1. Introduction and statement of results. Let \( \mathbb{R}^n \) be \( n \)-dimensional Euclidean space and let \( G \) be a finite group of orthogonal transformations of \( \mathbb{R}^n \) generated by reflections. Let \( V = \mathbb{C}^n \) be the complexification of \( \mathbb{R}^n \). Then \( G \) acts naturally in \( V \) and we say that \( G \) is a Euclidean reflection group in \( V \). Let \( S \) be the \( \mathbb{C} \)-algebra of complex-valued polynomial functions on \( V \), let \( \mathcal{I}(S) \) be the subalgebra of polynomials invariant under \( G \) and let \( F \) be the ideal of \( S \) generated by the homogeneous elements of positive degree in \( \mathcal{I}(S) \). Chevalley \([2]\) has proved that

(a) \( \mathcal{I}(S) \) is generated over \( \mathbb{C} \) by \( n \) algebraically independent homogeneous polynomials \( f_1, \ldots, f_n \) and the unit element.

(b) \( S \neq F \) as \( G \)-module affords the regular representation of \( G \).

In view of (b) every irreducible \( G \)-module \( M \) occurs in \( S/F \) with multiplicity equal to \( \dim M \). Since \( F \) is a homogeneous ideal, \( S/F = \sum_q (S/F)_q \) is naturally graded. We prove two theorems concerning the graded \( G \)-module structure of \( S/F \).

The symmetric group \( G \) of degree \( n \) acts naturally as a Euclidean reflection group in \( V \) by permuting the elements of a basis. The irreducible characters of \( G \) are in 1-1 correspondence with partition diagrams of \( n \) nodes \([6]\). In the set of partition diagrams there is a natural ordering. The evidence suggests that characters which occur early in this ordering occur early in the decomposition \( S/F = \sum_q (S/F)_q \) in the sense that they occur for small values of \( q \). On the other hand, a formula of Frobenius \([6, \text{p. 534}]\) indicates that if a character occurs early in the partition ordering, then a reflection (transposition) fixes a large part of the corresponding representation space. These observations led to the following.

**Theorem 1.** Let \( G \) be a Euclidean reflection group and let \( M \) be an irreducible \( G \)-module. Let \( \gamma \in G \) be a reflection, let \( M^-_{\gamma} \) be the subspace of all \( x \in M \) such that \( \gamma x = -x \) and suppose that \( \dim M^-_{\gamma} = \dim M^- \) is independent of \( \gamma \). If \( M \) is an irreducible constituent of \( \sum_q (S/F)_q \) for precisely the values \( q_1(M), \ldots, q_r(M) \), \( l = \dim M \), then the average of the \( q_i(M) \) is

\[
\frac{\dim M^-}{\dim M} r
\]

where \( r \) is the number of reflections in \( G \).

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The assumption that \( \dim M^- \) is independent of \( \gamma \) is satisfied for all \( M \) when the reflections in \( G \) form a single conjugate class, and is satisfied for the irreducible modules \( E_p \) of Theorem 2 when \( G \) is a Weyl group. The extreme cases in Theorem 1 are given by \( \dim M^- = 0 \), corresponding to the principal character, and \( \dim M^- = \dim M \), corresponding to the alternating character of \( G \).

**Theorem 2.** Let \( L \) be a complex simple Lie algebra and let \( V \) be a Cartan subalgebra of \( L \). Let \( G \) be the Weyl group of \( L \) acting in \( V \). Let \( E_p \) be the \( G \)-module of alternating multilinear \( p \)-forms over \( V \). Then \( E_p \) is irreducible and occurs as a constituent of \( (S/F)^q \) for precisely the values \( q = m_{i_1} + \cdots + m_{i_p}, \)

\[
i_1 < \cdots < i_p,
\]

where \( m_i + 1 \) is the degree of \( f_i \). A basis for the isotypic component of \( S/F \) of type \( E_p \) is given by the set of minors of order \( p \) of the Jacobian matrix \( J \) of \( f_1, \ldots, f_n \) reduced mod \( F \).

The irreducibility of the modules \( E_p \) for Weyl groups of the exceptional Lie algebras \( E_6, E_7 \) was noticed by Frame \cite{5}. It seems likely that Theorem 2 is true for all the Euclidean reflection groups. The proof we give depends on a theorem of Burnside \cite{1} on Weyl groups which allows us to compute certain invariants of \( G. \)

Theorem 2 and its proof have the following corollaries:

(2a) All minors of \( J \) are linearly independent over \( C \) and remain linearly independent after reduction mod \( F \). In particular, none of them vanish.

