NODAL NONCOMMUTATIVE JORDAN ALGEBRAS

BY

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1. A finite-dimensional power-associative algebra \( \mathfrak{A} \) is said to be nodal [6] if every element of \( \mathfrak{A} \) can be written as \( \alpha 1 + z \) where \( \alpha \in \mathbb{F} \), 1 is the unity element of \( \mathfrak{A} \) and \( z \) is nilpotent and if the set of all nilpotent elements is not a subalgebra of \( \mathfrak{A} \).

In [3; 4], Kokoris has shown that every simple nodal noncommutative Jordan algebra of characteristic \( p \neq 2 \) has the form \( \mathfrak{A} = \mathfrak{A}^1 + \mathfrak{A} \) with \( \mathfrak{A}^1 = \mathbb{F}[x_1, \ldots, x_n] \) for some \( n \) where the generators are all nilpotent of index \( p \) and the multiplication is associative. If \( f \) and \( g \) are two elements of \( \mathfrak{A} \) then the multiplication table of \( \mathfrak{A} \) is given by

\[
f g = f \circ g + \frac{1}{2} \sum_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \circ c_{ij},
\]

where the circle product is the product in \( \mathfrak{A}^1 \) and

\[
c_{ij} = x_i x_j - x_j x_i.
\]

In [7] Schäfer considers nodal noncommutative Jordan algebras defined by a skew-symmetric bilinear form (i.e., \( c_{ij} \in \mathbb{F} \)) and those with two generators. All of these algebras are Lie-admissible (i.e., \( \mathfrak{A}^- \) is a Lie algebra). Schäfer obtained the derivation algebras of these algebras defined by a skew-symmetric bilinear form.

Here, we examine all simple nodal noncommutative Jordan algebras that are Lie-admissible over a field \( \mathbb{F} \) of characteristic \( p \neq 2 \). First a set of generators is obtained having properties suitable for further study. This set of generators is then used to find the algebras \( D(\mathfrak{A}) \) of derivatives of \( \mathfrak{A} \) and the algebras \( \text{adj} \mathfrak{A}^- \) and \( (\text{adj} \mathfrak{A}^-)' \). Schäfer has shown that all of the simple Lie algebras defined by Block [1] can be realized as \( (\text{adj} \mathfrak{A}^-)' \) for some \( \mathfrak{A} \) that is simple, nodal noncommutative Jordan and Lie-admissible. Hence we have obtained a somewhat different formulation of these algebras. The question remains whether all of these algebras, \( (\text{adj} \mathfrak{A}^-)' \), are in the class defined by Block. It is our intention to investigate this question in a subsequent paper.

2. We define the mapping \( D_y = D(y) \) by

\[
x D_y = xy - yx.
\]

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Then $D_y = R_y - L_y$ where $R_y$ and $L_y$ are the right and left multiplications by $y$ on $\mathfrak{A}$.

A derivation of an algebra $\mathfrak{B}$ is a linear transformation $T$ on $\mathfrak{B}$ into $\mathfrak{B}$ such that for $x, y \in \mathfrak{B}$

$$(xy)T = (xT)y + x(yT).$$

Since $\mathfrak{A}^{\ominus}$ is a Lie algebra $D_y$ is the right multiplication by $y$ of $\mathfrak{A}^{\ominus}$ and is a derivation of $\mathfrak{A}^{\ominus}$. By expanding

$$2(x \circ y)D_z - 2(xD_z) \circ y - 2x \circ (yD_z)$$

in terms of the multiplication of $\mathfrak{A}$ and using the flexible law

$$(xy)z + (zy)x = x(yz) + z(yx)$$

we see that $D_z$ is also a derivation of $\mathfrak{A}^{+}$ and hence of $\mathfrak{A}$.

It is well known [2, p. 108] that any set of $n$ elements of $\mathfrak{A}$ whose cosets form a basis of the $n$-dimensional space $\mathfrak{A} - \mathfrak{A} \circ \mathfrak{A}$ can serve as a set of generators of $\mathfrak{A}^{+}$. This result shall be our chief tool in the proof of the following theorem.

**Theorem 1.** Let $\mathfrak{A}$ be a simple, Lie-admissible, nodal noncommutative Jordan algebra over a base field $\mathfrak{F}$ of characteristic $p \neq 2$. If $\mathfrak{A}^{+}$ has an even number of generators then a set of generators $x_1, \ldots, x_{2r}$ can be chosen for $\mathfrak{A}^{+}$ so that

$$x_iD(x_{i+r}) = 1 + \alpha_i x_{i+r}^{p-1} \circ x_{i+r-1}^{p-1}, \quad i = 1, \ldots, r,$$

$$x_iD(x_j) = 0, \quad j \neq i + r,$$

with $\alpha_i \in \mathfrak{F}$. If $\mathfrak{A}^{+}$ has an odd number of generators then a set of generators $x_1, \ldots, x_{2r+1}$ can be chosen for $\mathfrak{A}^{+}$ so that (1) is satisfied and

$$x_{2r+1}D(x_j) = 0, \quad j = 1, \ldots, 2r - 2,$$

$$x_{2r+1}D(x_{2r}) = x_{2r}^{p-1} \circ (1 + \beta x_{2r+1}^{p-1}),$$

$$x_{2r+1}D(x_{2r-1}) = x_{2r-1}^{p-1} \circ (1 + \beta x_{2r+1}^{p-1}),$$

with $\alpha$ and $\beta$ in $\mathfrak{F}$.

**Proof.** Since $\mathfrak{A}$ is simple $\mathfrak{A}$ can not be an ideal of $\mathfrak{A}^{\ominus}$. For if $\mathfrak{A}$ is an ideal of $\mathfrak{A}^{\ominus}$ then since it is an ideal of $\mathfrak{A}^{+}$ it would be closed under both the operations $R_y - L_y$ and $R_y + L_y$ for $y \in \mathfrak{A}$. Therefore it would be also an ideal of $\mathfrak{A}$. Hence there must be a pair of generators $x$ and $y$ such that $yD_x$ is nonsingular. Since $y$ can be replaced by $\alpha y$ for any $\alpha$ in $\mathfrak{F}$ we assume

$$yD_x = 1 + m \circ y^k = b^{-1}.$$  

We also assume $y$ has been chosen so that $k$ is a maximum. If $k < p - 1$ then letting $q = (y - 1/(k + 1))y^{k+1} \circ m \circ b$ we have
\[ qD_x = 1 - \frac{1}{k + 1} y^{k+1} \circ (m \circ b)D_x \]
\[ = 1 + q^{k+1} \circ m' \]

which contradicts the choice of \( k \). Hence we can assume in (3) that \( k = p - 1 \).

