INVARIANTS OF EUCLIDEAN REFLECTION GROUPS

BY

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1. **Introduction and statement of results.** Let $\boldsymbol{R}^n$ be n-dimensional Euclidean space and let G be a finite group of orthogonal transformations of $\boldsymbol{R}^n$ generated by reflections. Let $V = \boldsymbol{C}^n$ be the complexification of $\boldsymbol{R}^n$. Then G acts naturally in V and we say that G is a Euclidean reflection group in V. Let S be the $\boldsymbol{C}$-algebra of complex-valued polynomial functions on V, let $I(S)$ be the subalgebra of polynomials invariant under G and let F be the ideal of S generated by the homogeneous elements of positive degree in $I(S)$. Chevalley [2] has proved that

(a) $I(S)$ is generated over $\boldsymbol{C}$ by n algebraically independent homogeneous polynomials $f_1, \cdots, f_n$ and the unit element.

(b) S/F as G-module affords the regular representation of G.

In view of (b) every irreducible G-module M occurs in S/F with multiplicity equal to $\dim M$. Since F is a homogeneous ideal, $S/F = \sum_q (S/F)_q$ is naturally graded. We prove two theorems concerning the graded G-module structure of S/F.

The symmetric group G of degree n acts naturally as a Euclidean reflection group in V by permuting the elements of a basis. The irreducible characters of G are in 1-1 correspondence with partition diagrams of n nodes [6]. In the set of partition diagrams there is a natural ordering. The evidence suggests that characters which occur early in this ordering occur early in the decomposition $S/F = \sum_q (S/F)_q$ in the sense that they occur for small values of q. On the other hand, a formula of Frobenius [6, p. 534] indicates that if a character occurs early in the partition ordering, then a reflection (transposition) fixes a large part of the corresponding representation space. These observations led to the following.

THEOREM 1. *Let G be a Euclidean reflection group and let M be an irreducible G-module. Let $\gamma \in G$ be a reflection, let M_γ^- be the subspace of all $x \in M$ such that $\gamma x = -x$ and suppose that $\dim M_\gamma^- = \dim M^-$ is independent of γ. If M is an irreducible constituent of $(S/F)_q$ for precisely the values $q_1(M), \cdots, q_l(M)$, $l = \dim M$, then the average of the $q_i(M)$ is*

$$\frac{\dim M^-}{\dim M} r$$

where r is the number of reflections in G.

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The assumption that $\dim M_\gamma^-$ is independent of γ is satisfied for all M when the reflections in G form a single conjugate class, and is satisfied for the irreducible modules E_p of Theorem 2 when G is a Weyl group. The extreme cases in Theorem 1 are given by $\dim M^- = 0$, corresponding to the principal character, and $\dim M^- = \dim M$, corresponding to the alternating character of G.

THEOREM 2. *Let L be a complex simple Lie algebra and let V be a Cartan subalgebra of L. Let G be the Weyl group of L acting in V. Let E_p be the G-module of alternating multilinear p-forms over V. Then E_p is irreducible and occurs as a constituent of $(S/F)_q$ for precisely the values $q = m_{i_1} + \cdots + m_{i_n}$, $i_1 < \cdots < i_p$, where $m_i + 1$ is the degree of f_i. A basis for the isotypic component of S/F of type E_p is given by the set of minors of order p of the Jacobian matrix J of $f_1, \cdots, f_n$ reduced* mod F.

The irreducibility of the modules E_p for Weyl groups of the exceptional Lie algebras $\mathfrak{E}_6$, $\mathfrak{E}_7$ was noticed by Frame [5]. It seems likely that Theorem 2 is true for all the Euclidean reflection groups. The proof we give depends on a theorem of Burnside [1] on Weyl groups which allows us to compute certain invariants of G.(2) Theorem 2 and its proof have the following corollaries:

(2a) All minors of J are linearly independent over $\boldsymbol{C}$ and remain linearly independent after reduction mod F. In particular, none of them vanish.

(2b) The algebra of invariants $I(S/F \otimes E)$ is an exterior algebra over $\boldsymbol{C}$ on n generators.

(2c) If $E = \sum_p E_p$ is the Grassmann algebra of V, then the algebra of invariants $I(E \otimes E)$ is a truncated polynomial algebra over $\boldsymbol{C}$, generated by the unit and an element w such that $w^{n+1} = 0$. The generator w may be identified with the Killing form.

(2d) For each $p = 0, \cdots, n$ there exists a homogeneous isomorphism

$$I(S/F \otimes E_p) \simeq I(S/F \otimes E_{n-p})$$

of graded vector spaces. Existence of this isomorphism for $p = 1$ is equivalent to so-called double duality in the exponents m_i, the fact that if the m_i are arranged in increasing order $m_1 \leqq \cdots \leqq m_n$ then $m_i + m_{n-i+1}$ is independent of i.

