SUBSTITUTION MINIMAL SETS

BY

W. H. GOTTSCHALK[1]

Introduction. We show that every substitution on finitely many symbols, which replaces each symbol by a block of length 2 or more, effectively determines at least one almost periodic point under the shift transformation of symbolic dynamics. The orbit-closures of these almost periodic points, called *substitution* minimal sets, are analyzed topologically to some extent and in particular it is proved that under certain conditions their structure groups are n-adic groups. Several of the known symbolic minimal sets, such as the Morse minimal set, are definable by the present method of construction. Minimal sets exhibiting new properties also appear. The first two sections contain general theorems on the trace relation of transformation groups and on ψ-adic transformation groups, some of these results being used to study substitution minimal sets in the third section. A few particular examples of substitution minimal sets are described in the last section. As general references for notions, notation, and terminology occuring here, consult [3; 1].

1. **The trace relation.**

1.01. NOTATION. Groups are taken to be multiplicative unless otherwise stated in a particular instance. All topological spaces and topological groups considered are assumed to be Hausdorff. If X is a set, then $\Delta(X)$ or Δ_X or Δ denotes the diagonal of $X \times X$, that is, the subset $\{(x,x) \mid x \in X\}$ of $X \times X$. If X is a set and if R is a relation in X, then xR for $x \in X$ denotes $\{y \mid y \in X \ \& \ (x,y) \in R\}$, and X/R denotes $\{xR \mid x \in X\}$. If T is a topological group, then $\mathscr{H}(T)$ or $\mathscr{H}_T$ or $\mathscr{H}$ denotes the class of all closed syndetic invariant subgroups of T. If (X,T) is a transformation group and if $S \subset T$, then X/S denotes $\{\overline{xS} \mid x \in X\}$.

1.02. DEFINITION. Let (X,T) be a transformation group. The *trace relation of* (X,T), denoted $\Lambda(X,T)$ or $\Lambda(X)$ or Λ_X or Λ, is defined to be the set of all couples $(x,y) \in X \times X$ such that $y \in \bigcap_{H \in \mathscr{H}} \overline{xH}$. If $x \in X$, then the *trace of* x *under* (X,T) and the *trace of* (X,T) *at* x is defined to be the subset $x\Lambda = \bigcap_{H \in \mathscr{H}} \overline{xH}$ of X. The transformation group (X,T) is said to *have singleton traces* provided that $x \in X$ implies $x\Lambda = \{x\}$, or equivalently $\Lambda = \Delta$. The transformation group (X,T) is said to be *totally minimal* provided that $x \in X$ implies $x\Lambda = X$, or

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equivalently $\Lambda = X \times X$, or equivalently $x \in X$ and $H \in \mathscr{H}$ implies $\overline{xH} = X$, or equivalently $H \in \mathscr{H}$ implies the transformation group (X, H) is minimal.

1.03. Remark. Let (X, T) and (Y, T) be transformation groups, let ϕ be a homomorphism of (X, T) into (Y, T), and let $x \in X$. Then:

(1) $x\Lambda_X\phi \subset x\phi\Lambda_Y$.

(2) If X is compact and if T is discrete, then $x\Lambda_X\phi = x\phi\Lambda_Y$.

Proof. (1) $x\Lambda_X\phi = (\bigcap_{H \in \mathscr{H}} \overline{xH})\phi \subset \bigcap_{H \in \mathscr{H}} \overline{xH}\phi \subset \bigcap_{H \in \mathscr{H}} \overline{xH\phi} = \bigcap_{H \in \mathscr{H}} \overline{x\phi H}$ $= x\phi\Lambda_Y$.

(2) By (1), $x\Lambda_X\phi \subset x\phi\Lambda_Y$. We show $x\phi\Lambda_Y \subset x\Lambda_X\phi$. Let $y \in x\phi\Lambda_Y = \bigcap_{H \in \mathscr{H}} \overline{x\phi H}$. If $H \in \mathscr{H}$, then $y \in \overline{x\phi H} = \overline{xH\phi} = \overline{xH}\,\phi$ and $y\phi^{-1} \cap \overline{xH} \neq \emptyset$. Since T is discrete, $\{\overline{xH} \mid H \in \mathscr{H}\}$ is a closed filter base on X by [3, 5.04]. Hence

$$y\phi^{-1} \cap \bigcap_{H \in \mathscr{H}} \overline{xH} \neq \emptyset, \; y \in \left(\bigcap_{H \in \mathscr{H}} \overline{xH}\right)\phi = x\Lambda_X\phi,$$

and $y \in x\Lambda_X\phi$.

1.04. Remark. Let (X, T) and (Y, T) be transformation groups such that (X, T) is homomorphic to (Y, T). Then:

(1) If (X, T) is minimal, then (Y, T) is minimal.

(2) If (X, T) is totally minimal, then (Y, T) is totally minimal.

Proof. (1) By hypothesis there exists a homomorphism ϕ of (X, T) onto (Y, T). Let $y \in Y$. There exists $x \in X$ such that $x\phi = y$. Then $Y = X\phi = \overline{xT}\,\phi \subset \overline{xT\phi} = \overline{x\phi T} = \overline{yT} \subset Y$ and $\overline{yT} = Y$.

(2) is immediate from (1).

1.05. Definition. Let (X, T) be a transformation group where X is compact. The *structure relation of* (X, T), denoted $\Sigma(X, T)$ or $\Sigma(X)$ or Σ_X or Σ, is defined to be the least invariant closed equivalence relation R in X such that the partition transformation group $(X/R, T)$ is equicontinuous; such is known to exist [1, Theorem 1]. The star-closed partition $X/\Sigma = \{x\Sigma \mid x \in X\}$ of X is called the *structure partition of* (X, T). The partition transformation group $(X/\Sigma, T)$ is called the *structure transformation group of* (X, T). When convenient, as in the case of flows, the structure transformation group $(X/\Sigma, T)$ of (X, T) may also be denoted $\mathscr{S}(X, T)$. The canonical map of X onto X/Σ is a homomorphism of (X, T) onto $(X/\Sigma, T)$ and is called the *structure homomorphism of* (X, T). If G is the transition group of $(X/\Sigma, T)$, if Φ is the group of all homeomorphisms of X/Σ onto X/Σ, and if Φ is provided with its space-index topology, then Φ is a topological group and the *structure group of* (X, T), denoted $\mathscr{G}(X, T)$ or $\mathscr{G}(X)$ or $\mathscr{G}_X$ or $\mathscr{G}$, is defined to be the closure $\bar{G}$ of G in Φ. It is known [3, 4.45] that $\mathscr{G}(X, T)$ is a compact topological group.

1.06. Definition. Let (X, T) be a transformation group where X is a uniform space. If x, $y \in X$, then x and y are said to be *proximal* (the first *to* the second) *under* (X, T) provided that if α is an index of X, then there exists $t \in T$ such that $(xt, yt) \in \alpha$. The *proximal relation of* (X, T), denoted $\Pi(X, T)$ or $\Pi(X)$ or Π_X or Π

is defined to be the set of all couples $(x, y) \in X \times X$ such that x is proximal to y under (X, T).

1.07. Definition. Let (X, T) be a transformation group and let R be a relation in X. The *ancestral of R under (X, T)*, denoted R^*, is defined to be the least invariant closed equivalence relation in X which contains R.

1.08. Theorem. *Let (X, T) be a minimal transformation group where X is compact. Then*:

(1) *Λ is the least invariant closed equivalence relation R in X such that the partition transformation group $(X/R, T)$ has singleton traces.*

(2) *$(X/\Lambda, T)$ is equicontinuous.*

(3) $\Lambda \supset \Sigma \supset \Pi^* \supset \Pi \supset \Delta$.

(4) *If (X, T) is point regularly almost periodic, then $\Lambda = \Sigma = \Pi^* = \Pi$.*

(5) *If there exists $x_0 \in X$ such that $x_0\Lambda \subset x_0\Sigma$, then $\Lambda = \Sigma$.*

(6) *If (X, T) is equicontinuous and if T is locally compact connected abelian, then (X, T) has singleton traces and $\Lambda = \Sigma = \Pi^* = \Pi = \Delta$.*

Proof. (1) For each $H \in \mathscr{H}$ define R_H to be the set of all couples $(x, y) \in X \times X$ such that $\overline{xH} = \overline{yH}$. From [3, 2.32] it follows that for each $H \in \mathscr{H}$, R_H is an invariant closed equivalence relation in X. Hence $\Lambda = \bigcap_{H \in \mathscr{H}} R_H$ is an invariant closed equivalence relation in X.

We show $(X/\Lambda, T)$ has singleton traces. Let ϕ be the canonical map of X onto X/Λ whence $x \in X$ implies $x\phi = x\Lambda$. Now ϕ is a homomorphism of (X, T) onto $(X/\Lambda, T)$. Let $y_0 \in X/\Lambda$. Assume there exists an element y_1 of the trace of y_0 under $(X/\Lambda, T)$ such that $y_0 \neq y_1$. Then $y_1 \in \overline{y_0H}$ for all $H \in \mathscr{H}$. Since $(X/\Lambda, T)$ is minimal by 1.04 (1), it follows from [3, 2.24] that $\overline{y_0H} = \overline{y_1H}$ for all $H \in \mathscr{H}$. Choose $x_0 \in y_0\phi^{-1}$ and $x_1 \in y_1\phi^{-1}$. Then $x_0\phi = y_0$, $x_1\phi = y_1$; and $H \in \mathscr{H}$ implies $\overline{x_0H}\,\phi = \overline{x_0H\phi} = \overline{x_0\phi H} = \overline{y_0H} = \overline{y_1H} = \overline{x_1\phi H} = \overline{x_1H\phi} = \overline{x_1H}\phi$ and $\overline{x_0H}\,\phi = \overline{x_1H}\phi$. Since x_0 and x_1 have different traces y_0 and y_1 under (X, T), there exists $H \in \mathscr{H}$ such that $\overline{x_0H} \cap \overline{x_1H} = \emptyset$, again by [3, 2.24]. Since $x_0\phi \in \overline{x_0H}\,\phi = \overline{x_1H}\,\phi$, there exists $x_2 \in \overline{x_1H}$ such that $x_0\phi = x_2\phi$ whence $\overline{x_2H} = \overline{x_1H}$ and $x_0\Lambda = x_2\Lambda$. Yet $x_0\Lambda \subset \overline{x_0H}$ and $x_2\Lambda \subset \overline{x_2H} = \overline{x_1H}$, so that $\overline{x_0H} \cap \overline{x_1H} \neq \emptyset$. This is a contradiction.

Let R be an invariant closed equivalence relation in X such that $(X/R, T)$ has singleton traces. We show $\Lambda \subset R$. Let ϕ be the canonical homomorphism of (X, T) onto $(X/R, T)$ whence $x \in X$ implies $x\phi = xR$. If $x \in X$, then it follows from 1.03 (1) that $(x\Lambda)\phi \subset (x\phi)\Lambda(X/R, T) = \{x\phi\} = \{xR\}$ whence $(x\Lambda)\phi = \{xR\}$. If $(x, y) \in \Lambda$, then $y \in x\Lambda$, $y\phi \in (x\Lambda)\phi = \{xR\}$, $\{y\phi\} = \{xR\}$, $\{yR\} = \{xR\}$, $yR = xR$, and $(x, y) \in R$. Hence $\Lambda \subset R$.

The proof of (1) is completed.

(2) It is enough by (1) to prove the following lemma:

(L) Let (X, T) be a minimal transformation group with singleton traces where X is compact. Then (X, T) is equicontinuous.

