DERIVATIONS ON AN ARBITRARY VECTOR BUNDLE(1)

BY

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1. **Introduction.** The theory of derivations of real-valued forms has been developed by Frölicher and Nijenhuis [1] and Spencer and Kodaira [2, §1]. H. K. Nickerson [3] has dealt with derivations of all degrees on arbitrary vector bundles, treating connections as a special case. The present author, in his junior paper written at Princeton in 1961, simplified Nickerson's definition by considering a product bundle of which the given vector bundle is a quotient space; derivations are easily taken on the product bundle, and appropiate ones induce the derivations on the vector bundle.

This paper is based on the author's junior paper and conversations with H. K. Nickerson and D. C. Spencer(2). It aims at discovering in a simple way all the derivations of arbitrary degree on an arbitrary vector bundle over a differentiable manifold. §2 gives necessary results from Kodaira and Spencer [2]. §3 establishes the basic theorem about derivations on an arbitrary vector bundle.

The remainder of this section is concerned with notation. (See also [4].) All structures are assumed to be differentiable arbitrarily often; all algebraic operations such as tensor products are over the real numbers. If $B = B(M)$ is any fiber bundle over the n-dimensional manifold M, and π is the projection of B onto M, then by $B(x)$ we mean $\pi^{-1}(x)$, $x \in M$ and by $B(X)$ we mean $\pi^{-1}(X)$, $X \subseteq M$. We denote the presheaf of differentiable local sections of $B(M)$ by $\mathscr{P}B(M)$ and the sheaf of germs of $\mathscr{P}B(M)$ by $\mathscr{S}B(M)$. We assume that M has a covering of coordinate systems; on the region U_a the coordinates are $x_a^1, \cdots, x_a^n$ (most of the time we omit the subscripts). If F is the fiber of B, then the point corresponding to $(x,f) \in U_a \times F$ under the coordinate system on $B(U_a)$ will be denoted by $(x,f)_a$.

Given the manifold M as above, we let

$$\mathbf{A}(M) = \{A^0(M), A^1(M), \cdots\}, \qquad A^r(M) = \bigwedge^r T^*(M)$$

be the graded bundle of real-valued forms on M. We put $V^r(M) = A^r(M) \otimes T(M)$. We define a map $\overline{\wedge}: A^r(M) \otimes T(M) \to A^{r-1}(M)$ by

$$(1.1) \qquad \langle v, \phi \,\overline{\wedge}\, u \rangle = \langle u \wedge v, \phi \rangle, \qquad v \in \bigwedge^{r-1} T(M), \phi \in A^r(M), u \in T(M).$$

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More generally, if W is any finite-dimensional vector space, we extend $\overline{\wedge}$ to

$$\overline{\wedge}:[A^r(M)\otimes W]\otimes V^s(M)\to A^{r+s-1}(M)\otimes W:$$

$$(\phi\otimes w)\,\overline{\wedge}\,(\psi\otimes u)=[\psi\wedge(\phi\,\overline{\wedge}\,u)]\otimes w, \tag{1.2}$$

$$\phi\in A^r(M),\psi\in A^s(M),w\in W,\ u\in T(M).$$

When computing with indices, we shall let capital letters stand for arbitrary strings of indices; the length will be indicated by such notations as (I_s) after equations. By $dx^I\,(I_s)$ is meant $dx^{i(1)}\wedge\cdots\wedge dx^{i(s)}$. When we have "mixed" objects such as a vector-valued form χ in $V^s(M)$ we shall write either

$$\chi=\chi^i_{(a)}\otimes\frac{\partial}{\partial x^i_a};\qquad \chi^i_{(a)}\in A^s(M)$$

or

$$\chi=\frac{1}{s!}dx^I_a\otimes\chi_{(a)I};\quad (I_s),\quad \chi_{(a)I}\in T(M).$$

$\chi^i_{(a)}$ and $\chi_{(a)I}$ are well-defined forms and tangent vectors, respectively, given the coordinate system; their relations to the corresponding objects in another coordinate system are exactly like those of the components of vectors and s-forms, respectively. Thus, if

$$\phi=\frac{1}{r!}dx^I\otimes\phi_I\in A^r(M)\otimes W\quad (I_r)$$

and

$$\psi=\psi^j\otimes\frac{\partial}{\partial x^j}\in V^s(M)$$

then

$$\phi\,\overline{\wedge}\,\psi=\frac{1}{(r-1)!}\psi^j\wedge dx^I\otimes\phi_{jI}\quad (I_{r-1}). \tag{1.3}$$

When we resort to indices that do not refer to any coordinate system, we shall not use the summation convention.

