WEAKLY COMPACT SETS

BY

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It has been conjectured that a closed convex subset $C$ of a Banach space $B$ is weakly compact if and only if each continuous linear functional on $B$ attains a maximum on $C$ [5]. This reduces easily to the case in which $C$ is bounded, and will be answered in the affirmative [Theorem 4] after some preliminary results are established. Following suggestions by Namioka and Peck, the result is then generalized, first to weakly closed subsets of Banach spaces and then to weakly closed subsets of complete locally convex linear spaces.

The original motivation for this conjecture was the knowledge that it is true when $C$ is the unit sphere of a separable Banach space [2], which was later extended to arbitrary Banach spaces [3, Theorem 5]. Additional support was given the conjecture when V. L. Klee [5] proved a seemingly related theorem—namely, that if $C$ is a bounded closed non-weakly-compact convex subset of a Banach space, then there is a decreasing sequence $\{K_i\}$ of nonempty closed convex subsets of $C$ such that, for each $x \in C$ and each $m \in [0, 1)$, the set $x + m(C - x)$ meets only finitely many of the sets $\{K_i\}$. If $f$ is a continuous linear functional whose sup on $C$ is $M$ and if there is no $x$ in $C$ for which $f(x) = M$, then a suitable choice for Klee's sequence $\{K_i\}$ is to let $K_n = C \cap \{x: f(x) \geq M - 1/n\}$ for each $n$.

It is interesting to note that the conjecture can be verified easily for a bounded closed convex set that is symmetric about an interior point $x$, since if $x$ is translated to 0, then the convex set as a unit sphere induces a norm for which the new Banach space is isomorphic to the original space. More generally, if 0 is an interior point of a bounded convex set $C$ and $K$ is the closed convex span of $C \cup (-C)$, then $K$ is symmetric about 0 and the sup on $K$ of a continuous linear functional is the larger of its sups on $C$ and $-C$. Therefore for convex bodies the conjecture can be established by using the known theorem for unit spheres [3, Theorem 5, p. 215].

The following theorem is a generalization of a characterization of weak compactness of the unit sphere that was useful in [2]. In the proof of this theorem and thereafter, we shall use the convention that a sequence of nonoverlapping sequences $\{K_i\}$ is to let $K_n = C \cap \{x: f(x) \geq M - 1/n\}$ for each $n$.

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members of conv $\{y_n\}$ is a sequence $\{x_n\}$ for which there is an increasing sequence of integers $\{p_i\}$ such that $x_n$ belongs to conv $\{y_{p_n}, \ldots, y_{p_{n+1}-1}\}$ for all $n$.

**Theorem 1.** A bounded closed convex subset $C$ of a Banach space is not weakly compact if and only if there exist positive numbers $\mu$ and $\delta$ such that if $\delta < \mu < \Delta$, then there exists a sequence $\{z_i\}$ of members of $C$ for which the following are true for all positive integers $n$:

(i) $\delta < \|\xi\| < \Delta$ for all $\xi \in \text{conv} \{z_i\}$;

(ii) $d(\text{conv} \{z_1, \ldots, z_n\}, \text{conv} \{z_{n+1}, \ldots\}) \geq \sigma$.

**Proof.** It is known that a closed convex subset $C$ of a Banach space is weakly compact if and only if each decreasing sequence $\{K_i\}$ of closed convex sets contained in $C$ has a nonempty intersection [1; 6]. If $\{z_i\}$, $\delta$, $\Delta$, and $\sigma$ exist for which (i) and (ii) are satisfied, we can let $K_n = \text{cl} \text{conv} \{z_n, z_{n+1}, \ldots\}$ for each $n$. If $w \in K_n$ for all $n$, then there are nonoverlapping members $\eta$ and $\zeta$ of $\text{conv} \{z_i\}$ such that

$$\|w - \eta\| < \frac{1}{2}\sigma \text{ and } \|w - \zeta\| < \frac{1}{2}\sigma.$$  

Then $\|\eta - \zeta\| < \sigma$. But it follows from (ii) that $\|\eta - \zeta\| \geq \sigma$. Therefore the sequence $\{K_i\}$ has an empty intersection and $C$ is not weakly compact.

Now suppose that $C$ is bounded, closed, convex, and not weakly compact. Then there is a decreasing sequence $\{T_i\}$ of closed convex subsets of $C$ that have an empty intersection. Choose a sequence $\{t_i\}$ for which $t_i \in T_i$ for each $n$. We shall show first that if it is not possible to satisfy (ii) with $\sigma = \frac{1}{2}$ and $\{z_i\}$ some sequence of nonoverlapping members of $\text{conv} \{t_i\}$, then there is a member $t_1^i$ of $\text{conv} \{t_i\}$ for which there is a sequence $\{t_i^1\}$ of nonoverlapping members of $\text{conv} \{t_i\}$ such that