(2b) The algebra of invariants \( I(S/F \otimes E) \) is an exterior algebra over \( C \) on \( n \) generators.

(2c) If \( E = \sum E_p \) is the Grassmann algebra of \( V \), then the algebra of invariants \( I(E \otimes E) \) is a truncated polynomial algebra over \( C \), generated by the unit and an element \( w \) such that \( w^{n+1} = 0 \). The generator \( w \) may be identified with the Killing form.

(2d) For each \( p = 0, \ldots, n \) there exists a homogeneous isomorphism

\[
I(S/F \otimes E_p) \simeq I(S/F \otimes E_{n−p})
\]

of graded vector spaces. Existence of this isomorphism for \( p = 1 \) is equivalent to so-called double duality in the exponents \( m_i \), the fact that if the \( m_i \) are arranged in increasing order \( m_1 \leq \cdots \leq m_n \) then \( m_i + m_{n−i+1} \) is independent of \( i \).

The double duality in the exponents \( m_i \) was a long standing mystery for Weyl groups, explained a few years ago by Coleman \cite{3} and Kostant \cite{7}. Even if one assumes the double duality as known, the argument in (2d) does not furnish an explicit isomorphism \( I(S/F \otimes E_1) \simeq I(S/F \otimes E_{n−1}) \). We prove the existence of the isomorphism by computing the Poincaré series of both spaces. It would thus be interesting to give a direct invariant-theoretic proof for the double duality by exhibiting an isomorphism which is in some sense a natural one. We have not been able to do this, but present a line of argument which seems to lead in the right

(2) Added in proof. R. Steinberg has kindly shown me a proof of irreducibility which is independent of Burnside’s theorem. His argument is valid for all the Euclidean groups.
We study the space of those differential 1-forms on \( V \) which are skew invariant under \( G \), in the sense that they are invariant under the rotation subgroup \( H \) of \( G \) and change sign under the elements outside \( H \). It is not hard to show that this space is a free module over \( I(S) \) of rank \( n \). From the double duality and the fact that \( G \) has a unique invariant quadratic form one concludes that among the polynomials \( f_1, \ldots, f_n \) there is a unique polynomial \( f_n \) of greatest degree. Then assuming the double duality we prove the following.

**Theorem 3.** Let \( G \) be an irreducible Euclidean reflection group in \( V \). Choose coordinates in \( V \) and let \( u_1, \ldots, u_n \) be the minors of order \( n - 1 \) of \( J \) obtained by deleting the partial derivatives of \( f_n \). If \( u_1, \ldots, u_n \) are algebraically independent, then there exists a homogeneous derivation \( \delta: S \to S \otimes E_1 \) of \( S \)-modules such that \( \delta f_1, \ldots, \delta f_n \) are a basis for the module of skew invariant differential 1-forms over \( I(S) \).

Granted the existence of the map \( \delta \), we can construct an explicit isomorphism \( I(S/F \otimes E_1) \simeq I(S/F \otimes E_{n-1}) \). We have been able to verify the algebraic independence of \( u_1, \ldots, u_n \) in special cases but have no general argument. If the \( u_i \) are algebraically independent, then the Jacobian (determinant) of the \( u_i \) must be a constant multiple of \( (\det J)^{n-2} \).

We work over the complex field \( \mathbb{C} \) as a matter of convenience, and irreducibility of modules will mean irreducibility over \( \mathbb{C} \). The complex field is probably an alien here because a likely conjecture of Kostant states that all the absolutely irreducible representations of a Euclidean reflection group may be written with coefficients in \( \mathbb{R} \). In any case, a real linear group which contains a reflection and is irreducible over \( \mathbb{R} \) remains irreducible over \( \mathbb{C} \).

