COMPLETELY MONOTONE SEQUENCES
AS INVARIANT MEASURES(1)

BY
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Introduction. A completely monotone sequence $a = (a_0, a_1, a_2, \cdots)$ of real
numbers is a sequence which satisfies the conditions

$$(\Delta^n a)_k \geq 0 \quad (n = 0, 1, 2, \cdots; \quad k = 0, 1, 2, \cdots)$$

where the operator $\Delta$, acting on any sequence $a$, is the sequence $\Delta a$ defined by

$$(\Delta a)_k = a_k - a_{k+1} \quad (k = 0, 1, 2, \cdots).$$

During the first part of the paper we will suppose that the completely monotone
sequence $a$ also satisfies

$$(2) \quad \sum_{n=0}^{\infty} a_n = 1.$$  

Using the sequence $a$, we will construct a probability space $(X, \mathcal{B}, \mu)$ and a
measure-preserving transformation $T$ on it. This transformation will not, in gen-
eral, be ergodic. We then make use of the Kryloff-Bogoliouboff decomposition [4]
(as extended by Oxtoby [5]) of a measure into its ergodic parts. Writing $\mu$ as an
integral of measures for which $T$ is ergodic, and observing the effect of this de-
composition on the sequence $a$, we obtain the classical reduction of $a$ to the form

$$a_k = \int_0^1 t^{k} dF(t) \quad (k = 0, 1, 2, \cdots).$$

An inversion formula, similar to Feller's [2], giving $F$ in terms of $a$ can also be
obtained in this way. In the last part of the paper we investigate the transformation
$T$ itself, and leave several interesting questions unanswered.

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$(X, \mathcal{B}, \mu)$ and the transformation $T$. Let us assume, then, that a sequence $a$

satisfying (1) and (2) has been given and proceed with the construction of $(X, \mathcal{B}, \mu)$.

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Let $I_\infty$ be the one-point compactification of the non-negative integers, with $\infty$ the added point. We form the compact space

$$X' = \bigoplus_{i=1}^\infty X_i \quad (X_i = I_\infty, i = 1, 2, 3, \ldots)$$

and consider the closed subset $X'$ of $X$ consisting of all sequences in $X$ which are increasing (not necessarily strictly increasing), thinking of $\infty$ as being greater than every integer.

By a rectangle $(r_1, r_2, r_3, \ldots, r_k)$ where $0 \leq r_1 \leq r_2 \leq \cdots \leq r_k$, is meant all sequences in $X'$ beginning with $r_1, r_2, r_3, \ldots, r_k$, i.e., all sequences $\omega$ in $X$ for which $\omega_i = r_i (i = 1, 2, 3, \ldots, k)$. The measure of a rectangle $R = (r_1, r_2, \ldots, r_k)$ is defined as follows:

$$\mu(R) = \begin{cases} (\Delta^{k-1} a)_{r_k} & \text{if } r_k \neq \infty, \\ 0 & \text{if } r_k = \infty. \end{cases}$$  

(3)

Let $\mathcal{A}$ be the (finitely additive) algebra consisting of all sets of $X'$ which depend on only a finite number of coordinates, i.e., $\mathcal{A} = \bigcup \mathcal{A}_n$ where $\mathcal{A}_n$ consists of sets which can be written as the union of rectangles of length $n$. The measure of any set in $\mathcal{A}$ may now be defined by writing the set as a union of disjoint rectangles. This measure can be extended to a countably additive measure (again called $\mu$) defined on $\mathcal{B}$, the smallest $\sigma$-field containing $\mathcal{A}$. Since $\mu$ is a regular measure on $\mathcal{A}$, this can be accomplished, for instance, by means of a theorem of Alexandroff [2, p. 138].

We let $X$ be the subset of $X'$ consisting of those sequences in which $\infty$ does not appear. The complement of $X$ in $X'$ may be written

$$X' = \bigcup_{n=1}^\infty \bigcup_{i_1 \leq i_2 \leq \cdots \leq i_n} (i_1, i_2, \ldots, i_n, \infty).$$

This set clearly has measure zero, no matter what the sequence $a$ may be, so we see that $X$ has measure one. From now on when we speak of rectangles we mean rectangles restricted to the space $X$.

