SEQUENTIALLY 1-ULC TORI

BY

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1. **Introduction.** A closed set X in Euclidean 3-space E^3 is called *tame* if there exists a homeomorphism h of E^3 onto itself such that $h(X)$ is a polyhedron. A set which is not tame is called *wild*. In this paper, we investigate conditions which determine tameness of an arc in E^3. Examples of wild arcs in E^3 are abundant; see, for example, [3; 8]. Also abundant are conditions implying tameness of an arc; see [7; 10].

Consider the following conditions placed on an arc $\mathscr{A}$ in E^3:

(1) $\mathscr{A}$ lies on a 2-sphere S in E^3.

(2) $\mathscr{A}$ lies on a simple closed curve J in E^3 which is the intersection of a nested sequence of (two-dimensional) tori plus their interiors.

This paper was motivated by a belief that (1) and (2) implied that $\mathscr{A}$ is tame. This turns out not to be the case; the wild arc constructed in [1] is a counterexample. With this in mind, we make the following definition. A sequence $\{M_1, M_2, \cdots\}$ of 2-manifolds in E^3 is *sequentially* 1-ULC if, given $\varepsilon > 0$, there exists a $\delta > 0$ and integer N such that: Whenever $n > N$, and α is a simple closed curve on M_n of diameter less than δ which bounds a disk on M_n, then α bounds a disk of diameter less than ε on M_n.

We now add another condition.

(3) The sequence of tori of condition 2 is sequentially 1-ULC.

Our primary result is that these three conditions imply tameness of the arc $\mathscr{A}$. This theorem yields as a corollary an answer to a question raised by Bing in [3]: No subarc of the "Bing sling" [3] lies on a disk.

A simple closed curve J is said to pierce a disk D if J links Bd D (boundary of D) and $J \cap D$ is a single point. As the "Bing sling" is the only example in the literature of a simple closed curve that pierces no disk, one is now led to a natural question. Can a different simple closed curve $\mathscr{K}$ be constructed where $\mathscr{K}$ pierces no disk, yet lies on a disk? In §3, we show the existence of such a simple closed curve $\mathscr{K}$. That $\mathscr{K}$ lies on a disk will be immediate from its construction. To show that $\mathscr{K}$ pierces no disk, we will use the following. Define $P_{\mathscr{A}}$ to be the set of points of an arc $\mathscr{A}$ at which $\mathscr{A}$ pierces a disk. We set up an alternate condition to (3) given above.

(3′) $P_{\mathscr{A}}$ is dense in $\mathscr{A}$.

Received by the editors February 15, 1963.

(1) Work on this paper supported by NSF Research Grant 15984.

Conditions (1), (2), and (3′) are also shown to imply tameness. This result is then used to establish that $\mathscr{K}$ pierces no disk.

2. No subarc of the "Bing sling" lies on a disk.

THEOREM 1. *If $\mathscr{A}$ is an arc in E^3 such that*

(1) *$\mathscr{A}$ lies on a 2-sphere S in E^3;*

(2) *$\mathscr{A}$ lies on a simple closed curve J in E^3 which is the intersection of a decreasing sequence of tori plus their interiors;*

(3) *The sequence of tori of* (2) *is sequentially* 1-ULC;
then $\mathscr{A}$ is tame.

Proof. We assume without loss of generality that the 2-sphere S is locally polyhedral mod $\mathscr{A}$ [4]. We will use Theorem 6 of [5] to establish that S is locally tame at all non-endpoints of the arc $\mathscr{A}$. Toward this goal, we prove the following.

ASSERTION. Given a non-endpoint p of $\mathscr{A}$, and $\varepsilon > 0$, there exists a $\delta > 0$ such that: if β is a simple closed curve lying in a δ-neighborhood of p, $\beta \cap S = \varnothing$, and β bounds a disk B in E^3, then β bounds a disk B'' in $E^3 - S$ such that B'' lies in an ε-neighborhood of p. This Assertion is a bit weaker than the statement that $E^3 - S$ is locally simply connected at p, which is the hypothesis of Theorem 6 of [5]. However, the Assertion is sufficiently strong so that the proof of Theorem 6 of [5] still remains valid, showing that S is locally tame at p. We now prove the Assertion in six steps, numbered for convenience.