We now write (3) as

\[ yD_x = 1 + y^{p-1} \circ x^t \circ m' = b^{-1} \]

and assume that \( y \) and \( x \) have been chosen so that \( t \) is a maximum. If \( t < p - 1 \) then, as above, we can replace \( x \) by \( x - 1/(t + 1) \circ x^{t+1} \circ m' \circ y^{p-1} \circ b \) to obtain a contradiction to our choice of \( t \). Hence we can assume \( x \) and \( y \) have been chosen so that

\[ yD_x = 1 + m_x \circ y^{p-1} \circ x^{p-1}. \]

If \( z \) is a third generator, in the same way that we altered the generator \( y \), we can add an element \( q \) of \( y \circ \mathfrak{U} \) to \( z \) to obtain the property

\[ (z + q)D_x \in y^{p-1} \circ \mathfrak{U}. \]

Hence we assume that all generators \( z \) different from \( x \) and \( y \) have been chosen so that

\[ zD_x = y^{p-1} \circ m_z. \]

Since for any \( q \) in \( \mathfrak{U} \) we have \( D_q \) a derivation of both \( \mathfrak{U} \) and \( \mathfrak{U}^{-} \) then

\[ zD_yD_x = zD_xD_y - yD_xD_z. \]

If (4) and (5) are substituted in (6) we have

\[ zD_yD_x = y^{p-1} \circ m_yD_y + y^{p-2} \circ x^{p-1} \circ m_y \circ yD_z \]
\[ - y^{p-1} \circ x^{p-1} \circ m_yD_z. \]

But the right-hand side of (7) is in \( y^{p-2} \circ \mathfrak{U} \); so also is the left-hand side. From (4) and (5) the only possible way for this to happen is to have

\[ zD_y = n_0 + y^{p-1} \circ n_1 \]

in which \( n_i \) is independent of \( y \). (i.e., \( n_i \) is a polynomial in which \( y \) does not appear.) In (7) this implies

\[ n_0D_x - y^{p-2} \circ n_1 = y^{p-1} \circ m_yD_y - y^{p-2} \circ x^{p-1} \circ m_y \circ n_0 \]
\[ - y^{p-1} \circ x^{p-1} \circ m_yD_z \]

and

\[ n_1 = x^{p-1} \circ m_y \circ n_0. \]
Write \( n_0 = x^k \circ t \). If \( k < p - 1 \) we can replace the generator \( z \) by the generator \( z + 1/(k + 1) \circ x^{k+1} \circ t = z' \) to get

\[
z' D_y = n_0 + y^{p-1} \circ m_y \circ n_0 \circ x^{p-1} + x^k \circ t \circ x D_y + \frac{1}{k + 1} x^{k+1} \circ t D_y
\]

\[
= y^{p-1} \circ x^{p-1} \circ m_y \circ n_0 + \frac{1}{k + 1} x^{k+1} \circ t D_y
\]

\[
= n'_0 + y^{p-1} \circ x^{k+1} \circ m_y \circ n'_1;
\]

in which \( n'_1 \) is again independent of \( y \). Note that if (5) holds and \( z \) is replaced by a generator \( z + q \) in which \( q \) is independent of \( y \) then (5) will be retained.

Again arguing on the maximum value of \( k \) that can be obtained in the expression \( n_0 = x^k \circ t \) we can conclude that \( k = p - 1, n_1 = 0 \) and

\[
z D_y = x^{p-1} \circ n_z,
\]

\[
z D_x = y^{p-1} \circ m_z
\]

in which \( n_z \) is independent of \( y \).

Identity (7) can now be reduced to

\[
(10) \quad x^{p-1} \circ n_z D_x = y^{p-1} \circ m_z D_y - y^{p-1} \circ x^{p-1} \circ m_y D_z.
\]

For a particular choice of a set of generators including \( x \) and \( y \) satisfying (4) assume there are two distinct generators \( w \) and \( z \) (both satisfying (9)). Write

\[
(11) \quad m_z = \sum x^i \circ m_i, \quad m_0 = \sum w^i \circ n_i.
\]

(When obvious, we shall omit index and range of the summation.) Then

\[
m_z D_y = - \sum ix^{i-1} \circ m_i + \sum x^i \circ m_i D_y - m_0 \circ y^{p-1} \circ x^{p-1}.
\]

But from (10) \( y^{p-1} \circ m_z D_y \in x \circ \mathfrak{A} \). Therefore

\[
(12) \quad - y^{p-1} \circ m_1 + y^{p-1} \circ m_0 D_y \in x \circ \mathfrak{A},
\]

\[
- y^{p-1} \circ m_1 + y^{p-1} \circ \sum i w^{i-1} \circ n_i \circ w D_y + \sum w^i \circ n_i D_y \in x \circ \mathfrak{A}.
\]

If \( w \) is replaced as a generator by \( w' = w - x \) then (9) still holds for \( z \) and hence so do the corresponding relationships (12). Note that if \( P(w) \) is a polynomial in \( w \) then \( P(w) - P(w + x) \in x \circ \mathfrak{A} \) and \( w D_y - (w + x) D_y = 1 \in x \circ \mathfrak{A} \). If we write \( q' \) for \( q = q(w) \) with \( w \) replaced by \( w + x \) then \( w' D_z = y^{p-1} \circ m'_z; \ m'_z = \sum x^i \circ m'_i; \ m'_0 = \sum w^i \circ n'_i \) and from (11) we have

\[
0 \equiv - y^{p-1} \circ m'_1 + y^{p-1} \circ \sum i w^{i-1} \circ n'_i \circ w' D_y + \sum w^i \circ n'_i D_y
\]

\[
\equiv - y^{p-1} \circ m_1 + y^{p-1} \circ \sum i w^{i-1} \circ n_i \circ w D_y + \sum w^i \circ n_i D_y
\]

\[
\equiv - y^{p-1} \circ m_1 + y^{p-1} \circ \sum i w^{i-1} \circ n_i \circ (w D_y - 1) + \sum w^i \circ n_i D_y
\]

modulo \( x \circ \mathfrak{A} \).
But this implies $y^{p-1} \circ \sum_{i \leq 1} \circ m_i \in x \circ \mathbb{A}$. Therefore $y^{p-1} \circ n_i \in x \circ \mathbb{A}$ for $i > 0$.

Now assume that in (11) we have chosen the $m_i$ to be independent of $x$. Then since $m_z$ is independent of $y$ and $m_0$ is independent of $x$ we have $n_i = 0$ for $i > 0$. Hence $m_0$ is independent of $w$. Since $w$ was arbitrary we must have $m_0$ a polynomial in the single generator $z$. But then $y^{p-1} \circ m_0 D_y \in x \circ \mathbb{A}$ by (9) and $y^{p-1} \circ m_1 \in x \circ \mathbb{A}$ by (12). However $m_1$ is independent of $x$ and $y$. Hence $m_1 = 0$.