The transformation $T$ is defined at a point $\omega = (\omega_1, \omega_2, \omega_3, \ldots)$ of $X$ as follows:

$$T \omega = \begin{cases} (0,0,\ldots,0, \omega_n + 1, \omega_{n+1}, \ldots) & \text{if } \omega_1 = \omega_2 = \omega_3 = \cdots, \\ (0,0,0,\ldots) & \text{if } \omega_1 = \omega_2 = \omega_3 = \cdots. \end{cases}$$  

(4)

Suppose $R = (r_1, r_2, \ldots, r_n)$ is a rectangle in $X$. If $r_1 \neq 0$, the image of $R$ under $T^{-1}$ is again a rectangle: $(r_1 - 1, r_2, \ldots, r_n)$. If $0 = r_1 = r_2 = \cdots = r_k$ but $r_{k+1} \neq 0$ where $k < n$, the inverse image of $R$ is the rectangle

$$(r_{k+1} - 1, r_{k+1} - 1, \ldots, r_{k+1} - 1, r_{k+2}, \ldots, r_n).$$
If, finally, \( 0 = r_1 = r_2 = \cdots = r_n \), the inverse image of \( R \) is, except for the constant sequences, the disjoint union

\[
\bigcup_{p=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcup_{i=0}^{j-1} (i_1, \ldots, i, j).
\]

Hence in any case the inverse image under \( T \) of any rectangle \( R \) is essentially a union of rectangles and so in particular is measurable. It is possible to calculate the measure of \( T^{-1}R \) directly from formula (3) to show that \( T \) is measure-preserving, but the following "geometric" description of \( T \) is more enlightening but not essential to the development of the paper. This idea has been used before, for instance in [1].

We construct a sequence of line segments, \( B_0, B_1, B_2, \cdots \), one above the other as in Figure 1a so that the segment \( B_n \) has length \( a_n \). Let us call the object so obtained a building and also regard it as a measure space by giving to each linear Borel set in the building its ordinary (linear) Lebesgue measure. Now if there were a measure-preserving transformation, say \( S \), defined on \( B_0 \), we could construct a measure-preserving transformation \( T \) on the building \( B \) by sending any point to the point directly above on the next floor, if possible, otherwise by descending straight down to \( B_0 \) and moving by \( S \). Then \( S \) is simply the transformation induced by \( T \) on \( B_0 \), of which we will speak later. What remains then is to define the transformation \( S \) on \( B_0 \).

We can partially define \( S \) on \( B_0 \) by writing \( B_0 \) itself as a building as in Figure 1b so that the segment \( B'_n \) has length \( a_n - a_{n+1} \). \( S \) is now partially defined on \( B_0 \) simply by rising one floor, if possible. To continue its definition, we must write \( B'_0 \) as a building in a way similar to the way we wrote \( B_0 \) as a building. Eventually then, \( S \) and hence \( T \) will be completely defined except perhaps on a set of measure zero. That we continue to get a building at each stage is due to the requirement (1) on the sequence \( a \). The floors of the \( n \)th stage building will have lengths \( (\Delta^{n-1}a)_0, (\Delta^{n-1}a)_1, \cdots \).

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The transformation $T$ defined on $X$ is said to be conjugate to the transformation $S$ on $Y$ if there is a 1-1 measure-preserving map $\phi$ from (almost all of) $X$ onto (almost all of) $Y$ such that $\phi^{-1}$ is measurable and $T = \phi^{-1} \circ S \circ \phi$. We can now show the relation between the two measure-preserving transformations defined so far.

