the light open map \( g^w \) is topologically equivalent to a simplicial map [20, p. 198, (5.1)], and it follows from [17] that for some fixed natural number \( K \) depending only on \( H, B(g^w) \) contains at most \( K \) points. Let \( \alpha(w) \) be the number of branch points in \( H^w \) \((1 \leq \alpha(w) \leq K) \). Let \( Y \) be any open set in \( T^{n-2} \), and let \( \tilde{y} \) in \( Y \) be a point at which the function \( \alpha \) is maximal on \( Y \). Let \( p^j \) be the points of \( B(g^\tilde{y}) \), and let \( P^j \) be mutually disjoint sets open in \( H^\tilde{y} \) such that \( P^j \cap g^{-1}(g^j) = \{p^j\} \) \((i, j = 1, 2, \cdots, \alpha(\tilde{y})) \); note that \( p^j \) may be in \( g^{-1}(g^j) \) for \( i \neq j \). There exists a disk \( \{\tilde{y}\} \times D^i \) such that \( g(p^j) \subseteq \{\tilde{y}\} \times (\text{int } D^i) \); \( \{\tilde{y}\} \times D^j \subseteq \{\tilde{y}\} \times D \); and if \( J^j \) is the component of \( g^{-1}(\{\tilde{y}\} \times D^j) \) containing \( p^i \), then \( J^j \subseteq P^j [20, p. 131, (4.41)] \). Since \( g \) is a pseudo-covering map, \( J^j \) is a topological 2-disk, and \( g \mid J^j \) is topologically equivalent to the analytic map \( \mu(z) = z^d \) \((d = 2, 3, \cdots) \).

Since \( g(B_y) \) is compact and \( \bigcup_w B(g^w) = B_g \), there exists a closed \((n - 2)\)-cell \( W^i \subseteq Y, \tilde{y} \in \text{int } (W^i) \), such that \( g(B(g^\tilde{y})) \) is disjoint from \( \{y\} \times \text{bdy } (D^i) \) for all
If \( y \in W^i \). If \( S^i \) is the component of \( g^{-1}(W^i \times D^i) \) containing \( p^i \), we may suppose that \( W^i \) is chosen small enough that \( S^i \cap H^w \) is connected for all \( w \in W^i \) and that the \( S^i \) are mutually disjoint (\( i = 1, 2, \cdots, \alpha(y) \)). If \( W = \bigcap_i \text{int}(W^i) \), then \( B_g \cap H^w \subset \bigcup_i S^i \) (\( i = 1, 2, \cdots, \alpha(y) \); \( w \in W \)).

Suppose that for some \( w \in W^i \), \( B(g^w) \cap S^i = \emptyset \); then since \( g|S^i \) is a pseudo-covering map and \( \{w\} \times D^i \) is simply connected, \( g|(H^w \cap S^i) \), and thus \( g|S^i \), has degree 1 (i.e., is a homeomorphism). Since \( p^i \in B_g \cap S^i \), \( B(g^w) \cap S^i \neq \emptyset \), for all \( w \in W^i \). Because of the choice of \( y \) and the fact that the \( S^i \) are mutually disjoint, each set \( B(g^w) \cap S^i (i = 1, 2, \cdots, \alpha(y)) \) is a single point.

Let \( \rho^i: W^i \times D^i \to W^i \) be the projection map, and let \( \beta^i = \rho^i|B_g \cap S^i \). Then \( \beta^i \) is continuous, and one-to-one \(((\beta^i)^{-1}(w) \) is the single point of \( g(B_g \cap S^i) \cap \{w\} \times D^i) \). Since \( g(B_g \cap S^i) \) is compact, \( \beta^i \) is a homeomorphism onto \( W^i \). Let \( d^i \) be the distance from \( g(B_g \cap S^i) \) to \( W^i \times \text{bdy}(D^i) \), and let \( \Delta^i \) be the closed disk of radius \( d^i \) and center 0 in \( C \). Let \( \sigma^i: W^i \times \Delta^i \to W^i \times D^i \) be the map defined by \( \sigma^i(w, x) = ((\beta^i)^{-1}(w) + (0, x)) \), where + is vector addition, 0 is the origin of \( E^{n-2} \), and \( x \in D^i \) (in \( C = E^2 \)). Since \( \sigma^i \) is continuous and one-to-one, \( W^i \times \Delta^i \) is compact, \( \sigma^i \) is a homeomorphism (into). Since \( \sigma^i(W^i \times \{0\}) = g(B_g \cap S) \), it follows from the theorem on invariance of domain that \( g(B_g \cap S^i) \) is a tamely embedded \((n-2)\)-cell. By \([5, p. 533, (4.1)] \) \( g|S^i \) (i.e., \( f|S^i \)) is topologically equivalent to \( F_{n,d} \), for some \( d \) \((d = 2, 3, \cdots)\).