Once again looking at (12) we have

$$y^{p-1} \circ m_z D_x \equiv -y^{p-1} \circ \sum x^{i-1} \circ m_i + y^{p-1} \circ \sum x^i \circ m_i D_y \equiv 0$$

modulo $x^{p-1} \circ \mathbb{A}$. With $m_0 D_y \in x^{p-1} \circ \mathbb{A}$ and $m_1 = 0$ we see that $m_2 = \ldots = m_{p-1} = 0$ and $m_z = m_0$ is a polynomial in $z$ with coefficients in $\mathbb{F}$. Similarly we obtain $n_z$ as a polynomial in $z$ with coefficients in $\mathbb{F}$. Therefore if the number of generators is greater than or equal to 4 and they have been picked so that (4) and (9) hold then $m_z$ and $n_z$ in (9) are polynomials in the single generator $z$.

However if $z$ and $w$ are two generators distinct from $x$ and $y$ then $z$ can be replaced as a generator by $z + w$. Indentity (9) still holds, i.e.,

$$(z + w) D_x = y^{p-1} \circ m_{z+w},$$

$$(z + w) D_y = x^{p-1} \circ n_{z+w}$$

in which $m_{z+w}$ and $n_{z+w}$ are polynomials in the single generator $(z + w)$. But $m_{z+w} = m_z + m_w$ and $n_{z+w} = n_z + n_w$. For these sums to be polynomials in $(z + w)$, $m_z$ and $n_z$ must be of degree at most 1. If $z$ is replaced by $z + z^2$ then (9) still holds for the generator $(z + z^2)$. In particular $m_z + 2z \circ m_z$ is of degree at most 1 in $(z + z^2)$. Write $m_z$ as $\alpha + \beta z$ and $m_z + 2m_z \circ m_z$ as $\gamma + \delta(z + z^2)$. Then $\beta = 2\alpha$. Since $z$ was arbitrary we must also have $\delta = 2\gamma$. But the same relationships that gave us $\beta = 2\alpha$ also give us $\delta = 2\gamma$, i.e., $\delta = \gamma = \alpha = \beta = 0$. Hence $m_z = 0$ and in the same manner $n_z = 0$.

We still assume we have at least two generators $z$ and $w$ distinct from $x$ and $y$. We also assume that they have been chosen so that

$$(13) \quad zD_z = zD_x = wD_x = wD_y = 0.$$ \textbf{We must have} $$wD_z D_x = wD_x D_z - zD_x D_w$$

and therefore $(wD_z) D_x = 0$. This implies that $wD_z$ is independent of $y$. Similarly $(wD_z) D_y = 0$ and $wD_z$ is independent of $x$. Then if we assume that all the generators distinct from $x$ and $y$ have been chosen so that their product in $\mathbb{A}^-$ by either $x$ or $y$ is 0, we can assume that the polynomials over $\mathbb{F}$ in these generators is an ideal $\mathcal{I}$ of $\mathbb{A}^-$. But then $\mathcal{I} \circ \mathbb{A}$ is an ideal in both $\mathbb{A}^-$ and $\mathbb{A}^+$ and hence in $\mathbb{A}$. Therefore $\mathcal{I}$ must contain a nonsingular element. This means that there are two generators $w$ and $z$, distinct from $x$ and $y$, such that $wD_z$ is nonsingular.
At this point we reconsider the polynomial $m_y$ obtained in (4). If the generators $x, y, z, w$ have been chosen so that (4) and (13) hold and $z$ and $w$ are such that $wD_z$ is nonsingular then (7) reduces to

$$y^{p-1} \circ x^{p-1} \circ m_y D_z = 0.$$  

Therefore $m_y D_z$ is 0 since it is independent of both $x$ and $y$. But this implies that $m_y$ is independent of $w$ and by symmetry $m_y$ is independent of $z$. If $t$ is a fifth generator then either $tD_z$ or $(w + t)D_z$ is nonsingular. In either case we see that $m_y$ is also independent of $t$. Hence $m_y \in \mathfrak{F}$.

We can now proceed in $\mathfrak{S}$ (defined above) with the same argument as above to obtain the result of the theorem for the even-dimensional case.

In the odd dimensional case we can proceed with the above argument until we are presented with an $\mathfrak{S}$ which is the set of polynomials over $\mathfrak{F}$ in three generators, say $x, y$ and $z$. Again by the previous arguments we can assume that $x, y$ and $z$ have been chosen so that

$$yD_x = 1 + y^{p-1} \circ x^{p-1} \circ m_y,$$

$$zD_x = y^{p-1} \circ m_z,$$

$$zD_y = x^{p-1} \circ n_z.$$ 

Consider (7). We have

$$x^{p-1} \circ n_z D_x = y^{p-1} \circ m_z D_y - y^{p-1} \circ x^{p-1} \circ m_y D_z.$$ 

Since $m_y$ is a polynomial in $x, y$ and $z$ then $y^{p-1} \circ x^{p-1} \circ m_y D_z = 0$. Also since $m_z$ is independent of $y$ and by (9) $m_z D_y$ is independent of $y$ we must have $m_z D_y \in x^{p-1} \circ \mathfrak{F}$. This implies that $m_z$ is independent of $x$. Hence $y^{p-1} \circ m_z D_y = \partial m_z / \partial z \circ x^{p-1} \circ y^{p-1}$ and $x^{p-1} \circ m_z D_x = \partial n_z / \partial z \circ x^{p-1} \circ y^{p-1}$. From (14) we have

$$\frac{\partial n_z}{\partial z} = \frac{\partial m_z}{\partial z}.$$ 

If both $n_z$ and $m_z$ are singular then $zD_x$ and $zD_y$ are in $z \circ \mathfrak{F}$. Hence $z \circ \mathfrak{F}$ is an ideal of $\mathfrak{F}^-$ and $\mathfrak{F}^+$. Since this denies the simplicity of $\mathfrak{F}$ we must have either $m_z$ or $n_z$ nonsingular. Assume $m_z = 1 + q$ in which $q \in \mathfrak{F}$. Then if $l$ is a polynomial in $z$ over $\mathfrak{F}$ we have

$$(z + l)D_x = y^{p-1} \circ (1 + q) + y^{p-1} \circ \frac{\partial l}{\partial z} \circ (1 + q).$$ 

Clearly, $l$ can be chosen so that $\partial l / \partial z \circ (1 + q) \equiv w$ modulo $z^{p-1} \circ \mathfrak{F}$. Hence we can assume

$$zD_x = y^{p-1} + \beta y^{p-1} \circ z^{p-1},$$

where $\beta$ is a polynomial in $z$. If $\beta$ is a polynomial in $z$ of degree less than $p-1$ then $zD_x$ is in $z \circ \mathfrak{F}$.