For any $\chi\in\mathscr{P}V^s(M)$, defined on the open set U, we define an operator $\mathscr{L}_\chi$ by

$$\mathscr{L}_\chi:\mathscr{P}A^r(U)\to\mathscr{P}A^{r+s}(U):\sigma\to(d\sigma)\,\overline{\wedge}\,\chi+(-1)^s d(\sigma\,\overline{\wedge}\,\chi). \tag{1.4}$$

Explicitly, if

$$\chi=\frac{1}{s!}\chi^i_J dx^J\otimes\frac{\partial}{\partial x^i},\qquad \sigma=\frac{1}{r!}\sigma_K dx^K\quad (J_s,K_r)$$

then we find

$$\begin{aligned}\mathscr{L}_\chi\sigma&=\frac{1}{r!s!}dx^J\wedge dx^k\wedge dx^L\left[\chi^i_J\frac{\partial}{\partial x^i}\sigma_{kL}+r\sigma_{iL}\frac{\partial}{\partial x^k}\chi^i_J\right]\\&=\frac{1}{r!s!}dx^J\wedge dx^K(\chi_J\cdot\sigma_K)+\frac{(-1)^s}{(r-1)!}d(\chi^i)\wedge dx^L(\sigma_{iL})\quad (J_s,K_r,L_{r-1}).\end{aligned} \tag{1.5}$$

We shall now define two important fiber bundle constructions. Let M be an n-dimensional manifold and $P(M)$ a principal fiber bundle over M. Since P is a manifold, differentiable like everything in this paper, we can construct its tangent bundle $T(P)$. This can be seen as a principal bundle over $T(M)$ whose group is $T(G)$. The group structure on $T(G)$, and the transition functions on $T(P)$, are naturally derived from G and P. Since G is a subgroup of $T(G)$, we may form equivalence classes of vectors in $T(P)$ under right translation by elements of G. They form a bundle called $T(P)/G$; it is a bundle over $T(M)$, (weakly) associated with $T(P)$. The fiber is isomorphic to the Lie algebra of G, which we take to be the tangent space at the unit element. The action of $T(G)$ on $T(e)$ is found by considering the left action of $T(G)$ on right-invariant sections of $T(G)$; it is not linear but affine. We shall denote by β the canonical map

$$\beta: T(P) \to T(P)/G: (v, g, t)_a \to (v, t \cdot g^{-1})_a.$$

We can consider $T(P)/G$ as a vector bundle over M; then its structure group will in general be larger than $T(G)$.

2. **Derivations of real-valued forms.** Let $\mathscr{P}\mathbf{A}(M)$ stand for the graded sequence $\{\mathscr{P}A^0(M), \mathscr{P}A^1(M), \cdots\}$. It can obviously be regarded as a graded local algebra over the real numbers with respect to the exterior product.

DEFINITION 1. A derivation of degree s at the point $x_0 \in M$ is a linear function $u_0: \mathscr{P}\mathbf{A}(M) \to \mathbf{A}(M)$ defined for sections over neighborhoods of x_0, satisfying the following conditions:

$$u_0(\mathscr{P}A^r(M)) \subseteq A^{r+s}(x_0), \tag{2.1}$$

$$\begin{aligned} u_0(\sigma \wedge \tau) &= (u_0\sigma) \wedge \tau(x_0) + (-1)^{rs}\sigma(x_0) \wedge u_0\tau, \\ &\qquad \sigma \in \mathscr{P}A^r(M), \tau \in \mathscr{P}\mathbf{A}(M). \end{aligned} \tag{2.2}$$