$$\|t_1^i - t_n^i\| < \frac{1}{2} \text{ for all } n > 1.$$  

To show this, we can suppose that no such $t_1^i$ exists. Then we let $z_1 = t_1$. There is then an $n_1$ such that $\|z_1 - w\| \geq \frac{1}{2}$ for all $w$ in $\text{conv} \{t_{n_1}, t_{n_1+1}, \ldots\}$. Let $z_2 = t_{n_1}$. For any finite set $M$ of members of $\text{conv} \{z_1, z_2\}$, we can choose $n_2$ large enough that $\|z - w\| \geq \frac{1}{2}$ for all $z$ in $M$ and all $w$ in $\text{conv} \{t_{n_2+1}, t_{n_2+2}, \ldots\}$. Therefore we can choose $n_2$ so that $\|z - w\| \geq \frac{1}{2}$ for all $z$ in $\text{conv} \{z_1, z_2\}$ and all $w$ in $\text{conv} \{t_{n_2+1}, t_{n_2+2}, \ldots\}$. Continuing in this way, we could satisfy (ii) with $\sigma = \frac{1}{2}$. If it is not true that (ii) can be satisfied for some $\sigma > 0$ and some sequence of nonoverlapping members of $\text{conv} \{t_i\}$, then an argument similar to that just completed can be used inductively to obtain sequences $\{t_i^k\}$, $k = 1, 2, \ldots$, such that $\{t_i^k\}$ is a sequence of nonoverlapping members of $\text{conv} \{t_2^{k-1}, t_3^{k-1}, \ldots\}$ and

$$\|t_1^k - t_n^k\| < \frac{1}{k+1} \text{ for all } n.$$
Then \( \| t^*_p - t^*_k \| < 1/(k + 1) \) for all \( p > k \), so \( \{ t^*_k \} \) is a Cauchy sequence. Its limit is in the intersection of the sets \( \{ T_n \} \), contrary to assumption. Therefore we can conclude that there is a positive number \( \sigma \), and a sequence \( \{ z^*_n \} \) of nonoverlapping members of \( \text{conv} \{ t_n \} \), for which (ii) is satisfied.

Now let \( \mu = \lim_{n \to \infty} \inf \{ \| x \| : x \in \text{conv} \{ z^*_n, z^*_n + 1, \ldots \} \} \). Since 0 is not in the intersection of the sets \( \{ T_j \} \), we have \( \mu > 0 \). If \( \delta < \mu - \Delta \), we can choose \( N \) so that

\[
\delta < \inf \{ \| x \| : x \in \text{conv} \{ z^*_N, z^*_N + 1, \ldots \} \}.
\]

Now choose \( \{ z_i \} \) so that for each \( n \) we have \( \| z_n \| < \Delta \) and \( \{ z_i \} \) is a sequence of nonoverlapping members of \( \text{conv} \{ z^*_N, z^*_N + 1, \ldots \} \). If \( \xi \in \text{conv} \{ z_i \} \), then \( \delta < \| \xi \| < \Delta \), so that (i) is now satisfied. Also, (ii) is satisfied by \( \{ z_i \} \) since (ii) was satisfied by \( \{ z^*_i \} \).

It is possible to strengthen Theorem 1 by replacing \( \text{conv} \{ z_1, \ldots, z_n \} \) in (ii) by the linear span of \( \{ z_1, \ldots, z_n \} \), denoted by \( \text{lin} \{ z_1, \ldots, z_n \} \). This is a consequence of the following theorem.

**Theorem 2.** Suppose that for a sequence \( \{ z_i \} \) and positive numbers \( \delta \) and \( \sigma \), we have

(i) \( \delta < \| \xi \| \) for all \( \xi \in \text{conv} \{ z_i \} \),

(ii) \( d(\text{conv} \{ z_1, \ldots, z_n \}, \text{conv} \{ z_{n+1}, \ldots \}) \geq \sigma \) for all \( n \).

If \( \theta < \frac{1}{2} \sigma \), then there is a subsequence \( \{ z^*_i \} \) of \( \{ z_i \} \) such that

(ii)' \( d(\text{lin} \{ z^*_1, \ldots, z^*_n \}, \text{conv} \{ z^*_{n+1}, \ldots \}) \geq \theta \) for all \( n \).

**Proof.** Suppose that \( \theta < \frac{1}{2} \sigma \). We shall use induction to show the existence of the subsequence \( \{ z^*_i \} \). Suppose that a positive integer \( p_k \) and a finite set \( \{ z_1^*, z_2^*, \ldots, z_k^* \} \) have been chosen so that (ii)' is valid for \( n \leq k \) if we use for \( \{ z_i^* \} \) the sequence consisting of \( \{ z_1^*, \ldots, z_k^* \} \) and all \( z_i \) with \( i > p_k \). If the analogous statement is not true for \( n \leq k + 1 \) when \( z_{k+1}^* = z_{p_k} \) and \( p_{k+1} \) is sufficiently large, then for any \( p \) there are members \( u \) and \( z \) of \( \text{lin} \{ z_1^*, \ldots, z_k^* \} \) and \( \text{conv} \{ z_{k+2}^*, \ldots \} \) such that

\[
\| u - z \| < \theta.
\]

