INVARIANT AND REDUCING SUBALGEBRAS OF MEASURE PRESERVING TRANSFORMATIONS(1)

BY
ROY L. ADLER

0. **Introduction.** The classification according to spatial isomorphism of measure preserving transformations is one of the basic questions of ergodic theory. Several invariants have been investigated; some pertain to spectral properties while others deal with the behavior of subsets with respect to measure preserving transformations such as the various concepts of mixing. For measure preserving transformations with pure point spectrum their spectral nature is sufficient to provide a complete classification; however, in general this is not the case. The reason that spectral invariants and also invariants referring to the behavior of subsets fail seems to be that they ignore algebraic properties which measure preserving transformations possess. A new invariant called entropy, algebraic in the sense that it deals with subalgebras of a measure algebra, has been investigated by the Russian school with great success in distinguishing various unitarily equivalent transformations. In this work another spatial invariant, algebraic in nature, is introduced. It deals with the subalgebras of a measure algebra that are invariant under a given transformation. These subalgebras form a lattice which in a certain sense is a spatial invariant.

1. **Preliminaries.** Let $(X, \mathscr{X}, m)$ be a measure space(2); i.e., X, is a set of elements, $\mathscr{X}$ a sigma-algebra of subsets of X called measurable sets, and m a countably additive measure defined on $\mathscr{X}$. It is expedient to restrict $(X, \mathscr{X}, m)$ to a Lebesgue space(3). This avoids certain pathologies occurring in arbitrary measure spaces, and yet it is sufficiently general to include all finite measure spaces of interest in ergodic theory. Associated with a measure space $(X, \mathscr{X}, m)$ is a measure algebra $\mathscr{X}(m)$ which is the Boolean sigma-algebra formed by identifying sets in $\mathscr{X}$ whose symmetric difference has zero measure, and the measure m is induced on the elements of $\mathscr{X}(m)$ in the natural way. The symbol $L^2(X)$ will denote, as usual,

Received by the editors December 18, 1962.

(1) This work is a portion of a doctoral thesis submitted to Yale University, June, 1961. I am deeply grateful to Professor S. Kakutani for his guidance in carrying out this research.

(2) For standard notions in measure theory see [9].

(3) A Lebesgue space is a space of measure one which is the same, measure theoretically, as an interval on the real line with Lebesgue measure together perhaps with a finite or countable set of points with positive measure (for precise definition see [17]).

the Hilbert space of complex-valued square integrable functions defined on $(X, \mathscr{X}, m)$, but sometimes $L^2(\mathscr{X}(m))$ will be used instead to stress that the functions are measurable with respect to $\mathscr{X}$ and are identified when equal almost everywhere.

A measure preserving transformation T of a measure space $(X, \mathscr{X}, m)$ is a one-to-one mapping of X onto itself such that if $B \in \mathscr{X}$, then TB, $T^{-1}B \in \mathscr{X}$ and $m(TB) = m(T^{-1}B) = m(B)$(4). A measure preserving transformation induces in an obvious way an automorphism of the measure algebra $\mathscr{X}(m)$; i.e., a one-to-one set mapping of $\mathscr{X}(m)$ onto itself which preserves measure and set operations. Furthermore Halmos and von Neumann [10] have shown that on a Lebesgue space every one-to-one set mapping of $\mathscr{X}(m)$ onto itself which preserves measure and set operations is induced by a measure preserving transformation on $(X, \mathscr{X}, m)$. In accordance with this fact we can allow the same symbol to stand for either the measure preserving transformation on X or the associated automorphism of $\mathscr{X}(m)$. Finally a measure preserving transformation defines a unitary operator U_T on $L^2(X)$ by $U_T: f(\cdot) \to f(T\cdot)$. Such unitary operators display algebraic characteristics; for a unitary operator U on $L^2(X)$ to be induced by a measure preserving transformation in the above manner it is necessary and sufficient that both U and U^{-1} send every bounded function into a bounded function and $U(f \cdot g) = U(f) \cdot U(g)$ whenever f and g are bounded functions (see [8, p. 45]).

Two measure preserving transformations T_1 and T_2 on $(X, \mathscr{X}, m)$ are said to be unitarily equivalent if there exists a unitary operator W on $L^2(X)$ such that $WU_{T_1}W^{-1} = U_{T_2}$. They are said to be spatially isomorphic if there exists a measure preserving transformation S on $(X, \mathscr{X}, m)$ such that $ST_1S^{-1} = T_2$ a.e. Properties shared by all unitarily equivalent measure preserving transformations are called spectral invariants and those shared by spatially isomorphic ones are called spatial invariants. Every spectral invariant is a spatial invariant, but not conversely. The most primitive spatial invariant is ergodicity; a measure preserving transformation is said to be ergodic if $B \in \mathscr{X}$, $TB = B$ a.e. implies that $m(B) = 0$ or $m(X - B) = 0$. That ergodicity is a spectral invariant is revealed by the Proper Value Theorem: A measure preserving transformation T on a finite measure space is ergodic if and only if the number 1 is a simple proper value of the induced unitary operator U_T. Ergodic transformations display a mixing behavior on the measurable subsets: a measure preserving transformation is ergodic if and only if, as $N \to \infty$, $N^{-1} \sum_{n=1}^{N} m(T^nA \cap B) \to m(A)m(B)$, for every $A, B \in \mathscr{X}$. The concept of mixing can be used to formulate other spatial invariants: a measure preserving transformation is said to be weakly mixing if, as $N \to \infty$, $N^{-1} \sum_{n=1}^{N} |m(T^nA \cap B) - m(A)m(B)| \to 0$, for every $A, B \in \mathscr{X}$; it is said to be strongly mixing if, as $n \to \infty$ $m(T A \cap B) \to m(A)m(B)$, for every $A, B \in \mathscr{X}$.