Proof of (L). Let $\mathscr{U}$ be the compatible uniformity of X. Consider the squared transformation group $(X \times X, T)$ and define $Q = \bigcap_{\alpha \in \mathscr{U}} \overline{\alpha T}$. In order to show that (X, T) is equicontinuous it is enough to show that $Q = \Delta_X$. Let $x, y \in X$ such that $x \neq y$. It is enough to show $(x, y) \notin Q$. Since $x\Lambda = \{x\} \neq \{y\} = y\Lambda$, then by [3, 2.24] there exists $H \in \mathscr{H}$ such that $\overline{xH} \cap \overline{yH} = \emptyset$. Choose an open symmetric index β of X for which $\beta^3 \cap (\overline{xH} \times \overline{yH}) = \emptyset$. Define

$$U = \bigcup \{\overline{zH} \mid z \in X \ \& \ \overline{zH} \subset \overline{xH}\beta\}$$

and

$$V = \bigcup \{\overline{zH} \mid z \in X \ \& \ \overline{zH} \subset \overline{yH}\beta\}.$$

Since $\{\overline{zH} \mid z \in X\}$ is a star-closed partition of X by [3, 2.32], it follows that U and V are open neighborhoods of x and y. Choose a compact subset K of T such that $T = HK$. There exists an index α of X such that $\alpha K^{-1} \subset \beta$. In order to show that $(x, y) \notin Q$, it is enough to show there do not exist $x_1 \in U$, $y_1 \in V$, and $t \in T$ such that $(x_1 t, y_1 t) \in \alpha$. Let $x_1 \in U$, $y_1 \in V$, and $t \in T$. Assume $(x_1 t, y_1 t) \in \alpha$. Now $t = hk$ for some $h \in H$ and some $k \in K$. We have

$$(x_1 h, y_1 h) = (x_1 hk, y_1 hk)k^{-1} = (x_1 t, y_1 t)k^{-1} \in \alpha K^{-1} \subset \beta$$

and $(x_1 h, y_1 h) \in \beta$. Since $x_1 h \in \overline{x_1 H} \subset \overline{xH}\,\beta$ and $y_1 h \in \overline{y_1 H} \subset \overline{yH}\,\beta$, it follows that $x_1 h \in x_2 \beta$ for some $x_2 \in \overline{xH}$ and $y_1 h \in y_2 \beta$ for some $y_2 \in \overline{yH}$. Hence

$$(x_2, y_2) = (x_2, x_1 h)(x_1 h, y_1 h)(y_1 h, y_2) \in \beta^3, \ (x_2, y_2) \in \overline{xH} \times \overline{yH}$$

and $(x_2, y_2) \in \beta^3 \cap (\overline{xH} \times \overline{yH}) \neq \emptyset$. This is a contradiction. The proof of (L) is completed.

The proof of (2) is completed.

(3) This follows immediately from (2). Also see [1].

(4) This is immediate from (3) and [3, 10.05]. The hypothesis of (4) means by definition that some point of X is regularly almost periodic under (X, T).

(5) Let ϕ be the canonical homomorphism of (X, T) onto $(X/\Sigma, T)$.

We show $x \in X$ implies $x\Lambda\phi = \{x\phi\}$. Let $x \in X$ and let α be an index of X/Σ. It is enough to show $x\Lambda\phi \times x\Lambda\phi \subset \alpha$. Since $(X/\Sigma, T)$ is uniformly equicontinuous, there exists an index β of X/Σ such that $\beta T \subset \alpha$. Choose a neighborhood U of $x_0\phi$ so that $U \times U \subset \beta$. Since $U\phi^{-1}$ is a neighborhood of $x_0\phi\,\phi^{-1} = x_0\Sigma = x_0\Lambda$ and since X/Λ is star-closed by (1), there exists a neighborhood V of x_0 such that $V\Lambda \subset U\phi^{-1}$. Choose $t \in T$ so that $xt \in V$. Then $xt\Lambda \subset V\Lambda \subset U\phi^{-1}$, $x\Lambda\phi t = x\Lambda t\phi = xt\Lambda\phi \subset U$, $x\Lambda\phi \subset Ut^{-1}$, and

$$x\Lambda\phi \times x\Lambda\phi \subset Ut^{-1} \times Ut^{-1} = (U \times U)t^{-1} \subset \beta T \subset \alpha.$$

If $(x, y) \in \Lambda$, then $y \in x\Lambda$, $y\phi \in x\Lambda\phi = \{x\phi\}$, $y\phi = x\phi$, $y\Sigma = x\Sigma$, and $(x, y) \in \Sigma$. Thus $\Lambda \subset \Sigma$. Since $\Lambda \supset \Sigma$ by (3), it follows that $\Lambda = \Sigma$.

(6) Let x, $y \in X$ such that $x \neq y$. By [3, 4.48] there exists an abelian group structure in X which makes X a topological group with x the identity element and which makes π_x a continuous group homomorphism of T into X, where π is the action of (X, T). Let C be the circle group. Since X is a compact abelian topological group, there exists a continuous group homomorphism ϕ of X into C such that $1 = x\phi \neq y\phi$. Since $X\phi$ is a nondegenerate connected subgroup of C, it follows that $X\phi = C$. Since $T\pi_x\phi = (xT)\phi$ is a dense connected subgroup of C, it follows that $T\pi_x\phi = (xT)\phi = C$. Define $H = 1(\pi_x\phi)^{-1} = 1\phi^{-1}\pi_x^{-1}$. Now H is a closed invariant subgroup of T. Since T is locally compact and T/H is isomorphic to the compact group C, it follows that H is a syndetic subgroup of T. Clearly $xH = H\pi_x \subset 1\phi^{-1}$ whence $\overline{xH} \subset 1\phi^{-1}$. From $y \notin 1\phi^{-1}$ we conclude that $y \notin \overline{xH}$ and $(x, y) \notin \Lambda$. The proof of (6) is completed by use of (3).

1.09. Definition. Let (X, T) be a minimal transformation group where X is compact. The star-closed partition $X/\Lambda = \{x\Lambda \mid x \in X\}$ of X is called the *trace partition of* (X, T). The partition transformation group $(X/\Lambda, T)$ is called the *trace transformation group of* (X, T). The canonical map of X onto X/Λ is a homomorphism of (X, T) onto $(X/\Lambda, T)$ and is called the *trace homomorphism of* (X, T).

1.10. Remark. Let (X, T) be a transformation group where X is compact and T is discrete, and let $x \in X$. Then the following statements are equivalent:

(1) $x\Lambda$ is a singleton.

(2) x is a regularly almost periodic point under (X, T).

Proof. By [3, 5.04], H is a filter base on T and therefore $\{\overline{xH} \mid H \in \mathscr{H}\}$ is a filter base on X.

Of course, (2) implies (1) even when X is not assumed compact and T is not assumed discrete. However, (1) does not imply (2) when T is not assumed discrete. For an example, consider a "straight line" minimal continuous flow on the 2-torus. Compare 1.08 (6).

1.11. Remark. Let (X, T) be a transformation group where X is compact and T is discrete. Then the following statements are equivalent:

(1) (X, T) has singleton traces.

(2) (X, T) is pointwise regularly almost periodic.

Proof. Use 1.10.

The last paragraph of 1.10 applies here word for word.

1.12. Remark. Let (X, T) be an equicontinuous transformation group where X is compact, and let $\mathscr{E}$ be a partition of X into open-closed subsets of X. Then there exists $H \in \mathscr{H}$ such that $E \in \mathscr{E}$ implies $EH = E$.

Proof. We assume without loss of generality that T is discrete.

Define $\alpha = \bigcup_{E \in \mathscr{E}} E \times E$. Now α is an index of X such that $E \in \mathscr{E}$ implies $E\alpha = E$. By [3, 4.38] there exists a (left) syndetic subset A of T such that $x \in X$ implies $xA \subset x\alpha$. Then $EA \subset E\alpha = E$ for all $E \in \mathscr{E}$. If $t \in A$, then $\{Et \mid E \in \mathscr{E}\}$ is a partition of X such that $E \in \mathscr{E}$ implies $Et \subset E$. Therefore, if $t \in A$ and if $E \in \mathscr{E}$, then $Et = E$ and $Et^{-1} = E$. Thus $E \in \mathscr{E}$ implies $E(A \cup A^{-1}) = E$. Define H_0 to be the subgroup of T generated by A whence H_0 is a syndetic subgroup of T and $EH_0 = E$ for all $E \in \mathscr{E}$. By [3, 5.04] there exists $H \in \mathscr{H}$ such that $H \subset H_0$. Hence $E \in \mathscr{E}$ implies $EH = E$.

1.13. REMARK. Let (X, T) be a transformation group where X is compact zero-dimensional. Then (X, T) is regularly almost periodic if and only if (X, T) is equicontinous.

Proof. Use 1.12 and [3, 4.38].

1.14. REMARK. Let (X, T) be an equicontinuous totally minimal transformation group where X is compact. Then X is connected.

Proof. Use 1.12.

1.15. THEOREM. *Let (X, T) be a minimal transformation group where X is compact and T is discrete. Then*:

(1) *Λ is the least invariant closed equivalence relation R in X such that the partition transformation group $(X/R, T)$ is pointwise regularly almost periodic.*

(2) *Λ is the least invariant closed equivalence relation R in X such that the partition transformation group $(X/R, T)$ is regularly almost periodic.*

(3) *$\Lambda = \Sigma$ if and only if X/Σ is zero-dimensional.*

(4) *$\Lambda = X \times X$ if and only if X/Σ is connected.*

(5) *If there exists $x_0 \in X$ such that $x_0\Lambda \subset x_0\Pi^*$, then $\Lambda = \Sigma = \Pi^*$.*

(6) *If $\mathscr{F}$ is a filter base of syndetic invariant subgroups of T, and if there exists $x_0 \in X$ such that $x_0\Lambda \supset \bigcap_{F \in \mathscr{F}} \overline{x_0F}$, then $x \in X$ implies $x\Lambda = \bigcap_{F \in \mathscr{F}} \overline{xF}$.*

Proof. (1) Use 1.08 (1) and 1.11.

(2) Use (1) and [3, 5.18].

(3) Suppose X/Σ is zero-dimensional. By 1.13, $(X/\Sigma, T)$ is regularly almost periodic. By (2), $\Lambda \subset \Sigma$. Since $\Lambda \supset \Sigma$ by 1.08 (3), we have that $\Lambda = \Sigma$.

Suppose $\Lambda = \Sigma$. By (2), $(X/\Lambda, T)$ is regularly almost periodic. Since $(X/\Lambda, T)$ is minimal, it follows from [3, 5.08] that $X/\Sigma = X/\Lambda$ is zero-dimensional.

(4) Suppose X/Σ is connected. Since $\Sigma \subset \Lambda$ by 1.08 (3), X/Λ is a continuous image of X/Σ and X/Λ is therefore connected. Since $(X/\Lambda, T)$ is minimal, it follows from [3, 2.28] that $(X/\Lambda, T)$ is totally minimal so that X/Λ is a trace (indeed, the only trace) of $(X/\Lambda, T)$. By 1.08 (1), $(X/\Lambda, T)$ has singleton traces whence the cardinal of X/Λ is 1 and $\Lambda = X \times X$.

Suppose $\Lambda = X \times X$, that is, (X, T) is totally minimal. By 1.04 (2), $(X/\Sigma, T)$ is totally minimal. By 1.14, X/Σ is connected.

(5) Let ϕ be the canonical homomorphism of (X, T) onto the partion transformation group $(X/\Pi^*, T)$. By [1, Theorem 2], $(X/\Pi^*, T)$ is distal whence $\Pi(X/\Pi^*, T) = \Delta(X/\Pi^*)$. By 1.03 (2), $x_0\Lambda\phi = x_0\phi\Lambda(X/\Pi^*, T)$. Since $x_0\Lambda \subset x_0\Pi^*$ and $x_0\Pi^*\phi$ is a singleton in X/Π^*, it follows that $x_0\Lambda\phi$ is a singleton in X/Π^* and therefore the trace of $x_0\phi$ under $(X/\Pi^*, T)$ is a singleton.

By 1.10, $(X/\Pi^*, T)$ is regularly almost periodic at $x_0\phi$. By 1.08 (4), $\Sigma(X/\Pi^*, T) = \Pi(X/\Pi^*, T) = \Delta(X/\Pi^*)$ and thus $(X/\Pi^*, T)$ is equicontinuous. Hence, using 1.08 (3.5), $\Lambda = \Sigma \subset \Pi^* \subset \Lambda$ and $\Lambda = \Sigma = \Pi^*$.