Since \( p \) can be arbitrarily large, there are an infinite number of nonoverlapping choices for \( z \). It follows from this and \( k \) being fixed that for any positive number \( \varepsilon \) there are members \( u_1 \) and \( u_2 \) of \( \text{lin} \{ z_1^*, \ldots, z_k^* \} \), and nonoverlapping members \( \eta \) and \( \zeta \) of \( \text{conv} \{ z_{k+2}^*, \ldots \} \), for which

\[
\| u_1 - \eta \| < \theta, \quad \| u_2 - \zeta \| < \theta,
\]

and

\[
\| u_1 - u_2 \| < \varepsilon.
\]

Then \( \| \eta - \zeta \| < 2 \theta + \varepsilon \). It follows from this and (ii) that
Since $\epsilon$ was arbitrary, we have $\theta \geq \frac{1}{2}\sigma$. This contradicts the assumption that $\theta < \frac{1}{2}\sigma$, so we conclude that the induction can be completed.

The next theorem is the particular consequence of Theorems 1 and 2 that will be used in proving Theorem 4. It is a generalization of a similar theorem about nonreflexive Banach spaces—Theorem 1 of [3].

**Theorem 3.** A bounded closed convex subset $C$ of a Banach space is not weakly compact if and only if there is a positive number $r$ for which there exists a sequence $\{z_i\}$ of members of $C$, and a sequence $\{f_i\}$ of linear functionals with unit norms, such that $f_n(z_i) > r$ if $n \leq i$ and $f_n(z_i) = 0$ if $n > i$.

**Proof.** Suppose first that such sequences $\{z_i\}$ and $\{f_i\}$ exist. Let $K_n$ be $\text{cl} [\text{conv} \{z_n, z_{n+1}, \cdots\}]$. Then the decreasing sequence $\{K_n\}$ of closed convex subsets of $C$ has an empty intersection, since if $w$ belongs to the intersection, then $f_n(w) \geq r$ for all $n$ and $\lim_{n \to \infty} f_n(w) = 0$. Therefore $C$ is not weakly compact.

Suppose now that $C$ is bounded, closed, convex, and not weakly compact. Then it follows from Theorems 1 and 2 that there is a sequence $\{z_i\}$ of members of $C$, and positive numbers $\delta$, $\Delta$, and $r$, such that

(i)* $\delta < \| \xi \| < \Delta$ for all $\xi \in \text{conv} \{z_i\}$;

(ii)* $d(\text{lin} \{z_1, \cdots, z_n\}, \text{conv} \{z_{n+1}, \cdots\}) \geq r$ for all $n$.

As a first step toward defining the desired continuous linear functional $f_n$ for a particular $n$, let us define a new norm on $\text{lin} \{z_1, \cdots, z_{n-1}\}$ and, when $z = \sum a_i z_i$, let

$$\|\| z \|\| = \max \left[ d(z, S_{n-1}), \sum_{i=1}^{n-1} a_i \right].$$

If $z \in S_{n-1}$, then $\|\| z \|\| = \| z \|$. If $\|\| z \|\| = 0$, then we must have $d(z, S_{n-1}) = 0$ and $z \in S_{n-1}$, so that $\|\| z \|\| = \| z \|$. Therefore $\|\| z \|\| \neq 0$ if $z \neq 0$. The triangle inequality is satisfied by $\|\| z \|\|$, since it is satisfied by $p$ if either $d(z, S_{n-1})$ is used for $p(z)$ or $\sum_{i=1}^{n-1} a_i z_i$ is used for $p(z)$ when $z = \sum a_i z_i$. Now let

$$\frac{1}{\theta} = \inf \{\|\| z \|\| : z \in \text{conv} \{z_n, z_{n+1}, \cdots\}\}.$$

It follows from the definition of $\|\| z \|\|$ and (ii)* that

(2) $$\frac{1}{\theta} = d(S_{n-1}, \text{conv} \{z_n, z_{n+1}, \cdots\}) \geq r.$$

The definition of $\theta$ implies that $\text{conv} \{\theta z_n, \theta z_{n+1}, \cdots\}$ contains no points $x$ of $\text{lin} \{z_n, z_{n+1}, \cdots\}$ for which $\|\| x \|\| < 1$. Therefore [7], there is a linear functional $f_n^*$ whose domain is $\text{cl} [\text{lin} \{z_n, z_{n+1}, \cdots\}]$ and for which

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(3) \( f^*_n(z) \geq 1 \) for all \( z \in \text{conv} \{\theta z_n, \theta z_{n+1}, \ldots\} \),

and

\( f^*_n(u) \leq 1 \) if \( \|u\| \leq 1 \).