The double duality in the exponents m_i was a long standing mystery for Weyl groups, explained a few years ago by Coleman [3] and Kostant [7]. Even if one assumes the double duality as known, the argument in (2d) does not furnish an explicit isomorphism $I(S/F \otimes E_1) \simeq I(S/F \otimes E_{n-1})$. We prove the existence of the isomorphism by computing the Poincaré series of both spaces. It would thus be interesting to give a direct invariant-theoretic proof for the double duality by exhibiting an isomorphism which is in some sense a natural one. We have not been able to do this, but present a line of argument which seems to lead in the right

(2) *Added in proof.* R. Steinberg has kindly shown me a proof of irreducibility which is independent of Burnside's theorem. His argument is valid for all the Euclidean groups.

direction. The elements of $S \otimes E_1$ may be viewed as differential 1-forms on V. We study the space of those differential 1-forms on V which are skew invariant under G, in the sense that they are invariant under the rotation subgroup H of G and change sign under the elements outside H. It is not hard to show that this space is a free module over $I(S)$ of rank n. From the double duality and the fact that G has a unique invariant quadratic form one concludes that among the polynomials $f_1, \cdots, f_n$ there is a unique polynomial f_n of greatest degree. Then assuming the double duality we prove the following.

THEOREM 3. *Let G be an irreducible Euclidean reflection group in V. Choose coordinates in V and let $u_1, \cdots, u_n$ be the minors of order $n-1$ of J obtained by deleting the partial derivatives of f_n. If $u_1, \cdots, u_n$ are algebraically independent, then there exists a homogeneous derivation $\hat{d}: S \to S \otimes E_1$ of S-modules such that $\hat{d}f_1, \cdots, \hat{d}f_n$ are a basis for the module of skew invariant differential 1-forms over $I(S)$.*

Granted the existence of the map $\hat{d}$, we can construct an explicit isomorphism $I(S/F \otimes E_1) \simeq I(S/F \otimes E_{n-1})$. We have been able to verify the algebraic independence of $u_1, \cdots, u_n$ in special cases but have no general argument. If the u_i are algebraically independent, then the Jacobian (determinant) of the u_i must be a constant multiple of $(\det J)^{n-2}$.

We work over the complex field $\boldsymbol{C}$ as a matter of convenience, and irreducibility of modules will mean irreducibility over $\boldsymbol{C}$. The complex field is probably an alien here because a likely conjecture of Kostant states that all the absolutely irreducible representations of a Euclidean reflection group may be written with coefficients in $\boldsymbol{R}$. In any case, a real linear group which contains a reflection and is irreducible over $\boldsymbol{R}$ remains irreducible over $\boldsymbol{C}$.

2. **Notation.** In this section we introduce some notation and collect some elementary facts about invariants and characters. Let G be a finite group of order g. By a graded G-module we mean a G-module which is a graded vector space $M = \sum_{q \geqq 0} M_q$ over $\boldsymbol{C}$, in which each homogeneous component M_q is a G-module finite dimensional over $\boldsymbol{C}$. Let μ_q be the character of G corresponding to the module M_q. To the graded G-module M we let correspond the series

$$M(t,\gamma) = \sum_{q \geqq 0} \mu_q(\gamma) t^q, \qquad \gamma \in G.$$

For $\gamma = 1$ this becomes the Poincaré series

$$M(t) = \sum_{q \geqq 0} (\dim M_q) t^q$$

of the graded vector space M. All the tensor products we consider are tensor products over $\boldsymbol{C}$. If M, N are graded G-modules, then $M \otimes N$ has a natural structure of graded G-module with the grading and G-module structure defined by

$$(M \otimes N)_q = \sum_{a+b=q} M_a \otimes N_b,$$

$$\gamma(x \otimes y) = \gamma x \otimes \gamma y, \qquad x \in M,\ y \in N,\ \gamma \in G.$$

From the fact that the character of a tensor product (direct sum) of two G-modules is the product (sum) of the characters it follows that

$$(M \otimes N)(t,\gamma) = M(t,\gamma)N(t,\gamma).$$

We let $I(M)$ denote the submodule of invariants of M, elements $x \in M$ such that $\gamma x = x$ for all $\gamma \in G$. For a finite dimensional M with character μ the connection between invariants and characters is the formula

$$\dim I(M) = \frac{1}{g} \sum_{\gamma \in G} \mu(\gamma).$$

Thus for a graded M we have

$$I(M)(t) = \frac{1}{g} \sum_{\gamma \in G} M(t,\gamma).$$

The dual M^* of M has a natural G-module structure defined by

$$(\gamma f)(x) = f(\gamma^{-1}x), \qquad x \in M, f \in M^*, \gamma \in G,$$

and we may extend this action to the algebra of polynomial functions on M or the Grassmann algebra of M. If μ is the character of M then $\mu^*(\gamma) = \mu(\gamma^{-1})$ is the character of M^*. We have an isomorphism $M \cong M^*$ of G-modules if and only if the character of M is real. The space $\mathrm{Hom}_{\mathbf{C}}(M,N)$ has a natural G-module structure defined by

$$(\gamma\phi)(x) = \gamma(\phi(\gamma^{-1}x)), \quad x \in M,\ \phi \in \mathrm{Hom}_{\mathbf{C}}(M,N),\ \gamma \in G,$$

and the submodule $I(\mathrm{Hom}_{\mathbf{C}}(M, N))$ is just the space $\mathrm{Hom}_G(M, N)$ of G-module homomorphisms. The natural isomorphism of vector spaces $\mathrm{Hom}_{\mathbf{C}}(M,N) \simeq N \otimes M^*$ is an isomorphism of G-modules and induces an isomorphism $\mathrm{Hom}_G(M,N) \simeq I(N \otimes M^*)$. In particular we see that if M is irreducible then $\dim I(N \otimes M^*)$ is the multiplicity of M in N, and that M is irreducible if and only if $\dim I(M \otimes M^*) = 1$.