Lemma 1.1 of [2] states that if u_0 is any derivation at the point x_0, and, if σ and τ are members of $\mathscr{P}\mathbf{A}(M)$ coinciding on some neighborhood of x_0, them $u_0\sigma = u_0\tau$. Because of this we may consider a derivation at a point as acting on germs of sections. It is clear how to define a derivation on a neighborhood U of M as an operator on $\mathscr{P}\mathbf{A}(M)$; taking germs of derivations, we see that they are operators on $\mathscr{S}\mathbf{A}(M)$. In the rest of this paper we shall use the same notations for germs as for their members; all germs are assumed to be in stalks over a common point $x \in M$, or over the points of $P(x) \subseteq P(M)$; in the latter case the section will be continuous in the sheaf topology.

Proposition 1.1 and equation (1.12′) of [2] can be formulated as

PROPOSITION 2.1. *The sheaf of germs of derivations of degree s on M is isomorphic to $\mathscr{S}V^s(M) \oplus \mathscr{S}V^{s+1}(M)$, setting $V^s(M) = 0$ for $s < 0$, with (ϕ, ξ) corresponding to the operator*

$$u: \quad \sigma \to \mathscr{L}_\phi \sigma + \sigma \overline{\wedge} \xi. \tag{2.3}$$

Here ϕ *and* ξ *are the* ϕ_i *and* $\xi_i - d\phi_i$ *of* [2].

If W is any finite-dimensional vector space, then r-forms with values in W can be constructed. The only natural way to define derivations on them is by the appropriate extension of the structure of Proposition 2.1 and (2.3). W can be embedded in many ways in a commutative ring with unit; and then Proposition 2.1 can be shown to hold with coefficients in that ring.

3. **General derivations.** It is well known that the results of the last section cannot be directly applied to a nontrivial vector bundle. However, there is a certain product bundle which is closely related to the vector bundle, and we shalll take advantage of this to find the derivations on the vector bundle.

To fix notation, let $P(M)$ be an arbitrary principal bundle over M with (real Lie) group G and projection α onto M. Let $B(M)$ be any vector bundle strongly associated with $P(M)$, with fiber F, a finite-dimensional vector space. There is (see [3, p. 511]) a canonical map $j: P \times F \to B$ such that for $g \in G$

$$j(pg, f) = j(p, gf); \tag{3.1}$$

if $p = (x, g_a)_a$ and $f \in F$ then

$$j(p, f) = (x, g_a f)_a. \tag{3.2}$$

Because j is bijective from $\{p\} \times F$ to $B(\alpha p)$, we can "lift up" sections of $B(M)$ to sections of $P \times F$: a local section f of $B(M)$ induces a unique local section $\tilde{f}$ of $P \times F$ defined by

$$j(\tilde{f}p) = f(\alpha p). \tag{3.3}$$

It is easily seen that this implies

$$\tilde{f}(pg) = g^{-1}(\tilde{f}p). \tag{3.4}$$

Any section defined on a fiber of P and satisfying (3.4) we call *right-invariant.*

To extend the constructions of the last paragraph to forms with values in $B(M)$, we introduce some convenient notation. Let the kernel of α_* in $\bigwedge^r T(P)$ be denoted by $K^r(P)$ and the image of α^* in $A^r(P)$ by $L^r(P)$. Put $\tilde{d}x^I$ for $\alpha^* dx^I$. Then $\tau \in L^r(p)$ if and only if $\omega \in K^r(p)$ implies $\langle \omega, \tau \rangle = 0$. If $\eta \in L^r(p) \otimes Q(p)$, Q being any vector bundle at all over P, then for $w \in \bigwedge^r T(x)$, $x = \alpha p$, we can define ${}_w\eta \in Q(p)$ to be $\langle \omega, \eta \rangle$ for any $\omega \in \bigwedge^r T(p)$ with $\alpha_* \omega = w$. We can now extend j to a map

$$\begin{gathered} k: L^r(P) \otimes F \to A^r(M) \otimes B(M): \langle w, k\Phi' \rangle = j(p, {}_w\Phi'), \\ \Phi' \in L^r(p) \otimes F,\ w \in \bigwedge^r T(x),\ x = \alpha p. \end{gathered} \tag{3.5}$$