(6) By [3, 2.25], $H \in \mathscr{H}$ implies X/H is a finite partition of X. Let $x \in X$.

Let $H \in \mathscr{H}$. Now $\bigcap_{F \in \mathscr{F}} \overline{x_0 F} \subset \overline{x_0 H}$, $\{\overline{x_0 F} \mid F \in \mathscr{F}\}$ is a filter base on X, and $\overline{x_0 H}$ is an open subset of X. Hence there exists $F \in \mathscr{F}$ such that $\overline{x_0 F} \subset \overline{x_0 H}$. Since $\overline{x_0 F}$ is an open subset of X, there exists $t \in T$ such that $xt^{-1} \in \overline{x_0 F}$. Hence $x \in \overline{x_0 t F} \subset \overline{x_0 t H}$, $\overline{xF} = \overline{x_0 t F}$, $\overline{xH} = \overline{x_0 t H}$, and $\overline{xF} \subset \overline{xH}$, making use of [3, 2.24].

Therefore, if $H \in \mathscr{H}$, then there exists $F \in \mathscr{F}$ such that $\overline{xF} \subset \overline{xH}$. Hence $\bigcap_{H \in \mathscr{H}} \overline{xH} \supset \bigcap_{F \in \mathscr{F}} \overline{xF}$. Since $\mathscr{H} \supset \mathscr{F}$, it follows that $\bigcap_{H \in \mathscr{H}} \overline{xH} \subset \bigcap_{F \in \mathscr{F}} \overline{xF}$. Thus $x\Lambda = \bigcap_{H \in \mathscr{H}} \overline{xH} = \bigcap_{F \in \mathscr{F}} \overline{xF}$.

The proof is completed.

2. ψ-adic transformation groups.

2.01. Notation. Let I denote the set of all integers. For $n \in I$ let I_n denote $\{i \mid i \in I \ \& \ n \leqq i\}$. If A is a set, then crd A denotes the cardinal of A

2.02. Remark and definition. Let T be a discrete abelian group and let ψ be a group isomorphism of T into T such that $T\psi$ is syndetic in T and $T\psi \neq T$. Then:

(0) $T \supset T\psi \supset T\psi^2 \supset \cdots$.

(1) For each $i \in I_1$, $T\psi^i$ is a syndetic subgroup of T whence $T/T\psi^i$ is a finite abelian group.

Let K be a choice set of the quotient group $T/T\psi$ whence $2 \leqq \mathrm{crd}(T/T\psi) = \mathrm{crd}\, K < \aleph_0$. Then for each $i \in I_1$:

(2) $\prod_{j=0}^{i-1} K\psi^j$ is a choice set of $T/T\psi^i$.

(3) $\mathrm{crd}(T/T\psi^i) = (\mathrm{crd}\, K)^i$.

For each $i \in I_1$ let $T/T\psi^i$ be provided with its discrete topology. Let coordinatewise multiplication and the product topology be introduced in $\times_{i \in I_1} T/T\psi^i$. Then:

(4) $\times_{i \in I_1} T/T\psi^i$ is an abelian multiplicative topological group which is homeomorphic to the Cantor discontinuum.

For each $i \in I_1$ let h_i be the canonical group homomorphism of $T/T\psi^{i+1}$ onto $T/T\psi^i$ whence $\tau \in T$ implies $(T\psi^{i+1} \cdot \tau)h_i = T\psi^i \cdot \tau$. Then for each $i \in I_1$:

(5) If $\tau \in T$, then

$$(T\psi^i \cdot \tau)h_i^{-1} = \{T\psi^{i+1} \cdot k\psi^i \cdot \tau \mid k \in K\}.$$

(6) h_i is exactly $(\mathrm{crd}\, K)$-to-1.

Let

$$T/\psi =_{\mathrm{df}} \operatorname*{inv\,lim}_{i \in I_1} (T/T\psi^i \xleftarrow[h_i]{} T/T\psi^{i+1}).$$

That is to say, T/ψ is the set of all sequences $(a_i \mid i \in I_1) \in \times_{i \in I_1} T/T\psi^i$ such that $a_i = a_{i+1}h_i$ for all $i \in I_1$. Then:

(7) T/ψ is the set of all sequences $(T\psi^i \cdot \tau_i \mid i \in I_1)$ such that $\tau_i \in T$ for all $i \in I_1$ and $\tau_i \tau_{i+1}^{-1} \in T\psi^i$ for all $i \in I_1$.

(8) T/ψ is the closure in $\times_{i \in I_1} T/T\psi^i$ of the image of T under the canonical homomorphism of T into $\times_{i \in I_1} T/T\psi^i$, which maps each $\tau \in T$ into $(T\psi^i \cdot \tau \mid i \in I_1)$.

(9) T/ψ is a closed subgroup of $\times_{i \in I_1} T/T\psi^i$.

(10) T/ψ is an abelian multiplicative topological group which is homeomorphic to the Cantor discontinuum.

The topological group T/ψ is called the *ψ-adic group.*

For each $i \in I_1$ let $(T/T\psi^i, T)$ be the coset transformation group on $T/T\psi^i$ under T whence $\tau, t \in T$ implies $(T\psi^i \cdot \tau)t = T\psi^i \cdot (\tau t)$. Let $(\times_{i \in I_1} T/T\psi^i, T)$ be the product transformation group of the sequence $((T/T\psi^i, T) \mid i \in I_1)$ whence $(a_i \mid i \in I_1) \in \times_{i \in I_1} T/T\psi^i$ and $t \in T$ implies $(a_i \mid i \in I_1)\, t = (a_i t \mid i \in I_1)$. Then:

(11) $(\times_{i \in I_1} T/T\psi^i, T)$ is also the homomorphism transformation group on $\times_{i \in I_1} T/T\psi^i$ under T induced by the canonical homomorphism of T into $\times_{i \in I_1} T/T\psi^i$, which maps each $\tau \in T$ into $(T\psi^i \cdot \tau \mid i \in I_1)$.

(12) T/ψ is the orbit-closure of the identity element of the group $\times_{i \in I_1} T/T\psi^i$ under the regularly almost periodic product transformation group

$$(\times_{i \in I_1} T/T\psi^i, T).$$

(13) The subset transformation group $(T/\psi, T)$ of the regularity almost periodic product transformation group $(\times_{i \in I_1} T/T\psi^i, T)$ is a regularly almost periodic minimal transformation group.

(14) $(T/\psi, T)$ is also the homomorphism transformation group on T/ψ under T induced by the canonical homomorphism of T into T/ψ, which maps each $\tau \in T$ into $(T\psi^i \cdot \tau \mid i \in I_1)$.

(15) For each $i \in I_1$, h_i is a homomorphism of the transformation group $(T/T\psi^{i+1}, T)$ onto the transformation group $(T/T\psi^i, T)$.

(16) $(T/\psi, T)$ is also the inverse limit transformation group

$$\operatorname*{inv\,lim}_{i \in I_1}((T/T\psi^i, T) \xleftarrow[h_i]{} (T/T\psi^{i+1}, T)).$$

The regularly almost periodic minimal transformation group $(T/\psi, T)$ is called the *ψ-adic transformation group.*

2.03. Definition. Let $n \in I_2$, let T be the discrete additive group I of all integers and let ψ be the group isomorphism of T into T such that $t \in T$ implies $t\psi = nt$. Then, the ψ-adic group T/ψ is familiarly called the *n-adic group* and is denoted D_n; the ψ-adic transformation group $(T/\psi, T)$ is called the *n-adic minimal set* and is denoted (D_n, τ_n) where τ_n is the transition of $(T/\psi, T)$ corresponding to $1 \in T$.

2.04. Notation. Let (X, T) and (Y, T) be transformation groups. The statement that (X, T) is isomorphic to (Y, T) is denoted $(X, T) \simeq (Y, T)$. The statement that (X, T) is homomorphic to (Y, T), that is, that there exists a homomorphism of (X, T) onto (Y, T), is denoted $(X, T) \overset{\simeq}{\to} (Y, T)$.

2.05. THEOREM. *Let (X, T) be a minimal transformation group where X is compact and T is discrete abelian, let ψ be a group isomorphism of T into T such that $T\psi$ is syndetic in T and $T\psi \neq T$, let* crd $(X/T\psi) =$ crd $(T/T\psi)$, *let ϕ be a homeomorphism of X into X such that $(x, t) \in X \times T$ implies $xt\phi = (x\phi)(t\psi)$, and define $Y = \bigcap_{i \in I_0} X\phi^i$. Then*:

(1) $(X/\Lambda, T) \overset{\simeq}{\to} (T/\psi, T)$.

(2) *If $Y \times Y \subset \Lambda$, then $(X/\Lambda, T) \simeq (T/\psi, T)$ and $yt\Lambda = Yt$ for all $y \in Y$ and all $t \in T$.*

(3) *If $Y \times Y \subset \Sigma$, then $\Lambda = \Sigma$, $(X/\Sigma, T) \simeq (T/\psi, T)$, and $\mathscr{G}(X, T) \simeq T/\psi$.*

(4) *If $Y \times Y \subset \Pi^*$, then $\Lambda = \Sigma = \Pi^*$.*

Proof. Let K be a choice set of $T/T\psi$ and for each $i \in I_1$ let $K_i =_{\mathrm{df}} \prod_{j=0}^{i-1} K\psi^j$. We prove:

(i) For each $i \in I_1$, $X/T\psi^i = \{X\phi^i t \mid t \in T\} = \{X\phi^i k \mid k \in K_i\}$ is a partition of X.

Let $i \in I_1$. By 2.02 (1) and [3, 2.24], $X/T\psi^i$ is a partition of X. If $x \in X$, then $X\phi^i = \overline{xT}\phi^i = [(xT)\phi^i]^- = [x\phi^i(T\psi^i)]^-$. Hence $X\phi^i \in X/T\psi^i$. Clearly then

$$\{X\phi^i k \mid k \in K_i\} \subset \{X\phi^i t \mid t \in T\} \subset X/T\psi^i.$$

By 2.02 (2), $T = T\psi^i \cdot K_i$. It follows from [3, 1.18 (6)] that if $x \in X$, then $X = [x\phi^i T]^- = [x\phi^i(T\psi^i)K_i]^- = [x\phi^i(T\psi^i)]^- K_i = [(xT)\phi^i]^- K_i = \overline{xT}\phi^i K_i = X\phi^i K_i$. The proof of (i) is completed.

We prove by induction:

(ii) For each $i \in I_1$, $(X\phi^i k \mid k \in K_i)$ is a partition of X.

By hypothesis and (i), crd $K =$ crd $(T/T\psi) =$ crd $(X/T\psi) =$ crd $\{X\phi k \mid k \in K\}$ whence $(X\phi k \mid k \in K)$ is a partition of X. Let $i \in I_1$ and assume $(X\phi^i k_0 \mid k_0 \in K_i)$ is a partition of X. Since $(X\phi k_1 \mid k_1 \in K)$ is a partition of X and ϕ^i is one-to-one, it follows that $((X\phi k_1)\phi^i \mid k_1 \in K) = (X\phi^{i+1}(k_1\psi^i) \mid k_1 \in K)$ is a partition of $X\phi^i$. Thus if $k_0 \in K_i$, then $(X\phi^{i+1}k_0(k_1\psi^i) \mid k_1 \in K)$ is a partition of $X\phi^i k_0$. Since $(X\phi^i k_0 \mid k_0 \in K_i)$ is a partition of X by the induction assumption it follows that

$$(X\phi^{i+1}k_0(k_1\psi^i) \mid (k_0, k_1) \in K_i \times K)$$

is a partition of X and therefore $(X\phi^{i+1}k \mid k \in K_{i+1})$ is a partition of X. The proof of (ii) is completed.

We prove:

(iii) If $t_1, t_2 \in T$ and if $i \in I_1$, then $X\phi\, t_1 = X\phi^i t_2$ if and only if $T\psi^i \cdot t_1 = T\psi^i \cdot t_2$.

