

ON OBSTRUCTIONS IN SPHERE BUNDLES AND IMMERSION OF MANIFOLDS

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In this paper we study the following problem. Let ξ be an oriented n -plane bundle over a complex B with Euler class $X(\xi)$. If $\alpha \in H^n(B; \mathbb{Z})$, is it possible to find another oriented n -plane bundle κ stably-isomorphic to ξ such that $X(\kappa) = X(\xi) + \alpha$? If n is odd and $\alpha \neq 0$, it is well known that this is impossible. If $X(\xi) = 0$ we may ask a similar question about the *secondary obstruction*. Theorems 2 and 3 answer these questions using an operation on cohomology defined in §E in terms of the indeterminacy of obstruction theory.

These results can then be applied to some problems concerned with immersing a manifold in Euclidean space. This depends upon a fundamental theorem of M. Hirsch [5]. We obtain some results on the immersibility (and imbeddability) of manifolds and the possible normal classes of immersions.

It is also worth mentioning that an identical investigation of *nonorientable* bundles can be carried out, yielding analogous results.

A. Obstruction theory. Let ξ be a fibre bundle (see [13] for definitions) with base-space a complex B , fibre F and *connected* structural group. This implies $\Pi_1(B)$ acts trivially on the homotopy groups of F . Let s be a section of ξ on B^{r-1} ($= (r-1)$ -skeleton of B). We denote the *obstruction cocycle* of s by $z(s, \xi)$, an r -cocycle with coefficients in $\Pi_{r-1}(F)$. The *obstruction class* of s , we denote by $\zeta(s, \xi) \in H^r(B; \Pi_{r-1}(F))$. If s_1, s_2 are sections of ξ on B^r which agree on B^{r-1} , we denote the *difference cochain* of s_1 and s_2 by $d_r(s_1, s_2, \xi)$, an r -cochain with coefficients in $\Pi_r(F)$. Recall that

$$(1) \quad \delta d_r(s_1, s_2, \xi) = z(s_1, \xi) - z(s_2, \xi).$$

If s_1 and s_2 are extendible over B^{r+1} , $d_r(s_1, s_2, \xi)$ is a cocycle. We denote this *difference class* by $\delta_r(s_1, s_2, \xi)$.

If ξ is an oriented $(m-1)$ -sphere bundle, let ξ_k denote the *bundle of k -frames* of ξ , i.e., the associated bundle whose fibre is the Stiefel manifold $V_{m,k}$. We describe $z(s, \xi_k)$ and $d_r(s_1, s_2, \xi_k)$. For each r -simplex σ of B , let v_σ be a section of $\xi_m|_\sigma$, with proper orientation. If $x \in \partial\sigma$, $v_\sigma(x)$ induces an identification of $\xi|_x$ with S^{m-1} and $s(x)$ is identified with an element of $V_{m,k}$. If σ is oriented this defines the

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element $z(s, \xi_k) \cdot \sigma \in \Pi_{r-1}(V_{m,k})$. It is easy to see this is independent of the choice of v_σ . We refer to this as “framing $s | \partial\sigma$ with v_σ .”

If we have sections s_1 and s_2 on B^r , agreeing on B^{r-1} choose $v_\sigma = \{v_\sigma^1, \dots, v_\sigma^m\}$ so that $\{v_\sigma^1, \dots, v_\sigma^k\} = s_2 | \sigma$. Then “framing $s_1 | \sigma$ with v_σ ” defines a map $(\sigma, \partial\sigma) \rightarrow (V_{m,k}, e)$, where e is a base point of $V_{m,k}$. If σ is oriented, this defines $d_r(s_1, s_2, \xi_k) \cdot \sigma \in \Pi_r(V_{m,k})$; again it is independent of the choice of v_σ .

Finally we recall a well-known fact. Suppose ξ and η are bundles with fibres F and F' with *connected* structural groups. Then any fibre-preserving map $\Phi: \xi \rightarrow \eta$ induces homomorphisms $\phi: \Pi_i(F) \rightarrow \Pi_i(F')$ (see [13]).