Let \( \Omega \cap \text{int}(T^{n-2}) \) be the maximal open set (possibly empty) such that \( g|g^{-1}(\Omega \times D) \) is locally, at each point, topologically equivalent to one of the maps \( F_{n,d} \). To review, we have seen that, for every open set \( Y \) in \( \text{int}(T^{n-2}) \), there exists (of course) a point \( \tilde{y} \in Y \) such that \( \alpha(\tilde{y}) \geq \alpha(w) \) for all \( w \in Y \); moreover, that there is an open neighborhood \( W \) of \( \tilde{y} \) with \( W \subset \Omega \). Thus \( \Omega \cap Y \neq \emptyset \). Since \( Y \) is an arbitrary open set in \( \text{int}(T^{n-2}) \), \( \Omega \) is a dense open set in \( \text{int}(T^{n-2}) \). Therefore \([9, p. 44, \text{Theorem IV 3}] \) its complement \( F \) in \( \text{int}(T^{n-2}) \) has dimension at most \( n - 3 \).

Let \( E \) be the set of points of \( B_g \) in \( g^{-1}(\{w\} \times \text{int}D) \) for \( w \in F \), and let \( \pi: \text{int}V \to \text{int}(T^{n-2}) \) (\( \text{int}V = g^{-1}(\text{int}(T^{n-2}) \times \text{int}D) \)) be the projection map. Then \( \pi(E) = F \), and, since \( \dim(B(g^w)) = 0 \), \( \dim(\pi^{-1}(w)) = 0 \) for all \( w \in F \). Since \( g|\text{int}V \) is a pseudo-covering map, \( \pi \) is a closed map and by \([9, pp. 91-92] \) \( \dim E \leq n - 3 \). This completes the proof of the first conclusion.

For the second conclusion, we will suppose throughout that \( B_f \neq \emptyset \). If \( E \) is somewhere dense in \( B_f \), then there exists an open set \( \Lambda \) in \( M^n \) such that \( \Lambda \cap B_f \neq \emptyset \) and \( \Lambda \cap B_f \subset E \). By the preceding argument, \( E \) is nowhere dense in the set of branch points at which the Jacobian matrix has rank at least \( n - 2 \). Then \( \Lambda \cap B_f \subset R_{n-3} \). Thus, if we still denote \( f|\Lambda \) by \( f \), it suffices to prove that \( B_f \neq R_{n-3} \) (if \( B_f \neq \emptyset \)).

First suppose that \( B_f \subset R_n \). Given \( \tilde{x} \in B_f \), let \( g \) be a pseudo-covering map given by \([5, p. 530, (2.2) \) and \( p. 529, (1.4) \)] on a neighborhood \( V \) of \( \tilde{x} \), \( V \subset E^n \), such that \( g|g^{-1}(g(B_g)) \) is a homeomorphism and \( g(V) = E \). Then \( g(B_g) \neq g(\tilde{x}) \) \([5, p. 535, (5.6)] \). It follows (see the first paragraph of the proof of (1.3)) from A. P. Morse's
Theorem [10] that for every point \( u \) in \( g(B_q) - g(\bar{x}) \) (and therefore in \( g(R_0) \)), there exists a closed \( n \)-cube \( X \) such that: \( q(\bar{x}) \in \text{int } X, u \notin X \), its faces are parallel to the coordinate \((n - 1)\)-planes, and those faces are disjoint from \( g(R_0) \). The restriction of \( g \) to each component of \( g^{-1}(\text{bdy } X) \) is a covering map onto \( \text{bdy } X \), and therefore that map is a homeomorphism. Since \( \bar{x} \in B_q \), the degree of \( g \) is at least two; since \( V \subset E^n \), \( g^{-1}(\text{bdy } X) \) separates \( V \) into at least three components, each of which maps onto one of the components of \( E^n - \text{bdy } X \). This contradicts the fact that \( g|_{g^{-1}(g(\overline{B_q})))} \) is one-to-one, so that \( B_f \not\subset R_0 \).