If $T\psi^i \cdot t_1 = T\psi^i \cdot t_2$ and if $x \in X$, then

$$\begin{aligned} X\phi^i t_1 &= \overline{xT}\,\phi^i t_1 = [(xT)\phi^i]^- t_1 = [(x\phi^i)(T\psi^i)]^- t_1 \\ &= [(x\phi^i)(T\psi^i)t_1]^- = [(x\phi^i)(T\psi^i)t_2]^- = [(x\phi^i)(T\psi^i)]^- t_2 \\ &= [(xT)\phi^i]^- t_2 = \overline{xT}\,\phi^i t_2 = X\phi^i t_2. \end{aligned}$$

Now suppose $X\phi^i t_1 = X\phi^i t_2$. Since $T = T\psi^i \cdot K_i$ by 2.02 (2), there exists $s_1, s_2 \in T\psi^i$ and $k_1, k_2 \in K_i$ such that $t_1 = s_1 k_1$ and $t_2 = s_2 k_2$. Since $X\phi^i \in X/T\psi^i$, it follows that $X\phi^i k_1 = X\phi^i s_1 k_1 = X\phi^i t_1 = X\phi^i t_2 = X\phi^i s_2 k_2 = X\phi^i k_2$. By (ii), $k_1 = k_2$ whence $T\psi^i \cdot t_1 = T\psi^i \cdot s_1 k_1 = T\psi^i \cdot k_1 = T\psi^i \cdot k_2 = T\psi^i \cdot s_2 k_2 = T\psi^i \cdot t_2$. The proof of (iii) is completed.

It is immediate from (ii) and (i) that:

(iv) If $x \in X$ and if $i \in I_1$, then there exists a unique $f(x,i) \in K_i$ such that $x \in X\phi^i f(x,i)$, whence $\overline{xT\psi^i} = X\phi^i f(x,i)$.

It is immediate from 2.02 (2), (iii) and (iv) that:

(v) If $x, y \in X$ and if $i \in I_1$, then the following statements are pairwise equivalent:

$$f(x,i) = f(y,i),\ T\psi^i \cdot f(x,i) = T\psi^i \cdot f(y,i),$$

$$X\phi^i f(x,i) = X\phi^i f(y,i),\ \overline{xT\psi^i} = \overline{yT\psi^i}.$$

We prove:

(vi) There exists a unique map ξ of X into the ψ-adic group T/ψ such that $x \in X$ implies $x\xi = (T\psi^i \cdot f(x,i) \mid i \in I_1)$.

Let $x \in X$. It is enough to show $(T\psi^i \cdot f(x,i) \mid i \in I_1) \in T/\psi$. If $i \in I_1$, then $x \in X\phi^i f(x, i)$ and $x \in X\phi^{i+1} f(x, i+1) \subset X\phi^i f(x, i+1)$ whence $X\phi^i f(x,i) = X\phi^i f(x, i+1)$ by (i) and $f(x,i) f(x, i+1)^{-1} \in T\psi^i$ by (iii). The proof of (vi) is completed.

We prove:

(vii) ξ is a homomorphism of the transformation group (X, T) onto the ψ-adic transformation group $(T/\psi, T)$.

The continuity of ξ follows from (v) and the fact that for each $i \in I_1$, $X/T\psi^i$ is a finite partition of X into closed subsets of X.

We show ξ is onto T/ψ. Let $(\tau_i \mid i \in I_1)$ be a sequence in T such that $(T\psi^i \cdot \tau_i \mid i \in I_1) \in T/\psi$, that is, $\tau_i \tau_{i+1}^{-1} \in T\psi^i$ for all $i \in I_1$. Then it follows from (iii) that for each $i \in I_1$, $X\phi^i \tau_i = X\phi^i \tau_{i+1} \supset X\phi^{i+1} \tau_{i+1}$. Hence there exists $x \in \bigcap_{i \in I_1} X\phi^i \tau_i$. It follows from (i), (iii) and (iv) that for each $i \in I_1$, $x \in X\phi_i \tau_i$, $x \in X\phi^i f(x, i)$, $X\phi^i \tau_i = X\phi^i f(x, i)$, and $T\psi^i \cdot \tau_i = T\psi^i \cdot f(x, i)$. Thus $x\xi = (T\psi^i \cdot f(x,i) \mid i \in I_1) = (T\psi^i \cdot \tau_i \mid i \in I_1)$.

Let $x \in X$ and let $t \in T$. If $i \in I_1$, then $x \in X\phi^i f(x,i)$, $xt \in X\phi^i f(x,i)t$, $xt \in X\phi^i f(xt,i)$, $X\phi^i f(x,i)t = X\phi^i f(xt,i)$, and $T\psi^i \cdot f(x,i)t = T\psi^i \cdot f(xt,i)$ by (iii). Now

$$\begin{aligned} (xt)\xi &= T\psi^i \cdot f(xt,i) \mid i \in I_1) \\ &= (T\psi^i \cdot f(x,i)t \mid i \in I_1) \\ &= (T\psi^i \cdot f(x,i) \mid i \in I_1)t = (x\xi)t. \end{aligned}$$

The proof of (vii) is completed.

We prove:

(viii) If $x \in X$, then $x\Lambda \subset \bigcap_{i \in I_1} \overline{xT\psi^i} = x\xi\xi^{-1}$.

The inclusion in (viii) is obvious from 2.02 (1). If $y \in X$, then

$$\begin{aligned} y \in \bigcap_{i \in I_1} \overline{xT\psi^i} &\Leftrightarrow \forall\ i \in I_1 \cdot y \in \overline{xT\psi^i} \\ &\Leftrightarrow \forall\ i \in I_1 \cdot \overline{yT\psi^i} = \overline{xT\psi^i} \text{ by (i)} \\ &\Leftrightarrow \forall\ i \in I_1 \cdot T\psi^i \cdot f(y,i) = T\psi^i \cdot f(x,i) \text{ by (v)} \\ &\Leftrightarrow (T\psi^i \cdot f(y,i) \mid i \in I_1) = (T\psi^i \cdot f(x,i) \mid i \in I_1) \\ &\Leftrightarrow y\xi = x\xi \\ &\Leftrightarrow y \in x\xi\xi^{-1}. \end{aligned}$$

The proof of (viii) is completed.

From (vii) and (viii) it follows immediately that:

(ix) There exists a unique map η of the trace partition X/Λ into the ψ-adic group T/ψ such that $x \in X$ implies $(x\Lambda)\eta = x\xi$; η is a homomorphism of the trace transformation group $(X/\Lambda, T)$ onto the ψ-adic transformation group $(T/\psi, T)$.

The proof of (1) is completed.

We prove:

(x) If $y \in Y$, then $Y = \bigcap_{i \in I_1} \overline{yT\psi^i}$.

If $i \in I_1$, then there exists $x \in X$ such that $x\phi^i = y$ whence

$$(xT)\phi^i = (x\phi^i)(T\psi^i) = yT\psi^i \text{ and } X\phi^i = \overline{xT}\phi^i = \overline{(xT)\phi^i} = \overline{yT\psi^i}.$$

Hence, $Y = \bigcap_{i \in I_1} X\phi^i = \bigcap_{i \in I_1} \overline{yT\psi^i}$. The proof of (x) is completed.

We prove (2). Suppose $Y \times Y \subset \Lambda$.

Let $y \in Y$. By (x), $y\Lambda \supset y(Y \times Y) = Y = \bigcap_{i \in I_1} \overline{yT\psi^i} \supset y\Lambda$. Hence $y\Lambda = Y$ and $yt\Lambda = y\Lambda t = Yt$ for all $t \in T$. By 1.15 (6) and (viii), $x \in X$ implies $x\Lambda = \bigcap_{i \in I_1} \overline{xT\psi^i} = x\xi\xi^{-1}$. Hence η is one-to-one and (2) is proved.

We prove (3). Suppose $Y \times Y \subset \Sigma$. Choose $y \in Y$. By (x), $y\Lambda \subset Y = y(Y \times Y) \subset y\Sigma$. By 1.08 (5), $\Lambda = \Sigma$ and (3) follows from (2).

We prove (4). Suppose $Y \times Y \subset \Pi^*$. Choose $y \in Y$. By (x), $y\Lambda \subset Y = y(Y \times Y) \subset y\Pi^*$. By 1.15 (5), $\Lambda = \Sigma = \Pi^*$ and (4) is proved.

The proof of the theorem is completed.

3. Substitutions.

3.01. Notation. For this paragraph let S be a partially ordered set. For a, $b \in S$, let

$$S[a,b] = {}_{\mathrm{df}}\{x \mid x \in S \ \&\ a \leqq x \leqq b\},$$

let

$$S[a,\rightarrow] = {}_{\mathrm{df}}\{x \mid x \in S \ \&\ a \leqq x\},$$

and let

$$S[\leftarrow, b] = {}_{\mathrm{df}}\{x \mid x \in S \;\&\; x \leqq b\}.$$

For $a \in S$ we also denote $S[a, \rightarrow]$ by S_a.

Let I denote the set of all integers. An *interval of* I is defined to be a nonvacuous subset E of I such that $i, j \in E$ implies $I[i,j] \subset E$. Evidently, an interval of I is a finite interval $I[a,b]$ for unique a, $b \in I$ with $a \leqq b$, or a right ray $I[a,\rightarrow]$ for unique $a \in I$, or a left ray $I[\leftarrow, b]$ for unique $b \in I$, or I itself.

The cardinal of a set E is denoted $\mathrm{crd}\, E$. The domain of a function f is denoted $\mathrm{dmn} f$, and the range of f is denoted $\mathrm{rng}\, f$. If f is a function and if $x \in \mathrm{dmn} f$, then the value of f at x is variously denoted by $f(x)$ or fx or f_x or xf. If f is a function and if $E \subset \mathrm{dmn} f$, then $f \mid E$ denotes the restriction of f to E.

3.02. STANDING HYPOTHESIS 1. Let $m \in I_1$ and let P be a set such that $\mathrm{crd}\, P = m$.

3.03. DEFINITION (SYNTAX). Since only the cardinal of P is of significance here, a convenient choice for P is $P = I[0, m-1]$. The set P may be called the *symbol set*, and an element of P may be called a *symbol*.

A *string over* P is defined to be a function on an interval of I to P. Let $\mathscr{S}$ or $\mathscr{S}(P)$ denote the set of all strings over P. If $A \in \mathscr{S}$ and if $i \in \mathrm{dmn}\, A$, then the value of A at i is denoted A_i or $A(i)$. If $A \in \mathscr{S}$, then the *length of* A is defined to be $\mathrm{crd}\, A$ or equally $\mathrm{crd}\,\mathrm{dmn}\, A$. If $k \in I_1$, then a *k-string over P* is defined to be a string over P whose length is k. A string A over P is said to be *finite* or *infinite* according as the length of A is finite or infinite, that is, according as the set A is finite or infinite.

If $k \in I_1$, if $p_i \in P$ for each $i \in I[1,k]$, if $\kappa \in I[1,k]$, and if $\omega \in I$, then the (κ,ω)-*indexed juxtaposition* (or *concatenation*) *of* $p_1, \cdots, p_k$, denoted $p_1 \cdots p_\kappa(\uparrow\omega) \cdots p_k$,(2) is defined to be the k-string

$$\{(\omega - \kappa + 1, p_1), \cdots, (\omega, p_\kappa), \cdots, (\omega - \kappa + k, p_k)\}$$

over P; in case $\omega = 0$, we may write $p_1 \cdots \dot{p}_\kappa \cdots p_k$ for $p_1 \cdots p_\kappa(\uparrow 0) \cdots p_k$; in case $\kappa = 1$ and $\omega = 0$, we may write $p_1 \cdots p_k$ for $\dot{p}_1 \cdots p_k$. If A is a finite string over P and if $\mathrm{dmn}\, A = I[c,d]$ where c, $d \in I$ with $c \leqq d$, then we note that $A = A_c(\uparrow c) A_{c+1} \cdots A_d = A_c \cdots A_{d-1} A_d(\uparrow d)$.