LEMMA 1. *Let Φ, ϕ be as above.*

- (a) *If s is a section of $\xi | B^{r-1}$ and $s' = \Phi s$, then $\phi_* z(s, \xi) = z(s', \eta)$.*
- (b) *If s_1, s_2 are sections of $\xi | B^r$ agreeing on B^{r-1} and $s'_i = \Phi s_i$, then:*

$$\phi_* d_r(s_1, s_2, \xi) = d_r(s'_1, s'_2, \eta).$$

B. Stable and nonstable isomorphisms. Let r, t be integers; we define

$$G_r = \lim_{k \rightarrow \infty} \Pi_{r+k}(S^k); \quad F_r^t = \lim_{k \rightarrow \infty} \Pi_t(V_{t-r+k, k})$$

as the homotopy groups in the stable range $k \geq r + 2$. If we consider the fibration $V_{t-r+k, k} \rightarrow V_{t-r+k, k-1}$ for large k we have an exact sequence:

$$(2) \quad F_r^t \xrightarrow{j} F_{r-1}^t \xrightarrow{\partial} G_{r-1} \xrightarrow{i} F_{r-1}^{t-1} \xrightarrow{j} F_{r-2}^{t-1}$$

for $t > 2r$.

If ξ is an oriented $(n - 1)$ -sphere bundle, we denote by ξ^k the k -fold suspension of ξ (the Whitney join of ξ with the trivial $(k - 1)$ -sphere bundle); thus ξ_m^k is the bundle of m -frames of ξ^k . Two oriented sphere-bundles ξ, η are said to be *s-isomorphic* ($\xi \sim_s \eta$) if $\xi^k \approx \eta^k$ (isomorphism of oriented bundles) for some k .

If s is a section of ξ_k^k , the *orthogonal complement* of s is a well-defined oriented $(n - 1)$ -sphere bundle; the definition is obvious. We will say two sections of ξ_k^k are *equivalent* if there is an (orientation-preserving) automorphism of ξ^k (covering the identity map of the base B) which carries one section into the other. It is clear that two sections of ξ_k^k are equivalent if and only if their orthogonal complements are isomorphic. On the other hand if $\eta^k \approx \xi^k$, it is easy to construct a section of ξ_k^k whose orthogonal complement is isomorphic to η .

If B is a finite-dimensional complex, there is an integer k , dependent only upon B , with the property that any two oriented $(n - 1)$ -sphere bundles ξ, η are *s-isomorphic* if and only if $\xi^k \approx \eta^k$; this is well known (see, e.g., [14]). Summarizing, we have proved:

THEOREM 1. *If B is a finite-dimensional complex and ξ is a sphere-bundle over B , there is a one-one correspondence between isomorphism classes of bundles*

s-isomorphic to ξ and equivalence classes of sections of ξ_k^k , for large enough k , induced by taking orthogonal complements.

Let $\xi \sim_s \eta$ be oriented $(n - 1)$ -sphere bundles such that $\xi|B^{r-1} = \eta|B^{r-1}$. We say $\alpha \in H^r(B; F_{r-n}^r)$ is an r -difference of ξ and η if $\alpha = \delta_r(s_1, s_2, \xi_k^k)$, where ξ, η are orthogonal complements of s_1, s_2 , resp., satisfying $s_1|B^{r-1} = s_2|B^{r-1}$. Note that α is not uniquely determined by ξ, η . A representative cocycle of α will also be called an r -difference.

C. Two lemmas. Let ξ, η be oriented $(n - 1)$ -sphere bundles with base space B , satisfying $\xi \sim_s \eta$ and $\xi|B^{r-1} = \eta|B^{r-1}$. Let α be an r -difference (cocycle) of ξ and η , and s be a section of $\xi|B^{r-1} (= \eta|B^{r-1})$.