2. **Notation.** In this section we introduce some notation and collect some elementary facts about invariants and characters. Let \( G \) be a finite group of order \( g \). By a graded \( G \)-module we mean a \( G \)-module which is a graded vector space \( M = \sum_{q \geq 0} M_q \) over \( \mathbb{C} \), in which each homogeneous component \( M_q \) is a \( G \)-module finite dimensional over \( \mathbb{C} \). Let \( \mu_q \) be the character of \( G \) corresponding to the module \( M_q \). To the graded \( G \)-module \( M \) we let correspond the series

\[
M(t, \gamma) = \sum_{q \geq 0} \mu_q(\gamma)t^q, \quad \gamma \in G.
\]

For \( \gamma = 1 \) this becomes the Poincaré series

\[
M(t) = \sum_{q \geq 0} (\dim M_q)t^q
\]

of the graded vector space \( M \). All the tensor products we consider are tensor products over \( \mathbb{C} \). If \( M, N \) are graded \( G \)-modules, then \( M \otimes N \) has a natural structure of graded \( G \)-module with the grading and \( G \)-module structure defined by
(1) \( (M \otimes N)_q = \sum_{a+b=q} M_a \otimes N_b \),
\[
\gamma(x \otimes y) = \gamma x \otimes \gamma y, \quad x \in M, y \in N, \gamma \in G.
\]

From the fact that the character of a tensor product (direct sum) of two \( G \)-modules is the product (sum) of the characters it follows that
\[
(M \otimes N)(t, \gamma) = M(t, \gamma)N(t, \gamma).
\]

We let \( I(M) \) denote the submodule of invariants of \( M \), elements \( x \in M \) such that \( \gamma x = x \) for all \( \gamma \in G \). For a finite dimensional \( M \) with character \( \mu \) the connection between invariants and characters is the formula
\[
dim I(M) = \frac{1}{g} \sum_{\gamma \in G} \mu(\gamma).
\]

Thus for a graded \( M \) we have
\[
I(M)(t) = \frac{1}{g} \sum_{\gamma \in G} M(t, \gamma).
\]

The dual \( M^* \) of \( M \) has a natural \( G \)-module structure defined by
\[
(\gamma f)(x) = f(\gamma^{-1} x), \quad x \in M, f \in M^*, \gamma \in G,
\]
and we may extend this action to the algebra of polynomial functions on \( M \) or the Grassmann algebra of \( M \). If \( \mu \) is the character of \( M \) then \( \mu^*(\gamma) = \mu(\gamma^{-1}) \) is the character of \( M^* \). We have an isomorphism \( M \cong M^* \) of \( G \)-modules if and only if the character of \( M \) is real. The space \( \text{Hom}_C(M, N) \) has a natural \( G \)-module structure defined by
\[
(\gamma \phi)(x) = \gamma(\phi(\gamma^{-1} x)), \quad x \in M, \phi \in \text{Hom}_C(M, N), \gamma \in G,
\]
and the submodule \( I(\text{Hom}_C(M, N)) \) is just the space \( \text{Hom}_G(M, N) \) of \( G \)-module homomorphisms. The natural isomorphism of vector spaces \( \text{Hom}_C(M, N) \cong N \otimes M^* \) is an isomorphism of \( G \)-modules and induces an isomorphism \( \text{Hom}_G(M, N) \cong I(N \otimes M^*) \). In particular we see that if \( M \) is irreducible then \( \dim I(N \otimes M^*) \) is the multiplicity of \( M \) in \( N \), and that \( M \) is irreducible if and only if \( \dim I(M \otimes M^*) = 1 \).

3. Let \( G \) be a Euclidean reflection group in \( V \) and let \( S \) be the algebra of complex valued polynomial functions on \( V \). Then \( S = \sum_{q \geq 0} S_q \) has a natural structure of graded \( G \)-module. A formula of Molien, easy to verify by assuming \( \gamma \) in diagonal form, states that
\[
S(t, \gamma^{-1}) = \frac{1}{(1 - \omega_1(\gamma)t) \cdots (1 - \omega_n(\gamma)t)}
\]
where \( \omega_1(\gamma), \ldots, \omega_n(\gamma) \) are the eigenvalues of \( \gamma \) as linear transformation of \( V \).
If $M$ is a finite dimensional $G$-module we give $S \otimes M$ the grading defined by $(S \otimes M)_q = S_q \otimes M$. Then Molien's formula implies

\begin{equation}
I(S \otimes M)(t) = \frac{1}{g} \sum_{\gamma \in G} \frac{\mu(\gamma^{-1})}{(1 - \omega_1(\gamma)t) \cdots (1 - \omega_n(\gamma)t)}
\end{equation}

where $\mu$ is the character of $M$. From Theorem (a) of Chevalley we see that

\begin{equation}
I(S)(t) = \frac{1}{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}.
\end{equation}

Chevalley has also shown [2] that if $p_1, \ldots, p_k \in S$ form a $C$-basis for $S/F$ when reduced mod $F$, then $p_1, \ldots, p_k$ are a basis for $S$ as free module over $I(S)$. From this fact we readily deduce the following two lemmas.