(1) There exists an integer N_1 and a positive number γ such that if α is any simple closed curve on $T_n, n > N_1$, and if α lies in a neighborhood of p of radius γ (which we abbreviate $\mathscr{O}_\gamma(p)$), then either $\alpha \cap S \neq \varnothing$, or α bounds a disk on T_n. Step 1 is devoted to a justification of this statement.

The arc $\mathscr{A}$ is now extended to form a simple closed curve K, $\mathscr{A} \subset K \subset S$. We assume $K \cap J = A$, i.e., $K \cup J$ is a θ-curve. Call the end points of $\mathscr{A}$ a and b, and call the sequence of tori given by our hypothesis $\{T_1, T_2, T_3, \cdots\}$; these may be taken to be polyhedral [4] and in general position. Let P be a plane missing p and separating a from b in E^3, with P in general position with respect to $\{T_1, T_2, \cdots\}$. For fixed n, $T_n \cap P$ is a collection of simple closed curves $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_k$. Let L_i be the disk on P bounded by λ_i. There exists an integer N_1 such that if n were chosen above to be larger than N_1, then L_i would not intersect both $\mathscr{A}$ and $J - \mathscr{A}$, for $i = 1, 2, \cdots, k$. This fact can be used to show that at least one of the λ_i's links J (in the sense of §9 of [4]). Thus, if $n > N_1$, at least one λ_I will link J. Note that such a λ_I could not bound a disk on T_n, but must lie on T_n in a nontrvial way.

There exists a positive number γ such that if a simple closed curve α lies in $\mathscr{O}_\gamma(p)$, then α does not intersect the plane P, and α does not link the simple closed curve $(J \cup K) - \operatorname{Int} \mathscr{A}$. Now let us suppose that α lies on T_n, for $n > N_1$, and α does not bound a disk on T_n. Then α and the λ_I of the preceding paragraph can be joined by an annulus on T_n; since λ_I links J, α must also link J. Since α does not link $(J \cup K) - \operatorname{Int} \mathscr{A}$, it follows from Theorem 9 of [4] that α links K. Since K lies on the 2-sphere S, it follows that $\alpha \cap S \neq \varnothing$, which completes step 1.

(2) We assume that diameter $\mathscr{A} > \varepsilon/3$. There exists a $\delta_1 > 0$ such that any δ_1-simple closed curve (a δ_1-set is a set of diameter less than δ_1) on S bounds an $\varepsilon/3$-disk on S. In particular $\delta_1 < \varepsilon/3$, of course. We use the sequential 1-ULC hypothesis to select a $\delta_2 > 0$ and integer N_2 such that any δ_2-simple closed curve on T_n, $n > N_2$, which bounds a disk on T_n bounds a $\delta_1/3$-disk on T_n; in particular $\delta_2 < \delta_1/3$.

(3) We now select a disk U on S containing $\mathscr{A}$ on its interior, "thin" enough so U has the following property: If W is any open set containing $\mathscr{A}$, and X is an open set containing S, then there exists a homeomorphism H of E^3 onto itself such that $H(S) = S$, $H =$ identity on $E^3 - X$, $H(U) \subset W$, and H moves no point of E^3 more than the minimum of the two numbers $\delta_2/3$ and $\gamma/2$.

The existence of such a disk U follows from the fact that S is locally tame, mod $\mathscr{A}$. To see this, note that if we had asked in the preceding paragraph that H be defined only on the 2-sphere S, then it is clear how to select U. In fact, in this case, H could be defined to be the identity on a small disk D_w, with $\mathscr{A} \subset \operatorname{Int} D_w \subset D_w \subset \operatorname{Int} U$. On the set $S - D_w$, where H is not the identity, H is isotopic to the identity. Furthermore, this set is tame, since it misses $\mathscr{A}$; hence it is bicollared in E^3. We now extend H to the bicollar in the obvious way, so that $H =$ identity, except on this bicollar. By choosing the bicollar to lie in X, we find H satisfies all required properties.