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in which $\beta \in \mathfrak{g}$. Now since $m_z$ is nonsingular the solutions of (15) are of the form $n_z = \alpha m_z$. Hence

$$zD_x = \alpha x^{p-1} \circ (1 + \beta z^{p-1}).$$

Again, let $l$ be a polynomial in $z$ over $\mathfrak{g}$. Then

$$(y + x^{p-1} \circ l)D_x = 1 + y^{p-1} \circ x^{p-1} \circ \left[ m_x + \frac{\delta l}{\delta z} \circ (1 + \beta z^{p-1}) \right].$$

Write $m_x = y + z \circ t$ and $l = \delta z + z^2 \circ l'$ in which

$$\frac{\partial (z^2 \circ l')}{\partial z} + z \circ t$$

is a multiple of $z^{p-1}$. Then

$$\frac{\partial (z^2 \circ l')}{\partial z} \circ (1 + \beta z^{p-1}) + z \circ t$$

is also a multiple of $z^{p-1}$. Now choose $\delta$ so that

$$\delta(1 + \beta_z^{-1}) + \frac{\partial (z^2 \circ l')}{\partial z} \circ (1 + \beta z^{p-1}) + z \circ t$$

is a constant. We now have

$$(y + x^{p-1} \circ l)D_x = 1 + y \circ x^{p-1} \circ y^{p-1}.$$ \hspace{1cm} \text{(18)}$$

Since $(y + x^{p-1} \circ l)^{p-1} \circ x^{p-1} = y^{p-1} \circ x^{p-1}$ we can assume the generator $y$ can be chosen so that

$$yD_x = 1 + y \circ x^{p-1} \circ y^{p-1}.$$ \hspace{1cm} \text{(19)}$$

We can now repeat the construction of the $z$ in (9) to obtain

$$zD_x = y^{p-1} \circ m_z,$$

$$zD_y = x^{p-1} \circ n_z.$$ \hspace{1cm} \text{(20)}$$

From these we can obtain (16) and (17). Hence we have concluded the proof of the theorem.

3. Let $\mathfrak{A}$ and $\mathfrak{A}^\ast$ be two simple nodal algebras that are equal as vector spaces and have the same $+$ algebras. Let there be an even number of generators $x_1, \ldots, x_{2r}$ with the multiplication in $\mathfrak{A}$ given by the $c_{ij} = x_i D(x_j)$ obtained in Theorem 1 and the multiplication in $\mathfrak{A}^\ast$ given by

$$c_{i+i+r} = 2,$$

$$c'_{ij} = 0$$

for $i = 1, \ldots, r$ and $j \neq i + r$. The algebra $\mathfrak{A}^\ast$ then falls into the class of simple
nodal algebras defined by a skew-symmetric bilinear form and studied by Schäfer [7].

Every derivation of $\mathcal{A}$ must be a derivation of $\mathcal{A}^*$. The derivations of $\mathcal{A}^*$ have been given by Jacobson [2, p. 107] as

\begin{equation}
\sum_{k=1}^{2r} \frac{\partial f}{\partial x_k} \circ a_k.
\end{equation}

We shall denote this derivation by $(a_1, \ldots, a_{2r})$. Assume $(a_1, \ldots, a_{2r})$ is a derivation of $\mathcal{A}$ and consider the possibility that $(b_1, \ldots, b_{2r})$ is a derivation of $\mathcal{A}^*$ in which

\begin{equation}
b_i = c_{is}^{-1} \circ c_{is} \circ a_i
\end{equation}

and $s$ is $i + r$ if $i \leq r$ and is $i - r$ if $i > r$. In the same way we choose $t$ so $t = j + r$ or $j - r$ and $t \leq 2r$.

Consider the expression

\begin{equation}
\sum_{k=1}^{2r} \left( \frac{\partial c_{ij}}{\partial x_k} \circ b_k + \frac{\partial b_i}{\partial x_k} \circ c_{ik} + \frac{\partial b_j}{\partial x_k} \circ c_{ij} \right)
\end{equation}

obtained from Schäfer's criteria [7, p. 312] that $(b_1, \ldots, b_{2r})$ be a derivative of $\mathcal{A}^*$. We want to show that for all $i$ and $j$ (20) is 0. By the choice of the $c_{ij}$'s (20) can be reduced to

\begin{equation}
\frac{\partial b_i}{\partial x_t} \circ c_{is} + \frac{\partial b_j}{\partial x_s} \circ c_{j_i}
\end{equation}

and by substituting the expressions (19) we have

\begin{align*}
c_{jt} \circ c_{is} \circ & \left( \frac{\partial a_i}{\partial x_t} \circ c_{is}^{-1} - \frac{\partial a_j}{\partial x_s} \circ c_{jt}^{-1} \right) \\
& + c_{jt} \circ c_{is} \circ \left( \frac{\partial c_{is}^{-1}}{\partial x_t} \circ a_i - \frac{\partial c_{jt}^{-1}}{\partial x_s} \circ a_j \right).
\end{align*}

For our purposes we can drop the factor $c_{jt} \circ c_{is}$, use the fact that if $q \in \mathbb{N}$ then $(1 + q^{p-1})^{-1} = 1 - q^{p-1}$, and

\begin{equation}
c_{is} \circ \frac{\partial c_{jt}}{\partial x_s} = c_{is} \circ c_{jt} \circ \frac{\partial c_{is}}{\partial x_t} = 0
\end{equation}

to further reduce (20) to

\begin{equation}
\frac{\partial a_i}{\partial x_t} \circ c_{jt} - \frac{\partial a_j}{\partial x_s} \circ c_{is} + \frac{\partial c_{jt}}{\partial x_s} \circ a_j - \frac{\partial c_{is}}{\partial x_t} \circ a_i.
\end{equation}

But the criteria that must be satisfied for $(a_1, \ldots, a_{2r})$ to be a derivation of $\mathcal{A}$ is that (21) be zero. Hence $(b_1, \ldots, b_{2r})$ is a derivation of $\mathcal{A}^*$. From identities (14) of Schäfer [7] we can now conclude that there is a $g$ such that
\[ b_i = \left( \frac{\partial g}{\partial x_i} + \sigma_i \circ x_i^{p-1} \right) \circ c_i \]

in which \( \sigma_i \) is in \( \mathfrak{X} \). Therefore

\[ a_i = \left( \frac{\partial g}{\partial x_i} + \sigma_i \circ x_i^{p-1} \right) \circ c_i. \]

Schafer has already proved [7, Theorem 8] that if the \( a \)'s are defined as in (22) then they define a derivation.

We summarize as follows.

**Theorem 2.** If \( \mathfrak{A} \) is a simple, nodal, Lie-admissible noncommutative Jordan algebra of characteristic \( p \neq 2 \) such that \( \mathfrak{A}^+ \) has an even number \( n \) of generators then the derivation algebra \( \mathfrak{D}(\mathfrak{A}) \) of \( \mathfrak{A} \) is the set of all mappings

\[ f \mapsto \sum_1^n \frac{\partial f}{\partial x_i} \circ a_i \]

in which the \( a_i \) are defined as in (22). The dimension of \( \mathfrak{D}(\mathfrak{A}) \) is \( p^n + n - 1 \).

We now investigate the algebras \text{adj} \( \mathfrak{A}^- \), \( \text{adj} \mathfrak{A}^- \)' and \( \text{adj} \mathfrak{A}^- \)''.