Accordingly, a local section Φ of $A^r(M)\otimes B(M)$ induces a local section $\tilde{\Phi}$ of $L^r(P)\otimes F$ defined by

$$k(\tilde{\Phi}p)=\Phi(\alpha p). \tag{3.6}$$

Analogously to (3.4), $\tilde{\Phi}$ is right-invariant in that

$$_w\tilde{\Phi}(pg)=g^{-1}(_w\tilde{\Phi}p), \qquad w\in \bigwedge^r T(\alpha p). \tag{3.7}$$

The proof of this involves the observation that $\alpha(pg)=\alpha p$, and hence

$$\alpha_*(\omega\cdot g)=\alpha_*(\omega), \qquad \omega\in\bigwedge^r T(P). \tag{3.8}$$

Conversely,

PROPOSITION 3.1. *If Φ' is a section of $L^r(P)\otimes F$, defined on $P(x)$ and right-invariant in the sense of* (3.7), *then it is induced by a form $\Phi\in A^r(x)$ according to* (3.6).

Proof. It suffices to show that $k(\Phi'p)=k(\Phi'(pg))$ for all $g\in G$, for then the common value can be taken for $\Phi(x)$. But indeed,

$$\begin{aligned}\langle w,k(\Phi'(pg))\rangle &= j(pg,{}_w\Phi'(pg))=j(pg,g^{-1}({}_w\Phi'p))\\ &= j(p,{}_w\Phi'p)=\langle w,k(\Phi'p)\rangle, \qquad w\in\bigwedge^r T(x).\end{aligned}$$

The above notation and reasoning can easily be extended to germs of sections; we shall suppose this done, without making any notational change. Then to a germ $\Phi\in\mathscr{S}(A^r(M)\otimes B(M))$ in the stalk over x there corresponds a germ $\tilde{\Phi}[p]\in\mathscr{S}(L^r(P)\otimes F)$ for each $p\in P(x)$; square brackets denote the point in whose stalk a germ is located.

We now make the assumption that a derivation D on $\mathbf{A}(M)\otimes B(M)$ is induced by a derivation Δ on $\mathbf{A}(P(M))\otimes F$, that is,

$$D\Phi=k(\Delta\tilde{\Phi}), \qquad \Phi\in\mathscr{S}(A(M)\otimes B(M)). \tag{3.9}$$

A full proof of this would be difficult, because the $\tilde{\Phi}$ are a restricted class of forms in $A(P)$. Now by the remark at the end of §2, any derivation of degree s on $\mathbf{A}(P)\otimes F$ is of the form

$$\Delta:\ \Phi'\to\mathscr{L}_\eta\Phi'+\Phi'\overline{\wedge}\xi',\ \eta\in\mathscr{S}(A^s(P)\otimes T(P)),\ \xi'\in\mathscr{S}(A^{s+1}(P)\otimes T(P)). \tag{3.10}$$

We shall investigate the requirements that Δ must satisfy for (3.9) to be meaningful.

First of all, the $A^s(P)$ and $A^{s+1}(P)$ of (3.10) can be replaced by $L^s(P)$ and $L^{s+1}(P)$. The proof of Proposition 2.1 in [1] constructs the two forms which specify the derivation from its action on forms of degree 0 and 1. Since Δ must leave the algebra of the $L^r(P)$ invariant, the forms η and ξ' are in $L^s(P)\otimes T(P)$ and $L^{s+1}(P)\otimes T(P)$.