LEMMA 2. If $2n \geq r + 3$ and $\partial: F_{r-n}^r \rightarrow G_{r-n}$ is the homomorphism in (2), then

$$\partial_{\#}(\alpha) = z(\xi, s) - z(\eta, s).$$

LEMMA 3. If $2n \geq r + 4$ and the sequence from (2):

$$0 \rightarrow G_{r+1-n} \xrightarrow{i} F_{r+1-n}^r \xrightarrow{j} F_{r-n}^r \rightarrow 0$$

is exact, then:

- (i) s is extendible over B^r in ξ if and only if it is so in η .
- (ii) If s', s'' are extensions of s over B^r in ξ, η , resp. and β is the Bockstein with respect to the above short exact sequence, then (regarding α as a cohomology class)

$$\beta(\alpha) = \zeta(\xi, s') - \zeta(\eta, s'').$$

REMARK. The restrictions on n and r serve only to place the coefficient groups in the stable range. Similar results hold for any integers r, n .

Proof of Lemma 2. Let s_1, s_2 be sections of ξ_k^k , whose orthogonal complements are isomorphic to ξ, η , resp., such that $s_1|B^{r-1} = s_2|B^{r-1}$. For each r -simplex σ , let $v_{\sigma} = (v_{\sigma}^1, \dots, v_{\sigma}^{n+k})$ and $w_{\sigma} = (w_{\sigma}^1, \dots, w_{\sigma}^{n+k})$ be properly oriented sections of $\xi_{n+k}^k|_{\sigma}$, satisfying:

$$(v_{\sigma}^1, \dots, v_{\sigma}^k) = s_1|_{\sigma}, \quad (w_{\sigma}^1, \dots, w_{\sigma}^k) = s_2|_{\sigma}.$$

“Framing $s_2|_{\sigma}$ with v_{σ} ” defines $f: (\sigma, \partial\sigma) \rightarrow (V_{n+k,k}, e)$ representing $\alpha \cdot \sigma$; “framing $(w_{\sigma}^1, \dots, w_{\sigma}^{k+1})$ with v_{σ} ” defines g satisfying the following commutative diagram:

$$\begin{array}{ccc} & & V_{n+k,k+1} \\ & \nearrow g & \downarrow p \\ \sigma & \xrightarrow{f} & V_{n+k,k} \end{array}$$

where p is a projection on the initial k -frame. It follows from the definition of ∂ that $\partial(\alpha \cdot \sigma) = (\partial_* \alpha) \cdot \sigma$ is represented by "framing $w_\sigma^{k+1} | \partial\sigma$ with $(v_\sigma^{k+1}, \dots, v_\sigma^{k+n})$ ".

By identifying ξ, η with the orthogonal complements of s_1, s_2 , we see that $z(s, \xi) \cdot \sigma$ (resp. $z(s, \eta) \cdot \sigma$) is represented by "framing $s | \partial\sigma$ with $(v_\sigma^{k+1}, \dots, v_\sigma^{n+k})$ " (resp. $(w_\sigma^{k+1}, \dots, w_\sigma^{n+k})$). Now Lemma 2 follows easily.

Proof of Lemma 3. Statement (i) follows from Lemma 2, since $\partial = 0$. Let s'_1, s'_2 be sections of ξ_{k+1}^k induced by (s_1, s') and (s_2, s'') , resp. If p is as defined in the proof of Lemma 2, then from Lemma 1 we have

$$p_* d_r(s'_1, s'_2, \xi_{k+1}^k) = \alpha.$$

By (1), $\delta d_r(s'_1, s'_2, \xi_{k+1}^k) = z(s'_1, \xi_{k+1}^k) - z(s'_2, \xi_{k+1}^k)$, and, again by Lemma 1:

$$i_*(z(s', \xi) - z(s'', \eta)) = z(s'_1, \xi_{k+1}^k) - z(s'_2, \xi_{k+1}^k);$$

now Lemma 3 follows from the definition of β .