Using Schafer's result [7, Theorem 7] we have \( \mathfrak{A}^- / \mathfrak{Y} \cong \text{adj} \mathfrak{A}^- \) is of dimension \( p^{2r} - 1 \).

Since \( D_n D_m - D_m D_n = D(n D_m) \) we can consider \( \text{adj} \mathfrak{A}^- \)' as the set of all \( D_s, x \in \mathfrak{A}^- \) such that there are \( y \) and \( z \) in \( \mathfrak{A}^- \) with \( x \equiv y D_z \) modulo \( \mathfrak{Y} \). Also \( x_i^2 D(x_{i+r}) = 2x_i \) implies \( D(x_i) \in \text{adj} \mathfrak{A}^- \)'.

Before examining the dimension of \( \text{adj} \mathfrak{A}^- \)' we consider a slightly more general situation.

Let \( \mathfrak{S} \) be an ideal of \( \mathfrak{A}^- \) containing all of the generators \( x_1, \cdots, x_{2r} \). Let \( m \) be a monomial of \( \mathfrak{A}^- \) that is not in \( \mathfrak{Y} \), and in which the exponent of \( x_1 \) is \( i \) and \( 0 \leq i < p - 1 \). Write \( m = x_1^i \circ n \). Then

\[ D(x_{i+r}) = \left( \frac{1}{i+1} x_1^{i+1} \circ n \right) D(x_{i+r}) = x_1^i \circ n \circ c_{11+r}. \]

If \( i > 0 \), \( c_{11+r} \in \mathfrak{S} \), or \( x_{i+r} \) appears in \( m \) with nonzero exponent then \( x_1^i \circ n \circ c_{11+r} = x_1^i \circ n = m \in \mathfrak{S} \). Arguing on the arbitrariness of the choice of \( x_1 \) we see that all terms of degree greater than 0 are in \( \mathfrak{S} \) except possibly those in which:

1. every generator appears to either the 0 or \( p - 1 \) power,
2. \( x_1 \) has exponent \( p - 1 \) if and only if \( x_{i+r} \) has exponent \( p - 1 \) for \( i = 1, \cdots, r \) and
3. \( x_i \) and \( x_{i+r} \) have exponent \( p - 1 \) if \( c_{ii+r} \in \mathfrak{S} \).

However, assume such a term is \( m \), and assume \( x_1 \) has exponent 0 in \( m \) and \( c_{11+r} \notin \mathfrak{S} \). Then from (23) we see that \( m \equiv - x_1 m \circ x_1^{p-1} \circ x_1^{i+r} \) modulo \( \mathfrak{S} \).
This leaves us with at most two residue classes modulo $\mathcal{I}$; the class containing 1 and the class containing $x_i^{p-1} \circ x_i^{p-1} \circ \cdots \circ x_i^{p-1} \circ x_i^{p-1}$ in which $\mathcal{G} = \{i_1, \ldots, i_r\}$ is the set of all $i \leq r$ such that $c_{ii+r} \in \mathbb{F}_p$. If $\mathcal{G}$ is empty then since

$$x_i D(x_{i+r}) = 1 + x_i x_i^{p-1} \circ x_i^{p-1}$$

and $x_i \neq 0$ there is at most one residue class, that one containing 1.

We now let $\mathcal{I}$ be the ideal in $\mathcal{U}^-$ such that $\mathcal{I} \cong (\text{adj } \mathcal{U}^-)'$. If $\mathcal{G} = \emptyset$ by the above result we have $\mathcal{I} = \mathcal{U}^-$ and $(\text{adj } \mathcal{U}^-)' = \text{adj } \mathcal{U}$.

In case $\mathcal{G} \neq \emptyset$ we first note that we have shown that $\mathcal{I}$ contains all monomials and binomials of the form

$$(24) \quad n \circ c_{ii+r},$$

$i = 1, \ldots, r$, and $n$ is a monomial without the factor $x_i^{p-1} \circ x_i^{p-1}$. To show that these are the only terms in $\mathcal{I}$ we consider two monomials $n = x \circ x_i \circ x_i^{p-1}$ and $m = y \circ x_i \circ x_i^{p-1}$ in which $x$ and $y$ are independent of $x_i$ and $x_{i+r}$. Every element of $\mathcal{I}$ is a sum of terms of the form $nDm$ and every $nDm$ is a sum of terms of the form

$$\begin{align*}
(x_i^k \circ x_i^{p-1})D(x_i^k \circ x_i^{p-1}) \circ y \circ x \\
y \circ x \circ x_i^{u+k-1} \circ x_i^{u+j-1} \circ (v-k-uj) \circ c_{ii+r}.
\end{align*}$$

If $u+k-1 = v+j-1 = p-1$ then $v-k-uj = 0$. Hence every element of $\mathcal{I}$ is a sum of terms of the form (24).

Now let $q$ be the product of all $x_i^{p-1}$ such that $i \in \mathcal{G}$. If $q$ is in $\mathcal{I}$ then it must be a sum of terms of the form (24). In fact we must have

$$q = \sum q \circ n_i \circ c_{ii+r},$$

in which $i \notin \mathcal{G}$, $n_i$ is a polynomial independent of any of the generators in $q$. But this is a polynomial identity that holds in any scalar extension of $\mathcal{G}$. Hence we can substitute field elements $\delta_i, \delta_{ii+r}$ of some scalar extension $\mathcal{R}$ of $\mathcal{G}$ for $x_i$ and $x_{i+r}, i \notin \mathcal{G}$, so that $1 + \alpha_i \delta_i^{p-1} \circ \delta_{ii+r}^{p-1} = 0$. But then the polynomial identity $q = 0$ holds over $\mathcal{R}$. Hence $q \notin \mathcal{I}$.

We now show that $(\text{adj } \mathcal{U}^-)'$ is simple. Let $\mathcal{I}$ be an ideal of $(\text{adj } \mathcal{U}^-)'$. To simplify the notation we will again actually work with an ideal in $\mathcal{U}^-$ and assume everything is reduced modulo $\mathcal{G}^1$.