(4) For standard notions and theorems in ergodic theory see [8].

Weakly mixing can be characterized spectrally by the mixing theorem: A transformation T is weakly mixing if and only if the number 1 is a simple proper value and the only proper value of the induced unitary operator U_T. It follows therefore that weakly mixing is also a spectral invariant. For strongly mixing no similar spectral characterization is known; yet strongly mixing as well as ergodicity and weakly mixing have the following functional forms revealing at once that they are spectral invariants: T is ergodic if and only if $N^{-1}\sum_{n=1}^{N}(U_T^n f, g) \to (f, 1)(1, g)$, as $N \to \infty$, for every $f, g \in L^2(X)$; T is weakly mixing if and only if $N^{-1}\sum_{n=1}^{N}\left|(U_T^n f, g) - (f,1)(1,g)\right| \to 0$, as $N \to \infty$ for every $f, g \in L^2(X)$; and T is strongly mixing if and only if $(U_T^n f, g) \to (f,1)(1,g)$, as $n \to \infty$, for every $f, g \in L^2(X)$. Finally although the class of ergodic transformations includes the class of weakly mixing transformations which includes the strongly mixing transformations, the classes are known to be distinct.

The family of subalgebras [5] of $\mathscr{X}(m)$ is partially ordered (by inclusion) and is a complete lattice: the infimum of two subalgebras $\mathscr{A}$ and $\mathscr{B}$ of $\mathscr{X}(m)$ is their intersection and the supremum $\mathscr{A} \vee \mathscr{B}$ is the subalgebra they generate: similar assertions hold for the infimum and supremum of any family of subalgebras. For a measure preserving transformation T we shall single out for investigation two types of subalgebras; a subalgebra $\mathscr{A}$ of $\mathscr{X}(m)$ is called *T-invariant* if $T\mathscr{A} \subseteq \mathscr{A}$ and it is called *T-reducing* if $T\mathscr{A} \subseteq \mathscr{A}$ and $T^{-1}\mathscr{A} \subseteq \mathscr{A}$. The family $\Lambda_i(T)$ of T-invariant subalgebras and the family $\Lambda_r(T)$ of T-reducing subalgebras are sublattices of the lattice of all subalgebras of $\mathscr{X}(m)$. If two measure preserving transformations T_1 and T_2 are spatially isomorphic with the measure preserving transformation S satisfying the relation $ST_1S^{-1} = T_2$, then the mapping $\mathscr{A} \to S\mathscr{A}$ is a lattice isomorphism of $\Lambda_i(T_1)$ onto $\Lambda_i(T_2)$ and $\Lambda_r(T_1)$ onto $\Lambda_r(T_2)$; that is to say, these lattices are spatial invariants. Rather than deal with the subalgebras of $\mathscr{X}(m)$ it is often more convenient to work with the corresponding L^2-subspaces of $L^2(\mathscr{X}(m))$. The correspondence $\mathscr{A} \to L^2(\mathscr{A})$ is a lattice isomorphism between the lattice of subalgebras of $\mathscr{X}(m)$ and the lattice of L^2-subspaces of $L^2(\mathscr{X}(m))$ and accordingly we can speak of $\Lambda_i(T)$ and $\Lambda_r(T)$ as the lattice of *invariant L^2-subspaces* and the lattice of *reducing L^2-subspaces* respectively for the operator U_T. The spectral counterpart of the lattice of L^2-subspaces of $L^2(\mathscr{X}(m))$ is the lattice of closed subspaces of $L^2(\mathscr{X}(m))$ in which the supremum operation is to be distinguished from the former one; the supremum of two closed subspaces M and N is the closed subspaces they generate while in the former lattice the supremum of $L^2(\mathscr{A})$ and $L^2(\mathscr{B})$ is $L^2(\mathscr{A} \vee \mathscr{B})$ which is not necessarily equal to $\overline{\operatorname{span}(L^2(\mathscr{A}) \cup L^2(\mathscr{B}))}$. The lattice of invariant $(U_T M \subseteq M)$ and of reducing $(U_T M \subseteq M, U_T^{-1} M \subseteq M)$ closed subspaces of $L^2(\mathscr{X}(m))$ are spectral invariants of T which we shall denote by $\Lambda_i(U_T)$ and $\Lambda_r(U_T)$, respectively. Finally a useful fact in connection with these ideas is the algebraic distinction between closed

(5) Throughout we use the term "subalgebra" in the sense of sigma-subalgebra.

subspaces and L^2-subspaces of $L^2(X)$; a necessary and sufficient condition that a closed subspace M of $L^2(\mathscr{X}(m))$ be of the form $M = L^2(\mathscr{A})$ where $\mathscr{A}$ is the smallest sigma-algebra of $\mathscr{X}(m)$ with respect to which all functions in M are measurable is that M contain a dense subalgebra $\mathscr{D}$ consisting of bounded functions, constant functions, and their conjugate functions (see for instance [6]).