Thus \( B_f \not\subset R_{n-3} \), for \( n = 3 \). We continue by induction on \( n \). If \( n \geq 4 \) and \( B_f \subset R_{n-3} \), then there exists \( \bar{x} \) in \( B_f \) at which the Jacobian matrix has rank at least one. We may suppose, by restriction, that the rank is at least one everywhere, and that \( f \) is the \( g \) of (1.1) for \( q = 1 \). Let \( \gamma \) be the \((n - 1)\)-plane of (1.1) that contains \( \bar{x} \). Let \( h = f|_\gamma \), and let \( Q_{n-4} \) be the set of points in \( \gamma \) at which the Jacobian matrix of \( h \) has rank at most \( n - 4 \); then, by the second conclusion of (1.1), \( \gamma \cap R_{n-3} = Q_{n-4} \). Since \( B_h \subset \gamma \cap B_f, B_h \subset Q_{n-4} \), contradicting the inductive hypothesis. Thus \( B_f \not\subset R_{n-3} \) for \( n \geq 3 \) (unless \( B_f = \emptyset \)), yielding the second conclusion.

The following extension of the inverse function theorem was proved in [4].

2.2. Corollary. Suppose that \( f : E^n \to E^n, n \geq 3, f \in C^n \) and \( \dim(R_{n-1}) = 0 \) \((R_{n-1} \) is the set of zeros of the Jacobian determinant). Then \( f \) is a local homeomorphism.

2.3. Corollary. If \( f : E^n \to E^n \) is light and \( C^n \), then \( B_f = \emptyset, \dim(B_f) = n - 2 \), or \( \dim(B_f) = n - 1 \); the last case occurs if and only if \( f \) is not open.

Proof. Since \( B_f \subset R_{n-1} \), \( \dim(f(B_f)) \leq n - 1 \) (by (1.3)); since \( f \) is light, \( \dim(B_f) \leq n - 1 \). If \( \dim(f(B_f)) \leq n - 2 \), then \( f \) is open [5, p. 531, (2.4)], so that either \( B_f = \emptyset \) or \( \dim(B_f) = n - 2 \) (by (2.1)). Thus, \( \dim(f(B_f)) = n - 1 \) if and only if \( f \) is not open [5, p. 531, (2.3)]. If, in this case, \( \dim(B_f) < n - 1 \), then the Jacobian determinant of \( f \) would be either non-negative or nonpositive everywhere; thus (1.7) \( f \) would be open. As a result, \( \dim(B_f) = n - 1 \) if and only if \( f \) is not open.

2.4. Corollary. There exists a light open map \( f : E^5 \to E^5 \) which is not topologically equivalent to any \( C^5 \) map.

The map is that given by [6, p. 620, (4.3)], so that \( B_f \) is not a 3-manifold at any point. If \( f \) were equivalent to a \( C^5 \) map, then at a dense set of its points \( B_f \) would be locally a 3-manifold (2.1).

2.5. Remarks. Given a \( C \) map \( f : E^n \to E^n \), its directional derivative at \( x \) in the direction of the nonzero vector \( (a_1, a_2, \ldots, a_n) \) is the length of the vector whose \( j \)th component is \( \sum_{i=1}^n a_i D_{ij} f_j(x) \). If \( f \) is a homeomorphism, it is called quasi-
conformal if (*) there exists \( B > 0 \) such that, for every point \( x \) in \( E^n \) and pair of vectors (directions) at \( x \), the ratio of the directional derivatives is less than \( B \).