Let \( f^{**}_n \) be defined by \( f^{**}_n(x) = f^*_n(x) \) when \( x \) is in the domain of \( f^*_n \), and \( f^{**}_n(z_i) = 0 \) if \( i < n \). Then

\[
|f^{**}_n(\sum a_i z_i)| = \left| f^*_n\left( \sum_n a_i z_i \right) \right| \leq \left\| \sum_n a_i z_i \right\|.
\]

Since \( \left\| \sum_n a_i z_i \right\| = d(\sum_n a_i z_i, S_{n-1}) \leq \| \sum a_i z_i \| \), we have

\[
\left| f^{**}_n(\sum a_i z_i) \right| \leq \sum a_i z_i \|.
\]

Therefore \( \|f^{**}_n\| \leq 1 \). Also, if \( n \leq i \), it follows from (3) that \( f^*_n(\theta z_i) \geq 1 \), and from this and (2) that

\[
f^{**}_n(z_i) = f^*_n(z_i) \geq \frac{1}{\theta} \geq r.
\]

Now we can define \( f_n \) as a norm-preserving extension of \( f^{**}_n / \|f^{**}_n\| \) to the entire space.

The second of the following lemmas is similar to the lemma used in [3] to prove that if a Banach space is nonreflexive then there is a continuous linear functional that does not attain its sup on the unit sphere. However, that lemma and the associated theorem were constructed by analogy with Theorem 2 of [3], which states that there is a continuous linear functional \( g \) on \( l^1 \) which has the property that, if \( l^1 \) is a subspace of a Banach space \( B \), then there is a norm-preserving extension of \( g \) that does not attain its sup on the unit sphere of \( B \). Lemma 2 and Theorem 4 use an analogy to continuous linear functionals of type \( \sum \beta_i x_i \) defined on \( (c_0) \), where \( \beta_i > 0 \) for each \( i \) and \( \sum \beta_i < +\infty \). Such a functional does not attain its sup on the unit sphere of \( (c_0) \).

If \( B \) is separable, then the following lemmas and theorem can be simplified as follows. First choose a subsequence \( \{f_{p_i}\} \) of \( \{f_i\} \) for which \( \lim f_{p_i}(x) \) exists for each \( x \) of \( B \), and let \( f^*_n \) be defined as \( h_n / \|h_n\| \), where

\[
h_n(x) = f_{p_i}(x) - \lim [f_{p_i}(x)]
\]

for each \( x \) of \( B \). Then for the sequences \( \{f_i\} \) and \( \{z_i\} \) of Theorem 3, there is a subsequence \( \{z_i^*\} \) of \( \{z_i\} \) such that \( f^*_n(z_i^*) > \frac{1}{2} r \) if \( n \leq i \) and \( f^*_n(z_i^*) = 0 \) if \( n > i \). If \( \{f_i^*\} \), \( \{z_i^*\} \), and \( \frac{1}{2} r \) are used instead of \( \{f_i\} \), \( \{z_i\} \) and \( r \), then Lemma 1 can be discarded and every lim inf and lim sup used in the proofs of Lemma 2 and Theorem 4 is equal to zero.

**Lemma 1.** Let \( C \) be a bounded convex subset of a Banach space \( B \) and let \( g \) and \( G_1, G_2, \ldots \) be continuous linear functionals on \( B \). If \( \beta, \gamma, \) and \( s \) are nonnegative numbers, then there exists a sequence \( \{G_i\} \) of nonoverlapping members of \( \text{conv} \{G_i\} \) such that
\[
\sup_{x \in C} \left[ g(x) + \beta h_1(x) - \gamma \lim \inf h_i(x) \right] \\
< \varepsilon + \sup_{x \in C} \left[ g(x) + \beta h_1(x) - \gamma \lim \sup h_i(x) \right]
\]

for all sequences \( \{h_i\} \) of nonoverlapping members of conv \( \{G_i\} \).

**Proof.** Let us first arrange all members of conv \( \{G_i\} \) that have rational coefficients in a sequence \( \{\phi_i\} \). Choose \( \{F_i\} \) and an element \( x_1 \) of \( C \) so that \( \{F_i\} \) is a sequence of nonoverlapping members with rational coefficients of conv \( \{G_i\} \) which has the property that

\[
g(x_1) + \beta \phi_1(x_1) - \gamma \lim \inf F^*_1(x_1)
\]

is within \( \frac{\varepsilon}{2} \) of being as large as possible for all such sequences and elements \( x_1 \). Then let \( \{F^*_1\} \) be a subsequence of \( \{F_i\} \) for which \( \lim F^*_1(x_1) \) exists and equals \( \lim \inf F_i(x_1) \). Now use this process inductively to choose for each \( k \) a sequence \( \{F^*_k\} \) and an element \( x_k \) of \( C \) with the following properties:

(a) \( \{F^*_k\} \) is a sequence of nonoverlapping members with rational coefficients of conv \( \{F^*_{k-1}, F^*_{k-2}, \ldots\} \);

(b) \( g(x_k) + \beta \phi_k(x_k) - \gamma \lim \inf_{i \to \infty} F^*_i(x_k) \) is within \( \frac{\varepsilon}{2} \) of being as large as possible for all choices of \( \{F^*_i\} \) as a sequence of nonoverlapping members of conv \( \{F^*_{k-1}, F^*_{k-2}, \ldots\} \) and all choices of \( x_k \) from \( C \);

(c) \( \lim_{i \to \infty} F^*_i(x_k) \) exists.