**Lemma.** The transformation $T$ defined on $B$ is conjugate to the transformation $T$ defined on $X$ by (4).

**Proof.** We can give coordinates to a point $\omega$ in $B$. Let $\omega_1$ be the integer satisfying $\omega \in B_{\omega_1}$. Drop $\omega$ to the first floor of $B$ and call this point $\omega'$. Let $\omega_2$ be the integer satisfying $\omega' \in B'_{\omega_2}$. Drop $\omega'$ to the first floor of $B_0$ and call this point $\omega''$. Let $\omega_3$ satisfy $\omega'' \in B''_{\omega_3}$, etc. This correspondence $\omega \to (\omega_1, \omega_2, \cdots)$ from $B$ to $X$ is easily seen to establish the conjugacy of $T$ on $B$ and $T$ on $X$.

The measures $\mu$ for which $T$ is ergodic. Since we will have to speak of several completely monotone sequences and their corresponding measures, we now write $\mu_\alpha$ for the measure defined on $X$ by means of the sequence $\alpha$, and $T_\alpha$ for the corresponding transformation previously called simply $T$. Of course all the transformations $T_\alpha$ are identical, but we may wish to distinguish a particular measure $\mu_\alpha$ which $T$ preserves. In this case, we write $T_\alpha$. For instance we could ask are $T_\alpha$ and $T_\beta$ conjugate?

If $\alpha$ and $\beta$ are completely monotone sequences which also satisfy (2), clearly any convex combination $\alpha \alpha + \beta \beta$ is another. In fact, because of the linearity of $\Delta$, (5) $\alpha \mu_\alpha + \beta \mu_\beta = \mu_{\alpha \beta + \beta \beta}$.

Also, given a completely monotone sequence $\alpha \neq (1,0,0,\cdots)$, we can form two others, namely $\sigma \alpha$ and $\tau \alpha$ defined by

\begin{align}(\sigma \alpha)_n &= a_{n+1}/(1 - a_0), \\
(\tau \alpha)_n &= (a_n - a_{n+1})/a_0 \quad (n = 0,1,2,\cdots).
\end{align}

These sequences also satisfy (2) if $\alpha$ does.

Let $T_\alpha | C$ be the transformation induced by $T_\alpha$ on the subset $C$ of $X$ of all sequences in $X$ having first coordinate zero, and let $T_\alpha | \bar{C}$ be the induced transformation on $\bar{C} = X - C$. Hence for instance $T_\alpha | C$ is defined at a point $\omega$ of $C$ by $T_\alpha | C(\omega) = T^k \omega$ where $k$ is the first positive integer for which $T^k(\omega) \in C$. $T_\alpha | C$ is a measure preserving transformation on $C$. We call the measure $\rho$ defined on subsets $E$ of $C$ by $\rho(E) = \mu_\alpha(E)/\mu_\alpha(C)$ the induced measure on $C$.

**Lemma.** Suppose $\alpha \neq (1,0,0,\cdots)$. Then the transformation $T_\alpha | C$ on $C$ with its induced measure, is conjugate to the transformation $T_{\alpha \alpha}$ on $X$. Also, $T_\alpha | \bar{C}$ on $\bar{C}$ with its induced measure is conjugate to $T_{\alpha \alpha}$.
Proof. A glance at the buildings is probably enough to convince the reader of this fact, but we will construct mappings $\phi: C \to X$ and $\psi: \bar{C} \to X$ which give the conjugacy explicitly. Let $\phi(\omega) = (\omega_2, \omega_3, \cdots), \omega \in C$ and $\psi(\omega) = (\omega_1 - 1, \omega_2 - 1, \cdots), \omega \in \bar{C}$. Suppose $R = (r_1, r_2, \cdots, r_k)$ is a rectangle in $X$. We must show that

$$\mu_\phi(R) = \mu_\phi(\psi^{-1}R) / \mu_\phi(C)$$

and

$$\mu_\psi(R) = \mu_\psi(\psi^{-1}R) / \mu_\psi(\bar{C}).$$

Since $\phi^{-1}(R) = (0, r_1, r_2, \cdots, r_k)$, from (3) and (6) we get

$$\mu_\phi(R) = (\Delta^{k-1}a)_{r_k} = (1 / a_0) (\Delta^k a)_{r_k}$$

$$= (1 / \mu_\phi(C)) \mu_\phi(\psi^{-1}R).$$

Similarly, $\psi^{-1}(R) = (r_1 + 1, r_2 + 1, \cdots, r_k + 1)$ so that

$$\mu_\psi(R) = (\Delta^{k-1}a)_{r_k} = (1 / (1 - a_0)) (\Delta^{k-1}a)_{r_k+1} = (1 / \mu_\psi(\bar{C})) \mu_\psi(\psi^{-1}(R)).$$