(This definition is equivalent to that given in [8].) A nonconstant complex analytic function \( f \) satisfies condition (*) (for \( B = 1 \)) except on \( B_r \), which consists of isolated points. Thus, it would be natural to call quasi-conformal (or quasi-analytic) light maps in \( E^n (n > 2) \) which satisfy condition (*), except at those points at which all directional derivatives are zero, i.e., \( R_0 \). We now observe that the only such \( C^n \) maps are local homeomorphisms (for \( n > 2 \)).

Suppose that \( f \) is \( C^n \), light, and not a local homeomorphism. If \( R_{n-1} \subset R_0 \), then \( \dim(R_{n-1}) \leq 0 \) (by (1.3)). Thus \( f \) is a local homeomorphism (2.2), contradicting the supposition. If \( R_{n-1} \not\subset R_0 \), then there exists \( x \) at which the rank of Jacobian matrix is \( k \), where \( 0 < k < n \). It follows from the definition of rank that there exist two vectors at \( x \) for which one directional derivative is positive, and the other is zero. Thus \( f \) does not satisfy condition (*).

We also remark that, except for local homeomorphisms, no \( C^n \) light open map is generic in the sense of Thom [14].

3. Some examples. Examples are given now to show that the exceptional set of dimension \( n - 3 \) in (2.1) and the compactness hypothesis in (1.8) are necessary.

3.1. Lemma. Given \( \delta_q > 0 \) (\( q = 1, 2, \cdots \)), there exists a \( C^\infty \) map \( \psi : E^1 \to E^1 \) with the following properties:

(1) \( \psi \) is an even function,
(2) \( \psi(r) = 0 \) if and only if \( r = 0 \),
(3) \( \psi'(r) > 0 \) for \( r > 0 \), and
(4) the 4th derivative \( \psi^{(4)}(r) \leq \delta_q (0 \leq r \leq 1/q; i = 0, 1, \cdots, q) \), where \( \psi^{(0)} = \psi \).

The proof is omitted.

3.2. Lemma. Given \( \epsilon_q > 0 \) (\( q = 1, 2, \cdots \)), there exists a \( C^\infty \) homeomorphism \( h : E^n \to E^n \) such that on each set \( S(0, 1/q) \) (where 0 is the origin) all \( h_1 \) and all partial derivatives of order at most \( q \) are bounded by \( \epsilon_q \).

Proof. Consider the class \( \mathcal{F} \) of all functions \( h : E^n \to E^n \) such that \( h_i(x) = \psi(r) \cdot x_i \), where \( r = x_1^2 + x_2^2 + \cdots + x_n^2 \) and \( \psi : E^1 \to E^1 \) is any \( C^\infty \) function. On each set \( S(0, 1/q) \) there exists constants \( \lambda_j > 0 \) such that each \( h_i \) in \( \mathcal{F} \) and all its partials of order at most \( q \) are bounded by \( \sum_{j=0}^q \lambda_j |\psi^{(j)}(r(x))| \). Let \( \delta_q < \epsilon_q / \sum_{j=0}^q \lambda_j \) (\( q = 1, 2, \cdots \)), and let \( \psi_0 \) be given by (3.1) for \( \{\delta_q\} \). Let \( h_i(x) = \psi_0(r) \cdot x_i \). That \( h \) is a homeomorphism follows from conclusions (2) and (3) of (3.1).

3.3. Lemma. Let \( U \) and \( L \) be, respectively, open and closed subsets of \( E^n \). Let \( f : U \to E^n \) be continuous, \( C^\infty \) on \( U - L \), and constant on \( U \cap L \); let \( V \) be a bounded open subset of \( U \) such that \( \overline{V} \subset U \). Then there exists a homeomorphism \( h : E^n \to E^n \) such that the restriction \( hf \big| V \in C^\infty \).
Proof. Throughout, symbols such as \( \overline{V} \) refer to closure in \( E^n \). Suppose that \( f(L) = 0 \). Let \( X_q = V \cap f^{-1}(S(0, 1/q)) \), let \( A_q = (X_q \cap V) - X_{q+1} \), and choose \( \alpha_q (0 < \alpha_q \leq 1) \) less than the distance \( d(A_q, L) \) \( (q = 1, 2, \ldots) \). We will define \( h \) so that the partial derivatives (of all orders) of \( h f | V \), call it \( F \), are zero on \( L \cap V \); \( h \) will be given by (3.2), where we need now specify the \( \varepsilon_q \) \( (q = 1, 2, \ldots) \).