Now let \( G_k = F^*_k \) for each \( k \). For any \( \phi_k \) in conv \( \{G_i\} \), the expression

\[
g(x_k) + \beta \phi_k(x_k) - \gamma \lim_{i \to \infty} G_i(x_k) = g(x_k) + \beta \phi_k(x_k) - \gamma \lim_{i \to \infty} F^*_i(x_k),
\]

and this expression cannot be increased by more than \( \frac{\varepsilon}{2} \) by replacing \( \{G_i\} \) by some sequence \( \{h_i\} \) of nonoverlapping members of conv \( \{G_i\} \) and \( x_k \) by some member of \( C \). Also, \( \lim \inf_{i \to \infty} h_i(x_k) = \lim \sup_{i \to \infty} h_i(x_k) \) for any such sequence. Now for a choice of \( \{h_i\} \), we can choose \( \phi_k \) so that \( | h_i(x) - \phi_k(x) | < \frac{\varepsilon}{2} \) if \( x \in C \) and use \( \phi_k \) and the corresponding \( x_k \) to show that it is impossible to have

\[
\sup_{x \in C} \left[ g(x) + \beta h_1(x) - \gamma \lim \inf h_i(x) \right] \\
\geq \varepsilon + \sup_{x \in C} \left[ g(x) + \beta h_1(x) - \gamma \lim \sup h_i(x) \right],
\]

since we could then replace \( h_1 \) by \( \phi_k \) and obtain

\[
[g(x_k) + \beta \phi_k(x_k) - \gamma \lim_{i \to \infty} h_i(x_k)] > \sup_{x \in C} \left[ g(x) + \beta \phi_k(x) - \gamma \lim \sup h_i(x) \right].
\]

**Lemma 2.** Suppose that \( C \) is a bounded convex subset of a Banach space \( B \), \( r \) is a positive number, \( \{z_i\} \) is a sequence of members of \( C \), and \( \{f_i\} \) is a sequence of linear functionals with unit norms such that \( f_n(z_i) > r \) if \( n \leq i \) and \( f_n(z_i) = 0 \) if \( n > i \). If \( \{\beta_i\} \) is a sequence of positive numbers, then there is a sequence \( \{g_i\} \) of nonoverlapping members of conv \( \{f_i\} \) which has the following property K:
If $g_k(\xi) - \lim \inf g_k(\xi) \leq (\frac{1}{2} + 2^{-k})r$ for some $k$ and some $\xi \in C$, then there is a $y$ in $C$ such that

$$\left[ \sum_{i=1}^{k} \beta_i g_i(y) - \left( \sum_{i=1}^{k} \beta_i \right) \lim \sup_{i \to 0} g_i(y) \right]$$

$$> 2^{-(k+2)}r \beta_k + \left[ \sum_{i=1}^{k} \beta_i g_i(\xi) - \left( \sum_{i=1}^{k} \beta_i \right) \lim \inf_{i \to 0} g_i(\xi) \right].$$

**Proof.** We shall show first that $\{g_i\}$ has property K when $k = 1$, if $\{g_i\}$ is any sequence of nonoverlapping members of conv $\{f_i\}$. To do this, we suppose that $\{g_i\}$ has been chosen and that a member $\xi$ of $C$ has the property that

$$g_1(\xi) - \lim \inf g_1(\xi) \leq \frac{1}{2}r.$$

We have

$$\beta_1 g_1(z_n) = \beta_1 g_1(z_n) > r \beta_1$$

if $n$ is large enough. Therefore there is an $n$ large enough that we can satisfy (4) with $k = 1$ and $y = z_n$ if it is true that

$$r \beta_1 > \frac{1}{2} r \beta_1 + \beta_1[g_1(\xi) - \lim \inf g_1(\xi)].$$

This inequality is true, since $g_1(\xi) - \lim \inf g_1(\xi) \leq \frac{1}{2}r$.