3. Let G be a Euclidean reflection group in V and let S be the algebra of complex valued polynomial functions on V. Then $S = \sum_{q \geqq 0} S_q$ has a natural structure of graded G-module. A formula of Molien, easy to verify by assuming γ in diagonal form, states that

$$S(t,\gamma^{-1}) = \frac{1}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)} \tag{3.1}$$

where $\omega_1(\gamma),\cdots,\omega_n(\gamma)$ are the eigenvalues of γ as linear transformation of V.

If M is a finite dimensional G-module we give $S \otimes M$ the grading defined by $(S \otimes M)_q = S_q \otimes M$. Then Molien's formula implies

$$I(S \otimes M)(t) = \frac{1}{g} \sum_{\gamma \in G} \frac{\mu(\gamma^{-1})}{(1 - \omega_1(\gamma)t) \cdots (1 - \omega_n(\gamma)t)} \tag{3.2}$$

where μ is the character of M. From Theorem (a) of Chevalley we see that

$$I(S)(t) = \frac{1}{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}. \tag{3.3}$$

Chevalley has also shown [2] that if $p_1, \cdots, p_k \in S$ form a C-basis for S/F when reduced mod F, then $p_1, \cdots, p_k$ are a basis for S as free module over $I(S)$. From this fact we readily deduce the following two lemmas.

LEMMA 1. *Let τ_q be the character of $(S/F)_q$. Then*

$$\sum_q \tau_q(\gamma^{-1}) t^q = \frac{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}{(1 - \omega_1(\gamma)t) \cdots (1 - \omega_n(\gamma)t)}.$$

Proof. Let $p_1, \cdots, p_k \in S$ form a C-basis for S/F when reduced mod F. Then the map $\sum_i s_i p_i \to \sum_i s_i \otimes (p_i + F)$, $s_i \in I(S)$ defines an isomorphism $S \simeq I(S) \otimes S/F$ of graded G-modules. Since G acts trivially on $I(S)$ we have $S(t, \gamma) = I(S)(t)(S/F)(t, \gamma)$. Thus from (3.1) and (3.3)

$$(S/F)(t, \gamma^{-1}) = \frac{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}{(1 - \omega_1(\gamma)t) \cdots (1 - \omega_n(\gamma)t)}$$

which proves the lemma.

If we let $t \to 1$ we find $\sum_q \tau_q(\gamma) = 0$ if $\gamma \neq 1$ and $\sum_q \tau_q(1) = g$, so that $\sum_q \tau_q$ is the character of the regular representation of G. Thus S/F affords the regular representation of G. If M is an irreducible G-module we let $a_q(M)$ be the multiplicity of M in $(S/F)_q$. Since S/F contains M with multiplicity equal to $\dim M$ we have

$$\sum_q a_q(M) = \dim M.$$

We view $S \otimes M$ naturally as an S-module and then $I(S \otimes M)$ is an $I(S)$-module.

LEMMA 2. *Let M be an irreducible G-module. Then $I(S \otimes M)$ is a free module over $I(S)$. It has a basis over $I(S)$ consisting of homogeneous elements in which the number of elements of degree q is $a_q(M^*)$. The rank of $I(S \otimes M)$ as $I(S)$-module is equal to* $\dim M$.

Proof. In the proof of Lemma 1 we have remarked that $S \simeq I(S) \otimes (S/F)$ and hence $S \otimes M \simeq I(S) \otimes (S/F) \otimes M$. Since G acts trivially on $I(S)$ we see by averaging over the group that $I(S \otimes M) \simeq I(S) \otimes I(S/F \otimes M)$. Thus $I(S \otimes M)$ is free over $I(S)$ and we may choose as basisa C-basis for $I(S/F \otimes M)$. This may be chosen as a union of C-bases for the $I((S/F)_q \otimes M)$. But $\dim I((S/F)_q \otimes M)$ is the multiplicity of the irreducible M^* in $(S/F)_q$ so $\dim I((S/F)_q \otimes M) = a_q(M^*)$. The rank of $I(S \otimes M)$ as $I(S)$-module is thus $\sum_q a_q(M^*) = \dim M^* = \dim M$. The argument shows that $I(S \otimes M)$ is free over $I(S)$ for any G-module M.