D. Secondary obstructions. Let $\eta \geq 2$ and ξ an oriented $(n - 1)$ -sphere bundle. The Euler class $X(\xi) \in H^n(B; Z)$ is the first obstruction to the existence of a section. If $n \geq 4$ and $X(\xi) = 0$, let $\alpha(\xi) \subset H^{n+1}(B; Z_2)$ be the collection of obstructions to extending sections on B_n . If we define:

$$h(\xi): H^{n-1}(B; Z) \rightarrow H^{n+1}(B; Z_2)$$

by $h(\xi) \cdot \alpha = Sq^2 \alpha + w_2(\xi) \cdot \alpha$, it is known that $\alpha(\xi)$ is a coset of $I(\xi) = \text{Image } h(\xi)$.

If $r \geq 3$ we can similarly define $X(\xi_r) \in H^{n+1-r}(B; F_0^{n-r})$ and, if $X(\xi_r) = 0$, $\alpha(\xi_r) \subset H^{n+2-r}(B; F_1^{n-r+1})$. We define, for $n > r + 3$:

$$h_r(\xi): H^{n-r}(B; F_0^{n-r}) \rightarrow H^{n+2-r}(B; F_1^{n-r+1})$$

by $h_r(\xi) \cdot \alpha = i_*(Sq^2 \alpha + w_2(\xi) \cdot \alpha)$, where $i: G_1 \rightarrow F_1^{n-r}$ is the homomorphism in (2). Note that $h_r(\xi) = h_{r+k}(\xi^k)$ for any k .

LEMMA 4. *With notation as in the preceding paragraph, $\alpha(\xi_r)$ is a coset of $I_r(\xi) = \text{Image } h_r(\xi)$. More explicitly, if s_1 and s_2 are sections of $\xi_r | B^{n+1-r}$ agreeing on B^{n-1-r} , then, if $\alpha = \delta_{n-r}(s_1, s_2, \xi_r)$:*

$$\zeta(s_1, \xi_r) - \zeta(s_2, \xi_r) = h_r(\xi) \cdot \alpha.$$

Proof. If B is a differentiable manifold and ξ is its tangent bundle, this is proved in [1]. In the general case, let M be a differentiable manifold containing B^{2n} as a deformation retract (at least up to homotopy type) whose tangent bundle τ satisfies $\tau | B^{2n} \approx \xi^n$. The existence of M is established in [2]. The natural map $\xi_r \rightarrow \xi_{n+r}$ induces section s'_i from s_i . Since $\tau_{n+r} | B^{2n} \approx \xi_{n+r}^n$, it follows from [1] that:

$$\zeta(s'_1, \xi_{n+r}^n) - \zeta(s'_2, \xi_{n+r}^n) = h_{n+r}(\xi^n) \cdot \delta_{n-r}(s'_1, s'_2, \xi_{n+r}^r).$$

But by Lemma 1, $\zeta(s_i, \xi_r) = \zeta(s'_i, \xi_{n+r}^n)$ and $\alpha = \delta_{n-r}(s'_1, s'_2, \xi_{n+r}^r)$. Thus Lemma 4 is proved.

E. Some operations on cohomology. Let ξ be an oriented $(n - 1)$ -sphere bundle over a *finite-dimensional* complex B ; if $i > 0$ and $r \geq n > 0$, we define partial multi-valued operations:

$$\Phi_{r-n}^i(\xi): H^r(B; F_{r-n}^r) \rightarrow H^{r+i}(B; F_{r+i-1-n}^{r+i-1})$$

which will have the following properties:

LEMMA 5. (a) $\Phi_k^{i+1}(\xi)$ is defined on $\text{Ker } \Phi_k^i(\xi) = \{\alpha \mid \Phi_k^i(\xi) \cdot \alpha \text{ contains } 0\}$.

(b) $\Phi_k^1(\xi) = 0$, defined everywhere.

(c) $\Phi_r^2(\xi)$ is single-valued; $\Phi_0^2(\xi) = h_0(\xi)$ if $n > 3$.

The $\Phi_k^i(\xi)$ will be useful to us because of the following lemma. Let $K_k(\xi) = \lim_{i \rightarrow \infty} \text{Ker } \Phi_k^i(\xi)$.