(4) We now select the $\delta > 0$ required in the Assertion, by requiring that $\delta < \delta_2/6$, $\delta < \gamma/2$ and $\mathcal{O}_\delta(p) \cap [S - U] = \varnothing$. We now prove the Assertion. Let β be a simple closed curve in $\mathcal{O}_\delta(p)$ such that β bounds a disk B, and $\beta \cap S = \varnothing$. We may assume that B lies in $\mathcal{O}_\delta(p)$ simply by pushing it there without moving β.

(5) In this step, we show that β bounds a disk B' of diameter less than δ_1, and such that $B' \cap \mathscr{A} = \varnothing$. Let m be an integer, $m > N_1$, $m > N_2$, so that $[T_m \cup \operatorname{Int} T_m] \cap \beta = \varnothing$. Let $\operatorname{Int} T_m$ be the open set W of step 3, and let X of step 3 be sufficiently small so that $X \cap \beta = \varnothing$. Step 3 guarantees the existence of a homeomorphism H, and the disk $H(B)$ has certain nice properties: Firstly, its diameter is less than diameter $B + \delta_2/3 + \delta_2/3 < \delta_2$. Secondly, β bounds $H(B)$, by choice of X. Most important, $S \cap T_m \cap H(B) = \varnothing$. This follows since $B \cap S \subset U$ so $H(B) \cap H(S) \subset H(U)$, but since $H(S) = S$, we have $H(B) \cap S \subset H(U)$, and since $H(U) \subset \operatorname{Int} T_m$, we have $H(B) \cap S \cap T_m = \varnothing$.

We assume without loss of generality that $H(B)$ is polyhedral on its interior and in general position with respect to T_m. Thus, $H(B) \cap T_m$ is a collection of simple closed curves $\alpha_1, \alpha_2, \cdots, \alpha_r$. Since $H(B) \cap T_m \cap S = \varnothing$, each α_i does not intersect S. By step 1, each α_i bounds a disk on T_m. Since $H(B)$ has diameter less than δ_2, and $\alpha_i \subset H(B)$, it follows that each α_i bounds a $\delta_1/3$-disk on T_m, by step 2. The usual disk replacement process (see step 6 for details) is now performed on the disk $H(B)$, yielding a disk B', of diameter less than diameter $H(B) + \delta_1/3 + \delta_1/3 < \delta_1$. Furthermore $B' \cap \operatorname{Int} T_m = \emptyset$, so $B' \cap \mathscr{A} = \emptyset$.

(6) The disk B' is placed in general position with respect to S, and the usual disk replacement process used to modify B' into a new disk B''. That is, $B' \cap S$ is a collection of simple closed curves $l_1, l_2, \cdots, l_t$. Note that $l_i \cap \mathscr{A} = \emptyset, i = 1, 2, \cdots, t$. An "innermost" l_i on S is selected, and the disk it bounds on B' is replaced by the $\varepsilon/3$-disk it bounds on S. (See step 2 for why we have an $\varepsilon/3$-disk.) This new disk on B' is pushed slightly to one side of S; this can be done because the new disk cannot contain $\mathscr{A}$ on its interior, as diameter $\mathscr{A} > \varepsilon/3$. Thus, this new disk is polyhedral.

This process is continued with another l_i, until all intersection is eliminated, yielding B''. We have $B'' \cap S = \emptyset$, β is the boundary of B'', and diameter B'' $<$ diameter $B' + \varepsilon/3 + \varepsilon/3 < \varepsilon$. This establishes the Assertion.

It remains to show that $\mathscr{A}$ is tame at its end points a and b. Now that we know that $\mathscr{A}$ is locally tame mod $a \cup b$, it is easy to construct arbitrarily small 2-spheres around a (or b) out of the tori $\{T_i\}$, such that each 2-sphere intersects $\mathscr{A}$ in exactly one point. Thus $\mathscr{A}$ will be tame at its end points by satisfying Properties P and Q of [10]. We omit details of this construction as they are tedious, and similar to the proof of Theorem 2. Indeed, all that we really need to establish Corollary 1 is that $\mathscr{A}$ is locally tame on its interior.