Let $\mathcal{L}$ be the set of all polynomials in $\mathcal{I}$ with a minimal number of terms in them. If the generator $x_i$ appears in any of these polynomials in $\mathcal{L}$ choose one such polynomial $m$ in which $x_i$ appears to the minimal positive degree. Consider $mD(x_i^{2})$ which is in $\mathcal{I}$ and has fewer terms than $m$ unless $x_i$ appears with positive exponent in every term of $m$. Also, if any term is of degree greater than 1 in $x_i$ then we have a contradiction to our choice of $m$ to be of minimal degree in $x_i$. Hence we can assume $m = x_i \circ n$ in which $n$ is independent of $x_i$. By choosing $n$
to be of minimal positive degree in some second generator and avoiding the use of derivations $D_y$ for which $x_{1+r}$ appear in $y$ we can repeat the above argument finally obtaining a monomial $m$ in $\mathcal{S}$ which is the product of distinct generators. If both $x_i$ and $x_{i+r}$ are in $m$ we can replace $m$ by $mD_{x_i}$. Hence we can assume in addition that the subscripts $i$ of the generator in $m$ satisfy $i \leq r$. Write

$$m = x_{i_1} \circ x_{i_2} \circ \cdots \circ x_{i_t}$$

and apply successively the derivations

$$D(x_{i_1+r}), D(x_{i_2+r} \circ c_{i_1+i_2+r}), \ldots, D(x_{i_t+r} \circ c_{i_1+i_2+\cdots+i_t+r})$$

obtaining $x_{i_1+r} \in \mathcal{S}$ and $x_{i_2+r} \circ D(x_{i_1+r}) = 2x_i \in \mathcal{S}$. Hence we can conclude that any generator that appears in a monomial of $\mathcal{S}$ is in $\mathcal{S}$. If $x_i$ is one such generator then for $i \neq j$, $x_iD(x_{i+r} \circ x_j) = 2x_{i+r} \circ x_j$ is in $\mathcal{S}$ and $x_j \in \mathcal{S}$. Therefore $\mathcal{S}$ contains all generators. By the results above $\mathcal{S}$ must be all of (adj $\mathcal{S}$)$'$ and (adj $\mathcal{S}$)$'$ is simple. We summarize in the following theorem.

**Theorem 3.** If $\mathcal{S}$ is a simple, Lie-admissible nodal noncommutative Jordan algebra of characteristic $p \neq 2$ with $2r$ generators then (adj $\mathcal{S}$)$'$ is a simple Lie algebra of dimension either $p^{2r} - 1$ or $p^{2r} - 2$ in the cases $\mathbb{S} = \emptyset$ or $\mathbb{S} \neq \emptyset$ respectively.

4. Let $\mathcal{S}$ and $\mathcal{S}^*$ be two nodal algebras that are equal as vector spaces and have the same + algebra. Let there be an odd number $n = 2r + 1$, of generators $x_1, \ldots, x_n$ with the multiplication in $\mathcal{S}$ given by $c_{i+r} = 2$ for $i = 1, \ldots, r$ and all other $c_{ij} = 0$.

Let $(a_1, \ldots, a_n)$ be a derivation of $\mathcal{S}$. Just as in the previous section we can show $(b_1, \ldots, b_{2r-2}, 0, 0, 0)$ is a derivation of $\mathcal{S}^*$ if

$$b_i = c_{i+r}^{-1} \circ c_{i+r} \circ a_i$$

for $i = 1, \ldots, r - 2$. Therefore we must have

$$a_i = \left( \frac{\partial g}{\partial x_i} + \sigma_i \circ x_{i+r}^{-1} \right) \circ c_{i+r}$$

for $i = 1, \ldots, r - 1$. Here though, $\sigma_i$ can apparently be any polynomial in $\mathbb{S}[x_{2r-1}, x_{2r}, x_{2r+1}]$. To obtain further restrictions on the $\sigma_i$ we examine derivations of the form

$$(\sigma_1 \circ x_{1+r}^{-1} \circ c_{11+r}, \ldots, \sigma_{2r-2} \circ x_{r+1}^{-1} \circ c_{2r-2+r-2}, a_{2r-1}, a_{2r}, a_{2r+1}).$$

We now use identity (5) of Schäfer [7] with $i \leq r - 2$ and $j \geq 2r - 1$ to obtain

$$(26) \sum_{2r-1}^{n} \frac{\partial \sigma_i}{\partial x_k} \circ x_{i+r}^{-1} \circ c_{jk} + \frac{\partial a_j}{\partial x_{i+r}} \circ c_{i+r+1} = 0.$$
must have \( \frac{\partial a_j}{\partial x_{i+r}} = 0 \) and \( a_j \) independent of \( x_{i+r} \). Interchanging \( i \) and \( i + r \) in (26) we see \( a_j \) is also independent of \( x_i \). Hence \( a_j \) is a polynomial in \( \mathcal{K}[x_{2r-1}, x_{2r}, x_{2r+1}] \).

We now select \( j \) in (26) to be \( 2r \). Then
\[
\frac{\partial \sigma_i}{\partial x_{2r-1}} x_{i+r}^{2r-1} c_{2r,2r-1} + \frac{\partial \sigma_i}{\partial x_{2r+1}} o x_{i+r}^{2r} o c_{2r,2r+1} = 0.
\]
Since \( x_{2r}^{2r-1} \) is a factor of \( c_{2r,2r+1} \) and \( \sigma_i \) is independent of \( x_{i+r} \) we must have
\[
\frac{\partial \sigma_i}{\partial x_{2r+1}} o c_{2r,2r+1} = 0,
\]
(27)
\[
\frac{\partial \sigma_i}{\partial x_{2r-1}} = 0.
\]
Hence \( \sigma_i \) is independent of \( x_{2r-1} \). In the same way we see that \( \sigma_i \) is independent of \( x_{2r} \). Now by the first relationship in (27) we have \( \sigma_i \) independent of \( x_{2r+1} \) and \( \sigma_i \in \mathcal{K} \).

We can now confine our attention to finding the derivations of an algebra \( \mathfrak{A} \) with three generators \( x, y, z \) in which multiplication is defined by
\[
yD_x = 1 + yx \circ y^{-1} = d_{12},
\]
\[
zD_x = y^{-1} \circ (1 + \beta z^{-1}) = d_{13},
\]
\[
zD_y = \alpha x \circ (1 + \beta z^{-1}) = d_{23}.
\]
Let \( (a_1, a_2, a_3) \) be a derivation of \( \mathfrak{A} \). Since there are derivations of the form
\[
\frac{\partial g}{\partial x} o d_{11} + \frac{\partial g}{\partial y} o d_{12} + \frac{\partial g}{\partial z} o d_{13}
\]
[7, Theorem 8] and \( a_1 \circ d_{12}^{-1} = \frac{\partial g}{\partial y} \) can be solved to within a multiple \( y^{-1} \) \[7, Lemma 1\], we can subtract off the derivation induced by \( g \) and assume
\[
a_1 = \delta \circ y^{-1} \text{ in which } \delta \text{ is a polynomial in } x \text{ and } z.
\]
Using the same lemma we can solve \(- \mu^{-1} \circ \delta = \frac{\partial g}{\partial z} \) to within a multiple of \( z^{-1} \) and such that \( g \) is in \( \mathcal{K}[x, z] \). Subtracting off the derivation corresponding to this \( y \) leaves us with
\[
a_1 = \delta_0 \circ z^{-1} \circ y^{-1} \text{ in which } \delta_0 \text{ is a polynomial in } x.
\]
The three conditions \[7\] that \( (a_1, a_2, a_3) \) be a derivation can be written in the form
\[
-d(\frac{\partial (d_{12}^{-1} a_2)}{\partial y}) o d_{12}^{-1} + \frac{\partial a_1}{\partial x} o d_{21} + \frac{\partial a_1}{\partial z} o d_{31} + \frac{\partial a_1}{\partial z} o d_{23} = 0,
\]
(28)
\[
y^{-1} \circ \frac{\partial (\mu^{-1} a_3)}{\partial z} \circ \mu^2 + \frac{\partial d_{13}}{\partial y} o a_2 + \frac{\partial a_1}{\partial y} o d_{32} + \frac{\partial a_3}{\partial y} o d_{21} = 0,
\]
(29)
in which $\mu = 1 + \beta z^{p-1}$.