The component functions (e.g., \( F_i \)) will be considered partials of order zero.

Suppose \( h \) is given by (3.2) for \( \varepsilon_q = \varepsilon_q, 0 = 1/q \). For \( x \in A_q, f(x) \in S(0, 1/q) \), so that \( |\varepsilon_i(x)| < 1/q \) \( (i = 1, 2, \ldots, n) \).

Now suppose that numbers \( \varepsilon_{q,m} > 0 \) \( (q = 1, 2, \ldots; m = 0, 1, \ldots, k; k \text{ fixed}) \) have been defined so that

1. any homeomorphism given by (3.2) for \( \{\varepsilon_{q,k}\} \) will satisfy (a) \( |P(x)| < 1/q \), for all \( x \in A_q \) and all partials \( P \) of \( F \) with order \( m \) at most the minimum of \( k \) and \( q \), and (b) \( P(x^0) = 0 \), for \( x^0 \in L \);
2. \( \varepsilon_{q,m'} < \varepsilon_{q,m} \), whenever \( m < m' \); and
3. \( \varepsilon_{q,m} = \varepsilon_{q,q} \), whenever \( m > q \).

Let this property of the sequence \( \{\varepsilon_{q,m}\} \) \( (m = 0, 1, \ldots, k) \) be called \( \Psi_k \) \( (k = 0, 1, \ldots) \); we have seen that there exists \( \{\varepsilon_{q,0}\} \) satisfying \( \Psi_0 \). We proceed by induction. Assuming a sequence \( \{\varepsilon_{q,m}\} \) \( (m = 0, 1, \ldots, k) \) satisfying \( \Psi_k \), we will find numbers \( \varepsilon_{q,k+1} \) \( (q = 1, 2, \ldots) \) such that \( \{\varepsilon_{q,m}\} \) \( (m = 1, 2, \ldots, k + 1) \) satisfies \( \Psi_{k+1} \).

Given any partial \( P \) of \( F \) with order \( k \), a natural number \( j \) \( (j = 1, 2, \ldots, n) \), and \( x^0 \in L \),

\[
D_jP(x^0) = \lim_{x_j \to x^0} \frac{P(x) - 0}{x_j - x_j^0}
\]

(since \( P(x^0) = 0 \), by \( \Psi_k(1) \)). Given \( x \in A_q \),

\[
\frac{|\varepsilon_{i}P(x)|}{|x_j - x_j^0|} \leq \frac{|P(x)|}{\alpha_q}.
\]

Now each such \( P \) is on \( \overline{V} - L \); a sum of products of partials of \( f \) and of \( h \), all of orders at most \( k \), each term having at least one partial of \( h \) as a factor. Since \( \overline{A} - L = \emptyset \), there is a uniform bound on the partials of \( f \) of order at most \( k \). For \( q \leq k \), let \( \varepsilon_{q,k+1} = \varepsilon_{q,q} \); for \( q > k \), let \( \varepsilon_{q,k+1} \) be chosen small enough so that \( \varepsilon_{q,k+1} \leq \varepsilon_{q,k} \) and, for any \( h \) given by (3.2) for \( \varepsilon_{q,k+1} \), \( \|P(x)\|/\alpha_q < 1/q \) (for all \( x \in A_q \) and for all partials \( P \) of order at most \( k \), a finite number of choices required for each \( q \)). Since \( L \cup \bigcup_{q=1}^{\infty} A_q \) is a neighborhood of \( L \), and since \( P|L = 0 \), all the partials of \( F \) of order at most \( k + 1 \) are 0 on \( L \), for \( h \) given by \( \{\varepsilon_{q,k+1}\} \). It follows that \( \{\varepsilon_{q,m}\} \) \( (m = 1, 2, \ldots, k + 1) \) satisfies \( \Psi_{k+1} \).

Positive numbers \( \varepsilon_{q,m} \) \( (q, m = 1, 2, \ldots) \) are defined, and the desired \( h \) is the one given by (3.2) for \( \varepsilon_q = \varepsilon_{q,q} \); all its partials are zero on \( L \).