If $k \in I_1$, if A^i is a finite string over P for each $i \in I[1,k]$, if $\mathrm{dmn}\, A^i = I[c_i, d_i]$ where c_i, $d_i \in I$ with $c_i \leqq d_i$ for each $i \in I[1,k]$, if $\kappa \in I[1,k]$, and if $\omega \in I$, then the (κ,ω)-*indexed juxtaposition* (or *concatenation*) *of* $A^1, \cdots, A^k$, denoted $A^1 \cdots A^\kappa(\uparrow\omega) \cdots A^k$, is defined to be the $(\mathrm{crd}\, A^1 + \cdots + \mathrm{crd}\, A^k)$-string $A^1_{c_1} \cdots A^1_{d_1} \cdots A^\kappa_c (\uparrow\omega) \cdots A^\kappa_{d_\kappa} \cdots A^k_{c_k} \cdots A^k_{d_k}$ over P: in case $\omega = 0$, we may write $A^i \cdots \dot{A}^\kappa \cdots A^k$ for $A^1 \cdots A^\kappa(\uparrow 0) \cdots A^k$; in case $\kappa = 1$ and $\omega = 0$, we may write $A^1 \cdots A^k$ for $\dot{A}^1 \cdots A^k$.

(2) In the notation $p_\kappa(\uparrow\omega)$, $(\uparrow\omega)$ means ω should be written directly above the p.

Extensions of the above notation to certain (perhaps infinite) families of symbols and (perhaps infinite) strings over P are clear.

For this paragraph let A, B, $C \in \mathscr{S}$. If $k \in I$, then the *k-translate of A* or *the translate of A by k*, denoted $k + A$ or $A + k$, is defined to be the string $\{(k + i, p) \mid (i, p) \in A\}$ over P, whence $\operatorname{dmn}(k + A) = k + \operatorname{dmn} A$ and $i \in \operatorname{dmn} A$ implies $(k + A)_{k+i} = A_i$. The strings A and B are said to be *congruent* (the first *to* the second), this statement being denoted $A \sim B$, provided there exists $k \in I$ such that $k + A = B$. Congruence $\sim$ of strings over P is clearly an equivalence relation in $\mathscr{S}$. A *substring of A* is defined to be a string over P which is a subset of A. Thus the following statements are pairwise equivalent: (i) A is a substring of B; (ii) $A \subset B$; (iii) $\operatorname{dmn} A \subset \operatorname{dmn} B$ and $A = B \mid \operatorname{dmn} A$. The string A is said to *occur in* B, this statement being denoted $A \leqq B$, provided that A is congruent to a substring of B. In the present context the sign $\leqq$ is to be read "occurs in." An *occurrence of A in B* is defined to be a substring of B which is congruent to A. Thus A occurs in B if and only if there exists an occurrence of A in B. If $a =_{\mathrm{df}} \operatorname{crd} A < \aleph_0$, if $b =_{\mathrm{df}} \operatorname{crd} B$, and if $a \leqq b$, then the number of occurences of A in B is at most $b - a + 1$. Clearly,

(1) $A \leqq A$,

(2) $A \leqq B \;\&\; B \leqq C \Rightarrow A \leqq C$,

(3) $A \sim B \Rightarrow A \leqq B \;\&\; B \leqq A$,

(4) $\operatorname{crd} A < \aleph_0 \Rightarrow (A \leqq B \;\&\; B \leqq A \Leftrightarrow A \sim B)$.

Thus the relation $\leqq$ is reflexive transitive (but not antisymmetric) in $\mathscr{S}$.

A string A over P is said to be *medial* provided that $0 \in \operatorname{dmn} A$. Let $\mathscr{S}_0$ or $\mathscr{S}_0(P)$ denote the set of all medial strings over P. *A block over P* is defined to be a string A over P such that $\operatorname{dmn} A = I[0, k - 1]$ for some $k \in I_1$. Thus a block over P is exactly a finite medial string over P whose domain is contained in I_0. If $k \in I_1$, then *a k-block over P* is defined to be a k-string over P which is also a block over P, that is, a function on $I[0, k - 1]$ to P. For each $k \in I_1$ let P^k denote the set of all k-blocks over P. Thus if $k \in I_1$, then P^k is the set of all functions on k to P where $k =_{\mathrm{df}} \{0, 1, \cdots, k - 1\}$ according to a familiar definitional scheme of the natural numbers. If $k \in I_1$ and if $p_i \in P$ for each $i \in I[1, k]$, then according to previous notational agreement $p_1 \cdots p_k$ denotes the k-block

$$p_1 \cdots p_k = \{(0, p_1), \cdots, (k - 1, p_k)\}$$

over P. We observe that if $k \in I_1$ and if $A \in P^k$, then $A = A_0 \cdots A_{k-1}$. Let P^+ denote $\bigcup_{i \in I_1} P^i$ whence P^+ is the set of all blocks over P. We note that the relation $\leqq$ (occurs in) is reflexive antisymmetric transitive in P^+ (and thus, restricted to P^+, is a partial ordering in P^+. If $A \in P^+$, then a *subblock of A* is defined to be a block over P which occurs in A.

If $A \in \mathscr{S}$, if $i, j \in I$ with $i \leqq j$, and if $I[i,j] \subset \operatorname{dmn} A$, then $A \mid [i,j]$ denotes the substring $A \mid I[i,j]$ of A, and $A[i,j]$ denotes the $(j - i + 1)$-block $A_i\, A_{i+1} \cdots A_j$ over P.

A *bisequence over* P is defined to be a string over P whose domain is I, that is a function on I to P. Let X or $X(P)$ denote the set of all bisequences over P. The set X is called the *bisequence set over* P. In a customary notation $X = P^I$, namely, the set of all functions on I to P.

3.04. Definition. Let P^* denote $\bigcup_{i \in I_2} P^i$ whence P^* is the set of all blocks over P which are of length 2 or more. A *substitution over* P is defined to be a map of P into P^*. If θ is a substitution over P, then the *length of* θ is defined to be the family $(\operatorname{crd}(p\theta) \mid p \in P)$.

3.05. Standing hypothesis 2. Let $m \in I_1$, let P be a set such that $\operatorname{crd} P = m$, let θ be a substitution over P, and let $(n_p \mid p \in P)$ be the length of θ. (For convenience we repeat all earlier standing hypotheses in each standing hypothesis.)

3.06. Notation. Let λ or $\lambda(\theta)$ denote the map of P^2 into P^2 such that $pq \in P^2$ implies $pq\lambda = (p\theta)_{n_p - 1}(q\theta)_0$; here $(p\theta)_{n_p-1}$ is the last symbol in $p\theta$ and $(q\theta)_0$ is the first symbol in $q\theta$. Let L or $L(\theta)$ denote the set of all $pq \in P^2$ such that there exists $i \in I_1$ such that $pq\lambda^i = pq$.

3.07. Remark. The following statements are valid:

(1) $L \neq \emptyset$.

(2) L coincides with the set of all points of P^2 which are periodic under λ.

(3) $L\lambda = L$ and $\lambda \mid L$ is a one-to-one map of L onto L.

(4) $\lambda \mid L$ is periodic, that is, there exists $i \in I_1$ such that $pq \in L$ implies $pq\lambda^i = pq$.

(5) $P^2 \supset P^2\lambda \supset P^2\lambda^2 \supset P^2\lambda^3 \supset \cdots, L = \bigcap_{i \in I_0} P^2\lambda^i$, and there exists $k \in I_1$ such that $i \in I$ with $i \geqq k$ implies $P^2\lambda^i = L$.

Proof. The above statements all follow easily from the fact that P^2 is finite whence $pq \in P^2$ implies the semi-orbit $\{pq\lambda^i \mid i \in I_0\}$ of pq under λ is finite.

3.08. Notation. Let μ or $\mu(\theta)$ denote the period of $\lambda \mid L$, that is, μ is the least $i \in I_1$ such that $pq \in L$ implies $pq\lambda^i = pq$. Let ν or $\nu(\theta)$ denote the least $i \in I_0$ such that $pq \in P^2$ implies $pq\lambda^i \in L$.

3.09. Remark. If for each $pq \in L$ we let μ_{pq} denote the least $i \in I_1$ such that $pq\lambda^i = pq$, then $\mu = \operatorname{lcm}\{\mu_{pq} \mid pq \in L\}$. Here lcm means "least common multiple." If for each $pq \in P^2$ we let ν_{pq} denote the least $i \in I_0$ such that $pq\lambda^i \in L$, then $\nu = \max\{\nu_{pq} \mid pq \in P^2\}$.

3.10. Notation. If $A \in \mathscr{S}_0$, then $A\theta$ denotes the medial string $\cdots(A_{-1}\theta)\ (A_0\theta)^{\cdot}\ (A_1\theta)\cdots$ over P.

If $i \in I_1$, then θ^i is defined to be the map of P into P^* which is determined recursively as follows:

(1) If $p \in P$, then $p\theta^1 = p\theta$.

(2) If $p \in P$ and if $j \in I[2, i]$, then $p\theta^j = (p\theta^{j-1})\theta$.

Thus if $i \in I_1$, then θ^i is again a substitution over P. For the sake of technical convenience let θ^0 be the identity map of P.

If $i \in I_1$, then θ^i is a substitution over P and $A\theta^i$ is now defined as a medial string over P for every medial string A over P. If A is a medial string over P, then $A\theta^0$ denotes A by convention.

3.11. REMARK. The following statements are valid:

(1) If $A \in \mathscr{S}_0$ and if $i, j \in I_0$, then $(A\theta^i)\theta^j = A\theta^{i+j}$ and $A(\theta^i)^j = A\theta^{ij}$.

(2) If $A, B \in \mathscr{S}_0$, $A \subset B$, and if $i \in I_0$, then $A\theta^i \subset B\theta^i$.

(3) If $A, B \in \mathscr{S}_0$, if $A \leqq B$, and if $i \in I_0$, then $A\theta^i \leqq B\theta^i$.

(4) If $A \in \mathscr{S}_0$, if $i \in I_0$, and if $A \subset A\theta^i$, then

$$A \subset A\theta^i \subset A\theta^{2i} \subset A\theta^{3i} \subset \cdots.$$

(5) If $A \in \mathscr{S}_0$, if $i \in I_0$, and if $A \leqq A\theta^i$, then

$$A \leqq A\theta^i \leqq A\theta^{2i} \leqq A\theta^{3i} \leqq \cdots.$$

(6) If $A \in P^+$, then $A\theta \in P^* \subset P^+$.

(7) If $pq, rs \in P^2$, and if $i \in I_0$, then $pq\lambda^i = rs$ if and only if $p\dot{q}\theta^i \supset r\dot{s}$.

(8) If $pq \in P^2$ and if $i \in I_0$, then $pq\lambda^i \leqq pq\theta^i$.

3.12. NOTATION. If $pq \in L$, then $pq\lambda^\mu = pq$ whence $p\dot{q} \subset p\dot{q}\theta^\mu \subset p\dot{q}\theta^{2\mu} \subset p\dot{q}\theta^{3\mu} \subset \cdots$ and we may define w_{pq} or $w_{pq}(\theta)$ to be $\bigcup_{i \in I_0} p\dot{q}\theta^{i\mu} \in X$. Let L_0 or $L_0(\theta)$ denote the set of all $pq \in L$ such that for each $r \in \operatorname{rng} w_{pq}$ there exists $i \in I_1$ such that $pq \leqq r\theta^i$. Let W or $W(\theta)$ denote $\{w_{pq} \mid pq \in L\}$. Let W_0 or $W_0(\theta)$ denote $\{w_{pq} \mid pq \in L_0\}$.

3.13. REMARK. If $pq, rs \in L$, then $w_{pq} = w_{rs}$ if and only if $pq = rs$. Thus $\operatorname{crd} W = \operatorname{crd} L$ and $\operatorname{crd} W_0 = \operatorname{crd} L_0$.