A. N. Kolmogorov [12] suggested a spatial invariant called entropy derived from information theory. The definition according to the improved version of Ja. G. Sinaĭ [19] is the following: for any finite subalgebra $\mathscr{A}$ of $\mathscr{X}(m)$ define the entropy $H(\mathscr{A})$ of $\mathscr{A}$ by $H(\mathscr{A}) = -\sum m(A_k)\log m(A_k)$ where the sum is taken over the finite atoms A_k of $\mathscr{A}$; the entropy $h(T,\mathscr{A})$ of a measure preserving transformation T with respect to a finite subalgebra $\mathscr{A}$ is defined by $h(T,\mathscr{A}) = \operatorname{limit}_{N\to\infty} N^{-1} H({}_T^N\mathscr{A})$ where the symbol ${}_T^N\mathscr{A}$ denotes the subalgebra $\bigvee_{n=0}^{N-1} T^n\mathscr{A}$; and the entropy $h(T)$ of T is defined as

$$h(T) = \sup\{h(T,\mathscr{A}) \colon \mathscr{A} \text{ finite}, \mathscr{A} \subseteq \mathscr{X}(m)\}.$$

The standard facts concerning the entropy of a measure preserving transformation are summarized in [16]. The entropy $h(T)$ is defined with respect to the algebra $\mathscr{X}(m)$; we could, of course, consider T restricted to one of its reducing subalgebras $\mathscr{C}$ and obtain a corresponding entropy $h_{\mathscr{C}}(T) = \sup\{h(T,\mathscr{A})\colon \mathscr{A} \text{ finite}, \mathscr{A} \subseteq \mathscr{C}\}$ for $\mathscr{C} \in \Lambda_r(T)$. The lattice $\Lambda_r(T)$ along with this function defined on its elements is again a spatial invariant. Although no extensive use in this work will be made of this concept, it is possibly an important consideration in the classification of measure preserving transformations. M. S. Pinsker [14] has introduced the notion of completely positive entropy wherein $h_{\mathscr{C}}(T) > 0$ for $\mathscr{C} \in \Lambda_r(T)$.

2. **General remarks.** We first observe that $\Lambda_r(T) \subseteq \Lambda_i(T)$ and $\Lambda_r(T)$ is non-empty, for it contains the trivial T-reducing subalgebra $\mathscr{X}(m)$ and the smallest subalgebra $\{\phi, X\}$ of $\mathscr{X}(m)$ which is denoted by 2. In general that $\Lambda_r(T) \neq \Lambda_i(T)$ is easily demonstrated by choosing T to be the shift transformation defined as follows: let $(X,\mathscr{X},m)$ be a Lebesgue space and $X_i = X$, $\mathscr{X}_i = \mathscr{X}$, and $m_i = m$; the shift transformation T, T: $x^* \to Tx^*$ on the two-sided infinite direct product measure space $(X^*,\mathscr{X}^*,m^*) = \prod_{i=-\infty}^{\infty}(X_i,\mathscr{X}_i,m_i)$ is the measure preserving transformation defined by $(Tx^*)_n = (x^*)_{n+1}$ where $(x^*)_n$ denotes the nth coordinate of the element $x^* \in X^*$. The measure space $(X^*,\mathscr{X}^*,m^*)$ is again a Lebesgue space and T is strongly mixing. Let the subalgebra $\mathscr{X}_i^*$ of $\mathscr{X}^*(m^*)$ be defined by $\mathscr{X}_i^* = \prod_{j=-\infty}^{i-1} 2_j x \mathscr{X}_i x \prod_{j=i+1}^{\infty} 2_j$ where $2_j = \{\phi, X_j\}$ and consider $\mathscr{A}^* = \bigvee_{i=-\infty}^{0} \mathscr{X}_i^*$. It is clear that $T\mathscr{A}^* = \bigvee_{i=-\infty}^{-1} \mathscr{X}_i^* \subseteq \mathscr{A}^*$ whereas $T^{-1}\mathscr{A}^* = \bigvee_{i=-\infty}^{1} \mathscr{X}_i^*$ which is not contained in $\mathscr{A}^*$. In fact we have (i) $\mathscr{A}^* \subseteq T^{-1}\mathscr{A}^*$, (ii) $\bigwedge_{n=-\infty}^{\infty} T^n\mathscr{A}^* = 2^*$ and (iii) $\bigvee_{n=-\infty}^{\infty} T^n\mathscr{A}^* = \mathscr{X}^*$. Transformations satisfying such conditions are called Kolmogorov transformations [14]. They are known to have completely positive entropy, and it is known that transformations with completely positive

entropy are Kolmogorov transformations [18]. An interesting question is to what extent do shift transformations characterize them.

A condition for the equality of the two lattices is expressed in the following

PROPOSITION 1. $\Lambda_l(T) = \Lambda_r(T)$ *if and only if* $h(T) = 0$.