Now suppose that $\{g_1, g_2, \ldots, g_n, F_n, F_{n+1}, \ldots\}$ has been chosen as a sequence of nonoverlapping members of conv $\{f_i\}$ and that, for each sequence $\{g_n, g_{n+1}, \ldots\}$ of nonoverlapping members of conv $\{F_n, F_{n+1}, \ldots\}$, property K is satisfied when $k \geq n$. Let

$$\theta_n = \inf_{\substack{G_n \subseteq \text{conv}\{g_1, \ldots, g_n\} \\cap \subseteq \text{conv}\{F_n, F_{n+1}, \ldots\}} \sup_{x \in C} \left[ \sum_{i=1}^{n} \beta_i g_i(x) + \beta_n G_n(x) \right] - \left( \sum_{i=1}^{n} \beta_i \right) \lim \inf G_i(x)],$$

where $\{G_i\}$ denotes a sequence $\{G_n, G_{n+1}, \ldots\}$ of nonoverlapping members of conv $\{F_n, F_{n+1}, \ldots\}$. Choose a particular $\{G_i^*\}$ so that

$$\sup_{x \in C} \left[ \sum_{i=1}^{n} \beta_i g_i(x) + \beta_n G_n^*(x) \right] - \left( \sum_{i=1}^{n} \beta_i \right) \lim \inf G_i^*(x)] < \theta_n + 2^{-(n+3)}r \beta_{n+1}.$$

If $G_n^* = G_n$ and $\{G_{n+1}^*, \ldots\}$ is a sequence of nonoverlapping members of conv $\{G_{n+1}^*, \ldots\}$, then for all sequences $\{h_{n+1}, \ldots\}$ of nonoverlapping members of conv $\{G_{n+1}^*, \ldots\}$ we have $\lim \inf h_{n+1}(x) \subseteq \lim \inf G_i^*(x)$ for all $x$ in $B$. Therefore it follows from Lemma 1 that we can let $G_n^* = G_n$ and choose $\{G_{n+1}^*, \ldots\}$ so that

$$\sup_{x \in C} \left[ \sum_{i=1}^{n} \beta_i g_i(x) + \beta_n G_n^*(x) \right] - \left( \sum_{i=1}^{n} \beta_i \right) \lim \inf G_i^*(x)] < \theta_n + 2^{-(n+3)}r \beta_{n+1},$$

where $\lim \inf G_i^*(x) \subseteq \lim \inf G_i(x)$ for all $x$ in $B$.
\[
\sup_{x \in \mathcal{C}} \left[ \sum_{i=1}^{n-1} \beta_i g_i(x) + \beta_n g_n^*(x) + \beta_{n+1} h_{n+1}(x) - \left( \sum_{i=1}^{n+1} \beta_i \right) \liminf h_i(x) \right] < 2^{-(n+3)} r \beta_{n+1}
\]

\[
+ \sup_{x \in \mathcal{C}} \left[ \sum_{i=1}^{n-1} \beta_i g_i(x) + \beta_n g_n^*(x) + \beta_{n+1} h_{n+1}(x) - \left( \sum_{i=1}^{n+1} \beta_i \right) \limsup h_i(x) \right],
\]

for all sequences \( \{h_{n+1}, \ldots\} \) of nonoverlapping members of \( \text{conv}\{G^*_{n+1}, \ldots\} \).

Now suppose there exists a \( G^{**} \) in \( \text{conv}\{G^*_{n+1}, \ldots\} \) and a sequence \( \{h_{n+1}, \ldots\} \) of nonoverlapping members of \( \text{conv}\{G^*_{n+1}, \ldots\} \) for which

\[
\sup_{x \in \mathcal{C}} \left[ \sum_{i=1}^{n-1} \beta_i g_i(x) + \beta_n g_n^*(x) + \beta_{n+1} G^{**}(x) - \left( \sum_{i=1}^{n+1} \beta_i \right) \liminf h_i(x) \right] < \theta_n + (\frac{1}{4} + 2^{-n}) r \beta_{n+1}.
\]

Let \( H = (\beta_n G_n^* + \beta_{n+1} G^{**})/ (\beta_n + \beta_{n+1}) \). Then

\[
\sup_{x \in \mathcal{C}} \left[ \sum_{i=1}^{n-1} \beta_i g_i(x) + (\beta_n + \beta_{n+1}) H(x) - \left( \sum_{i=1}^{n+1} \beta_i \right) \liminf h_i(x) \right] < \theta_n + (\frac{1}{4} + 2^{-n}) r \beta_{n+1}.
\]

We know that property \( K \) is valid for \( k = n \) with \( h_i \) substituted for \( g_i \) when \( i > n \) and \( H \) substituted for \( g_n \). Therefore if \( \xi \in \mathcal{C} \) is such that the expression \( S \),

\[
S = \sum_{i=1}^{n-1} \beta_i g_i(x) + \beta_n H(x) - \left( \sum_{i=1}^{n+1} \beta_i \right) \liminf h_i(x),
\]

is within \( 2^{-\eta-2} r \beta_{n+1} \) of its sup when \( x = \xi \), then

\[
H(\xi) - \liminf h_i(\xi) > (\frac{1}{4} + 2^{-n}) r
\]

and \( S \) is increased by more than \( (\frac{1}{4} + 2^{-n}) r \beta_{n+1} \) if the quantity

\[
\beta_{n+1} [H(\xi) - \liminf h_i(\xi)]
\]

is added, so it follows from (9) that

\[
\sup_{x \in \mathcal{C}} \left[ \sum_{i=1}^{n-1} \beta_i g_i(x) + \beta_n H(x) - \left( \sum_{i=1}^{n+1} \beta_i \right) \liminf h_i(x) \right] < \theta_n + (\frac{1}{4} + 2^{-n}) r \beta_{n+1} - (\frac{1}{4} + 2^{-n}) r \beta_{n+1} = \theta_n.
\]