The last three terms of (28) are in $y^{p-1} \circ \mathfrak{H}$ since $a_1$, and $d_{31}$ are. Hence both $-d_{12}^2 \circ \partial (d_{12}^{-1} \circ a_2) / \partial y$ and $\partial (d_{12}^{-1} \circ a_2) / \partial y$ are in $y^{p-1} \circ \mathfrak{H}$. But the second polynomial is of degree at most $p - 2$ in $y$ and hence is 0. Therefore there is a polynomial $\delta_1$ independent of $y$ and such that $a_2 = \delta_1 \circ d_{12}$.

Identity (28) now reduces to

$$
\left( - \alpha x^{p-1} \circ \frac{\partial (\mu^{-1} a_3)}{\partial z} \circ \mu^2 + \frac{\partial d_{23}}{\partial x} \circ a_1 + \frac{\partial a_2}{\partial y} \circ d_{32} + \frac{\partial a_2}{\partial x} \circ d_{31} + \frac{\partial a_3}{\partial x} \circ d_{12} = 0 \right)
$$

(30)

Arguing on the degree of $z$ in each term of (31) we can conclude $\delta_0 / \delta x = 0$ and $\delta_0$ is independent of $x$. But $\delta z^{p-1} \circ y^{p-1} = \delta z^{p-1} \circ d_{13}$ and

$$
(\delta z^{p-1} \circ d_{13}, \delta z^{p-1} \circ d_{23}, 0)
$$

is a derivation of $A$. Subtracting off this derivation we can assume $a_1 = \delta_0 = 0$.

From (31), since $\delta_1$ is independent of $y$, we also get $\delta_1$ independent of $z$, i.e., $\delta_1$ is a polynomial in $\mathfrak{G}[x]$. Therefore we can find a polynomial $g$ in $\mathfrak{G}[x]$ that is a solution of $\delta_1 \circ d_{12} = d_{21} \circ \partial g / \partial x$ to within a constant multiple of $x^{p-1}$, say $\eta x^{p-1}$.

Subtracting off the derivation

$$
\left( 0, d_{21} \circ \frac{\partial g}{\partial x} - \eta x^{p-1} \circ d_{21}, d_{31} \circ \frac{\partial g}{\partial x} - \eta x^{p-1} \circ d_{31} \right)
$$

we can assume $a_1 = a_2 = 0$. Equations (29) and (30) now reduce to

$$
- y^{p-1} \circ \frac{\partial (\mu^{-1} a_3)}{\partial z} \circ \mu^2 + \frac{\partial a_3}{\partial y} \circ d_{21} = 0,
$$

(32)

$$
- \alpha x^{p-1} \circ \frac{\partial (\mu^{-1} a_3)}{\partial z} \circ \mu^2 + \frac{\partial a_3}{\partial x} \circ d_{12} = 0.
$$

Since $d_{21}$ is nonsingular we can argue on the degree of $y$ to get $\partial a_3 / \partial y = 0$ and $a_3$ is independent of $y$. In the same manner $a_3$ is independent of $x$. But then $\mu^{-1} a_3$ is independent of $z$. Hence $a_3 = \eta \mu$ for $\eta \in \mathfrak{G}$.

By direct substitution in (32) it can be seen that $(0, 0, \eta \mu)$ is a derivation of $\mathfrak{H}$. We investigate to see if it is of the form $(a_1, a_2, a_3)$ in which
If \( a_1 = a_2 = 0 \) then \( a_3 \) modulo \( y^{p-1} \) is independent of \( y \). In the same way \( g \) is independent of \( x \). Therefore

\[ a_3 = -x^{p-1} \circ y^{p-1} \circ \mu \circ (\alpha_1 + \alpha_2) \]

which is not of the form \( \eta \mu \) for \( \eta \in \mathfrak{g} \).

We can now conclude:

**Theorem 4.** Let \( \mathfrak{g} \) be a simple, nodal, Lie-admissible noncommutative Jordan algebra of characteristic \( p \neq 2 \) with \( 2r + 1 \) generators; then the derivation algebra \( \mathcal{D}(\mathfrak{g}) \) of \( \mathfrak{g} \) is the set of all mappings

\[
 f \rightarrow \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \circ a_i \right)
\]

in which

\[
 a_i = \sum_{j=1}^{n} \left( \frac{\partial g}{\partial x_j} + \alpha_j x_j^{p-1} \right) \circ c_{ij},
\]

\[
 a_{2r+1} = \sum_{2r-1}^{2n} \left( \frac{\partial g}{\partial x_i} + \alpha_i x_i^{p-1} \right) \circ c_{2r+1} + \eta \mu
\]

for \( i = 1, \ldots, 2r \). (In case \( i < 2r - 1 \) then \( a_i \) reduces to a single summand.) The dimension of \( \mathcal{D}(\mathfrak{g}) \) is \( p^{2r+1} + 2r + 1 \).

To determine the dimension of \( (\text{adj } \mathfrak{g})' \) we proceed as in the even-dimensional case. Let \( \mathfrak{I} \) be an ideal of \( \mathfrak{g} \) containing all of the generators \( x_1, \ldots, x_{2r} \). Using only the generators \( x_1, \ldots, x_{2r-2} \) we have the result from the even-dimensional case that the only possible residue classes modulo \( \mathfrak{I} \) are the classes determined by \( 1 \) and the polynomials of the form \( m \) in which \( q = x_1^{p-1} \circ \cdots \circ x_{2r-2}^{p-1} \) and \( m \) is a polynomial in \( x_{2r-1}, x_{2r} \), and \( x_{2r+1} \). We adopt the notation above using \( x, y \) and \( z \), \( x_{2r-1}, x_{2r} \), and \( x_{2r+1} \) respectively. Assume \( m \) is a monomial and \( m = x^i \circ n \), \( n \) independent of \( x \) and \( i < p - 1 \); then

\[
 \left( \frac{1}{i+1} q \circ x^{i+1} \circ n \right) D_y = -q \circ m \circ d_{12}.
\]