4. To prove Theorem 1 we simply compute the Poincaré series $I(S \otimes M^*)(t)$ in two ways and compare the results for $t = 1$. Set $a(t) = \sum_q a_q(M)t^q$. From Lemma 2 with M replaced by M^* we have

$$I(S \otimes M^*)(t) = \frac{a(t)}{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}$$

and thus from (3.2) we see that

$$\frac{1}{g} \sum_{\gamma \in G} \frac{\mu(\gamma)}{(1 - \omega_1(\gamma)t) \cdots (1 - \omega_n(\gamma)t)} = \frac{a(t)}{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})} \tag{4.1}$$

where μ is the character of M. Let G_1 be the set of elements of G, distinct from the identity, which fix an $n-1$ dimensional subspace of V. For $\gamma \in G_1$ the eigenvalues $\omega_i(\gamma)$ are $1, 1, \cdots, 1, \omega$ where ω is a root of unity. Now the fact that G may be written as a real orthogonal group implies $\omega = -1$. The left-hand side of (4.1) becomes

$$\frac{1}{g}\left[\frac{\mu(1)}{(1-t)^n} + \frac{1}{(1-t)^{n-1}(1+t)} \sum_{\gamma \in G_1} \mu(\gamma) + \cdots\right]$$

where $\cdots$ denotes terms which have at most $(1-t)^{n-2}$ in the denominator. Since $a(1) = \sum_q a_q(M) = \dim M = \mu(1)$ we have

$$\frac{1}{(1-t)^{n-1}(1+t)} \sum_{\gamma \in G_1} \mu(\gamma) + \cdots = \frac{g a(t)}{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})} - \frac{a(1)}{(1-t)^n}.$$

Now multiply both sides by $(1-t)^{n-1}$ and let $t \to 1$. Using the known formula [4; 8] $g = \prod_i (m_i + 1)$ for the group order we find

$$\frac{1}{2} \sum_{\gamma \in G_1} \mu(\gamma) = -a'(1) + \frac{1}{2}\left(\sum_i m_i\right) a(1) \tag{4.2}$$

where a' denotes the derivative with respect to t. For $\gamma \in G_1$, $\mu(\gamma) = \dim M_\gamma^+ - \dim M_\gamma^-$ is, by assumption, independent of γ. The number of elements in G_1 is the number of reflections in G which is known [4; 8] to be $\sum_i m_i$. From the definition of $a(t)$ we have $a'(1) = \sum_q q a_q(M)$. If we insert this information in (4.2) we find

$$\frac{1}{2}\left(\sum_i m_i\right)(\dim M^+ - \dim M^-) = -\sum_q q a_q(M) + \frac{1}{2}\left(\sum_i m_i\right)(\dim M^+ + \dim M^-)$$

so that

$$\sum_q qa_q(M) = \left(\sum_i m_i\right)\dim M^- .$$

The average of the $q_i(M)$ is then

$$\frac{\sum_q qa_q(M)}{\sum_q a_q(M)} = \frac{\dim M^-}{\dim M}\, r$$

with $r = \sum_i m_i$. The significance of the integer $r = \sum_i m_i$ in this formula becomes clear if one observes, with Lemma 1, that r is the largest integer q for which $(S/F)_q \neq 0$.

5. To prove Theorem 2 we shall need some results from an earlier article [9] together with a lemma concerning certain invariants of the symmetric group. Let $E = \sum_p E_p$ be the Grassman algebra of V. The homogeneous component E_p of degree p is the space of all p-linear alternating functions on V. We identify E_0 with C and E_1, as vector space, with S_1. The group G acts naturally on E and on $S \otimes E$. Choose a coordinate system $x_1, \cdots, x_n$ in V and let $d: S \otimes E \to S \otimes E$ be the C-linear map defined by

$$d: s \otimes x_{i_1} \wedge \cdots \wedge x_{i_n} \to \sum_{j=1}^{n} \frac{\partial s}{\partial x_j} \otimes x_j \wedge x_{i_1} \wedge \cdots \wedge x_{i_n}, \qquad s \in S.$$

If we identify S with $S \otimes C$ then $dx_i = d(x_i \otimes 1) = 1 \otimes x_i$ so that the elements of $S \otimes E_p$ may be written in the form

$$\sum_{i_1 < \cdots < i_n} s_{i_1 \cdots i_n} dx_{i_1} \cdots dx_{i_n}, \qquad s_{i_1 \cdots i_n} \in S.$$

It is clear that $S \otimes E$ is just the algebra of differential forms on V and that d is exterior differentiation. Since d commutes with the action of G on $S \otimes E$ it follows that d maps $I(S \otimes E)$ into $I(S \otimes E)$. In particular, the differentials df_i are invariants of $S \otimes E$. We have shown in [9] that the C-algebra $I(S \otimes E)$ of invariant differential forms is an exterior algebra on n generators over the C-algebra $I(S)$ of invariant polynomials, and is in fact generated over $I(S)$ by the differentials $df_1, \cdots, df_n$ of the polynomial invariants $f_1, \cdots, f_n$ and the unit element. It follows that

$$\text{(5.1)} \qquad I(S \otimes E_p)(t) = \frac{\sigma_p(t^{m_1}, \cdots, t^{m_n})}{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}, \qquad p = 1, \cdots, n,$$

where $\sigma_p(t_1, \cdots, t_n)$ is the pth elementary symmetric function in the indeterminates $t_1, \cdots, t_n$.