This contradicts (5), so we can conclude that (8) is false for all \( G^{**} \) and all sequences \( \{h_{n+1}, \ldots\} \) of nonoverlapping members of \( \text{conv}\{G^*_{n+1}, \ldots\} \). It follows from this and (6) that
\[
\sup_{x \in C} \left[ \sum_{i=1}^{n-1} \beta_i g_i(x) + \beta_n G_n^*(x) + \beta_{n+1} G_{n+1}^*(x) - \left( \sum_{i=1}^{n+1} \beta_i \right) \lim \inf h_i(x) \right]
\]

\[
(10) \quad > \left[ \frac{1}{4} + 2^{-n} - 2^{-(n+3)} \right] r \beta_{n+1}
\]

\[
+ \sup_{x \in C} \left[ \sum_{i=1}^{n-1} \beta_i g_i(x) + \beta_n G_n^*(x) - \left( \sum_{i=1}^{n} \beta_i \right) \lim \inf h_i(x) \right],
\]

whatever the choice of \(G^*\) and \(\{h_{n+1}, \cdots\}\) as nonoverlapping members of conv\{\(G_{n+1}^*, \cdots\}\}. Now let \(G_n^* = g_n\). It then follows from (7) and (10) that

\[
\sup_{x \in C} \left[ \sum_{i=1}^{n} \beta_i g_i(x) + \beta_{n+1} h_{n+1}(x) - \left( \sum_{i=1}^{n+1} \beta_i \right) \lim \sup h_i(x) \right]
\]

\[
> -2^{-(n+3)} r \beta_{n+1}
\]

\[
(11) \quad + \sup_{x \in C} \left[ \sum_{i=1}^{n} \beta_i g_i(x) + \beta_{n+1} h_{n+1}(x) - \left( \sum_{i=1}^{n+1} \beta_i \right) \lim \inf h_i(x) \right]
\]

\[
> \left[ \frac{1}{4} + 2^{-n} - 2^{-(n+2)} \right] r \beta_{n+1} + \sup_{x \in C} \left[ \sum_{i=1}^{n} \beta_i g_i(x) - \left( \sum_{i=1}^{n} \beta_i \right) \lim \inf h_i(x) \right].
\]

Now we are prepared to attack property K when \(k = n + 1\). Suppose that \(\{h_{n+1}, \cdots\}\) is a sequence of nonoverlapping members of conv\{\(G_{n+1}^*, \cdots\}\} and that \(\xi\) belongs to \(C\) and is such that \(h_{n+1}(\xi) - \lim \inf h_i(\xi) \leq [\frac{1}{4} + 2^{-(n+1)}] r\). Then it is easy to verify directly that

\[
\left( \frac{1}{4} + 2^{-(n+1)} \right) r \beta_{n+1}
\]

\[
(12) \quad \geq \left[ \sum_{i=1}^{n} \beta_i g_i(\xi) + \beta_{n+1} h_{n+1}(\xi) - \left( \sum_{i=1}^{n+1} \beta_i \right) \lim \inf h_i(\xi) \right]
\]

\[
- \left[ \sum_{i=1}^{n} \beta_i g_i(\xi) - \left( \sum_{i=1}^{n} \beta_i \right) \lim \inf h_i(\xi) \right].
\]

From (11), it follows that there is a \(y\) in \(C\) such that

\[
\left( \sum_{i=1}^{n} \beta_i g_i(y) + \beta_{n+1} h_{n+1}(y) - \left( \sum_{i=1}^{n+1} \beta_i \right) \lim \sup h_i(y) \right]
\]

\[
(13) \quad > \left[ \frac{1}{4} + 2^{-n} - 2^{-(n+2)} \right] r \beta_{n+1}
\]

\[
+ \sup_{x \in C} \left[ \sum_{i=1}^{n} \beta_i g_i(x) - \left( \sum_{i=1}^{n} \beta_i \right) \lim \inf h_i(x) \right].
\]

It follows from (12), (13), and the equality of \([\frac{1}{4} + 2^{-n} - 2^{-(n+2)}]\) and \(2^{-(n+2)} + [\frac{1}{4} + 2^{-(n+1)}]\), that
Therefore property K is valid when \( k = n + 1 \), for all choices of \( \{g_{n+1}, \ldots\} \) as a sequence of nonoverlapping members of \( \text{conv}\{G^{*+1}, G^{*+2}, \ldots\} \).

**Theorem 4.** Let \( C \) be a bounded, closed, non-weakly-compact, convex subset of a Banach space \( B \). Then there is a continuous linear functional defined on \( B \) that does not attain its sup on \( C \).