3.14. LEMMA. *If $pq \in L_0$ and if $r \in \operatorname{rng} w_{pq}$, then there exists $k \in I_1$ such that $pq \leqq r\theta^k$ and $r \leqq r\theta^k$.*

Proof. There exists $i \in I_1$ such that $pq \leqq r\theta^i$. Since $w_{pq} = \bigcup_{j \in I_1} p\dot{q}\theta^{j\mu}$, there exists $j \in I_1$ such that $r \leqq pq\theta^{j\mu}$. Thus $pq \leqq pq\theta^{j\mu} \leqq r\theta^{i+j\mu}$ and $r \leqq pq\theta^{j\mu} \leqq r\theta^{i+j\mu}$ Defining $k = i + j\mu \in I_1$ it follows that $pq \leqq r\theta^k$ and $r \leqq r\theta^k$. The proof is completed.

3.15. LEMMA. *If $pq \in L_0$, then there exists $k \in I_1$ such that $r \in \operatorname{rng} w_{pq}$ and $j \in I_1$ implies $pq \leqq r\theta^{jk}$.*

Proof. By 3.14 for each $r \in \operatorname{rng} w_{pq}$ there exists $k_r \in I_1$ such that $pq \leqq r\theta^{k_r}$ and $r \leqq r\theta^{k_r}$. Define $k = \operatorname{lcm}\{k_r \mid r \in \operatorname{rng} w_{pq}\}$. Here lcm means "least common multiple." Then $r \in \operatorname{rng} w_{pq}$ implies $pq \leqq r\theta^k$ and $r \leqq r\theta^k$. The conclusion follows.

3.16. REMARK. If $a \in I_1$, then:

(0) θ^a is a substitution over P.

(1) $\lambda(\theta^a) = \lambda(\theta)^a$.

(2) $L(\theta^a) = L(\theta)$.

(3) $\mu(\theta^a)$ divides $\mu(\theta)$.

(4) If $pq \in L(\theta^a) = L(\theta)$, then $w_{pq}(\theta^a) = w_{pq}(\theta)$.

(5) $L_0(\theta^a) = L_0(\theta)$.

(6) $W(\theta^a) = W(\theta)$.

(7) $W_0(\theta^a) = W_0(\theta)$.

3.17. REMARK. If Q is a nonvacuous subset of P such that $Q\theta \subset Q^*$, then:

(0) $\theta \mid Q$ is a substitution over Q.

(1) $\lambda(\theta \mid Q) = \lambda(\theta) \mid Q^2$.

(2) $L(\theta \mid Q) = L(\theta) \cap Q^2$.

(3) $\mu(\theta \mid Q)$ divides $\mu(\theta)$.

(4) If $pq \in L(\theta \mid Q) = L(\theta) \cap Q^2$, then $w_{pq}(\theta \mid Q) = w_{pq}(\theta)$.

(5) $L_0(\theta \mid Q) = L_0(\theta) \cap Q^2$.

(6) $W(\theta \mid Q) = W(\theta) \cap Q^I$.

(7) $W_0(\theta \mid Q) = W_0(\theta) \cap Q^I$.

3.18. DEFINITION. We shall find the following notions useful:

(1) θ is said to be *reflective* provided that $p \in P$ implies $p \leqq p\theta$.

(2) θ is said to be *laced* (that is, *interlaced*) provided that $p, q \in P$ implies $p \leqq q\theta$.

(3) θ is said to be *intersective* provided that $\bigcap_{p \in P} \operatorname{rng}(p\theta) \neq \emptyset$.

(4) θ is said to be *irreducible* provided that if Q is a nonvacuous subset of P such that $Q\theta \subset Q^*$, then $Q = P$.

3.19. LEMMA. *If $A \in P^*$, then there exists $pq \in L$ such that $pq \leqq A\theta^\nu$.*

Proof. There exists $rs \in P^2$ such that $rs \leqq A$. Let $pq =_{\mathrm{df}} rs\lambda^\nu$. Then $pq \in L$ and $pq = rs\lambda^\nu \leqq rs\theta^\nu \leqq A\theta^\nu$ by 3.11 (8). The proof is completed.

3.20. REMARK. If θ is reflective and if $i, j \in I_0$ with $i \leqq j$, then $p \in P$ implies $p\theta^i \leqq p\theta^j$.

3.21. LEMMA. *If θ is intersective, then there exists $pq \in L$ such that $r \in P$ implies $pq \leqq r\theta^{\nu+2}$.*

Proof. There exists $s \in \bigcap_{r \in P} \operatorname{rng}(r\theta)$. By 3.19 there exists $pq \in L$ such that $pq \leqq (s\theta)\theta^\nu$. If $r \in P$, then $s \leqq r\theta$ and $pq \leqq (s\theta)\theta^\nu = s\theta^{\nu+1} \leqq (r\theta)\theta^{\nu+1} = r\theta^{\nu+2}$ whence $pq \leqq r\theta^{\nu+2}$. The proof is completed.

3.22. LEMMA. *If θ is intersective, then $L_0 \neq \emptyset$.*

Proof. Use 3.21.

3.23. LEMMA. *If θ is irreducible, then for each $pq \in P^2$ there exists $i \in I_1$ such that $p \leqq q\theta^i$.*

Proof. Let $pq \in P^2$. Let $Q =_{\mathrm{df}} \bigcup_{i \in I_1} \operatorname{rng}(q\theta^i)$. It follows that $\emptyset \neq Q \subset P$ and $Q\theta \subset Q^*$ whence $Q = P$, $p \in Q$, and $p \leqq q\theta^i$ for some $i \in I_1$. The proof is completed.

3.24. LEMMA. *If θ is reflective irreducible, then there exists $k \in I_1$ such that θ^k is laced and therefore intersective.*

Proof. By 3.23 for each $pq \in P^2$ there exists $k_{pq} \in I_1$ such that $p \leqq q\theta^{k_{pq}}$. Let $k =_{\mathrm{df}} \max\{k_{pq} \mid pq \in P^2\}$. If $pq \in P^2$, then $q\theta^{k_{pq}} \leqq q\theta^k$ by 3.20. Hence $pq \in P^2$ implies $p \leqq q\theta^k$. The proof is completed.

3.25. Lemma. *If θ is reflective irreducible then $L_0 \neq \emptyset$.*

Proof. By 3.24 there exists $k \in I_1$ such that θ^k is intersective. By 3.22, $L_0(\theta^k) \neq \emptyset$. Since $L_0(\theta^k) = L_0(\theta)$ by 3.16 (5), it follows that $L_0(\theta) \neq \emptyset$. The proof is completed.

3.26. Lemma. *There exists a nonvacuous subset Q of P such that $Q\theta \subset Q^*$ and $\theta \mid Q$ is an irreducible substitution over Q.*

Proof. Let $\mathscr{A}$ be the class of all nonvacuous subsets Q of P such that $Q\theta \subset Q^*$. Now $\mathscr{A} \neq \emptyset$ since $P \in \mathscr{A}$. Let $\mathscr{A}$ be partially ordered by subset inclusion. Since P is finite, it follows that $\mathscr{A}$ is finite. Since $\mathscr{A}$ is finite nonvacuous, it follows there exists a minimal element Q_0 of $\mathscr{A}$. The set Q_0 has the desired properties. The proof is completed.

3.27. Lemma. *If θ is reflective, then $L_0 \neq \emptyset$.*

Proof. By 3.26 there exists a nonvacuous subset Q of P such that $Q\theta \subset Q^*$ and $\theta \mid Q$ is an irreducible substitution over Q. Clearly $\theta \mid Q$ is a reflective substitution over Q. By 3.25, $L_0(\theta \mid Q) \neq \emptyset$. Since $L_0(\theta \mid Q) = L_0(\theta) \cap Q^2$ by 3.17, it follows that $L_0(\theta) \neq \emptyset$. The proof is completed.

3.28. Lemma. *If θ is irreducible, then there exists $k \in I_1$ such that θ^k is reflective.*

Proof. By 3.23 for each $p \in P$ there exists $k_p \in I_1$ such that $p \leqq p\,\theta^{k_p}$. Let $k =_{\mathrm{df}} \mathrm{lcm}\,\{k_p \mid p \in P\}$. Here lcm means "least common multiple." By 3.11 (5), $p \in P$ implies $p\theta^{k_p} \leqq p\theta^k$. Hence $p \in P$ implies $p \leqq p\theta^k$; and θ^k is reflective. The proof is completed.

3.29. Lemma. *If θ is irreducible, then $L_0 \neq \emptyset$.*

Proof. By 3.28 there exists $k \in I_1$ such that θ^k is reflective. By 3.27, $L_0(\theta^k) \neq \emptyset$. Since $L_0(\theta^k) = L_0(\theta)$ by 3.16 (5), it follows that $L_0(\theta) \neq \emptyset$. The proof is completed.

3.30. Lemma. $L_0 \neq \emptyset$.

Proof. By 3.26 there exists a nonvacuous subset Q of P such that $Q\theta \subset Q^*$ and $\theta \mid Q$ is an irreducible substitution over Q. By 3.29, $L_0(\theta \mid Q) \neq \emptyset$. Since $L_0(\theta \mid Q) = L_0(\theta) \cap Q^2$ by 3.17 (5), it follows that $L_0(\theta) \neq \emptyset$. The proof is completed.

3.31. Notation. Let ϕ or $\phi(\theta)$ denote the map of X into X such that $x \in X$ implies $x\phi = x\theta$.

For each $pq \in P^2$ let

$$X_{pq} =_{\mathrm{df}} \{x \mid x \in X \;\&\, x_{-1} = p \,\&\, x_0 = q\} = \{x \mid x \in X \,\&\, x[-1,0] = pq\}.$$

3.32. Remark. The following statements are valid:

(1) $(X_{pq} \mid pq \in P^2)$ is a partition of X.

(2) If $pq \in P^2$ and if $i \in I_0$, then $X_{pq}\phi^i \subset X_{pq\lambda^i}$.

(3) If $pq \in L$, then:

(i) $X_{pq} \supset X_{pq}\phi^\mu \supset X_{pq}\phi^{2\mu} \supset X_{pq}\phi^{3\mu} \supset \cdots$.

(ii) $\bigcap_{i \in I_0} X_{pq}\phi^{i\mu} = \{w_{pq}\}$.

(4) If $pq \in L$ and if $i \in I_0$, then $w_{pq}\phi^i = w_{pq\lambda^i}$.

(5) $W\phi = W$ and $\phi \mid W$ is a one-to-one map of W onto W.

(6) $X \supset X\phi \supset X\phi^2 \supset X\phi^3 \supset \cdots$.

(7) $W = \bigcap_{i \in I_0} X\phi^i = \{x \mid x \in X \,\&\, x\phi^\mu = x\} = \{x \mid x \in X \,\&\, \exists\, i \in I_1 \cdot x\phi^i = x\}$.

3.33. Notation. Let $\mathscr{R}$ denote the set of all real numbers. Let d or $d(P)$ denote the function on $X \times X$ to $\mathscr{R}$ such that:

(1) If $x \in X$, then $d(x,x) = 0$.

(2) If $x,\ y \in X$ with $x \neq y$, then $d(x,y) = (1+k)^{-1}$ where k is the least nonnegative integer such that $x_k \neq y_k$ or $x_{-k} \neq y_{-k}$. It may be verified that d is a metric of X such that the metric topology of X induced by d coincides with the product topology of $X = P^I$ induced by the discrete topology of P. Let X be provided with its metric d so that X is a metric space. The metric space X is easily seen to be homeomorphic to the Cantor discontinuum if $m =_{\mathrm{df}} \operatorname{crd} P \geqq 2$. Of course, if $m = 1$, then $\operatorname{crd} X = 1$. The metric space X is called the *bisequence space over* P.

3.34. Remark. If $pq \in P^2$, then X_{pq} is a nonvacuous open-closed subset of X.

3.35. Notation. Let σ or $\sigma(P)$ denote the map of X into X such that $x \in X$ implies $x\sigma = (x_{i+1} \mid i \in I)$. It is readily proved that σ is a homeomorphism of X onto X. The map σ is called the *shift* (*transformation*) of X. The discrete flow (X,σ) is called the *symbolic flow over* P or the *m-ary symbolic flow*. It is well known that (X,σ) is expansive mixing.