The other requirement for (3.9) to give a well-defined operator is that Δ must send right-invariant forms into right-invariant forms. To be punctilious, we would have to say "right-invariant continuous sections $\Phi'[p]$ of $\mathscr{S}(A(P)\otimes F)$, defined on $P(x)$." To see what the implications are for Δ, let us first take the case where Φ' is of degree 0. Then (3.9) reduces to $\Delta\Phi' = d\Phi' \overline{\wedge} \eta$. This implies that for $w \in \mathscr{S} \wedge^s T(x)$, $\alpha_* \omega = w$,

$$_w(\Delta\Phi') = \langle \omega, d\Phi' \overline{\wedge} \eta \rangle.$$

But suppose

$$\eta = \Sigma_j \eta_j \otimes \tau_j, \qquad \eta_j \in \mathscr{S}L^s(P), \qquad \tau_j \in \mathscr{S}T(P).$$

Then

$$\begin{aligned} \langle \omega, d\Phi' \overline{\wedge} \eta \rangle &= \Sigma_j \langle \omega, \eta_j \wedge (d\Phi' \overline{\wedge} \tau_j) \rangle \\ &= \Sigma_j \langle \omega, \eta_j \rangle \langle \tau_j, d\Phi' \rangle = \langle {}_w\eta, d\Phi' \rangle = {}_w\eta \cdot \Phi'. \end{aligned}$$

Therefore $_w\eta \cdot \Phi'$ must be right-invariant, or

$$(_w\eta \cdot \Phi')[pg] = g^{-1}(_w\eta \cdot \Phi')[p]. \tag{3.11}$$

PROPOSITION 3.2. *A necessary and sufficient condition that* (3.11) *be satisfied for all right-invariant* Φ' *of degree zero, is*

$$(_w\eta[p])\, g = {}_w\eta[pg]. \tag{3.12}$$

Proof. The right-hand side of (3.11) is equal to $g^{-1}(_w\eta[p] \cdot \Phi'[p]) = {}_w\eta[p] \cdot (g^{-1}\, \Phi'[p])$ because g^{-1} acts linearly on F. Letting R_g denote the right translation by g, this in turn equals

$$_w\eta[p] \cdot (\Phi' \circ R_g)[p] = ((_w\eta[p])\, g) \cdot \Phi'[R_g p]$$

by the right-invariance of Φ' and the definition of the right action of g on the tangent bundle $T(P)$. Thus it is necessary and sufficient that

$$((_w\eta[p])\, g) \cdot \Phi'[pg] = {}_w\eta[pg] \cdot \Phi'[pg].$$

(3.12) is sufficient to guarantee this; its necessity is less obvious since Φ' is not perfectly arbitrary. However, suppose there is a vector $u \in T(p)$ such that $u \cdot \Phi' = 0$ for all right-invariant local sections Φ'. Then u must belong to $K^1(p)$, since the variation of Φ' with respect to the point αp in the base space is quite arbitrary. Now consider a coordinate system on the neighborhood of the fiber containing p. Without loss of generality we suppose that the G-coordinate of p is e, so that u is represented by a tangent vector $u_0 \in T(e)$. Then, restricted to the fiber of p, Φ' is an F-valued function on G:

$$\phi : f \to g^{-1} f,$$

where f is some member of F. By hypothesis $u_0 \cdot \phi = 0$. But the function $g \to g \cdot \phi g$ is constant, so

$$0 = u_0 \cdot (g \cdot \phi g) = u_0(\phi) + (u_0 \cdot \phi)$$

which implies

$$(u_0 \cdot \phi) = -u_0(f). \tag{3.13}$$

Here $u_0(\ )$ denotes the action of $u_0 \in T(e)$ on F, induced by the representation of G on F. We now have $u_0(f) = 0$ for all $f \in F$; therefore, for any element g in the one-parameter group generated by u_0, $gf = f$. But the hypothesis at the beginning of the section was that $B(M)$ was strongly associated with $P(M)$ and hence the representation of G on F is faithful. Therefore, the one-parameter group is trivial; $u_0 = 0$; $u = 0$; and an element of $T(p)$ is uniquely determined by its action on right-invariant sections of $P \times F$. This completes the proof.