**Lemma 1.** Let $\tau_q$ be the character of $(S/F)_q$. Then

\[\sum_q \tau_q(\gamma)t^q = \frac{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}{(1 - \omega_1(\gamma)t) \cdots (1 - \omega_n(\gamma)t)}.
\]

**Proof.** Let $p_1, \ldots, p_k \in S$ form a $C$-basis for $S/F$ when reduced mod $F$. Then the map $\sum_is_i \rightarrow \sum_is_i \otimes (p_i + F)$, $s_i \in I(S)$ defines an isomorphism $S \simeq I(S) \otimes S/F$ of graded $G$-modules. Since $G$ acts trivially on $I(S)$ we have $S(t, \gamma) = I(S)(t)(S/F)(t, \gamma)$. Thus from (3.1) and (3.3)

\[(S/F)(t, \gamma^{-1}) = \frac{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}{(1 - \omega_1(\gamma)t) \cdots (1 - \omega_n(\gamma)t)}
\]

which proves the lemma.

If we let $t \rightarrow 1$ we find $\sum_q \tau_q(\gamma) = 0$ if $\gamma \neq 1$ and $\sum_q \tau_q(1) = g$, so that $\sum_q \tau_q$ is the character of the regular representation of $G$. Thus $S/F$ affords the regular representation of $G$. If $M$ is an irreducible $G$-module we let $a_q(M)$ be the multiplicity of $M$ in $(S/F)_q$. Since $S/F$ contains $M$ with multiplicity $\dim M$ we have

\[\sum_q a_q(M) = \dim M.
\]

We view $S \otimes M$ naturally as an $S$-module and then $I(S \otimes M)$ is an $I(S)$-module.

**Lemma 2.** Let $M$ be an irreducible $G$-module. Then $I(S \otimes M)$ is a free module over $I(S)$. It has a basis over $I(S)$ consisting of homogeneous elements in which the number of elements of degree $q$ is $a_q(M^*)$. The rank of $I(S \otimes M)$ as $I(S)$-module is equal to $\dim M$.

**Proof.** In the proof of Lemma 1 we have remarked that $S \simeq I(S) \otimes (S/F)$ and hence $S \otimes M \simeq I(S) \otimes (S/F) \otimes M$. Since $G$ acts trivially on $I(S)$ we see by averaging over the group that $I(S \otimes M) \simeq I(S) \otimes (I(S/F) \otimes M)$. Thus $I(S \otimes M)$ is free over $I(S)$ and we may choose as bases a $C$-basis for $I(S/F \otimes M)$. This may be
chosen as a union of \( C \)-bases for the \( I((S/F)_q \otimes M) \). But \( \dim I((S/F)_q \otimes M) \) is the multiplicity of the irreducible \( M^* \) in \( (S/F)_q \) so \( \dim I((S/F)_q \otimes M) = a_q(M^*) \). The rank of \( I(S \otimes M) \) as \( I(S) \)-module is thus \( \sum a_q(M^*) = \dim M^* = \dim M \). The argument shows that \( I(S \otimes M) \) is free over \( I(S) \) for any \( G \)-module \( M \).

4. To prove Theorem 1 we simply compute the Poincaré series \( I(S \otimes M^*)(t) \) in two ways and compare the results for \( t = 1 \). Set \( a(t) = \sum a_q(M)t^q \). From Lemma 2 with \( M \) replaced by \( M^* \) we have

\[
I(S \otimes M^*)(t) = \frac{a(t)}{(1 - t^{m_1 + 1}) \ldots (1 - t^{m_n + 1})}
\]

and thus from (3.2) we see that

\[
\left( 1 + \sum_{y \in G_1} \mu(y) t^{a_1} \right) \prod_{y \in G_1} \frac{1}{(1 - t^{a_1} + \cdots + t^{a_n})} = \frac{a(t)}{(1 - t^{m_1 + 1}) \ldots (1 - t^{m_n + 1})}
\]

where \( \mu \) is the character of \( M \). Let \( G_1 \) be the set of elements of \( G \), distinct from the identity, which fix an \( n - 1 \) dimensional subspace of \( V \). For \( \gamma \in G_1 \) the eigenvalues \( \omega_\gamma \) are \( 1, 1, \ldots, 1, \omega \) where \( \omega \) is a root of unity. Now the fact that \( G \) may be written as a real orthogonal group implies \( \omega = -1 \). The left-hand side of (4.1) becomes