3.36. Remark. If $x \in X$ and if $i,j,k \in I$ with $i \leqq j$, then

$$x\sigma^k[i,j] = x[i+k, j+k].$$

3.37. Definition. If (E,h) is a discrete flow, that is, if E is a topological space and if h is a homeomorphism of E onto E, and if $x \in E$, then the *orbit of x under h*, denoted $O(x,h)$, is defined to be $\{xh^i \mid i \in I\}$, and the *orbit-closure of x under h*, denoted $\bar{O}(x,h)$, is defined to be the closure $\{xh^i \mid i \in I\}^-$ of the orbit of x under h

3.38. Theorem. *The following statements are valid*:

(1) W_0 *coincides with the set of all* $x \in W$ *such that* x *is almost periodic under* σ.

(2) $W_0 \neq \emptyset$.

(3) *If* $w \in W_0$, *then* $\bar{O}(w,\sigma)$ *is minimal under* σ.

Proof. (1) Let $pq \in L_0$. Since $w_{pq}\theta^\mu = w_{pq}$, in order to show that w_{pq} is almost periodic under σ it is enough to show that there exists $k \in I_1$ such that $r \in \operatorname{rng} w_{pq}$ implies $pq \leqq r\theta^{k\mu}$. Now use 3.15.

Let $pq \in L$ such that w_{pq} is almost periodic under σ. Since $pq \leqq w_{pq}$, it follows readily that for each $r \in \operatorname{rng} w_{pq}$ there exists $i \in I_1$ such that $pq \leqq r\theta^i$. Hence $pq \in L_0$ and $w_{pq} \in W_0$.

(2) Use 3.30.

(3) Use (1) and [3, 4.07].

3.39. DEFINITION. A *substitution minimal set over P generated by θ* is defined to be a minimal set $\bar{O}(w_{pq}, \sigma)$ under σ where $pq \in L_0$.

3.40. REMARK. The class of all substitution minimal sets over $P = I[0, m-1]$ generated by θ, for all m and θ, is denumerable.

3.41. REMARK. The following statements are valid:

(1) If $x \in X$ and if x is almost periodic under σ, then $x\phi$ is almost periodic under σ.

(2) $L_0\lambda = L_0$ and $\lambda \mid L_0$ is a one-to-one map of L_0 onto L_0.

(3) $W_0\phi = W_0$ and $\phi \mid W_0$ is a one-to-one map of W_0 onto W_0.

3.42. REMARK. Let pq, $rs \in L$. Then:

(1) If $p = r$, the w_{pq} is negatively-asymptotic to w_{rs}.

(2) If $q = s$, then w_{pq} is positively-asymptotic to w_{rs}.

3.43. STANDING HYPOTHESIS 3. Let $m \in I_1$, let P be a set such that $\operatorname{crd} P = m$, let $n \in I_2$, and let θ be a substitution over P of constant length $(n \mid p \in P)$ whence $\theta: P \to P^n$.

3.44. REMARK. The following statements are valid:

(1) If $x \in X$ and if $i \in I$, then

$$x\phi[ni, n(i+1)-1] = x_i\theta.$$

(2) If x, $y \in X$, if $i, j \in I$ with $i \leqq j$, and if $x[i,j] = y[i,j]$, then

$$x\phi[ni, n(j+1)-1] = y\phi[ni, n(j+1)-1].$$

(3) ϕ is a homomorphism of (X, σ) into (X, σ^n); that is, ϕ is a continuous map of X into X such that $\sigma\phi = \phi\sigma^n$.

(4) If $i \in I$ and if $j \in I_0$, then $\sigma^i\phi^j = \phi^j\sigma^{in^j}$.

(5) If $pq \in P^2$, if x, $y \in X_{pq}$, and if $i \in I_0$, then $x\phi^i[-n^i, n^i-1] = y\phi^i[-n^i, n^i-1]$ and $d(x\phi^i, y\phi^i) \leqq (1+n^i)^{-1}$.

(6) If $pq \in P^2$ and if $i \in I_0$, then

$$\operatorname{diam} X_{pq}\phi^i < n^{-i}.$$

(7) If $pq \in P^2$, if $i \in I_0$, if $j \in I$, and if $0 \leqq j < n^i$, then

$$\operatorname{diam} X_{pq}\phi^i\sigma^j < (n^i - j)^{-1}.$$

3.45. REMARK. Let pq, $rs \in L$. Then:

(1) w_{pq} is negatively-proximal to w_{rs} if and only if there exists $i \in I_0$ such that $p\theta^{i\mu} \cap r\theta^{i\mu} \neq \emptyset$. If $p\theta \cap r\theta \neq \emptyset$, then w_{pq} is negatively-proximal to w_{rs}.

(2) w_{pq} is positively-proximal to w_{rs} if and only if there exists $i \in I_0$ such that $q\theta^{i\mu} \cap s\theta^{i\mu} \neq \emptyset$. If $q\theta \cap s\theta \neq \emptyset$, then w_{pq} is positively-proximal to w_{rs}.

(3) If $p \neq r$, and if $t, u \in P$ with $t \neq u$ implies $t\theta \cap u\theta = \emptyset$, then w_{pq} is negatively-distal from w_{rs}.

(4) If $q \neq s$, and if $t, u \in P$ with $t \neq u$ implies $t\theta \cap u\theta = \emptyset$, then w_{pq} is positively-distal from w_{rs}.

3.46. Standing Hypothesis 4. Let $m \in I_1$, let P be a set such that $\operatorname{crd} P = m$, let $n \in I_2$, and let θ be a one-to-one substitution over P of constant length $(n \mid p \in P)$ whence $\theta: P \to P^n$ is one-to-one.

3.47. Remark. Let $pq, rs \in L$. Then:

(1) w_{pq} is negatively-asymptotic to w_{rs} if and only if

$$p\theta^{\mu}[0, n^{\mu} - 2] = r\theta^{\mu}[0, n^{\mu} - 2].$$

(2) w_{pq} is positively-asymptotic to w_{rs} if and only if

$$q\theta^{\mu}[1, n^{\mu} - 1] = s\theta^{\mu}[1, n^{\mu} - 1].$$

3.48. Remark. The map ϕ is an isomorphism of (X, σ) into (X, σ^n); that is ϕ is a homeomorphism of X into X such that $\sigma\phi = \phi\sigma^n$.

3.49. Theorem. *Let $\bar{O}(w, \sigma)$ be one and the same subset M of X for all $w \in W_0$, let $\Lambda, \Sigma, \Pi, \Pi^*$ refer to (M, σ), and let $W_0 \times W_0 \subset \Pi^*$. Then:*

(1) *(M, σ) is minimal.*

(2) *$\Lambda = \Sigma = \Pi^*$.*

(3) *If $w \in W_0$ and if $i \in I$, then $w\sigma^i\Lambda = W_0\sigma^i$.*

(4) *If $\operatorname{crd} M/\sigma^n = n$, then $\mathscr{S}(M, \sigma) \simeq (D_n, \tau_n)$ and $\mathscr{G}(M, \sigma) \simeq D_n$.*

(5) *If M is infinite and if n is prime, then $\operatorname{crd} M/\sigma^n = n$.*

(6) *If M is infinite and if $m = \operatorname{crd} P$ is less than or equal to the least prime factor of n, then $\operatorname{crd} M/\sigma^n = n$.*

(7) *If M is infinite and if $m = \operatorname{crd} P = 2$, then $\operatorname{crd} M/\sigma^n = n$.*

(8) *If $\operatorname{crd} L_0 \geqq 2$, then M is infinite and therefore M is homeomorphic to the Cantor discontinuum.*

Proof. It is immediate from 3.38 (3) that (M, σ) is minimal.

If $w \in W_0$, then $M\phi = \bar{O}(w, \sigma)\phi = \bar{O}(w\phi, \sigma^n) \subset \bar{O}(w\phi, \sigma) \subset M$ by 3.41 (3). Thus $M \supset M\phi \supset M\phi^2 \supset \cdots$.

By 3.32 (7), $\bigcap_{i \in I_0} M\phi^i \subset \bigcap_{i \in I_0} X\phi^i = W$. By 3.41 (3) and 3.38 (1), $W_0 \subset \bigcap_{i \in I_0} M\phi^i \subset M \cap W = W_0$. Hence

$$W_0 = \bigcap_{i \in I_0} M\phi^i.$$

Statements (2), (3) and (4) are now immediate from 1.15 and 2.05.

We prove (5). Suppose n is prime and $\operatorname{crd} M/\sigma^n \neq n$. Let $w \in W_0$. Let a be the least positive integer such that $\bar{O}(w\sigma^a, \sigma^n) = \bar{O}(w, \sigma^n)$. Since $1 \leqq a < n$ and a divides n, it follows that $a = 1$ and

$$\bar{O}(w, \sigma^n) = \bar{O}(w\sigma, \sigma^n) = \bar{O}(w\sigma^2, \sigma^n) = \cdots = \bar{O}(w\sigma^{n-1}, \sigma^n) = \bar{O}(w, \sigma)$$

whence

$$M = M\phi = M\phi^2 = \cdots, W_0 = \bigcap_{i \in I_0} M\phi^i = M,$$

and M is finite. This proves (5).

We prove (6). Suppose $m = \operatorname{crd} P$ is less than or equal to the least prime factor of n and $\operatorname{crd} M/\sigma^n \neq n$. Let $x \in M$. Let a be the least positive integer such that $\bar{O}(x\phi\sigma^a, \sigma^n) = \bar{O}(x\phi, \sigma^n)$. Now $1 \leqq a < n$ and a divides n. Define $b = n/a$ whence $b \in I$ and $2 \leqq b \leqq n$. We have $M\phi = M\phi\sigma^a = M\phi\sigma^{2a} = \cdots$. Hence $x\phi$ can be partitioned into a-strings over P to yield a new bisequence whose b-blocks are at most m in number. Since $m \leqq b$, it follows from [5, Theorem 7.4] that $x\phi$ is periodic, x is periodic, and M is finite. This proves (6).

Statement (7) is immediate from (6).

If M is finite, then σ is periodic on M, $\Pi = \Delta_M$, $\Pi^* = \Delta_M$, $W_0 \times W_0 \subset \Delta_M$, $\operatorname{crd} W_0 = 1$, and $\operatorname{crd} L_0 = 1$. This proves (8).

The writer has a strong suspicion that this theorem (3.49) can be considerably improved.

4. **Examples.**

4.01. Notation. We adopt the notation of §3 here in so far as applicable. If $a \in I_1$, then p^a denotes the constant block $pp \cdots p$ of length a.

4.02. Example 1. Let $P = \{0, 1\}$ and let $\theta: 0 \to 00$, $1 \to 11$. Then:

(1) The hypotheses of 3.46 with $m = 2$ and $n = 2$ are satisfied.

(2) λ is the identity map of P^2.

(3) $L = P^2$, $L_0 = \{00, 11\}$, $\mu = 1$,

$$W = \{w_{pq} \mid pq \in P^2\}, \; W_0 = \{w_{00}, w_{11}\}.$$

(4) The hypotheses of 3.49 are not satisfied since

$$\bar{O}(w_{00}, \sigma) = \{w_{00}\} \neq \{w_{11}\} = \bar{O}(w_{11}, \sigma).$$

(5) w_{00} and w_{11} are each fixed under σ; $w_{01} \downarrow w_{00}$; $w_{01} \uparrow w_{11}$; $w_{10} \downarrow w_{11}$; $w_{10} \uparrow w_{00}$. Here $\downarrow$ means "negatively-asymptotic to" and $\uparrow$ means "positively-asymptotic to."

4.03. Example 2. Let $P = \{0, 1\}$ and let θ: $0 \to 01$, $1 \to 10$. Then:

(1) The hypotheses of 3.46 with $m = 2$ and $n = 2$ are satisfied.

(2) $\lambda: 00 \to 10$, $01 \to 11$, $10 \to 00$, $11 \to 01$.

(3) $L = L_0 = P^2$, $\mu = 2$, $W = W_0 = \{w_{pq} \mid pq \in P^2\}$.

(4) The hypotheses of 3.49 are satisfied and $\operatorname{crd} M/\sigma^2 = 2$.

(5) $\mathscr{S}(M,\sigma) \simeq (D_2, \tau_2)$.

(6) $\mathscr{G}(M,\sigma) \simeq D_2$.