Also if \( m = y^i \circ n \), \( n \) independent of \( y \) and \( i < p - 1 \) then

\[
 \left( \frac{1}{i+1} q \circ y^{i+1} \circ n \right) D_x = q \circ m \circ d_{12}.
\]
Hence the only remaining residue classes of \( \mathfrak{S} \) to examine are those determined by \( q \circ x^{p-1} \circ y^{p-1} \circ n \) in which \( n \) is a polynomial in \( z \). However the equation
\[
( q \circ x^{p-1} \circ t ) D_x = q \circ x^{p-1} \circ y^{p-1} \circ \frac{\partial t}{\partial y} \circ \mu = q \circ x^{p-1} \circ y^{p-1} \circ n
\]
can be solved for \( t \), a polynomial in \( \mathfrak{R}[z] \), to within a scalar multiple of \( q \circ x^{p-1} \circ y^{p-1} \circ z^{p-1} \). Hence the only possible residue class of \( \mathfrak{S} \) is that containing \( q \circ x^{p-1} \circ y^{p-1} \circ z^{p-1} \). If \( S = \emptyset \) (the set of all \( i = 1, \ldots, r \) such that \( c_{i+r} \in \mathfrak{S} \)) and \( \beta \neq 0 \) then as we have seen in the even-dimensional case \( q \circ x^{p-1} \circ y^{p-1} \circ \mu \in \mathfrak{S} \) and \( (q \circ x^{p-1} \circ y^{p-1} \circ z^{p-1} ) D_x = (q \circ x^{p-1} \circ y^{p-1} \circ \mu) \in \mathfrak{S} \). Therefore \( q \circ x^{p-1} \circ y^{p-1} \circ z^{p-1} \in \mathfrak{S} \).

If \( \mathfrak{S} \) is the ideal in \( \mathfrak{U}^- \) such that \( \mathfrak{U}^- / \mathfrak{S} \) is isomorphic to \( (\text{adj } \mathfrak{U}^-)' \), then we can show, exactly as in the even-dimensional case, that \( q \circ x^{p-1} \circ y^{p-1} \circ z^{p-1} \) is not in \( \mathfrak{S} \) if either \( S \neq \emptyset \) or \( \beta = 0 \). Hence (adj \( \mathfrak{U}^- \)' is of dimension \( p^{2r-1} - 1 \) or \( p^{2r+1} - 2 \). We now examine the ideals of (adj \( \mathfrak{U}^- \)' . Let \( \mathfrak{S} \) be an ideal of (adj \( \mathfrak{U}^- \)' . (We again use the notation of \( \mathfrak{U}^- \)). As in the even-dimensional case we can assume there are polynomials of the form \( x_i \circ m \) for any \( i \leq 2r - 1 \) and in which \( m \) is a polynomial in \( \mathfrak{R}[x,y,z] \).

Consider those polynomials \( x \circ m \). If \( m \) is in \( \mathfrak{R}[x] \) we choose a \( k \) so that
\[
(x_1 \circ m) D(x_1 \circ x_1 + r \circ x^k) = x_1 \circ x^{p-1}.
\]
If \( m \notin \mathfrak{R}[x] \), write
\[
m = m_1 + \sum_k x^1 \circ n_i
\]
in which \( m_1 \) is a polynomial in \( x \), every term of every nonzero \( n_i \) has either a \( y \) or \( z \) in it and some \( n_i \neq 0 \). If \( k \neq 0 \) then
\[
(x_1 \circ m) D(x^{p-k}) = -k x^{p-1} \circ n_1 D_x \circ x_1 \neq 0
\]
is in \( \mathfrak{S} \). If \( k = 0 \) then
\[
(x_1 \circ m) D(x^{p-1}) = (-n_0 D_x \circ x^{p-2} - x^{p-1} \circ n_1 D_x) \circ x_1 \neq 0
\]
is in \( \mathfrak{S} \). If \( n_0 D_x \circ \) and \( n_1 D_x \) are in \( \mathfrak{R} \) then as above we can conclude \( x_1 \circ x^{p-1} \in \mathfrak{S} \).

If \( n_0 D_x \) is in \( \mathfrak{R} \) but \( n_1 D_x \) is not then
\[
(x_1 \circ m) D(x^{p-1} D_x) = -x^{p-1} \circ x_1 \circ n_1 D_x D_x \neq 0
\]
is in \( \mathfrak{S} \). If \( n_0 D_x \notin \mathfrak{R} \) then
\[
(x_1 \circ m) D(x^{p-1} D(x^2)) = -2 n_0 D_x^2 \circ x^{p-1} \circ x_1 \neq 0
\]
is in \( \mathfrak{S} \). In any case, there is a polynomial \( x_1 \circ x^{p-1} \circ m \) in \( \mathfrak{S} \) in which \( m \in \mathfrak{R}[y,z] \). If \( m \) is in \( \mathfrak{R} \) we can proceed as in the even-dimensional case to show that \( x_i, \ldots, x_{2r} \) are in \( \mathfrak{S} \).
If \( m \) is independent of \( y \) then assume \( m \) is such a polynomial of minimal degree in \( z \). We have

\[
(x_1 \circ x^{p-1} \circ m)D^p_z = x_1 \circ x^{p-1} \circ \frac{\partial m}{\partial z}.
\]

By the minimality of the degree of \( z \) in \( m \) we have \( \partial m/\partial z = 0 \), \( m \in \mathcal{I} \) and \( x_1 \circ x^{p-1} \in \mathfrak{I} \).

If \( m \) is not independent of \( y \) then

\[
(x_1 \circ x^{p-1} \circ m)D^{p-2}_y = x_1 \circ x \circ m
\]
is in \( \mathfrak{I} \). Let \( k \) be the smallest exponent of \( y \) in \( m \). If \( k = 0 \) then \( (x_1 \circ x \circ m)D_z = x_1 \circ y^{p-1} \circ m \circ \mu = x_1 \circ y^{p-1} \circ n \) is in \( \mathfrak{I} \) for some polynomial \( n \) in \( \mathbb{F}[z] \). If \( k \neq 0 \) then \( (x_1 \circ x \circ m)D(y^{p-k}) = kx_1 \circ y^{p-1} \circ n \) is in \( \mathfrak{I} \) for some polynomial \( n \) in \( \mathbb{F}[z] \). Choose \( n \) to be of minimal degree in \( z \). Then as above we can show \( n \) is in \( \mathfrak{I} \) and \( x_1 \circ y^{p-1} \in \mathfrak{I} \).

Thus either \( x_1 \circ x^{p-1} \) or \( x_1 \circ y^{p-1} \) is in \( \mathfrak{I} \). As in the even-dimensional case this implies \( x_1, \ldots, x_{2r} \), are in \( \mathfrak{I} \). Hence from our conclusion above on such ideals \( \mathfrak{I} \) we have \( (\text{adj } \mathfrak{A})' \) is simple. Thus

**Theorem 5.** If \( \mathfrak{A} \) is a simple, Lie-admissible nodal noncommutative Jordan algebra of characteristic \( p \neq 2 \) with \( 2r + 1 \) generators then \( (\text{adj } \mathfrak{A})' \) is a simple Lie-algebra. The dimension of \( (\text{adj } \mathfrak{A})' \) is \( p^{2r-1} - 1 \) if \( \mathcal{G} = \emptyset \) and \( \beta \neq 0 \) and is \( p^{2r+1} - 2 \) if either \( \mathcal{G} \neq 0 \) or \( \beta = 0 \).

**References**


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