These equations show that $\phi$ and $\psi$ are indeed measure-preserving isomorphisms between $C$ with its induced measure and $X$, and between $\bar{C}$ with its induced measure and $X$, resp. If $\omega = (0, \omega_2, \omega_3, \cdots)$ is any point in $C$, we see that $\phi \circ T_a | C(\omega) = T(\omega_2, \omega_3, \cdots) = T \circ \phi(\omega) = T_{\phi a} \circ \phi(\omega)$ and in the same way, we can show that $\psi \circ T_a | \bar{C} = T_{\psi a} \circ \psi$, proving the lemma.

**Lemma.** If $T_a$ is ergodic, then $a$ is of the form

$$(7) \quad a_n = \theta^n (1 - \theta) \quad (n = 0, 1, 2, \cdots)$$

for some $\theta$, $0 \leq \theta < 1$. (We set $\theta^0 = 1$).

**Proof.** Suppose $T_a$ is ergodic. If $a = (1, 0, 0, \cdots)$ then certainly $a$ has the form required in the lemma. We exclude this case from what follows. Hence the induced transformations $T_a | C$ and $T_a | \bar{C}$ are defined, and they also are ergodic, since any transformation induced by an ergodic transformation is itself ergodic. Now by the previous lemma these induced transformations are conjugate to $T_{\phi a}$ and $T_{\psi a}$ respectively so that $T_{\phi a}$ and $T_{\psi a}$ are ergodic. Or what comes to the same thing $T$ is ergodic with respect to the three measures $\mu_{\phi a}, \mu_{\phi a}$ and $\mu_{\psi a}$. But by (5)

$$\mu_a = (1 - a_0) \mu_{\phi a} + a_0 \mu_{\psi a}.$$ This implies, as is well known, that $\mu_a = \mu_{\phi a} = \mu_{\psi a}$. In particular, $a_n = a_{n+1} / (1 - a_0)$, so that $a_n = (1 - a_0)^n a_0 (n = 0, 1, 2, \cdots)$. Setting $\theta = 1 - a_0$, we get the lemma. The measure $\mu_a$, where $a$ is given by (7), will sometimes be written $\mu_\theta$.

We conclude this section with one more lemma.

**Lemma.** If $\mu$ is a probability measure on $X$ for which $T$ is measure-preserving, then $\mu$ is of the form $\mu_a$ for some $a$ satisfying (1) and (2).
Proof. We must set \( a_n = \mu(n) \), where \((n)\) is the rectangle consisting of all sequences in \( X \) beginning with \( n \). Given any rectangle \((r_1, r_2, \ldots, r_k)\), we may write it as the image, under an appropriate power of \( T^{-1} \), of the rectangle \((r_k, r_k, \ldots, r_k)\).

Hence
\[
\mu(r_1, r_2, \ldots, r_k) = \mu(r_k, r_k, \ldots, r_k).
\]

Now by induction on \( n \) we can prove that
\[
(A^a)_k = \mu(k, k, \ldots, k) \quad (k = 0, 1, 2, \ldots).
\]

For \( n = 0 \), this is just the definition of \( a_k \). Suppose \((8)\) is true for \( n = m - 1 \). Then
\[
(A^a)_k = (A^{m-1}a)_k - (A^{m-1}a)_{k+1}
\]
\[
= \mu(k, k, \ldots, k) - \mu(k + 1, k + 1, \ldots, k + 1)
\]
\[
= \sum_{i=k}^{\infty} \mu(k, k, \ldots, k, i) - \mu(k + 1, k + 1, \ldots, k + 1)
\]
\[
= \sum_{i=k+1}^{\infty} \mu(k, k, \ldots, k + 1, i) - \mu(k + 1, k + 1, \ldots, k + 1) + (k, k, \ldots, k)
\]
\[
= \sum_{i=k+1}^{\infty} \mu(k + 1, k + 1, \ldots, k + 1, i) - \mu(k + 1, k + 1, \ldots, k + 1)
\]
\[
+ \mu(k, k, \ldots, k) = \mu(k, k, \ldots, k).
\]

This proves \((8)\) for all \( n \) and hence proves the lemma.