LEMMA 6. Let ξ be an oriented $(n - 1)$ -sphere bundle. If $\alpha \in H^r(B; F_{r-n}^r)$, there exists a bundle $\eta \sim_s \xi$, satisfying $\xi \mid B^{r-1} = \eta \mid B^{r-1}$, with r -difference α if and only if $\alpha \in K_{r-n}(\xi)$.

We define $\Phi_{r-n}^i(\xi)$ as follows. Let s be a section of ξ_k^k (for large k) whose orthogonal complement is isomorphic to ξ . Given $\alpha \in H^r(B; F_{r-n}^r)$, $\Phi_{r-n}^i(\xi) \cdot \alpha$ is defined if there is a section s' of $\xi_k^k \mid B^{r+i-1}$ which agrees with s on B^{r-1} , such that $\alpha = \delta_r(s, s', \xi_k^k)$. We define $\Phi_{r-n}^i(\xi) \cdot \alpha$ to be the set of $\zeta(s', \xi_k^k)$, for all such s' . It follows immediately that:

(3) There is a section s' of $\xi_k^k \mid B^{r+i}$ which agrees with s on B^{r-1} , such that $\alpha = \delta_r(s, s', \xi_k^k)$, if and only if $\alpha \in \text{Ker } \Phi_{r-n}^i(\xi)$.

This implies (a) of Lemma 5. Statement (b) follows directly from the definition and (1). To prove (c), let s', s'' be sections of $\xi_k^k \mid B^{r+1}$ which agree with s on B^{r-1} , such that $\delta_r(s', s'', \xi_k^k)$ is a coboundary. Then s' is homotopic to s'' on B^r and $\zeta(s', \xi_k^k) = \zeta(s'', \xi_k^k)$. The second part of (c) follows from Lemma 4.

Lemma 6 follows directly from Theorem 1 and (3), taking $i > \dim B$.

F. The main theorems. As usual, let ξ be an oriented $(n - 1)$ -sphere bundle over a *finite-dimensional* base B , with Euler class $X(\xi)$ and, if $X(\xi) = 0$, secondary obstruction $\alpha(\xi)$. The following theorems measure the dependence of $X(\xi)$ and $\alpha(\xi)$ on the stable class of ξ .

THEOREM 2. If $n \geq 2$ is even, $\alpha \in H^n(B; Z)$, there is an $\eta \sim_s \xi$ with $X(\eta) = X(\xi) + \alpha$ if and only if $\alpha \in 2K_0(\xi)$.

THEOREM 3. If $n \geq 4$, $X(\xi) = 0$ and $\alpha \in H^{n+1}(B; Z_2) / I(\xi)$ then there is $\eta \sim_s \xi$ with $\alpha(\eta) = \alpha(\xi) + \alpha$:

(a) for $n \equiv 1$ or $2 \pmod{4}$, if and only if $\alpha = 0$,

(b) for $n \equiv 3 \pmod{4}$, if and only if $\alpha \in rSq^1 K_0(\xi)$,

(c) for $n \equiv 0 \pmod{4}$, if $\alpha \in r f_* K_1(\xi)$,

where $r: H^{n+1}(B; Z_2) \rightarrow H^{n+1}(B; Z_2) / I(\xi)$ is the quotient map and $f: Z_2 + Z_2 \rightarrow Z_2$ is nonzero.

REMARKS. (a) If n is odd, $X(\xi)$ is an integral Stiefel-Whitney class and therefore a stable invariant of ξ .

(b) If $n < 4$, results analogous to Theorem 3 can be easily obtained.

Proof of Theorem 2. First note that, if $\xi \sim_s \eta$, then $\xi|B^{n-1} = \eta|B^{n-1}$. If $n > 2$, Theorem 2 is a direct consequence of Lemmas 2 and 6, since $\partial:F_0^n \rightarrow G_0$ is multiplication by 2 for n even. But if $r = n = 2$, Lemma 2 is still valid, since $\Pi_k(S^k) = G_0$ for $k \geq 1$; thus Theorem 2 holds for $n = 2$.