\[
\frac{1}{g} \left[ \frac{\mu(1)}{(1 - t)^n} + \frac{1}{(1 - t^{a_1} + \cdots + t^{a_n})} \sum_{y \in G_1} \mu(y) + \cdots \right]
\]

where \( \cdots \) denotes terms which have at most \( (1 - t)^{n-2} \) in the denominator. Since \( a(1) = \sum a_q(M) = \dim M = \mu(1) \) we have

\[
\frac{1}{(1 - t)^{a_1} + \cdots + t^{a_n}} \sum_{y \in G_1} \mu(y) + \cdots = \frac{ga(t)}{(1 - t^{m_1 + 1}) \ldots (1 - t^{m_n + 1})} - \frac{a(1)}{(1 - t)^n}.
\]

Now multiply both sides by \( (1 - t)^{n-1} \) and let \( t \to 1 \). Using the known formula [4; 8] \( g = \prod_i (m_i + 1) \) for the group order we find

\[
\frac{1}{2} \sum_{y \in G_1} \mu(y) = -a'(1) + \frac{1}{2} \left( \sum m_i \right) a(1)
\]

where \( a' \) denotes the derivative with respect to \( t \). For \( \gamma \in G_1 \), \( \mu(\gamma) = \dim M_\gamma^- \) - \( \dim M^- \gamma \) is, by assumption, independent of \( \gamma \). The number of elements in \( G_1 \) is the number of reflections in \( G \) which is known [4; 8] to be \( \sum m_i \). From the definition of \( a(t) \) we have \( a'(1) = \sum_q qa_q(M) \). If we insert this information in (4.2) we find

\[
\frac{1}{2} \left( \sum m_i \right) (\dim M^+ - \dim M^-) = - \sum_q qa_q(M) + \frac{1}{2} \left( \sum m_i \right) (\dim M^+ + \dim M^-)
\]
so that
\[ \sum_q q a_q(M) = \left( \sum_i m_i \right) \dim M^{-}. \]
The average of the \( q_i(M) \) is then
\[ \frac{\sum_q q a_q(M)}{\sum_q a_q(M)} = \frac{\dim M^{-}}{\dim M} r \]
with \( r = \sum_i m_i \). The significance of the integer \( r = \sum_i m_i \) in this formula becomes clear if one observes, with Lemma 1, that \( r \) is the largest integer \( q \) for which \( (S/F)_q \neq 0 \).

5. To prove Theorem 2 we shall need some results from an earlier article [9] together with a lemma concerning certain invariants of the symmetric group. Let \( E = \sum_p E_p \) be the Grassman algebra of \( V \). The homogeneous component \( E_p \) of degree \( p \) is the space of all \( p \)-linear alternating functions on \( V \). We identify \( E_0 \) with \( C \) and \( E_1 \), as vector space, with \( S_1 \). The group \( G \) acts naturally on \( E \) and on \( S \otimes E \). Choose a coordinate system \( x_1, \ldots, x_n \) in \( V \) and let \( d : S \otimes E \to S \otimes E \) be the \( C \)-linear map defined by
\[ d : s \otimes x_{i_1} \wedge \cdots \wedge x_{i_p} \to \sum_{j=1}^n \frac{\delta s}{\delta x_j} \otimes x_{i_1} \wedge \cdots \wedge x_{i_p}, \quad s \in S. \]
If we identify \( S \) with \( S \otimes C \) then \( dx_i = d(x_i \otimes 1) = 1 \otimes x_i \) so that the elements of \( S \otimes E_p \) may be written in the form
\[ \sum_{i_1 < \cdots < i_n} s_{i_1} \cdots i_n dx_{i_1} \cdots dx_{i_n}, \quad s_{i_1} \cdots i_n \in S. \]
It is clear that \( S \otimes E \) is just the algebra of differential forms on \( V \) and that \( d \) is exterior differentiation. Since \( d \) commutes with the action of \( G \) on \( S \otimes E \) it follows that \( d \) maps \( I(S \otimes E) \) onto \( I(S \otimes E) \). In particular, the differentials \( df_i \) are invariants of \( S \otimes E \). We have shown in [9] that the \( C \)-algebra \( I(S \otimes E) \) of invariant differential forms is an exterior algebra on \( n \) generators over the \( C \)-algebra \( I(S) \) of invariant polynomials, and is in fact generated over \( I(S) \) by the differentials \( df_1, \ldots, df_n \) of the polynomial invariants \( f_1, \ldots, f_n \) and the unit element. It follows that
\[ I((S \otimes E_p)(t)) = \frac{\sigma_p(t^{m_1}, \ldots, t^{m_n})}{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}, \quad p = 1, \ldots, n, \]
where \( \sigma_p(t_1, \ldots, t_n) \) is the \( p \)th elementary symmetric function in the indeterminates \( t_1, \ldots, t_n \).