Proof. It is known that $h(T) = 0$ if and only if

$$ {}_T^\infty\mathscr{A} = \bigvee_{n=0}^{\infty} T^n\mathscr{A} = {}_{T^{-1}}^{\infty}\mathscr{A} $$

for every finite subalgebra $\mathscr{A}$ of $\mathscr{X}(m)$(see [16]). Thus $\Lambda_l(T) = \Lambda_r(T)$ implies $h(T) = 0$ follows immediately since

$$ {}_T^\infty\mathscr{A} \in \Lambda_l(T). $$

The converse is almost as immediate; suppose $h(T) = 0$ and $\mathscr{B} \in \Lambda_l(T)$. For any $A \in \mathscr{B}$ denote the subalgebra $\{\phi, A, X - A, X\}$ by $\mathscr{A}$. Because $\mathscr{A}$ is finite, the above criterion is applicable and

$$ {}_T^\infty\mathscr{A} = {}_{T^{-1}}^{\infty}\mathscr{A}. $$

However $\mathscr{B}$ is T-invariant; therefore

$$ {}_T^\infty\mathscr{A} \subseteq \mathscr{B} $$

and thus $T^{-1}A \in \mathscr{B}$, i.e., $T^{-1}\mathscr{B} \subseteq \mathscr{B}$.

For S a measure preserving transformation on $(X, \mathscr{X}, m)$ let $\mathscr{A}_S$ denote the subalgebra $\{A : SA = A, A \in \mathscr{X}(m)\}$ of $\mathscr{X}(m)$ consisting of measurable S-invariant sets. S is ergodic if and only if $\mathscr{A}_S = 2$. The following proposition shows that certain reducing subalgebras have a special representation.

PROPOSITION 2. *If S commutes with T almost everywhere, then $\mathscr{A}_S$ is a T-reducing subalgebra.*

Proof. If $A \in \mathscr{A}_S$, then $STA = TSA = TA$ and $ST^{-1}A = T^{-1}SA = T^{-1}A$; therefore $TA, T^{-1}A \in \mathscr{A}_S$.

The question immediately arises whether every T-reducing subalgebra has the form $\bigwedge_{S\in\mathscr{S}} \mathscr{A}_S$ where $\mathscr{S}$ is some family of measure preserving transformations which commute with T almost everywhere. A counterexample to this conjecture will be given in §4.

3. **Reducing subalgebras and pure point spectrum.** A measure preserving transformation is said to have pure point spectrum if $L^2(X)$ is spanned by its proper functions. P. R. Halmos and J. von Neumann [10] showed that an ergodic measure preserving transformation with pure point spectrum on a Lebesgue space can be considered measure theoretically the same as a rotation

$T_a : x \to ax$ which is Haar measure preserving on some compact separable Abelian group(6). They also proved that a rotation $T_a : x \to ax$ on a compact separable Abelian group X is ergodic if and only if $\{a^n : n = 0, \pm 1, \pm 2, \cdots\}$ is dense in X. Such groups admitting ergodic rotations are called monothetic and are discussed in [5] and [11].

We shall discuss reducing subalgebras of ergodic measure preserving transformations with pure point spectrum on Lebesgue spaces in light of the representation theorem. Thus throughout this section the measure space $(X, \mathscr{X}, m)$ will consist of a compact separable monothetic (hence Abelian) group X with generating element a, i.e., $X = \overline{\{a^n : n = 0, \pm 1, \pm 2, \cdots,\}}$, the family $\mathscr{X}$ of Borel measurable subsets of X, and Haar measure m, and T_a will denote the ergodic measure preserving transformation $T_a : x \to ax$.

Proposition 3. *The sequence $\{U^n_{T_a} : n = 0, 1, 2, \cdots\}$ contains a subsequence converging to $U^{-1}_{T_a}$ in the strong operator topology in the space of bounded linear transformations on $L^2(X)$.*

Proof. Since a is a generating element of X there exists a subsequence n_i such that $a^{n_i} \to a^{-1}$. Then for any continuous function g we have $g(a^{n_i}x) \to g(a^{-1}x)$ for all x and by bounded convergence we have $U^{n_i}_{T_a} g \to U^{-1}_{T_a} g$ in the L^2-norm. For $f \in L^2(X)$ there exists a sequence of continuous functions g_k on X such that $g_k \to f$ in L^2-norm. Because $U^k_{T_a}$ is unitary, we obtain the following:

$$\begin{aligned} \| U^{n_i}_{T_a} f - U^{-1}_{T_a} f \|_{L^2} &\leqq \| U^{n_i}_{T_a} f - U^{n_i}_{T_a} g_k \|_{L^2} \\ &\quad + \| U^{n_i}_{T_a} g_k - U^{-1}_{T_a} g_k \|_{L^2} + \| U^{-1}_{T_a} g_k - U^{-1}_{T_a} f \|_{L^2} \\ &\leqq 2 \| f - g_k \|_{L^2} + \| U^{n_i}_{T_a} g_k - U^{-1}_{T_a} g_k \|_{L^2} ; \end{aligned}$$

therefore

$$\operatorname*{limit\,sup}_{a^{n_i} \to a^{-1}} \| U^{n_i}_{T_a} f - U^{-1}_{T_a} f \|_{L^2} \leqq 2 \| f - g_k \|_{L^2}$$

for all k. Since g_k can be chosen arbitrarily close to f in the L^2-norm, we have

$$\| U^{n_i}_{T_a} f - U^{-1}_{T_a} f \|_{L^2} \to 0.$$

Corollary. $\Lambda_i(U_{T_a}) = \Lambda_r(U_{T_a})$.