LEMMA 3. *Let $x_1, \cdots, x_n$ be indeterminates, let G be the symmetric group on $x_1, \cdots, x_n$ and let E be the exterior algebra on $x_1, \cdots, x_n$ over C. The group G acts naturally on the commutative algebra $E \otimes E$. Set*

$$u = \sum_i x_i \otimes x_i, \qquad v = \sum_{i<k} x_i \otimes x_k + x_k \otimes x_i.$$

Then

$$I(E_p \otimes E_p) = Cu^p \oplus Cu^{p-1}v, \qquad p = 1, \cdots, n-1.$$

Proof. Clearly both u^p and $u^{p-1}v$ are in $I(E_p \otimes E_p)$. To hold the indices in check we let Ω denote the set of increasing sequences $i_1 < \cdots < i_p$ of p integers chosen from $1, \cdots, n$, we let (i) denote an element of Ω, let $\{i\}$ be the corresponding unordered set, and write $x_{(i)} = x_{i_1} \wedge \cdots \wedge x_{i_p}$. Suppose $y = \sum c_{(i),(k)} x_{(i)} \otimes x_{(k)} \in I(E_p \otimes E_p)$ where $c_{(i),(k)} \in C$ and the sum is over all pairs $(i), (k)$ of elements of Ω. If for given sets $\{i\}, \{k\}$ the intersection $\{i\} \cap \{k\}$ contains fewer than $p-1$ elements, then there exist two distinct indices, say k_α, k_β, which are distinct from all elements of $\{i\}$. Now apply the transposition $(k_\alpha k_\beta)$ of G to each term in the sum y. The invariance of y shows that $c_{(i),(k)} = -c_{(i),(k)}$ and hence $c_{(i),(k)} = 0$. Thus we may write $y = y_1 + y_2$ where y_1 is a linear combination of elements $x_{(i)} \otimes x_{(k)}$ such that $\{i\} \cap \{k\}$ contains p elements, in other words $\{i\} = \{k\}$, and y_2 is a linear combination of elements $x_{(i)} \otimes x_{(k)}$ such that $\{i\} \cap \{k\}$ contains $p-1$ elements. Invariance of y implies the invariance of y_1 and y_2. We thus have $y_1 = \sum_{(i)} b_{(i)} x_{(i)} \otimes x_{(i)}$ with $b_{(i)} \in C$ and invariance of y_1 shows that all the $b_{(i)}$ are equal, say $b_{(i)} = b$. Then $y_1 = b \sum_{(i)} x_{(i)} \otimes x_{(i)}$ is a C-multiple of $u^p = p! \sum_{(i)} x_{(i)} \otimes x_{(i)}$. Similarly with slightly more effort one sees that y_2 is a C-multiple of $u^{p-1}v$. Thus the elements u^p, $u^{p-1}v$ span $I(E^p \otimes E^p)$. For $p < n$ both u^p and $u^{p-1}v$ are not zero and hence linearly independent over C. This proves the lemma. The argument breaks down for $p = n$ only because $u^{n-1}v = 0$ and in that case we have $I(E_n \otimes E_n) = Cu^n$. The elements u, v also satisfy the relations $u^{n+1} = 0$ and $v^2 = 0$.

We are now in position to prove Theorem 2. Since L is simple, G acting in V is an irreducible group. A theorem of Burnside [1] states that there exists a coordinate system $x_1, \cdots, x_n$ in V such that G acting on $V^* \simeq E_1$ includes the symmetric group H on $x_1, \cdots, x_n$. In this coordinate system the Killing form must be

$$\frac{a}{2} \sum_i x_i^2 + b \sum_{i<k} x_i x_k \in I(S_2)$$

where a, b are real numbers. We cannot have $a = 0$ because the form is positive definite. We let $I(E \otimes E)$ denote the elements of $E \otimes E$ invariant under G and $I_H(E \otimes E)$ the elements invariant under H. Under the map $f \to \sum_i (\partial f / \partial x_i) \otimes x_i$ of $S_2 \to E_1 \otimes E_1$ the Killing form maps into $au + bv \in I(E_1 \otimes E_1)$ where $u, v \in I_H(E_1 \otimes E_1)$ are the invariants of Lemma 3. Since $I(E \otimes E) \subseteq I_H(E \otimes E)$, Lemma 3 shows that $\dim I(E_p \otimes E_p) \leqq 2$. Suppose $\dim I(E_p \otimes E_p) = 2$ for some $p = 1, \cdots, n-1$. We prove that $\dim I(E_{n-1} \otimes E_{n-1}) = 2$. If $p = n-1$ there is nothing to prove so assume $p < n-1$. From Lemma 3 we see that $u^p \in I(E_p \otimes E_p)$ and $u^{p-1}v \in I(E_p \otimes E_p)$. Since $v^2 = 0$ it follows that both $au^p v = u^{p-1}v(au + bv)$ and $au^{p+1} + bu^p v = u^p(au + bv)$ are in $I(E_{p+1} \otimes E_{p+1})$. Since $p < n-1$ we have $u^p v \neq 0$ and since $a \neq 0$ it follows that $\dim I(E_{p+1} \otimes E_{p+1}) = 2$. We conclude by induction that $\dim I(E_{n-1} \otimes E_{n-1}) = 2$. Let Z be the 1-dimensional G-module defined by the homomorphism $\gamma \to \det \gamma$, $\gamma \in G$. Then $E_{n-1} \simeq E_1 \otimes Z$ as G-modules. Since $\det \gamma = \pm 1$, $Z \otimes Z \simeq C$ is the trivial G-module. Then $E_{n-1} \otimes E_{n-1} \simeq E_1 \otimes E_1$ as G-modules so that $\dim I(E_1 \otimes E_1) = 2$ which contradicts the irreducibility of E_1. Thus $\dim I(E_p \otimes E_p) = 1$ for all p and hence E_p is irreducible for all p.