Proof of Theorem 3. For (a) note that $i:G_1 \rightarrow F_1^n$ is a split monomorphism; therefore, $i_*:H^{n+1}(B;G_1) \rightarrow H^{n+1}(B;F_1^n)$ is a monomorphism and so is the induced homomorphism:

$$i'_*: H^{n+1}(B;G_1)/I(\xi) \rightarrow H^{n+1}(B;F_1^n)/I_1(\xi).$$

By Lemmas 1 and 4, $\alpha(\xi_{k+1}^k) = i'_* \alpha(\xi)$ and similarly for η , if k is large. If $\xi^k \approx \eta^k$, since i'_* is a monomorphism, $\alpha(\xi) = \alpha(\eta)$.

To prove (b) and (c) recall that $\eta \sim_s \xi$ implies $\eta|B^{n-1} = \xi|B^{n-1}$. By Lemma 6, the n -difference $\alpha' \in K_0(\xi)$. If $n \equiv 3 \pmod 4$, it follows from Lemma 3 that $\alpha(\eta) - \alpha(\xi) = rSq^1(\alpha')$. If $\eta|B^n = \xi|B^n$, then the $(n+1)$ -difference $\alpha'' \in K_1(\xi)$, again by Lemma 6. From Lemma 2, $\alpha(\eta) = \alpha(\xi)$ if $n \equiv 3 \pmod 4$, and $\alpha(\eta) - \alpha(\xi) = rf_*(\alpha'')$ if $n \equiv 0 \pmod 4$. If $\eta|B^{n+1} = \xi|B^{n+1}$, clearly $\alpha(\xi) = \alpha(\eta)$. This completes the proof of Theorem 3.

Note that we have not obtained a *complete* answer if $n \equiv 0 \pmod 4$.

G. Imbeddability and immersibility of manifolds. We will now present some applications of our results as mentioned in the introduction. We will often need some restriction on the dimensions of the manifold and the Euclidean space. For this purpose we will say a pair of integers $(r+s, r)$ is *proper* if the homomorphism $i:G_r \rightarrow F_r^{s-1}$ is zero, for $r, s \geq 1$.

LEMMA 7. *The pair (m, r) is proper if:*

- (i) $r = 1, m \equiv 1 \pmod 4$,
- (ii) $r = 2, m \equiv 1 \text{ or } 2 \pmod 4$,
- (iii) $r = 3, m \equiv 3 \pmod 8$,
- (iv) $r = 4 \text{ or } 5, \text{ any } m$.

This follows directly from considerations of the exact sequence (2) and the results of [11].

For the remainder of this paper let M be a *closed orientable $(q-1)$ -connected differentiable n -manifold*. All imbeddings and immersions are differentiable unless otherwise stated. Haefliger has shown in [3] that M is imbeddable in $(2n - q + 1)$ -space if $n > 2q$ and immersible in $(2n - q + 1)$ -space if $n > 2q - 2$. The following two theorems improve these results by a few dimensions with suitable restrictions on M and the integers n and q .

The normal Stiefel-Whitney classes of M will be denoted by \bar{W}_i (in dimension i); if i is odd > 1 (resp. even or $i = 1$), it is an integral (resp. mod 2) cohomology class. If ξ is the normal bundle of an immersion, $X(\xi)$ will be referred to as the *normal class* of the immersion. If $X(\xi) = 0$, $\alpha(\xi)$ will be the *second normal class* of the immersion. Let M_0 be the complement of an open disk in M .

THEOREM 4. *Let M be as above with $\bar{W}_{n-q} = 0$.*

- (a) *If $q = 1$ and $n > 4$, then M is immersible in $(2n - 2)$ -space if and only if:*
- (i) $n \equiv 1 \pmod{4}$ or
 - (ii) $\bar{W}_2 \cdot \bar{W}_{n-2} = 0$.
- (b) *If $q > 1$ and $n > 2q + 2$, M is imbeddable in $(2n - q)$ -space. If, in addition, (n, q) is proper, M is immersible in $(2n - q - 1)$ -space.*