COROLLARY. 1. *No subarc of the "Bing sling" [3] lies on a disk.*

Proof. If some subarc does lie on a disk, then a smaller subarc lies on a 2-sphere S, by §5 of [5]. We observe that the "Bing sling" satisfies Properties 2 and 3 of Theorem 1, with the necessary tori being provided by its very construction. Thus, this small subarc is tame, by Theorem 1, which is a contradiction to the fact that it pierces no disk.

3. **The simple closed curve $\mathscr{K}$ which pierces no disk, yet lies on a disk.** Using a technique developed by Bing [2], one can construct a 2-sphere $\mathscr{S}$ in E^3 whose wild points from a wild, cellular arc in E^3, which we call ξ. For an exact description, see [1]. The arc ξ can be completed to a simple closed curve Z on $\mathscr{S}$, and the same argument which shows that ξ is cellular (see [9]) will establish that Z is the intersection of a decreasing sequence of tori plus their interiors (note: these tori cannot be sequentially 1-ULC, by Theorem 1).

If any non-endpoint x of ξ had the property that ξ pierced a disk at x, then one could use the symmetry given by the construction of ξ to show that ξ pierces a disk at a dense subset of itself, i.e., P_ξ is dense in ξ. This, however, gives us a contradiction on account of the following.

THEOREM 2. *If $\mathscr{A}$ is an arc in E^3 such that*
(1) *$\mathscr{A}$ lies on a 2-sphere S in E^3;*

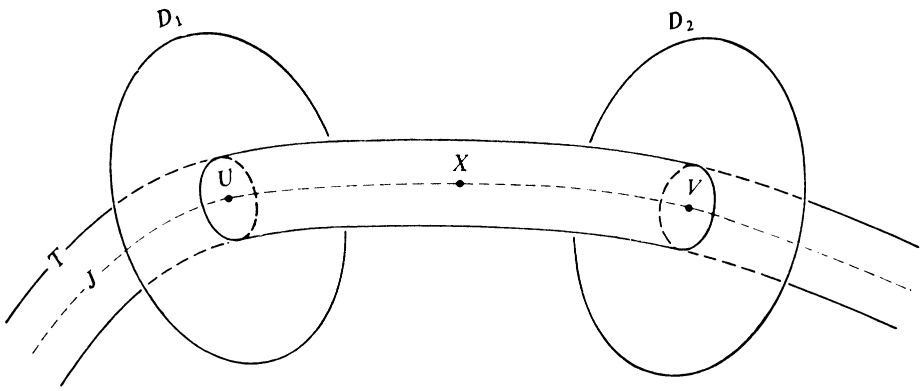
(2) *$\mathscr{A}$ lies on a simple closed curve J in E^3 which is the intersection of a nested sequence of tori plus their interiors;*

(3) *$P_{\mathscr{A}}$ is dense in $\mathscr{A}$;*

then $\mathscr{A}$ is tame.

Proof. Let x be any non-endpoint of $\mathscr{A}$. Given $\varepsilon > 0$, we will show that there exists a 2-sphere $\mathfrak{S}$ of diameter less than ε, such that $x \in \operatorname{Int} \mathfrak{S}$, and $\mathfrak{S} \cap J$ is a set of two points. This will establish that $\mathscr{A}$ is locally tame at all non-endpoints [10]. The endpoints of $\mathscr{A}$ can then be taken care of by the same method, with only slight changes necessary in the construction of $\mathfrak{S}$.

The 2-sphere $\mathfrak{S}$ will be constructed as shown in the figure. That is, an annulus of the torus T will connect two disks D_1 and D_2 which are pierced by J on opposite sides of x. The 2-sphere $\mathfrak{S}$ will consist of the annulus plus one subdisk of each D_i, $i = 1,2$. Of course, it must be justified that there is a torus and two disks which intersect as nicely as shown in the figure. This is done in eight steps.