Let $\theta: S \to S/F$ be the natural map and extend θ to a map, denoted again θ, of $S \otimes E \to S/F \otimes E$ by letting it be the identity on E. Then θ is a homomorphism of G-modules and of C-algebras. Since θ is a homomorphism of G-modules we certainly have $\theta I(S \otimes E) \subseteq I(S/F \otimes E)$. Actually we have $\theta I(S \otimes E) = I(S/F \otimes E)$ because, using complete reducibility of the representations of G, we may choose a graded G-module T such that $S = F \oplus T$ and then $\theta: I(T \otimes E) \to I(S/F \otimes E)$ is an isomorphism. Set $z_i = \theta(df_i)$. Since $I(S \otimes E)$ is generated over $I(S)$ by the df_i and the unit element, and since every element $s \in I(S)$ may be written as $s = s_0 + s_1$ with $s_0 \in C$ and $s_1 \in F$, it follows that the z_i together with the unit element generate $I(S/F \otimes E)$ as algebra over C. Thus the $\binom{n}{p}$ elements $z_{i_1} \cdots z_{i}$ $i_1 < \cdots < i_p$ generate $I(S/F \otimes E_p)$ as vector space over C. In fact they form a basis for $I(S/F \otimes E_p)$ because Lemma 2 shows that $\dim I(S/F \otimes E_p) = \dim E_p = \binom{n}{p}$. Thus the $z_{i_1} \cdots z_{i_p}$ are linearly independent over C.

The Killing form induces a natural isomorphism $E_p \simeq E_p^*$ of G-modules and hence a natural isomorphism $S/F \otimes E_p \simeq \mathrm{Hom}(E_p, S/F)$ of G-modules. Under this isomorphism the invariants $I(S/F \otimes E_p)$ correspond to $\mathrm{Hom}_G(E_p, S/F)$. For $(i) \in \Omega$, let $\phi_{(i)}$ be the image in $\mathrm{Hom}_G(E_p, S/F)$ of $z_{i_1} \cdots z_{i_p}$. The linear independence of the $z_{i_1} \cdots z_{i}$ implies linear independence of the $\phi_{(i)}$. Now if M is an irreducible G-module and $\phi_1, \cdots, \phi_s$ are linearly independent G-module homomorphisms of M into a G-module N then the sum $\sum_i \phi_i(M)$ is direct. This follows at once from Schur's lemma by induction on the number of summands. In the case at hand this means that the sum $\sum_{(i)} \phi_{(i)}(E_p)$ is direct. Since the number of summands is $\binom{n}{p} = \dim E_p$, the sum $\sum_{(i)} \phi_{(i)}(E_p)$ is the isotypic component of S/F of type E_p. Thus E_p occurs as an irreducible constituent of $(S/F)_q$ for precisely the values $q = m_{i_1} + \cdots + m_{i_p}$, $i_1 < \cdots < i_p$. From the definition of $\phi_{(i)}$ as the image of $\theta(df_{i_1} \cdots df_{i_p})$ it follows that a basis for $\phi_{(i)}(E_p)$ is given by the $\binom{n}{p}$ minors of J which involve $f_{i_1}, \cdots, f_{i_p}$, reduced mod F. This completes the proof of Theorem 2.

For the corollaries we argue as follows:

(2a) The linear independence over C of the minors of J after reduction mod F amounts to the linear independence of the elements $\theta(df_{i_1} \cdots df_{i_p})$ over C. This we have shown.

(2b) The proof of the theorem shows that $I(S/F \otimes E)$ is generated as algebra over C by the $z_i = \theta(df_i)$ and the unit element. Since $z_i z_j = -z_j z_i$, $I(S/F \otimes E)$ is a homomorphic image of an exterior algebra on n generators. But

$\dim I(S/F \otimes E) = \dim E = 2^n$ so $I(S/F \otimes E)$ is in fact an exterior algebra on the $z_i = \theta(df_i)$.

(2c) The proof of the theorem shows that $I(E_p \otimes E_p) = \boldsymbol{C}w^p$ for all $p = 1, \cdots, n$ where $w = au + bv$ may be identified with the Killing form. Suppose we have an isomorphism $E_p \simeq E_q$ of G-modules. Then $\binom{n}{p} = \dim E_p = \dim E_q = \binom{n}{q}$ so $q = p$ or $q = n - p$. Suppose $q = n - p$. Let χ_p be the character of E_p and let $\gamma \in G$ be a reflection. Then $\chi_p(\gamma)$ is the pth elementary symmetric function of the eigenvalues $1, 1, \cdots, 1, -1$ so that $\chi_p(\gamma) = \binom{n-1}{p} - \binom{n-1}{p-1}$. Now $\chi_p(\gamma) = \chi_{n-p}(\gamma)$ shows $n - p = p$. Thus in any case $q = p$. It follows that $I(E_p \otimes E_q) = 0$ for $q \neq p$ and hence $I(E \otimes E) = \sum_p I(E_p \otimes E_p)$ is generated over $\boldsymbol{C}$ by the unit element and an element $w = au + bv$ which satisfies $w^{n+1} = 0$ and which may be identified with the Killing form.