Proof. From the proposition there is a sequence $U^{n_i}_{T_a}$ converging to $U^{-1}_{T_a}$ in the strong operator topology. Since a subspace $M \in \Lambda_i(U_{T_a})$ is closed and U_{T_a}-invariant, it follows that $U^{n_i}_{T_a} M \subseteq M$ and therefore $U^{-1}_{T_a} M \subseteq M$.

Corollary. $\Lambda_i(T_a) = \Lambda_r(T_a)$.

(6) The standard facts concerning topological groups used in this work are contained in [13] and [15]. By separable we mean satisfying the second axiom of countability.

Proof. It is obvious that $T_a\mathscr{A} = \mathscr{A}$ if and only if $U_{T_a}L^2(\mathscr{A}) = L^2(\mathscr{A})$. We consider $L^2(\mathscr{A})$ as a closed U_T-invariant subspace and apply the previous corollary.

COROLLARY. $h(T_a) = 0$.

Proof. The result follows from Proposition 1 and the previous corollary.

PROPOSITION 4. *If* $M \in \Lambda_r(U_{T_a})$

$$M = \overline{\text{span}\ \{x^* \ :\ x^* \in M,\ x^* \in X^*\}},$$

where X^* *denotes the character group of* X; *Conversely if* $M = \overline{\text{span}\ H^*}$ *where* H^* *is a subset of* X^* *then* $M \in \Lambda_r(U_{T_a})$.

Proof. Suppose M is not the closed subspace spanned by the characters which it contains; then letting

$$L = \overline{\text{span}\ \{x^*:\ x^* \in M,\ x^* \in X^*\}}$$

we have that $M' \neq 0$ where $M' = M \cap L^{\perp}$. The closed subspace M' is U_T-invariant, for let $f \in M'$ and $x^* \in L$. Since $M' \perp L$, we have

$$(U_{T_a}f, x^*) = (f, U_{T_a}^{-1}x^*) = x^*(a)(f, x^*) = 0;$$

that is, $U_{T_a}f \in L$. This fact along with the U_{T_a}-invariance of M establishes that $M' \in \Lambda_i(U_{T_a})$. Applying Proposition 3 we have $M' \in \Lambda_r(U_{T_a})$ or $U_T M' = M'$. Because M' is not the zero subspace, there exists a character $x^* \in X^*$ not orthogonal to it and x^* can be written $x^* = f_1 + f_2$ where $f_1 \in M'$ and $f_2 \in M'^{\perp}$. Consequently

$$U_{T_a}f_1 + U_{T_a}f_2 = U_{T_a}x^* = x^*(a)x^* = x^*(a)(f_1 + f_2)$$

whereupon

$$U_{T_a}f_1 = x^*(a)f_1.$$

But T_a is ergodic and x^* is a proper function of T_a with proper value $x^*(a)$, so by the Proper Value Theorem f_1 is almost everywhere equal to a constant times the character x^*; that is to say, $x^* \in M'$ which is a contradiction.

The converse is immediate since all characters are proper functions of U_{T_a}.

COROLLARY. *If* $\mathscr{A} \in \Lambda_r(T_a)$ *then* $L^2(\mathscr{A}) = \overline{\text{span}\, H^*}$ *where* H^* *is some subgroup of* X^*; *conversely if* H^* *is a subgroup of* X^* *there is a subalgebra* $\mathscr{A} \in \Lambda_r(T_a)$ *such that* $\overline{\text{span}\, H^*} = L^2(\mathscr{A})$. *In addition the lattice* $\Lambda_r(T_a)$ *is isomorphic to* $\Lambda(X^*)$ *the lattice of subgroups of the character group* X^* *of* X.

Proof. Consider $\mathscr{A} \in \Lambda_r(T_a)$. From the above proposition

$$L^2(\mathscr{A}) = \overline{\text{span}\ \{x^* \ :\ x^* \in L^2(\mathscr{A}),\ x^* \in X^*\}}.$$

Since the product of bounded functions and the reciprocal of functions bounded away from zero in $L^2(\mathscr{A})$ are still in it, it is clear that $\{x^* : x^* \in L^2(\mathscr{A})\}$ is a subgroup of X^*. Conversely assuming that H^* is a subgroup of X^* the span H^* is an algebra of bounded functions which includes the constant functions and is closed under conjugation; therefore $\overline{\text{span}\, H^*} = L^2(\mathscr{A})$ where $\mathscr{A}$ is the smallest subalgebra of $\mathscr{X}(m)$ with respect to which all characters $x^* \in L^*$ are measurable. It is clear that $L^2(\mathscr{A})$ is U_{T_a}-invariant. Finally the correspondence $\mathscr{A} \leftrightarrow \{x^* : x^* \in L^2(\mathscr{A})\}$ supplies the lattice isomorphism of $\Lambda_r(T_a)$ onto $\Lambda(X^*)$.