To prove that \( F \in C^\infty \), it is sufficient to prove that each partial \( P \) is continuous
on $L$; let $k$ be the order of $P$. By $Q_k(1)$, $|P(x)| < 1/q$ $(x \in A_q; q = k, k + 1, \ldots)$, so that $P(x) \to 0$ as $x \to x^0$, $x^0 \in L$.

3.4. Corollary. There exists $f : E^3 \to E^3$ $f C^\infty$, light and open, such that $B_f$ has a point component.

The map given in [6, p. 614, (3.3)] is topologically equivalent to a map simplicial except at the origin 0, and thus it is equivalent to a map $C^\infty$ except at 0. From (3.3) we have the desired result.

Although $B_f$ need not be locally connected, it follows from (2.1) that for $f : E^3 \to E^3$, $f C^3$, light and open, each component $K$ of $B_f$ is locally connected. (Suppose that $K$ is not locally connected; then it contains [20, p. 19, (12.3)] a subcontinuum $H$ such that $K$ is not locally connected at any point of $H$. At each point $x$ of $H - E$, where $E$ is the exceptional set of (2.1), there exists a neighborhood $U$ such that the restriction $f|U$ is a canonical map $F_{n,d}$. Since $H \cap U$ is a tame arc, we have a contradiction.)

The example whose branch set has a Cantor set of point components [6, p. 614] is also equivalent to a $C^\infty$ map. (Appropriate modifications of (3.2) and (3.3) are required.)

For another example of a $C^\infty$ (3-to-1) open map, let $z$ be a complex variable, $t$ real, and let $f : E^3 \to E^3$ be defined by

$$f(z,t) = (z^3 - 3ze^{-2t^2} \sin^2 t^{-1}, t).$$

Then (with $z = x + iy$) $B_f$ is the union of the curves $x = \pm e^{-t^2} \sin t^{-1}$ in the $(x - t)$-plane. Still another example is given in (2.4).

The following remark answers in the negative question II of [13, p. 266].

3.5. Remark. There exists a $C^\infty$ 3-to-1 open map $f : E^3 \to E^3$ which is not topologically equivalent to any real analytic map.

We use the map above for which $B_f$ has a Cantor set $X$ of point components, or one with a sequence of point components converging to a point. It follows from (1.1) (see the proof of (2.1)) that $X \subset R_0$. Suppose that $f$ is real analytic. Then $R_0$ is an analytic set (the zeros of $\Sigma_{j,i} (D^3 f)^2$), and thus [3, p. 141] is locally connected. Since $\dim(R_0) = 0$ (1.3), we have a contradiction.

3.6. Theorem. There exists a $C^\infty$ open map $f : E^2 \to E^2$ which is not light.

Proof. The domain of $f$ will actually be the square $S$ given by $|x| < 1$ and $|y| < 1$; let $L$ be the intersection of the $y$-axis with $S$. Let $r^j$ $(j = 1, 2, \ldots)$ be any countable dense subset of $L - \{0\}$, and let $h : S \to S$ be given by $h(x, y) = (x, xy)$.

If $X_{j,k}$ $(j, k = 1, 2, \ldots)$ are the subsets of $h(S)$ defined by

$$2^{-j-1}(2k-1) \leq x \leq \left(\frac{3}{2}\right)^j 2^{-j-1}(2k-1),$$

then their closures are mutually disjoint and each
Let $g: h(S) \to E^2$ be a map such that

1. $g | (h(S) - \{0\})$ is a $C^\infty$ local homeomorphism,
2. $g(h(S) - \bigcup_{j,k} X_{j,k})$ is the identity map,
3. $g(X_{j,k}) \subset S(0, 3 \cdot 2^{-2^{j-1}(2k-1)})$, and
4. there exists a point $p_{j,k}$ common to $n_{-1}(A_{j,k})$ and the line $y = r^j$ such that $g(h(p_{j,k})) = 0$ $(j,k = 1, 2, \ldots)$.

(By 1) and 3 is continuous at 0.)

Given any neighborhood $U$ of $r^j$, there exists $p_{j,k} \in U$; since $g(h(r^j)) = 0$, it follows from conditions (1) and (4) that $g(n_{-1}(r^j)) \in \text{int}(g(n_{-1}(U)))$. Since the $r^j$ are dense in $L$, and since $gh(S - L)$ is a $(C^\infty)$ local homeomorphism, $gh$ is open. The result follows from (3.3).