The decomposition of \( \mu \). Let us first discuss the continuity properties of the transformation \( T \).

Lemma. The transformation \( T \) on the set \( X \) is continuous. Moreover, \( T \) restricted to the subset \( U \) of \( X \) consisting of unbounded points

\[
U = \{ \omega \in X : \lim_{n \to \infty} \omega_n = \infty \}
\]

is one-to-one and onto.

Proof. The second statement of the lemma is clear. To prove the first, suppose \( \rho \) is a point in \( X \) and that \( \rho^n \) is a sequence of points in \( X \) converging to \( \rho \). If \( \rho \) is
not a constant sequence, \( T \rho \) is determined by a finite number \( k \) of coordinates of \( \rho \), say \( \rho_1, \cdots, \rho_k \). But for large \( n \) the first \( k \) coordinates of \( \rho^n \) must also be \( \rho_1, \cdots, \rho_k \), and hence \( T(\rho^n) \) converges to \( T \rho \). On the other hand, if \( \rho \) is a constant sequence, the first \( l \) coordinates of \( \rho^n \) are eventually all the same so that \( T(\rho^n) \) converges to \((0,0,\cdots)\) which is of course \( T \rho \).

Thus we can say that the system \((T,U)\) is a Borel system. As Oxtoby has shown, the essential features of the Kryloff-Bogoliouboff decomposition remain valid in such a system. Let us emphasize the fact that the results of the Kryloff-Bogoliouboff paper do not apply to our situation directly, although perhaps through a modification of the space \( X \) (or of its topology) they could be made applicable.

The result that we make use of is this: to almost every point \( \omega \) of \( X \) (i.e., to every point in a set which has measure one under any invariant measure) we can assign an ergodic measure \( \mu_\omega \) in such a way that, for any bounded Borel-measurable function \( f \) on \( X \), the function

\[
\int fd\mu_\omega
\]

is measurable and, moreover, for any invariant measure \( \mu_\alpha \) on \( X \), we have

\[
\mu_\alpha(R) = \int_X \mu_\omega(R)d\mu_\alpha(\omega)
\]

for any rectangle \( R \) in \( X \).

Since \( \mu_\omega \) is ergodic, we have seen (7) that \( \mu_\omega = \mu_\theta \) for some \( \theta \) between 0 and 1. Let this number be \( \theta(\omega) \). If, in equation (9), we put for \( f \) the characteristic function of the rectangle \( (0) \), we see that

\[
\theta(\omega) = 1 - \int_X fd\mu_\omega
\]

is a measurable function on \( X \). If we let \( R \) be the rectangle \( (n) \), equation (10) yields

\[
a_n = \int_X \mu_{\theta(\omega)}(n)d\mu_\omega(\omega) = \int_X \theta(\omega)^n(1-\theta(\omega))d\mu_\omega(\omega).
\]

Let \( G_\omega(x) \) be the distribution function for \( \theta(\omega) \) with respect to the measure \( \mu_\omega \), i.e.,

\[
G_\omega(x) = \mu_\omega\{\theta(\omega) \leq x\}
\]

so that (11) becomes

\[
a_n = \int_0^1 t^n(1-t)dG_\omega(t) \quad (n = 0,1,2,\cdots).
\]

Setting \( F_\omega(x) = \int_0^x (1-t)dG_\omega(t) \) we get the usual form of the representation:

\[
a_n = \int_0^1 t^n dF_\omega(t) \quad (n = 0,1,2,\cdots).
\]
We are now in a position to prove the following theorem.