The foregoing has a graphic interpretation in one dimension: let X be the unit interval and $T : x \to x + \gamma$ (modulo 1) where γ is irrational. If $\mathscr{A} \in \Lambda_r(T)$ then there exists an integer n such that $\mathscr{A}$ is the family of measurable sets which are periodic of period $1/n$ in the unit interval(7).

Finally we have a representation in terms of commuting transformations.

COROLLARY. *If* $\mathscr{A} \in \Lambda_r(T_a)$, *then* $\mathscr{A} = \bigwedge_{b \in G} \mathscr{A}_{T_b}$ *where* G *is some subgroup of* X.

Proof. From the preceding corollary let H^* be the subgroup of X^* such that $L^2(\mathscr{A}) = \overline{\text{span}\, H^*}$. Let G be the closed subgroup of X annihilated by H^*, i.e., $G = \{b : b \in X, x^*(b) = 1 \text{ for all } x^* \in H^*\}$. Consider $\bigwedge_{b \in G} \mathscr{A}_{T_b}$; it is a reducing subalgebra of T_a and hence $L^2(\bigwedge_{b \in G} \mathscr{A}_{T_b})$ is spanned by the characters in it. Furthermore if $x^* \in L^2(\bigwedge_{b \in G} \mathscr{A}_{T_b})$ it is clear that $U_{T_b} x^* = x^*$ for all $b \in G$ which implies that $x^*(b) = 1$ for all $b \in G$. In other words $L^2(\bigwedge_{b \in G} \mathscr{A}_{T_b}) = \overline{\text{span}\, K^*}$ where $K^* = \{x^* : x^*(b) = 1 \text{ for all } b \in G\}$. A theorem of Pontrjagin [15, p. 136] asserts that $K^* = H^*$ from which the desired conclusion that $L^2(\mathscr{A}) = L^2(\bigwedge_{b \in G} \mathscr{A}_{T_b})$ follows.

4. **Invariant subalgebras and Anzai's skew-product transformations.** Let $(X, \mathscr{X}, m)$ and $(Y, \mathscr{Y}, m)$ be unit intervals with Borel measurability and Lebesgue measure and $(Z = X \times Y, \mathscr{Z} = \mathscr{X} \times \mathscr{Y}, m^* = m \times m)$ be the unit square with the usual direct product measurability and measure. Let us consider Anzai's skew-product transformations $T_{\gamma,\alpha} : (x, y) \to (x + \gamma, y + \alpha(x))$ (additions modulo 1) where γ is an irrational number and $\alpha(\cdot)$ is a real valued measurable function on X for which $T_{\gamma,\alpha}$ is ergodic (see [3]). Some partial results are known; for instance, $\Lambda_r(T_{\gamma,\alpha}) = \Lambda_i(T_{\gamma,\alpha})$ and $\Lambda_r(U_{T_{\gamma,\alpha}}) \subsetneqq \Lambda_i(U_{T_{\gamma,\alpha}})$. The first results from Proposition 1 and the fact that $h(T_{\gamma,\alpha}) = 0$ (see [1]); the second from the fact that $L^2(Z)$ can be decomposed into a direct sum $L^2(Z) = H_0 + H_0^{\perp}$ of Hilbert spaces where U_T restricted to $H_0^{\perp}$ is unitarily equivalent to the unitary operator induced by a shift on an infinite dimensional torus (see [3]). Also we have

PROPOSITION 5. *Let* $\mathscr{A}$ *be a subalgebra such that* $\mathscr{X} \times 2 \subseteq \mathscr{A} \subseteq \mathscr{Z}$. $\mathscr{A} \in \Lambda_r(T_{\gamma,\alpha})$

(7) The material in this section is based on this result which was shown to the author by Professor S. Kakutani. His proof employed an interesting application of the mean ergodic theorem. A modification of it is adopted in §4.

if and only if $\mathscr{A} = \mathscr{A}_S$ *where* $S:(x,y)\to(x,y+\rho)$ (*addition modulo* 1) *for some real number* ρ.

Proof. Since S commutes with T the "if" part is obvious. For the "only if" part let $f \nsim 0$ be a function in $L^2(\mathscr{A})$. In the Fourier expansion

$$f \sim \sum_{p,q} c_{pq} e^{2\pi i(px+qy)}$$

there exists a coefficient $c_{p_0q_0} \neq 0$. One implication of the hypothesis is that $e^{-2\pi i(p_0\gamma+q_0\alpha(x))} \in L^2(\mathscr{A})$; hence the unitary operator defined by

$$V: g(x,y) \to e^{-2\pi i(p_0\gamma+q_0\alpha(x))} g(x+\gamma, y+\alpha(x)), \quad g \in L^2(\mathscr{Z}),$$

maps $L^2(\mathscr{A})$ into itself, in particular $V_k f \in L^2(\mathscr{A})$, $k = 1,2,\cdots$. By the mean ergodic theorem l.i.m. $N^{-1}\sum_{k=1}^N V^k f$ exists; and because $L^2(\mathscr{A})$ is a closed subspace of $L^2(\mathscr{Z})$, the limit is a member of $L^2(\mathscr{A})$. We will show this limit is $e^{2\pi i(p_0x+q_0y)}$.