(2d) From Theorem 2 or directly from (5.1) we conclude that $I(S/F \otimes E_p)(t) = \sigma_p(t^{m_1}, \cdots, t^{m_n})$. Thus one has a homogeneous isomorphism

$$I(S/F \otimes E_1) \simeq I(S/F \otimes E_{n-1}) \tag{5.2}$$

of graded vector spaces if and only if there exists an integer k such that $t^k \sigma_1(t^{m_1}, \cdots, t^{m_n}) = \sigma_{n-1}(t^{m_1}, \cdots, t^{m_n})$. Comparing coefficients on both sides shows that this condition is equivalent to the existence of an integer k such that $k + m_i + m_{n-i+1} = m_1 + \cdots + m_n$. This is equivalent in turn to the statement that $m_i + m_{n-i+1}$ is independent of i, the double duality. The same kind of coefficient comparison shows that double duality implies the isomorphisms

$$I(S/F \otimes E_p) \simeq I(S/F \otimes E_{n-p}).$$

For later use we remark that (5.2) is equivalent to the existence of a homogeneous isomorphism

$$I(S \otimes E_1) \simeq I(S \otimes E_{n-1}) \tag{5.3}$$

of graded vector spaces.

6. Let M, N be G-modules. We say that M and N are skew isomorphic if there exists a 1-1 $\boldsymbol{C}$-linear map θ of M onto N such that $\theta\gamma x = (\det\gamma)\gamma\theta x$ for all $x \in M$ and all $\gamma \in G$. We call θ a skew isomorphism between M and N. Since $\det\gamma = \pm 1$ the relation of skew isomorphism is symmetric. Again we let Z denote the 1-dimensional G-module defined by the homomorphism $\gamma \to \det\gamma$ and let z be a generator of Z. If we set $\hat{M} = M \otimes Z$ we see that $x \to x \otimes z$, $x \in M$, defines a skew isomorphism between M and $\hat{M}$. Since $Z \otimes Z \simeq \boldsymbol{C}$, $\hat{M}\hat{}$ and M are isomorphic as G-modules.

From Lemma 1 we conclude that $t^r(S/F)(t^{-1}, \gamma) = \det\gamma(S/F)(t, \gamma)$ where $r = \sum_i m_i$. Thus $\tau_{r-q}(\gamma) = (\det\gamma)\tau_q(\gamma)$ for all $\gamma \in G$ and all $q = 0, \cdots, r$ so that $(S/F)_q$ and $(S/F)_{r-q}$ are skew isomorphic. If M is an irreducible G-module and N

is any G-module, then $\hat{M}$ is irreducible and the multiplicity of M in N is equal to the multiplicity of $\hat{M}$ in $\hat{N}$. Hence the

THEOREM. *If M is an irreducible G-module, then the multiplicity of M in $(S/F)_q$ is equal to the multiplicity of $\hat{M}$ in $(S/F)_{r-q}$ where $r = \sum_i m_i$. Thus in the notation of Theorem 1, with suitable ordering, $q_i(\hat{M}) = r - q_i(M)$.*

We say that $x \in M$ is a skew invariant if $\gamma x = (\det \gamma)x$ for all $\gamma \in G$. Let $\hat{I}(M)$ denote the subspace of skew invariant elements of M. Then the map $x \to x \otimes z$ defines a natural isomorphism $\hat{I}(M) \simeq I(\hat{M})$ of vector spaces. It follows that we have an isomorphism

$$\hat{I}(S \otimes M) \simeq I(S \otimes \hat{M}) \tag{6.1}$$

of graded vector spaces which is homogeneous of degree zero. From Lemma 2 we see that $\hat{I}(S \otimes M)$ is free over $I(S)$ of rank equal to $\dim M$.

Since $E_{n-1} \simeq \hat{E}_1$ is an isomorphism of G-modules, the isomorphism (5.3) equivalent to the double duality becomes

$$\hat{I}(S \otimes E_1) \simeq I(S \otimes E_1). \tag{6.2}$$

Now $I(S \otimes E_1)$ is generated freely over $I(S)$ by the df_i. If we can construct the homogeneous derivation $\hat{d}$ of Theorem 3 then $\hat{I}(S \otimes E_1)$ is generated freely over $I(S)$ by the $\hat{d}f_i$ and the homogeneous isomorphism (6.2) is defined by $df_i \to \hat{d}f_i$. Thus in this formulation the double duality is equivalent to the existence of the map $\hat{d}$.

In connection with (6.2) it is worth noting that the homogeneous isomorphism $\hat{I}(S \otimes E_0) \simeq I(S \otimes E_0)$ amounts to the familiar fact that every skew invariant polynomial may be written as an invariant polynomial multiplied by $\det J$.