**Theorem.** If \( b = (b_0, b_1, b_2, \cdots) \) is any completely monotone sequence, there is an increasing function \( F_b(x) \) defined on the unit interval such that

\[
(12) \quad b_n = \int_0^1 t^n dF_b(t) \quad (n = 0, 1, 2, \cdots).
\]

**Proof.** Let \( b_* = \lim b_n \). The sequence \( a \) defined by \( a_n = (b_n - b_{n+1})/(b_0 - b_*) \) \((n = 0, 1, 2, \cdots)\) is of course completely monotone, but also satisfies (2) (unless \( b_0 = b_1 = \cdots = b_* \), but this case can easily be handled separately). Hence, from the remarks above, we have \( a_n = \int_0^1 t^n dF_a(t) \) \((n = 0, 1, 2, \cdots)\). Let \( F_b(x) = (b_0 - b_*)L(x) + b_*\delta(x) \) where \( L(x) = \int_0^x (1/(1 - t))dF_a(t) \) and \( \delta(x) = 1 \) if \( x = 1 \) and zero otherwise. Then

\[
\int_0^1 t^n dF_b(t) = (b_0 - b_*) \int_0^1 (t^n/(1 - t))dF_a(t) + b_* \int_0^1 t^n d\delta(t)
\]

\[
= (b_0 - b_*) \left( \int_0^1 (1/(1 - t))dF_a(t) - \sum_{k=0}^{n-1} \int_0^1 t^k dF_a(t) \right) + b_*
\]

\[
= b_* + (b_0 - b_*) \left( \int_0^1 (1/(1 - t))dF_a(t) - \sum_{k=0}^{n-1} (b_k - b_{k+1})/(b_0 - b_*) \right)
\]

\[
= b_* + b_n - b_0 + (b_0 - b_*) \int_0^1 (1/(1 - t))dF_a(t)
\]

\[
= b_* + b_n - b_0 + (b_0 - b_*) \int_0^1 dG_a(t)
\]

\[
= b_n.
\]

Hence (12) holds and this proves the theorem.

**An inversion formula.** We will now try to express the function \( G_a \) explicitly in terms of the sequence \( a \).

**Lemma.** There is a subset \( V \) of \( X \) which has measure one under any \( T \)-invariant probability measure and is such that if \( \omega \) is any point of \( V \),

\[
(13) \quad \lim_{n} (\omega_n / n) = \theta(\omega)/(1 - \theta(\omega)).
\]

**Proof.** We will represent the space \( X \) in yet another way as the space \( Y \) of all sequences of non-negative integers. We map a point \( \omega \) in \( X \) to the sequence \((\omega_1, \omega_2, -\omega_1, \omega_3, -\omega_2, \cdots)\) in \( Y \). If \( \mu_\theta \) is an ergodic measure on \( X \), we find that the corresponding measure \( v_\theta \) on \( Y \) is given by

\[
v_\theta(s_1, s_2, \cdots, s_k) = \prod_{i=1}^k (1 - \theta)\theta^{s_i}
\]
for any rectangle \( S = (s_1, s_2, \cdots, s_k) \) in \( Y \). Hence \( \nu_\theta \) is a product measure in \( Y \) which is invariant under the shift transformation \((s_1,s_2,s_3,\cdots) \to (s_2,s_3,s_4,\cdots)\). We apply the ergodic theorem, for the shift transformation, to the function \( f \) defined on \( Y \) by \( f(s_1,s_2,\cdots) = s_1 \). Then for almost every \((\nu_\theta)\) point \((s_1,s_2,\cdots)\) in \( Y \) we have

\[
\lim_{n} \frac{s_1 + s_2 + \cdots + s_n}{n} = \int_{Y} f(s) d\nu_\theta(s)
\]

\[
= \sum_{i=0}^{\infty} i(1 - \theta)^i = \theta/(1 - \theta)
\]

or, in \( X \),

\[
\lim_{n} \frac{\omega_n}{n} = \theta/(1 - \theta) \quad \mu_\theta \text{-a.e.}
\]

Clearly \( \theta(\omega) = \theta \) on a set \( V_\theta \) of \( \mu_\theta \) measure one. Hence the set \( V \) on which the two measurable functions \( \theta(\omega)/(1 - \theta(\omega)) \) and \( \lim (\omega_n/n) \) coincide is a set which has measure one for any \( T \)-invariant measure. This proves the lemma.