**Proof.** It follows from Theorem 3 that the number \( r \) and the sequences \( \{z_i\} \) and \( \{f_i\} \) hypothesized for Lemma 2 actually exist. For the resulting sequence \( \{g_i\} \), it will be convenient to have

\[
\left| \sum_{k=1}^{\infty} \beta_k g_k(y) - \sum_{k=1}^{\infty} \beta_k \limsup g_k(y) \right| + \left| \sum_{k=1}^{\infty} \beta_k g_k(\xi) - \sum_{k=1}^{\infty} \beta_k g_k(\xi) \right| 
\leq 2^{-(k+2)}r\beta_k,
\]

for all \( k \) and all \( y \) and \( \xi \) in \( C \). For this, it is sufficient to have

\[
4 \sum_{k=1}^{\infty} \beta_k \leq 2^{-(k+2)}r\beta_k \quad \text{or} \quad \sum_{k=1}^{\infty} \beta_k \leq \frac{r\beta_k}{2k+4M},
\]

where \( M \) is an upper bound for \( \| x \| : x \in C \). This will be satisfied if we let \( \beta_1 = 1 \) and require that \( \beta_{n+1} \leq 2^{-(n+5)}r\beta_n/M \) and \( \beta_{n+1} \leq \frac{1}{2}\beta_n \) for all \( n \). Now let \( \Phi \) be a linear functional of unit norm defined on the space \( (m) \) of bounded sequences and such that

\[
\liminf x_i \leq \Phi(x_1, x_2, \cdots) \leq \limsup x_i.
\]

For example, we can let \( \Phi \) be any linear functional of unit norm such that \( \Phi(x_1, x_2, \cdots) = \lim_{n \to \infty} x_n \) whenever this limit exists (or we could let \( \Phi \) be a “Banach limit”—but we do not need the “translation invariance” of a Banach limit). Now for the sequence \( \{g_i\} \) of the lemma, define a continuous linear functional \( g \) on \( B \) by letting \( g(z) = \Phi[g_1(z), g_2(z), \cdots] \). Then \( \| g \| \leq 1 \) and we also have

\[
\liminf g_i(z) \leq g(z) \leq \limsup g_i(z)
\]

for all \( z \) in \( B \). Let \( G \) be defined by

\[
G(x) = \sum_{i=1}^{\infty} \beta_i g_i(x) - \sum_{i=1}^{\infty} \beta_i g(x).
\]

It follows from property K of the lemma that, if
for some $\xi \in C$ and some $k$, then there is a $y$ in $C$ such that

$$\left[ \sum_{i=1}^{k} \beta_i g_i(y) - \left( \sum_{i=1}^{k} \beta_i \right) \limsup g_i(y) \right]
> 2^{-(k+2)} r \beta_k + \left[ \sum_{i=1}^{k} \beta_i g_i(\xi) - \left( \sum_{i=1}^{k} \beta_i \right) \liminf g_i(\xi) \right].$$

From this and (14) we have

$$\left[ \sum_{i=1}^{\infty} \beta_i g_i(y) - \left( \sum_{i=1}^{\infty} \beta_i \right) \limsup g_i(y) \right] > \left[ \sum_{i=1}^{\infty} \beta_i g_i(\xi) - \left( \sum_{i=1}^{\infty} \beta_i \right) \liminf g_i(\xi) \right],$$

and

$$\sum_{i=1}^{\infty} \beta_i g_i(y) - \left( \sum_{i=1}^{\infty} \beta_i \right) g(y) > \sum_{i=1}^{\infty} \beta_i g_i(\xi) - \left( \sum_{i=1}^{\infty} \beta_i \right) g(\xi),$$

or $G(y) > G(\xi)$. Therefore if $G$ attains its sup on $C$ at $u$, then

$$g_k(u) > \left( \frac{1}{4} + 2^{-k} \right) r + \liminf g_i(u) \quad \text{for all } k.$$
linear homeomorphism \( h \) that carries \( E \) onto a closed linear subspace \( E' \) of the product space \( P = \prod_{a \in A} B_a \). If \( w \) indicates the weak topology, then \((E', w)\) is a closed subspace of \((P, w)\) and \((P, w) = \prod_{a \in A} (B_a, w)\).

For each \( a \in A \), let \( \pi_a \) denote the canonical projection of \( P \) onto \( B_a \). If \( f_a \) is continuous linear functional on \( B_a \), then \( f_a \pi_a h \) is a continuous linear functional on \( E \). Since \( f_a \pi_a h \) attains its sup on \( S \), \( f_a \) attains its sup on \( \pi_a h S \). From Theorem 5 it follows that \( \text{wcl}(\pi_a h S) \) is weakly compact, whence of course the set \( \prod_{a \in A} \text{wcl}(\pi_a h S) \) is a weakly compact subset of \( P \) and its weakly closed subset \( \text{wcl}(h S) \) is also weakly compact. But then \( \text{wcl}(S) \) is weakly compact and the proof is complete.

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