THEOREM 5. *Let M be as above with $\bar{W}_{n-q} = \bar{W}_{n-q-1} = 0$.*

- (a) *If $q > 2$, $n > 2q + 4$ and M_0 is immersed in $(2n - q - 1)$ -space with second normal class α , then M is imbeddable in $(2n - q - 1)$ -space if:*
- (i) $n \equiv q + 1 \pmod{4}$ or
 - (ii) $n \equiv q \pmod{4}$ and $\alpha \in r Sq^1 H^{n-q-1}(M)$ or
 - (iii) $\alpha = 0$.
- (b) *If $q \neq 2$, $n > 2q + 4$, $(n, q + 1)$ is proper and M is immersed in $(2n - q - 1)$ -space with second normal class α , then M is immersible in $(2n - q - 2)$ -space if and only if (i), (ii) or (iii) hold.*

We will also prove the following.

THEOREM 6. *Suppose $(n, q + 1)$ is proper and M is imbeddable in $(2n - q - 1)$ -space. Then, if $W_2 = 0$, M is immersible in $(2n - q - 2)$ -space.*

From Theorems 3, 4 and 5 we can derive some results about projective spaces.

COROLLARY. (a) *The following imbeddings are possible:*

- (i) CP^n in $(4n - 2)$ -space if $n \neq$ power of two.
 - (ii) QP^n in $(8n - 4)$ -space if $n \neq$ power of two.
- (b) *The following immersions are possible:*
- (i) RP^n in $(2n - 3)$ -space if $n \equiv 1 \pmod{4}$.
 - (ii) QP^n in $(8n - 5)$ -space if $n \neq$ power of two.
 - (iii) QP^n in $(8n - 6)$ -space if QP^n is imbeddable in $(8n - 5)$ -space.
- (c) *The following immersion is impossible: CP^n in $(4n - 2)$ -space if $n =$ power of two.*

We obtain (a) and (b-ii) from Theorem 4 (b), (b-i) from Theorems 4(b) and 5(b-ii), (b-iii) from Theorem 6 and (c) from Theorem 4(a).

REMARKS. (a) There is considerable overlap of these results with those of [4], [12] and [8].

(b) Assertion (c) of the corollary contradicts a result of [10].

(c) It will be clear that one might improve Theorems 4 and 5 with more information about the operations of §E.

H. The normal class of an immersion. In [7] R. Lashof and S. Smale propose the problem of determining which elements α of $H^r(M; Z)$ can be realized as the normal class of an immersion of M in $(n+r)$ -space. If r is odd, only \bar{W}_r itself is possible; therefore we only consider even r . One restriction is that $\alpha \equiv \bar{W}_r \pmod{2}$; are there other restrictions?

If $r = n$, it is proved in [7] (see also [5]) that this is the *only* restriction, i.e. α may be any *even* element of $H^n(M; Z)$. We shall prove:

THEOREM 7. *Suppose M is immersible in $(2n-t)$ -space. If $t < 3$ and $n-t$ is even > 0 , every element α of $H^{n-t}(M; Z)$ satisfying $\alpha \equiv \bar{W}_{n-t} \pmod{2}$ is realizable as the normal class of an immersion.*

I. Proofs of Theorems 4-7. We first prove a lemma.

LEMMA 8. *Suppose $n > 2r$ and $(n, r-1)$ is proper. If:*

- (i) M_0 is imbedded in $(2n-r)$ -space or,
- (ii) M is immersed in $(2n-r+1)$ -space with a normal field on M_0 , then M is immersible in $(2n-r)$ -space.

Proof. Let f be an imbedding of M_0 in $(2n-r)$ -space. By "adding a cap" we may extend f to a *topological* imbedding of M in $(2n-r+1)$ -space. According to [3] there is a *differentiable* imbedding of M in $(2n-r+1)$ -space which agrees with f outside some open disk. Clearly this reduces us to (ii).