**THEOREM.** If \( a \) is any completely monotone sequence satisfying (2), \( G_a \) is given by

\[
G_a(x) = \lim_{n} \sum_{i=0}^{[nx/(1-x)]} \binom{n+i-1}{n-1} (\Delta^a)_i
\]

at every point \( x \) at which \( G_a \) is continuous.

**Proof.** First, from (13), we see that \( \theta(\omega) = \lim_n \omega_n/(n + \omega_n) \), so that

\[
\mu_a[\theta(\omega) < x] \leq \lim_n \mu_a[\omega_n/(n + \omega_n) < x] \\
\leq \mu_a[\theta(\omega) \leq x].
\]

We calculate the center member of this inequality:

\[
\mu_a[\omega_n/(n + \omega_n) < x] = \mu_a[\omega_n < nx/(1 - x)]
\]

\[
= \sum_{i=0}^{[nx/(1-x)]} \mu_a[\omega_n = i] = \sum_{i=0}^{[nx/(1-x)]} \binom{n+i-1}{n-1} (\Delta^a)_i
\]

where

\[
\binom{n+i-1}{n-1}
\]

is the number of rectangles of length \( n \) ending in \( i \). Now, passing to the limit, we get

\[
G_a(x-) \leq \sum_{i=0}^{[nx/(1-x)]} \binom{n+i-1}{n-1} (\Delta^a)_i \leq G_a(x),
\]

proving the theorem.
The transformations $T_\theta$. We can obtain some information about the spectra of the measure-preserving transformations $T_\theta$, although the results in this section are incomplete and probably not the best possible. Let us begin with the following theorem.

**Theorem.** If $\alpha$ is an eigenvalue of the transformation $T_\theta$, then

$$\lim_{n} \alpha^{B(n)} = 1 \quad \mu_\theta-a.e.$$ 

where $B(n)$ is the binomial coefficient

$$\binom{n + \omega_n}{n}.$$

**Proof.** Let $f$ be the eigenfunction having eigenvalue $\alpha$. For each $\omega$ in $X$, let $\omega^n$ be the rectangle $(\omega_1, \omega_2, \ldots, \omega_n)$ of length $n$; let $R(\omega, i, j)$ be the rectangle $(\omega_n, \omega_{n+1}, \ldots, \omega_n, i, j)$ of length $n + 2$, and $S(\omega, i, j)$ be the rectangle $(\omega_1, \omega_2, \ldots, \omega_n, i, j)$ of length $n + 2$. It is known that

$$\lim_{n} \left( \frac{1}{n^\omega} \right) \int_{\omega^n} f d\mu_\theta = f(\omega) \quad a.e.$$ 

and hence

$$\lim_{n} \left| \frac{1}{n^\omega} \right| \int_{\omega^n} f d\mu_\theta = 1 \quad a.e.$$ 

Estimating the value of $\left| \int_{\omega^n} f d\mu_\theta \right|$, we find

$$\left| \int_{\omega^n} f d\mu_\theta \right| = \left| \sum_{j \geq 1 \geq \omega_n} \int_{S(\omega, i, j)} f d\mu_\theta \right| = \left| \sum_{j \geq 1 \geq \omega_n} \int_{R(\omega, i, j)} f d\mu_\theta \right|$$

$$\leq \sum_{j = \omega_n}^\infty \sum_{i = \omega_n}^\infty \int_{R(\omega, i, j)} f d\mu_\theta \leq \sum_{j = \omega_n}^\infty \theta^j (1 - \theta)^{n+2} G_n(j)$$

where $G_n(j)$ is a sum of powers of $\alpha$, the powers being those powers of $T$ which send the rectangle $R(\omega, \omega_n, j)$ into the rectangle $R(\omega, i, j)$ ($i = \omega_n, \omega_n + 1, \ldots, j$).