$$\text{l.i.m.} \frac{1}{N}\sum_{k=1}^N V^k f = \text{l.i.m.} \frac{1}{N}\sum_{k=1}^N \sum_{p,q} c_{pq} V^k e^{2\pi i(px+qy)}.$$

We can interchange limits because mean convergence of the Fourier series is uniform with respect to k whereupon

$$\begin{aligned}\text{l.i.m.} &\frac{1}{N}\sum_{k=1}^N V^k f \\ &= \sum_{p,q} c_{p,q}\, \text{l.i.m.} \frac{1}{N}\sum_{k=1}^N V^k e^{2\pi i(px+qy)} \\ &= \sum_{p,q} c_{p\,q} e^{2\pi i(px+qy)}\, \text{l.i.m.} \frac{1}{N}\sum_{k=1}^N e^{2\pi i\{k(p-p)\gamma+(q-q_0)[\alpha(x)+\alpha(x+\gamma)+\ldots+\alpha(x+k\gamma-\gamma)]\}}.\end{aligned}$$

Applying the mean ergodic theorem to the unitary operator

$$W: g(x) \to e^{2\pi i[(p-p)\gamma+(q-q_0)\alpha(x)]} g(x+\gamma)$$

we have

$$\begin{aligned}\text{l.i.m.} &\frac{1}{N}\sum_{k=1}^N W^k 1 \\ &= \text{l.i.m} \frac{1}{N}\sum_{k=1}^N e^{2\pi i\{k(p-p_0)\gamma+(q-q_0)[\alpha(x)+\alpha(x+\gamma)+\ldots+\alpha(x+k\gamma-\gamma)]\}} \\ &= G(x) \text{ exists}\end{aligned}$$

and satisfies the relation $WG = G$ a.e., that is,

$$(*) \qquad G(x) = e^{2\pi i[(p-p_0)\gamma+(q-q_0)\alpha(x)]} G(x+\gamma) \quad \text{a.e.,}$$

whereupon $|G(x+\gamma)| = |G(x)|$ a.e. The transformation $x \to x+\gamma$ (mod 1) is also ergodic which implies $|G(x)| = a$ a.e.; or in other words, $G(x) = ae^{2\pi i\theta(x)}$ where $\theta(\cdot)$ is some real-valued function on X. From (*) it follows that if $a \neq 0$ then

$$(q - q_0)\alpha(x) = [\theta(x) + (p - p_0)x] - [\theta(x + \gamma) + (p - p_0)(x + \gamma)] \pmod 1 \text{a.e.}$$

In order to comply with Anzai's conditions [3] of ergodicity of $T_{\gamma\,\alpha}$ we must have $a = 0$ if $q - q_0 \neq 0$. Thus

$$\text{l.i.m.}\ \frac{1}{N}\sum_{k=1}^{N} W^k 1 = \begin{cases} 0, & q \neq q_0, \\ 0, & q = q_0,\ p \neq p_0, \\ 1, & q = q_0,\ p = p_0. \end{cases}$$

Therefore l.i.m. $N^{-1}\sum_{k=1}^{N} V^k f = c_{p_0q_0} e^{2\pi i(p_0x + q_0y)} \in L^2(\mathscr{A})$. Since $e^{2\pi ipx} \in L^2(\mathscr{A})$ for all integers p, we have: $e^{2\pi iq_0y} \in L^2(\mathscr{A})$ and we can assume that q_0 is the smallest such positive integer. It is then clear that

$$L^2(\mathscr{A}) = \overline{\text{span}\{e^{2\pi i(px + mq_0y)} : p = 0, \pm 1, \pm 2, \cdots, m = 0, \pm 1, \pm 2, \cdots\}}$$

and $\mathscr{A} = \mathscr{A}_S$ where $S : (x, y) \to (x, y + 1/q_0) \pmod 1$.

In the above case we note that S commutes with $T_{\gamma.\alpha}$. In general not every reducing subalgebra can be represented as the family invariant sets of some commuting transformation or even as the infimum of such families. Consider $T = T_{\gamma,\alpha}$ where $\alpha(x) = x$ and let $L^2(\mathscr{A})$ be spanned by a subgroup of the character group generated by $e^{2\pi ip_0x}$ and $e^{2\pi ip_0q_0y}$ where p_0 and q_0 are two integers. Since $U_T : e^{2\pi ip_0x} \to e^{2\pi ip_0\gamma} e^{2\pi ip_0x}$ and U_T: $e^{2\pi ip_0q_0y} \to e^{2\pi ip_0q_0x} e^{2\pi ip_0q_0y}$, it is obvious that $U_T L^2(\mathscr{A}) = L^2(\mathscr{A})$, i.e., $\mathscr{A} \in \Lambda_r(T)$. The measure preserving transformations which commute with T are of the form $T^n R$ where $R : (x, y) \to (x, y + \rho) \pmod 1$ for some real number ρ (see [2]) and R is the form of the nonergodic ones. Since $L^2(\mathscr{A}_R) = \text{span}\{e^{2\pi i(px + qm_0y)} : p = 0 \pm 1, \pm 2, \cdots, q = 0, \pm 1, \pm 2, \cdots$, and m_0 is some integer$\}$, it is clear that $\mathscr{A}$ is not equal to $\mathscr{A}_R$ or the infimum of any family of such subalgebras.