Now (3.12) says that ${}_w\eta[p]$ is right-invariant: hence there is a unique ${}_w\chi \in \mathscr{S}T(P)/G$ such that

$$\beta({}_w\eta[p]) = {}_w\chi[x], \qquad x = \alpha p.$$

Here we are taking $T(P)/G$ as a vector bundle over M. Since ${}_w\eta[p]$ depends linearly on $w \in \mathscr{S} \bigwedge^s T(M)$, there is a $\chi \in \Sigma^s = \mathscr{S}(A^s(M) \otimes T(P)/G)$ such that

$$\beta({}_w\eta[p]) = \langle w, \chi[\alpha p] \rangle, \qquad w \in \mathscr{S} \bigwedge^s T(P). \tag{3.14}$$

PROPOSITION 3.3. *If $\eta \in S(L^s(P) \otimes T(P))$ satisfies* (3.14), *then the derivation $\Delta = \mathscr{L}_\eta$ sends right-invariant forms of arbitrary degree into right-invariant forms.*

Proof. The form

$$\Phi' = \frac{1}{r!} \tilde{d}x^I \otimes \tilde{\Phi}_I \quad (I_r)$$

is right-invariant if and only if its coefficients $\tilde{\Phi}_I$ are right-invariant. Now if

$$\eta = \frac{1}{s!} dx^J \otimes \eta_J = \eta^i \otimes \frac{\partial}{\partial x^i} + \Sigma_h \bar{\eta}^h \otimes \lambda_h \quad (J_s), \qquad \lambda_h \in \mathscr{S}K^1(P)$$

then from (1.5) we get

$$\begin{aligned} \mathscr{L}_\eta \tilde{\Phi} &= \frac{1}{r!s!} \tilde{d}x^J \wedge \tilde{d}x^I \otimes (\eta_J \cdot \tilde{\Phi}_I) \\ &\quad + \frac{(-1)^s}{(r-1)!} d(\eta^i) \wedge \tilde{d}x^K \otimes \tilde{\Phi}_{iK} \qquad (I_r, J_s, K_{r-1}). \end{aligned}$$

But if η_J and $\tilde{\Phi}_I$ are a right-invariant vector field and section, respectively, then $\eta_J \cdot \tilde{\Phi}_I$ is also a right-invariant section, by Proposition 3.2, so that the first term is right-invariant. As for the second term, we note that $\partial/\partial x^i$ is a right-invariant vector field in $T(P)$, from which it follows that if η_J is right-invariant then η^i_J is constant along the fiber; thus the second term is right-invariant, and so the whole expression is.

We have now obtained the basic operation of Nickerson (cf. [3, equation 20]), which is a map

$$\Sigma^s(M) \otimes \mathscr{S}(A^r(M) \otimes B(M)) \to \mathscr{S}(A^{r+s}(M) \otimes B(M))$$

given by

$$\chi \cdot \Phi = k(\mathscr{L}_\eta \tilde{\Phi}), \tag{3.15}$$

where η is the germ of $\mathscr{S}V^s(P)$ satisfying (3.14) and $\tilde{\Phi}$ is the germ of $\mathscr{S}(A^r(P) \otimes F)$ satisfying (3.6). We shall display this operation in local coordinates. If

$$\chi = \frac{1}{s!} dx^I \otimes \left(\chi_I^j \frac{\partial}{\partial x^j} + \omega_I \right) \qquad (I_s),$$

where $\omega_I \in T(e)$ (an abuse of language), then

$$\eta[x, e] = \frac{1}{s!} \tilde{d}x^I \otimes \left(\chi_I^j \frac{\partial}{\partial x^j} + \omega_I \right).$$

Now as we found in proving Proposition 3.2 and (3.13),

$$\omega_I \cdot \tilde{\Phi}_K[x, e] = -\omega_I(\tilde{\Phi}_K),$$

where $\omega_I(\ \ )$ is defined as was $u_0(\ \ )$ in (3.13). It is now simple to compute

$$\begin{aligned} \chi \cdot \Phi = {} & \frac{1}{r!s!} dx^I \wedge dx^K \otimes \left(\chi_I^j \frac{\partial}{\partial x^j} \Phi_K - \omega_I(\Phi_K) \right) \\ & + \frac{1}{(r-1)!s!} dx^I \wedge dx^j \wedge dx^L \otimes \left(\frac{\partial}{\partial x^j} \chi_I^k \right) \Phi_{kL} \\ & (I_s, K_r, L_{r-1}). \end{aligned} \tag{3.16}$$