(7) The minimal set (M,σ) is the well-known Morse minimal set.

4.04. EXAMPLE 3. Let $P = \{0,1\}$ and let θ: $0 \to 01$, $1 \to 00$. Then:

(1) The hypotheses of 3.46 with $m = 2$ and $n = 2$ are satisfied.

(2) $\lambda : 00 \to 10$, $01 \to 10$, $10 \to 00$, $11 \to 00$.

(3) $L = L_0 = \{00, 10\}$, $\mu = 2$, $W = W_0 = \{w_{00}, w_{10}\}$.

(4) The hypotheses of 3.49 are satisfied and $\operatorname{crd} M/\sigma^2 = 2$.

(5) $\mathscr{S}(M,\sigma) \simeq (D_2, \tau_2)$.

(6) $\mathscr{G}(M,\sigma) \simeq D_2$.

(7) The minimal set (M,σ) was constructed in a different way in [3, 12.52]. Actually, the right-hand sequence in w_{00} and w_{10} may be constructed recursively in many ways. For example, write down a column with 0 and 1 alternating, and at any level both prefix and suffix the previous block. Again, construct the sequence $010201030102010\cdots$ by suffixing the next integer and repeating the previous block, and then reduce modulo 2.

4.05. EXAMPLE 4. Let $P = \{0,1\}$ and let θ: $0 \to 0010$, $1 \to 1011$. Then:

(1) The hypotheses of 3.46 with $m = 2$ and $n = 4$ are satisfied.

(2) λ is the identity map of P^2.

(3) $L = L_0 = P^2$, $\mu = 1$, $W = W_0 = \{w_{pq} \mid pq \in P^2\}$.

(4) The hypotheses of 3.49 are satisfied and $\operatorname{crd} M/\sigma^4 = 4$.

(5) $\mathscr{S}(M,\sigma) \simeq (D_4, \tau_4) \simeq (D_2, \tau_2)$.

(6) $\mathscr{G}(M,\sigma) \simeq D_4 \simeq D_2$.

(7) The points w_{01} and w_{10} of M are bilaterally proximal under σ but not unilaterally asymptotic under σ.

(8) If ρ: $X \to X$ is the map such that $x \in X$ implies $x\rho = (x'_{-i} \mid i \in I)$, then:

(i) ρ is a reversor of (X,σ), that is, ρ is a homeomorphism of X onto X such that $x \in X$ implies $x\sigma\rho = x\rho\sigma^{-1}$;

(ii) $M\rho = M$ whence $\rho \mid M$ is a reversor of (M,σ) and (M,σ) is reversible.

(9) No bisequence in M is symmetric. A bisequence $x \in X$ is said to be *symmetric* provided there exists $a \in I$ such that $i \in I$ implies $x(i) = x(-i + a)$. If a bisequence $x \in X(P)$ is symmetric for any P, then its orbit-closure $\bar{O}(x,\sigma)$ is reversible. Examples 2, 3, 8 contain symmetric bisequences and are therefore reversible.

We prove directly that $\operatorname{crd} M/\sigma^4 = 4$. This will illustrate a certain technique. For $pq \in P^2$ define $M_{pq} = M \cap X_{pq}$. Then:

$$(-1) M = M_{00} \cup M_{01} \cup M_{10} \cup M_{11},$$

$$M_{00} \neq \emptyset,\ M_{01} \neq \emptyset,\ M_{10} \neq \emptyset,\ M_{11}, \neq \emptyset$$

$(M_{00}, M_{01}, M_{10}, M_{11})$ is a partition of M.

(0) $M_{00}\phi \subset M_{00}, M_{01}\phi \subset M_{01}, M_{10}\phi \subset M_{10}, M_{11}\phi \subset M_{11}$

$$M\phi \subset M_{00} \cup M_{01} \cup M_{10} \cup M_{11}$$

$$M\phi \cap M_{00} \neq \emptyset,\ M\phi \cap M_{01} \neq \emptyset,\ M\phi \cap M_{10} \neq \emptyset,\ M\phi \cap M_{11} \neq \emptyset$$

(1) $M_{00}\phi\sigma \subset M_{00},\ M_{01}\phi\sigma \subset M_{10},\ M_{10}\phi\sigma \subset M_{00},\ M_{11}\phi\sigma \subset M_{10},$

$$M\phi\sigma \subset M_{00} \cup M_{10}$$

$$M\phi\sigma \cap M_{00} \neq \emptyset,\ M\phi\sigma \cap M_{10} \neq \emptyset$$

(2) $M_{00}\phi\sigma^2 \subset M_{01},\ M_{01}\phi\sigma^2 \subset M_{01},\ M_{10}\phi\sigma^2 \subset M_{01},\ M_{11}\phi\sigma^2 \subset M_{01},$

$$M\phi\sigma^2 \subset M_{01}$$

$$M\phi\sigma^2 \cap M_{01} \neq \emptyset$$

(3) $M_{00}\phi\sigma^3 \subset M_{10},\ M_{01}\phi\sigma^3 \subset M_{11},\ M_{10}\phi\sigma^3 \subset M_{10},\ M_{11}\phi\sigma^3 \subset M_{11},$

$$M\phi\sigma^3 \subset M_{10} \cup M_{11}$$

$$M\phi\sigma^3 \cap M_{10} \neq \emptyset,\ M\phi\sigma^3 \cap M_{11} \neq \emptyset$$

Therefore $M\phi$, $M\phi\sigma$, $M\phi\sigma^2$, $M\phi\sigma^3$ are pairwise different and consequently disjoint. Since $M/\sigma^4 = \{M\phi, M\phi\sigma, M\phi\sigma^2, M\phi\sigma^3\}$, we conclude that crd $M/\sigma^4 = 4$. Actually it is enough to recognize that $M\phi \neq M\phi\sigma$ and $M\phi \neq M\phi\sigma^2$ since 1 and 2 are the only positive divisors of 4 which are less than 4.

4.06. EXAMPLE 5. Let $P = \{0,1\}$ and let $\theta: 0 \to 0010,\ 1 \to 1101$. Then:

(1) The hypotheses of 3.46 with $m = 2$ and $n = 4$ are satisfied.

(2) λ is the identity map of P^2.

(3) $L = L_0 = P^2$, $\mu = 1$, $W = W_0 = \{w_{pq} \mid pq \in P^2\}$.

(4) The hypotheses of 3.49 are satisfied and crd $M/\sigma^4 = 4$.

(5) $\mathscr{S}(M,\sigma) \simeq (D_4, \tau_4) \simeq (D_2, \tau_2)$.

(6) $\mathscr{G}(M,\sigma) \simeq D_4 \simeq D_2$.

(7) The minimal set (M,σ) is irreversible [2].

4.07. EXAMPLE 6. Let $P = \{0,1\}$, let $a \in I_1$, and let $\theta: 0 \to 00^a1,\ 1 \to 01^a1$. Then:

(1) The hypotheses of 3.46 with $m = 2$ and $n = a + 2$ are satisfied.

(2) If $pq \in P^2$, then $pq\lambda = 10$.

(3) $L = L_0 = \{10\}$, $\mu = 1$, $W = W_0 = \{w_{10}\}$.

(4) The hypotheses of 3.49 are satisfied and crd $M/\sigma^{a+2} = a + 2$.

(5) $\mathscr{S}(M,\sigma) \simeq (D_{a+2}, \tau_{a+2})$.

(6) $\mathscr{G}(M,\sigma) \simeq D_{a+2}$.

(7) The point w_{10} of M is regularly almost periodic under σ but not periodic under σ.

(8) To reverse the minimal set (M,σ), dualize and reflect the bisequences between the indices -1 and 0.

4.08. EXAMPLE 7. Let $a \in I_2$, let $P = I[0, a-1]$, and let

$$\theta: 0 \to 012 \cdots (a-2)(a-1),\ 1 \to 123 \cdots (a-1)0, \cdots, (a-1) \to (a-1)01 \cdots (a-3)(a-2).$$

Then:

(1) The hypotheses of 3.46 with $m = a$ and $n = a$ are satisfied.

(2) If $pq \in P^2$, then $pq\lambda = rq$ where $r \in P$ such that $r \equiv p - 1 \pmod a$.

(3) $L = L_0 = P^2$, $\mu = a$, $W = W_0 = \{w_{pq} \mid pq \in P^2\}$.

(4) The hypotheses of 3.49 are satisfied and $\operatorname{crd} M/\sigma^a = a$.

(5) $\mathscr{S}(M,\sigma) \simeq (D_a, \tau_a)$.

(6) $\mathscr{G}(M,\sigma) \simeq D_a$.

(7) If $\rho: M \to M$ is the map defined by leaving 0 unchanged but interchanging 1 and $a-1$, 2 and $a-2$, and so on, and subsequently reflecting between -1 and 0, then ρ is a reversor of (M,σ) whence (M,σ) is reversible.

(8) The minimal set (M,σ) may be called the *a-cyclic* (substitution) minimal set and may be regarded as a generalization of Example 2 to any integer greater than 1.

4.09. Example 8. Let $P = \{0,1,2\}$ and let

$$\theta: 0 \to 0121021201210,\ 1 \to 1202102012021,\ 2 \to 2010210120102.$$

Then:

(1) The hypotheses of 3.46 with $n = 3$ and $n = 13$ are satisfied.

(2) λ is the identity map of P^2.

(3) $L = P^2$, $L_0 = P^2 - \{00, 11, 22\} = \{01, 02, 10, 12, 20, 21\}$, $\mu = 1$, $W = \{w_{pq} \mid pq \in P^2\}$, $W_0 = \{w_{01}, w_{02}, w_{10}, w_{12}, w_{20}, w_{21}\}$.

(4) The hypotheses of 3.49 are satisfied and $\operatorname{crd} M/\sigma^{13} = 13$.

(5) $\mathscr{S}(M,\sigma) \simeq (D_{13}, \tau_{13})$.

(6) $\mathscr{G}(M,\sigma) \simeq D_{13}$.

(7) No bisequence in M contains a substring congruent to a block of the form AA where $A \in P^+$ [4].

4.10. Example 9. Let $P = \{0,1,2\}$ and let

$$\theta: 0 \to 012010210121,\ 1 \to 021201020121,\ 2 \to 020102120121.$$

Then:

(1) The hypotheses of 3.46 with $m = 3$ and $n = 12$ are satisfied.

(2) If $pq \in P^2$, then $pq\lambda = 10$.

(3) $L = L_0 = \{10\}$, $\mu = 1$, $W = W_0 = \{w_{10}\}$.

(4) The hypotheses of 3.49 are satisfied and $\operatorname{crd} M/\sigma^{12} = 12$.

(5) $\mathscr{S}(M,\sigma) \simeq (D_{12}, \tau_{12})$.

(6) $\mathscr{G}(M,\sigma) \simeq D_{12} \simeq D_2 \oplus D_3$.

(7) No bisequence in $M = \bar{O}(w_{10}, \sigma)$ contains a substring congruent to a block of the form AA where $A \in P^+$ [6].

(8) The point w_{10} of M is regularly almost periodic under σ but not periodic under σ.

4.11. Example 10. Let $P = \{0,1,2\}$ and let $\theta: 0 \to 02,\ 1 \to 0121,\ 2 \to 012021$.

Then:

(1) The hypotheses of 3.05, but not of 3.43, are satisfied.

(2) λ: $00 \to 20$, $01 \to 20$, $02 \to 20$, $10 \to 10$, $11 \to 10$, $12 \to 10$, $20 \to 10$, $21 \to 10$, $22 \to 10$.

(3) $L = L_0 = \{10\}$, $\mu = 2$, $W = W_0 = \{w_{10}\}$.

(4) No bisequence in $\bar{O}(w_{10}, \sigma)$ contains a substring congruent to a block of the form AA where $A \in P^+$. [The writer learned of this substitution and this property of it from Marshall Hall.]

4.12. REMARK. In at least certain particular instances of substitution minimal sets, their traces may be calculated in the manner of [2].

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UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PENNSYLVANIA,
YALE UNIVERSITY,
NEW HAVEN, CONNECTICUT