Let s be a section of $\xi|_{M_0}$, where ξ is the normal bundle of an immersion of M in $(2n-r+1)$ -space. Let s' be the induced section of $\xi_{k+1}^k|_{M_0}$, for large k . Then, by Lemma 1, $\zeta(s', \xi_{k+1}^k) = i_*\zeta(s, \xi)$, where $i: G_{r-1} \rightarrow F_{r-1}^{n-r}$; but $i_* = 0$ if $(n, r-1)$ is proper. Therefore ξ_{k+1}^k has a section. We now appeal to the following result of M. Hirsch, contained in [5].

(4) *If any n -manifold V is immersed in $(n+k+1)$ -space, where $k > 0$, with normal bundle ξ , then V is immersible in $(n+k)$ -space if and only if there exists $\eta \sim_s \xi$ with a section.*

This completes the proof of Lemma 8.

Proof of Theorem 4. According to [4], M is imbeddable in $(2n-1)$ -space. If ξ is the normal bundle $\alpha(\xi)$ is the single element $\bar{W}_2 \cdot \bar{W}_{n-2}$, by [9]. Since Sq^1 is zero on $H^{n-1}(M; Z_2)$ and $K_1(\xi) = H^n(M; Z_2)$, (a) follows from Theorem 3 and (4).

To prove (b) we first note that M_0 can be immersed in $(2n-q-1)$ -space. In fact if M_0 is immersed in $(2n-q-1+k)$ -space, the first obstruction to a normal k -field is $\bar{W}_{n-q} = 0$ and all higher obstructions are in zero groups; we then apply

(4). By [6] M_0 can be imbedded in $(2n - q - 1)$ -space if $q > 1$. Now (b) follows by "adding a cap" and smoothing (as in Lemma 8) and by Lemma 8.

Proof of Theorem 5. Suppose M_0 is immersed in $(2n - q - 1)$ -space with normal bundle ξ and $\alpha = \alpha(\xi)$. Now $K_1(\xi) = H^{n-q}(M; Z_2)$ and $K_0(\xi) = H^{n-q-1}(M; Z)$ by Lemma 5 and $\alpha(\xi)$ is the only obstruction to a section. By Theorem 3 and (4), M_0 is immersible in $(2n - q - 2)$ -space if and only if (i), (ii) or (iii) hold. If $q > 2$, M_0 can be imbedded in $(2n - q - 2)$ -space by [6]. Now (a) follows by "adding a cap" and smoothing and (b) follows from Lemma 8 for $q > 2$.

Suppose $q = 1$ and M is immersed in $(2n - 2)$ -space with normal bundle ξ and $\alpha = \alpha(\xi)$. By Theorem 3 and (4), there is an immersion of M in $(2n - 2)$ -space with normal field on M_0 if and only if (i), (ii) or (iii) hold, since $K_0(\xi) = H^{n-2}(M; Z)$ and $K_1(\xi) = H^{n-1}(M; Z_2)$ by Lemma 5, and $\alpha(\xi)$ is the only obstruction to a normal field on M_0 . Now (b) follows for $q = 1$ by Lemma 8.

Proof of Theorem 6. Suppose M is imbedded in $(2n - q - 1)$ -space with normal bundle ξ . Let N be the total space of the sphere-bundle ξ . The following facts about the cohomology of N is due to Massey (see [9]). There is a subring A of $H^*(N; Z_2)$, closed under cohomology operations, which is disjoint from $p^*H^*(M; Z_2)$, where $p: N \rightarrow M$ is the bundle projection. Furthermore, there is an element $a \in A$ and $\alpha \in \alpha(\xi)$ satisfying:

$$Sq^2(a) = p^*(\alpha) + a \cdot p^*(\bar{W}_2).$$

If $\bar{W}_2 = 0$, then $p^*(\alpha) = 0$; since p^* is a monomorphism, $\alpha = 0$. Therefore $\xi|_{M_0}$ has a section and Theorem 6 follows from Lemma 8.

Proof of Theorem 7. Suppose M is immersed in $(2n - t)$ -space with normal bundle ξ . It follows from Lemma 5 that $K_0(\xi) = H^{n-t}(M; Z)$, if $t < 3$, since $h_0(\xi) = 0$. Now Theorem 7 follows from Theorem 2 and (4).

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