7. Let $q = m_1 + \cdots + m_{n-1}$. We have a sequence of natural maps

$$S_q \otimes E_{n-1} \xrightarrow{\psi_1} S_q \otimes E_1^* \xrightarrow{\psi_2} \mathrm{Hom}(E_1, S_q) \xrightarrow{\psi_3} \mathrm{Hom}(S_1, S_q)$$

where ψ_1 is a skew isomorphism of G-modules induced by the natural duality in the Grassman algebra, where ψ_2 is an isomorphism of G-modules induced by the natural isomorphism of vector spaces, and ψ_3 is the isomorphism of G-modules induced by the identification of E_1 with S_1. The composite map

$$\psi : S_q \otimes E_{n-1} \to \mathrm{Hom}(S_1, S_q)$$

is a skew isomorphism of G-modules. Let $\eta = \psi(df_1 \cdots df_{n-1})$. Since $df_1 \cdots df_{n-1} \in I(S_q \otimes E_{n-1})$ we have $\eta \in \hat{I}(\mathrm{Hom}(S_1, S_q))$ so that η is a skew homomorphism of S_1 into S_q. Since S_1 is irreducible, η must be injective, and from the definition of the map ψ we see that $\eta x_i = u_i$ where u_i is the minor of order $n-1$ of J obtained by deleting the derivatives of f_n and the derivatives with respect to x_i.

Our map $d: S \to S \otimes E_1$ is homogeneous of degree -1 and commutes with the action of G. Let $\hat{d} = d \circ \eta$. Then $\hat{d}$ is a skew isomorphism of S_1 into $S_{q-1} \otimes E_1$. Since S is a polynomial ring we may extend $\hat{d}$ to a derivation $\hat{d}: S \to S \otimes E_1$ of S-modules. By induction on the degree of a homogeneous element one sees that $\hat{d}\gamma s = (\det \gamma)\gamma \hat{d}s$ for all $\gamma \in G$ and all $s \in S$. Thus $\hat{d}$ maps $I(S)$ into $\hat{I}(S \otimes E_1)$. Since $\hat{d}x_i = du_i = \sum_k (\partial u_i / \partial x_k) dx_k$ we have

$$\hat{d}f = \sum_i \sum_k \frac{\partial f}{\partial x_i} \frac{\partial u_i}{\partial x_k} dx_k. \tag{7.1}$$

We claim that the elements $\hat{d}f_1, \cdots, \hat{d}f_n$ are linearly independent over S. If not, then we have a relation $\sum_i s_i \hat{d}f_i = 0$ where $s_i \in S$ and where s_1, say, is not zero. Then multiplication by $\hat{d}f_2 \cdots \hat{d}f_n$ shows that $\hat{d}f_1 \cdots \hat{d}f_n = 0$. On the other hand, computing directly from (7.1) shows that

$$\hat{d}f_1 \cdots \hat{d}f_n = \det\left(\frac{\partial f_i}{\partial x_k}\right) \det\left(\frac{\partial u_i}{\partial x_k}\right) dx_1 \cdots dx_n$$

which is not zero in view of our assumption about the algebraic independence of the u_i. Thus the $\hat{d}f_i$ are linearly independent over S and the sum $P = \sum_i I(S)\hat{d}f_i$ is direct. Since G is a real group it has an invariant quadratic form f_1 and hence $m_1 = 1$. The degree of the map $\hat{d}$ is thus $q - 1 = m_1 + \cdots + m_{n-1} - 1 = m_2 + \cdots + m_{n-1}$ and the Poincaré series for the graded vector space P is thus

$$P(t) = \frac{t^{m_2 + \cdots + m_{n-1}}(t^{m_1} + \cdots + t^{m_n})}{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}.$$

On the other hand, using (5.1), (6.1) and the double duality we see that $P(t) = I(S \otimes E_{n-1})(t) = \hat{I}(S \otimes E_1)(t)$. Since $P \subseteq \hat{I}(S \otimes E_1)$ is an inclusion of graded vector spaces we have $P = \hat{I}(S \otimes E_1)$. Thus $\hat{I}(S \otimes E_1)$ is freely generated over $I(S)$ by $\hat{d}f_1, \cdots, \hat{d}f_n$ and Theorem 3 is proved.

8. For the symmetric group on n letters $x_1, \cdots, x_n$ we can give the following construction for the skew invariant differential 1-forms. Let $\sigma_1, \cdots, \sigma_n$ be the elementary symmetric functions of $x_1, \cdots, x_n$ and let $\Delta(x_1, \cdots, x_{n-1}) = \prod_{1 \leqq i < j \leqq n-1} (x_i - x_j)$ be the fundamental skew invariant polynomial for the symmetric group on the letters $x_1, \cdots, x_{n-1}$. Then a basis for the skew invariant differential 1-forms over the algebra of symmetric functions is given by the forms

$$\omega_k = \sum_i (-1)^{i+1} \Delta(x_1, \cdots, \tilde{x}_i, \cdots, x_n) \frac{\partial \sigma_k}{\partial x_i} dx_i, \qquad k = 1, \cdots, n,$$

where $\tilde{x}_i$ means that the letter x_i is to be omitted.

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