5. **Concluding remarks.** A conjecture naturally arises concerning the present circle of ideas. For two measure preserving transformations T_1 and T_2 let W be a unitary operator on $L^2(X)$ such that $WU_{T_1}W^{-1} = U_{T_2}$ and let $L : \Lambda_r(T_1) \to \Lambda_r(T_2)$ be a lattice isomorphism of $\Lambda_r(T_1)$ onto $\Lambda_r(T_2)$ such that $h_{\mathscr{L}}(T_1) = h_{L\mathscr{L}}(T_2)$ for each $\mathscr{L} \in \Lambda_r(T_1)$. Is T_1 spatially isomorphic to T_2? The following discussion yields a negative answer. Frampton [7] has shown that any measure preserving transformation is unitarily equivalent to its inverse. On the other hand, Anzai [4] has apparently constructed an ergodic skew-product transformation

$$T_{\gamma\,\alpha}(x, y) \to (x + \gamma, y + \alpha(x)) \quad \text{(additions modulo 1)}$$

where the number γ and the real-valued measurable function α are chosen in such a way that $T_{\gamma.\alpha}$ is not spatially isomorphic to its inverse $T_{\gamma.\alpha}^{-1}$. It is clear that

$\Lambda_i(T_{\gamma,\alpha}) = \Lambda_r(T_{\gamma,\alpha}) = \Lambda_r(T_{\gamma,\alpha}^{-1}) = \Lambda_i(T_{\gamma,\alpha}^{-1})$. The lattice isomorphism L called for in the conjecture can be taken to be the identity. Also $h_{\mathscr{C}}(T_{\gamma,\alpha}) = h_{L\mathscr{C}}(T_{\gamma,\alpha}^{-1}) = 0$ for all $\mathscr{C} \in \Lambda_r(T_{\gamma,\alpha})$. The conjecture, however, is still open for the class of measure preserving transformations with completely positive entropy. Furthermore, the determination of the lattice of reducing subalgebras for any transformation besides those of pure point spectrum is still to be accomplished. To date no transformations have been distinguished by their lattice of reducing subalgebras which could not be distinguished by other means more effectively; nevertheless, the purpose of the present investigation is to introduce a concept which perhaps is fundamental to the isomorphism problem in ergodic theory.

Bibliography

1. L. M. Abramov, *On entropy of flows*, Dokl. Akad. Nauk SSSR **128** (1959), 873–875. (Russian)

2. R. L. Adler, *Generalized commutativity properties of measure preserving transformations* (to appear).

3. H. Anzai, *Ergodic skew product transformations on the torus*, Osaka Math. J. **3** (1951), 83–99.

4. ———, *On an example of a measure preserving transformation which is not conjugate to its inverse*, Proc. Japan Acad. **27** (1951), 517–522.

5. H. Anzai and S. Kakutani, *Bohr compactifications of a locally compact Abelian group*. I, II, Proc. Imp. Acad. Tokyo **19** (1943), 476–480, 533–539.

6. R. R. Bahadur, *Measurable subspaces and subalgebras*, Proc. Amer. Math. Soc. **6** (1955), 565–570.

7. J. Frampton, *Unitary equivalence of a measure preserving transformation and its inverse*, oral communication.

8. P. R. Halmos, *Lectures on ergodic theory*, Math. Soc. of Japan, Tokyo, No. 3, 1956.

9. ———, *Measure theory*, Van Nostrand, New York, 1950.

10. P. R. Halmos and J. von Neumann, *Operator methods in classical mechanics*. II, Ann. of Math. (2) **2** (1942), 332–350.

11. P. R. Halmos and H. Samelson, *On monothetic groups*, Proc. Nat. Acad. Sci. U. S. A. **28** (1942), 254–258.

12. A. N. Kolmogorov, A *new metric invariant of transient dynamical systems and automorphisms in Lebesgue spaces*, Dokl. Akad. Nauk SSSR **119** (1958), 861–864. (Russian)

13. L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953.

14. M. S. Pinsker, *Dynamical systems with completely positive or zero entropy*, Dokl. Akad. Nauk SSSR **133** (1960), 1025–1026. (Russian)

15. L. Pontrjagin, *Topological groups*, Princeton Univ. Press, Princeton, N. J., 1948.

16. V. A. Rokhlin, *Entropy of metric automorphism*, Dokl. Akad. Nauk SSSR **124** (1959), 980–983.

17. ———, *On fundamental ideas of measure theory*, Amer. Math. Soc. Transl. No. 71, 1952.

18. V. A. Rokhlin and Ja. G. Sinaĭ, *Construction and properties of invariant measurable partitions*, Dokl. Akad. Nauk SSSR **141** (1961), 1038–1041.

19. Ja. G. Sinaĭ, *On the notion of entropy of a dynamical system*, Dokl. Akad. Nauk SSSR **124** (1959), 768–771.

International Business Machines Corporation,
Yorktown Heights, New York