We now turn our attention to the second term on the right side of (3.10), viz., $\tilde{\Phi} \overline{\wedge} \xi'$. Actually, we are only interested in the image of this under k; and we shall prove

PROPOSITION 3.4. *If $\xi' \in L^{s+1}(p) \otimes T(p)$, then there is a $\xi \in V^{s+1}(x)$, $x = \alpha p$ such that*

$$\Phi \overline{\wedge} \xi = k(\tilde{\Phi} \overline{\wedge} \xi'), \qquad \Phi \in A^r(x) \otimes B(M).$$

Proof. Suppose

$$\xi' = \Sigma_i \xi_i' \otimes \tau_i, \quad \tau_i \in T(p); \qquad \tilde{\Phi} = \frac{1}{r!} \tilde{d}x^J \otimes \tilde{\Phi}_J \quad (J_r).$$

Then

$$\tilde{\Phi} \overline{\wedge} \xi' = \Sigma_i \frac{1}{r!} (\xi_i' \wedge (\tilde{d}x^J \overline{\wedge} \tau_i)) \otimes \tilde{\Phi}_J$$

and

$$k(\tilde{\Phi} \overline{\wedge} \xi') = \Sigma_i \frac{1}{r!} \xi_i \wedge k((\tilde{d}x^J \overline{\wedge} \tau_i) \otimes \tilde{\Phi}_J),$$

where $\xi_i \in A^{s+1}(M)$ is such that $\alpha_* \xi_i = \xi_i'$. Now by (3.5), if $w \in \bigwedge^{r-1} T(M)$ and $\alpha_* \omega = w$, then

$$\begin{aligned}\langle w, k((\tilde{d}x^J \overline{\wedge} \tau_i) \otimes \tilde{\Phi}_J)\rangle &= j(p, \langle \omega, (\tilde{d}x^J \overline{\wedge} \tau_i) \otimes \tilde{\Phi}_J\rangle) \\ &= \langle \tau_i \wedge \omega, \tilde{d}x^J\rangle \cdot j(p, \tilde{\Phi}_J) = \langle \alpha_* \tau_i \wedge w, dx^J\rangle \Phi_J \\ &= \langle w, (dx^J \overline{\wedge} \alpha_* \tau_i) \otimes \Phi_J\rangle .\end{aligned}$$

Therefore, the proposition is true, if we take $\xi = \Sigma_i \xi_i \otimes \alpha_* \tau_i$. Although our construction of ξ was not canonical, it is easy to see that ξ is uniquely determined by ξ'. The requirement of right-invariance, which is that $k(\tilde{\Phi} \overline{\wedge} \xi')$ should not depend on the position of p on the fiber, merely says that ξ is well defined at $\alpha p \in M$ and determines the second term of the derivation.

THEOREM. *The sheaf of germs of degree s on the vector bundles associated with the principal bundle $P(M)$ is isomorphic to $\Sigma^s(M) \oplus \mathscr{S}V^{s+1}(M)$, with (χ, ξ) corresponding to*

$$D: \Phi \longrightarrow \chi \cdot \Phi + \Phi \overline{\wedge} \xi. \tag{3.17}$$

Indeed, every derivation D is determined by (3.9) and (3.10) for some η and ξ'. Consideration of the case where Φ is of degree zero shows (Proposition 3.2) that η must be right-invariant, and Proposition 3.3 shows that that implies the right-invariance of the first term in (3.10). Therefore the second term must also be right-invariant. We know, from (3.15) and Proposition 3.4 that the two terms in $D\Phi$ must be $\chi \cdot \Phi$ and $\Phi \wedge \xi$.

Nickerson [3, §5] proves, as we really have not, that if F is extended to a commutative ring F' with unit, then there are no other derivations on the bundle $B'(M) = P \times_G F'$ than those of the form of (3.17).

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