(1) J cannot possibly be expressed as the intersection of a decreasing sequence of 3-cells in E^3. This follows easily from the fact that J is an absolute neighborhood retract. Thus, there exists a positive number ε_1 such that any 2-sphere lying completely in an ε_1-neighborhood of J cannot contain J in its interior. We assume that $\varepsilon < \varepsilon_1$, and that diameter $J > \varepsilon$.

(2) Select points u and v of $\mathscr{A}$ such that the subarc $\overline{uxv}$ of J with endpoints u and v and non-endpoint x has diameter less than $\varepsilon/2$. The other subarc of J with endpoints u and v will be denoted by $\overline{uyv}$. The points u and v are also selected so that J pierces a disk D_1 at u and a disk D_2 at v. We assume that D_1 and D_2 are sufficiently small so that diameter $(\overline{uxv} \cup D_1 \cup D_2) < \varepsilon/2$ and so that D_1 and D_2 are disjoint.

(3) In this step we select the torus T of the figure. Select a ray R starting at x, such that R and $[\overline{uxv} \cup D_1 \cup D_2]$ are disjoint. R may be taken to be locally polyhedral mod x. There is a positive number η, such that if A is an arc in E^3 of diameter less than η, and if A intersects both $\overline{uxv}$ and $\overline{uyv}$, then A intersects $D_1 \cup D_2$. This follows from the fact that J pierces the disks D_1 and D_2 at u and v, respectively. We also assume that $\eta <$ dist $(R, \overline{uyv})$ and $\eta < \varepsilon$.

Let D_1' be an $\eta/8$-disk with $u \in \text{Int } D_1' \subset D_1' \subset \text{Int } D_1$; let D_2' be similarly situated in D_2. We choose γ sufficiently small so that a γ-neighborhood of J intersects D_i only in a subset of D_i', $i = 1,2$. We have $\gamma < \eta/8$, of course. The torus T is now selected from our sequence so T lies in this γ-neighborhood of J. By applying [4], we may assume that T is polyhedral, that D_1 is locally polyhedral mod u, that D_2 is locally polyhedral mod v; furthermore, we assume that T, D_1, D_2 and R are in general position.

(4) At the present time, T may intersect D_1 and D_2 very differently from the way indicated in the figure. We now simplify this intersection.

Let us examine a simple closed curve L of $T \cap [D_1 \cup D_2]$. L may be classified thus:

1. L bounds a disk on T;
2. L does not bound a disk on T.

L may also be classified in a different way. Assume for convenience that $L \subset D_1$.

1′. L bounds a subdisk E_1 of D_1 which does not contain u.

2′. The subdisk E_1 of D_1 bounded by L does contain u.

We show that L is of Type 1 if and only if L is of Type 1′, that is, these classifications are really the same.

If L is of Type 1, then L does not link J. Thus, L is also of Type 1′.

If L is of Type 1′, then using techniques of Theorem 1 of [6], one can show that L bounds a disk which does not intersect J, and whose interior does not intersect T. If L were of Type 2, then by cutting T along L and inserting two copies of this disk, one could construct a 2-sphere in contradiction to step 1. Thus, if L is of Type 1′, then L must also be of Type 1.

(5) All L of Type 1 are now removed. That is, we suppose that L is an "innermost" simple closed curve of Type 1 in D_1. The subdisk E_1 of D_1 bounded by L will not contain any simple closed curves of Type 2. This is obvious from the equivalence of Types 1 and 2 with Types 1′ and 2′. Thus, T may be altered by removing the disk bounded by L on T, replacing it by E_1, then pushing to one side slightly. This process is repeated until all Type 1 simple closed curves have been removed, forming a new torus T'.

We now show that $J \subset \text{Int } T'$. The first stage of the alteration of T consisted of interchanging two disks. This will change Int T only by adding to or subtracting from it the 3-cell bounded by these two disks. J cannot lie in this 3-cell, by step 1. The same line of reasoning is continued during each alteration in the construction of T', showing that $J \subset \text{Int } T'$.