Let $S(n, r)$ be the number of rectangles of length $n$ which end in $r$. Let $P(n, s) = S(n, 0) + S(n, 1) + \cdots + S(n, s)$. Note that the number $B(n)$, appearing in the statement of the theorem, is $P(n, \omega_n)$.

We leave to the reader the laborious task of showing that

$$(15) \quad \left| G_n(j) \right| = \left| 1 + \alpha^{P(n, \omega_n)} + \cdots + \alpha^{P(n, \omega_n) + \cdots + P(n, j-1)} \right|.$$ 

Notice that this implies $\left| G_n(j) \right| \leq j - \omega_n + 1$. Now we have

$$\left| \left( \frac{1}{\mu_\theta(n)} \right) \int_{\omega^n} f d\mu \right| \leq \left( \frac{1}{\theta^n (1 - \theta)^n} \right) \sum_{j = \omega_n}^\infty \theta^j (1 - \theta)^{n+2} \left| G_n(j) \right| \leq 1.$$
Both sides of this inequality approach 1 as \( n \) approaches infinity, so, for the middle,

\[
\lim_{n} (1 - \theta)^2 \sum_{j=0}^{\infty} \theta^{j} |G_{n}(\omega_{n} + j)| = 1 \quad \text{a.e.}
\]

But this can happen only if \( |G_{n}(\omega_{n} + j)| \) approaches \( j + 1 \) almost everywhere. Referring to (15), we see that \( \alpha^{R(n)} \) approaches 1, almost everywhere. This is what we wished to prove.

**Theorem.** If \( \theta < 1/2 \), the transformation \( T_{\theta} \) has no prime roots of unity as eigenvalues.

**Proof.** Suppose \( \alpha \) is a \( p \)th root of unity and is an eigenvalue. It is easy to see that the binomial coefficients

\[
\binom{p^{k} + i}{p^{k}} \quad (k = 1, 2, \cdots; i = 0, 1, \cdots, p^{k} - 1)
\]

are all equal to 1 (mod \( p \)). From the previous theorem we have

\[
\lim_{k} \alpha^{(p^{k} + \omega_{p^{k}})} = 1 \quad \text{a.e.}
\]

(16)

On the other hand, from (13), we have

\[
\lim_{k} \omega_{p^{k}} / p^{k} = \theta/(1 - \theta) < 1 \quad \text{a.e.}
\]

since \( \theta < 1/2 \). That is, eventually \( \omega_{p^{k}} < p^{k} \), so that eventually

\[
\binom{p^{k} + \omega_{p^{k}}}{p^{k}} = 1 \quad \text{(mod p)}.
\]

(17)

Hence (16) and (17) together form the contradiction which proves the theorem.

We obtain the immediate

**Corollary.** If \( \theta < 1/2 \), all powers of \( T_{\theta} \) are ergodic.

It is interesting to notice that the infinite product:

\[
f(\omega) = \alpha^{(1+\omega_{1}-1)} \alpha^{(2+\omega_{2}-1)} \cdots \alpha^{(k+\omega_{k}-1)}
\]

if it converges, is an eigenfunction with eigenvalue \( \alpha \). We have seen already that if \( \alpha \) is an eigenvalue, then

\[
\lim_{n} \alpha^{(n+\omega_{n})} = 1 \quad \text{a.e.}
\]

(18)
Integration of (18) over $X$ yields
\[ \lim_{n \to \infty} (1 - \theta)^n \sum_{j=0}^{\infty} \theta^j \binom{n + j - 1}{n - 1} \alpha^{n+j} = 1. \]

Also left open is the question of the conjugacy of $T_\theta$ and $T_\phi^{-1}$, and the question mentioned earlier, of the conjugacy of $T_\theta$ and $T_\phi$, $\theta \neq \phi$.

BIBLIOGRAPHY