Remark. Given any three natural numbers $j$, $k$, and $n$ such that (1) $0 \leq j \leq \min(k - 1, n - 2)$, (2) $1 \leq k \leq n - 1$, and (3) $n \geq 2$, modifications of the above argument yield a nonlight $C^\infty$ open map $f: E^n \to E^n$ for which $\dim(f(B_j)) = k$ and $\dim(B_j) = k$.

3.7. Remark. If $f: E^n \to E^n$ is $C^n$ open, but not light, then for every $k$ ($k = 1, 2, \ldots$) there exists $x$ such that $f^{-1}(x)$ consists of isolated points, at least $k$ in number. The proof is similar to that of (1.8).

4. A counterexample to a statement of Stoilow. In [13] S. Stoilow states that, if $f: E^3 \to E^3$ is light open, then $\dim(B_j) \leq 1$. His proof employs the following lemma [13, pp. 263-264]: Let $x \in E^3$, and let $B_\rho$ be the geometric ball of radius $\rho$ and center $f(x)$. Then there exists $r > 0$ and a compact neighborhood $D$ of $x$ such that $f|D$ is open and $f(D) = B_r$. There exist positive numbers $\epsilon_1$ and $\epsilon_2$ such that the number of components of $f^{-1}(B_\rho)$ is the same for all $\rho$ with $0 < \epsilon_1 < \rho < \epsilon_2$. Moreover, for any such set of numbers $\epsilon_1, \epsilon_2$, and $\rho$, each component of $f^{-1}(\text{bdy}(B_\rho))$ is a 2-manifold. The last statement is false in general.

It appears that a modification of the proof of (2.1) using this statement would yield (2.1) for $n = 3$ and $f$ light open but not necessarily differentiable. For this reason it seems worthwhile to give a counterexample here.

We write $E^3$ as $E^1 \times C$, where $C$ is the complex plane, and let $X_m$ be the set of $(t, z)$ such that either $|t| \leq 2^{-m}$ and $|z| \leq 2^{-m}$, or $2^{-m} \leq t \leq 3 \cdot 2^{-m-1}$ and $|z - 2^{-m-1}| \leq 2^{-m-1}$ $(m = 1, 2, \ldots)$. Then $X_{m+1} \subset \text{int}(X_m)$, and there exists a homeomorphism $h: E^3 \to E^3$ such that $K_m = h(\text{bdy}(X_m))$ is a geometric 2-sphere about the origin 0. The map $hF_{3,2}$ is the desired counterexample $f$, since $f^{-1}(K_m) (= F_{3,2}^{-1}(\text{bdy}(X_m)))$ is not a 2-manifold while $f^{-1}(h(X_m))$ is connected $(m = 1, 2, \ldots)$.

Stoilow uses a characterization of compact 2-manifolds in $E^3$ due to Wilder [18, Theorem 21], and the sets $f^{-1}(K_m)$ fail to satisfy the first conclusion of that theorem. With a suitable modification of the sets $X_m$, the sets $f^{-1}(K_m)$ also fail to satisfy the second conclusion.
Added in Proof. J. Väisälä has kindly pointed out to the author the following simple example of a $C^\infty$ map $f : E^2 \to E^2$ which is open but not light (cf. 3.6). For $z = x + iy$ and $x \neq 0$, $f(z) = \exp(-z/x^3); f(iy) = 0$. Except on the imaginary axis $f$ is a local homeomorphism.

In *Images of critical sets*, Ann. of Math. (2) 68 (1958), 247–259, Arthur Sard considers maps $f$ of $U \to E^n$ into $E^n$. He proves under very general differentiability hypotheses that if $R_k$ is the countable union of sets of finite Hausdorff $(k + 1)$-measure, then the $(k + 1)$-measure of $f(R_k)$ is 0. It follows [9, p. 104] that $\dim(f(R_k)) \leq k$. Thus, in this case (1.3) is the consequence of a more general result. In general, however, $R_k$ need not be the countable union of sets of finite $(k + 1)$-measure.

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**References**