(6) If t is a point of $T' - D_1 - D_2$, then t can be joined to J by an arc $A(t)$ which is disjoint from $D_1 \cap D_2$, and which is of diameter less than $\eta/4$. To see this we examine two cases: either t lies on T, or t lies very close D_1' or D_2'. The latter case is clear from the choice of D_1' and D_2', in fact, the arc will have diameter less than $\eta/8$. In the former case, we begin by joining t to J with an $\eta/8$ arc, which may intersect D_1' (or D_2'). If it does, we modify it by bending it just before it hits D_1' so it instead runs down the side of D_1' to J. This bent arc will have diameter less than $\eta/8 + \eta/8 = \eta/4$, as desired.

(7) We now look at the components $C_1, C_2, \cdots, C_m$ of $T' - D_1 - D_2$. If t and t' are both points of the same component, say C_1, then $A(t)$ and $A(t')$ both intersect the same component of $J - u - v$. Otherwise, let $\overline{tt'}$ be an arc of C_1 joining t and t'. Let s and s' be two points of this arc such that $A(s)$ and $A(s')$ intersect different components of $J - u - v$, and such that the subarc $\overline{ss'}$ of $\overline{tt'}$ has diameter less than $\eta/2$. The path $[A(s) \cup \overline{ss'} \cup A(s')]$ has diameter less than η, contradicting the definition of η. Thus, each C_i lies in an $\eta/4$-neighborhood of either $\overline{uxv}$ or $\overline{uyv}$.

(8) The desired 2-sphere $\mathfrak{S}$ may now be selected. Since the ray R hits T' an odd number of times, and since $R \cap [D_1 \cup D_2] = \emptyset$, R hits some component C_N of $T' - D_1 - D_2$ an odd number of times. C_N cannot lie in an $\eta/4$-neighborhood of $\overline{uyv}$, since $C_N \cap R \neq \emptyset$, and $\eta < \operatorname{dist}(R, \overline{uyv})$. Thus, C_N lies in an $\eta/4$-neighborhood of $\overline{uxv}$.

C_N cannot be all of T', as

$$\operatorname{diam}(C_N) < \operatorname{diam}(\overline{uxv}) + \eta/4 + \eta/4 < \varepsilon,$$

whereas Int T' contains J, a set of diameter larger than ε, so T' has diameter larger than ε. Thus, C_N will be an annulus of T', with two boundary simple closed curves of Type 2', by steps 4 and 5. Furthermore, both of these simple closed curves do not lie on D_1. If they did, the annulus lying between them on D_1 could be added to C_N to produce a torus T'' disjoint from J. Since $R \cap T''$ would contain an odd number of points. J would lie in Int T''. Thus diameter $(T'') > \varepsilon$. But diameter $(T'') <$ diameter $(\overline{uxv} \cup D_1 \cup D_2) + \eta/4 + \eta/4 < \varepsilon$ which gives us a contradiction. By similar reasoning, both of these simple closed curves do not lie on D_2.

Let $\mathfrak{S}$ be the 2-sphere composed of C_N plus the subdisk of D_1 bounded by a boundary simple closed curve of C_N, plus the subdisk of D_2 bounded by the other boundary simple closed curve of C_N. Then,

$$\operatorname{diam}(\mathfrak{S}) < \operatorname{diam}(\overline{uxv} \cup D_1 \cup D_2) + \eta/4 + \eta/4 < \varepsilon, \text{ and}$$

$\mathfrak{S} \cap J$ will be just the two points u and v. That x lies in Int $\mathfrak{S}$ follows from the fact that $\mathfrak{S} \cap R$ consists of an odd number of points. This completes the proof of Theorem 2.

It is a simple matter to construct a simple closed curve $\mathscr{K}$ such that $\mathscr{K}$ looks locally just like the arc ξ, and with $\mathscr{K}$ lying on a 2-sphere in E^3. To do this the construction of [1] is simply performed with eyebolts hooking in a circular fashion at each stage. Thus, $\mathscr{K}$ lies on a 2-sphere in E^3, yet pierces no disk in E^3.

QUESTION. Is $\mathscr{K}$ homogeneously embedded in E^3? Precisely, given points p and q in $\mathscr{K}$, is there a homeomorphism h of E^3 onto itself such that $h(\mathscr{K})=\mathscr{K}$ and $h(p) = q$?

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