**Lemma 3.** Let \( x_1, \ldots, x_n \) be indeterminates, let \( G \) be the symmetric group on \( x_1, \ldots, x_n \) and let \( E \) be the exterior algebra on \( x_1, \ldots, x_n \) over \( C \). The group \( G \) acts naturally on the commutative algebra \( E \otimes E \). Set
Then

\[ I(E_p \otimes E_p) = C u^p \oplus C u^{p-1} v, \quad p = 1, \ldots, n - 1. \]

**Proof.** Clearly both \( u^p \) and \( u^{p-1} v \) are in \( I(E_p \otimes E_p) \). To hold the indices in check we let \( \Omega \) denote the set of increasing sequences \( i_1 < \cdots < i_p \) of \( p \) integers chosen from \( 1, \ldots, n \), we let \( \{ i \} \) be the corresponding unordered set, and write \( x_{(i)} = x_{i_1} \wedge \cdots \wedge x_{i_p} \). Suppose \( y = \sum c_{(i),(k)} x_{(i)} \otimes x_{(k)} \in I(E_p \otimes E_p) \) where \( c_{(i),(k)} \in C \) and the sum is over all pairs \( (i),(k) \) of elements of \( \Omega \). If for given sets \( \{ i \}, \{ k \} \) the intersection \( \{ i \} \cap \{ k \} \) contains fewer than \( p - 1 \) elements, then there exist two distinct indices, say \( k_a, k_b \), which are distinct from all elements of \( \{ i \} \). Now apply the transposition \( (k_a, k_b) \) of \( G \) to each term in the sum \( y \). The invariance of \( y \) shows that \( c_{(i),(k)} = -c_{(k),(i)} \) and hence \( c_{(i),(k)} = 0 \). Thus we may write \( y = y_1 + y_2 \) where \( y_1 \) is a linear combination of elements \( x_{(i)} \otimes x_{(k)} \) such that \( \{ i \} \cap \{ k \} \) contains \( p \) elements, in other words \( \{ i \} = \{ k \} \), and \( y_2 \) is a linear combination of elements \( x_{(i)} \otimes x_{(k)} \) such that \( \{ i \} \cap \{ k \} \) contains \( p - 1 \) elements. Invariance of \( y \) implies the invariance of \( y_1 \) and \( y_2 \). We thus have \( y_1 = \sum b_{(i)} x_{(i)} \otimes x_{(i)} \) with \( b_{(i)} \in C \) and invariance of \( y_1 \) shows that all the \( b_{(i)} \) are equal, say \( b_{(i)} = b \). Then \( y_2 = b \sum x_{(i)} \otimes x_{(i)} \) is a \( C \)-multiple of \( u^p = \sum x_{(i)} \otimes x_{(i)} \). Similarly with slightly more effort one sees that \( y_2 \) is a \( C \)-multiple of \( u^{p-1} v \). Thus the elements \( u^p, u^{p-1} v \) span \( I(E^p \otimes E^p) \). For \( p < n \) both \( u^p \) and \( u^{p-1} v \) are not zero and hence linearly independent over \( C \). This proves the lemma. The argument breaks down for \( p = n \) only because \( u^{n-1} v = 0 \) and in that case we have \( I(E_n \otimes E_n) = C u^n \). The elements \( u, v \) also satisfy the relations \( u^{n+1} = 0 \) and \( v^2 = 0 \).

We are now in position to prove Theorem 2. Since \( L \) is simple, \( G \) acting in \( V \) is an irreducible group. A theorem of Burnside [1] states that there exists a coordinate system \( x_1, \ldots, x_n \) in \( V \) such that \( G \) acting on \( V^* \sim E_1 \) includes the symmetric group \( H \) on \( x_1, \ldots, x_n \). In this coordinate system the Killing form must be

\[ \frac{a}{2} \sum_i x_i^2 + b \sum_{i<k} x_i x_k \in I(S_2) \]

where \( a, b \) are real numbers. We cannot have \( a = 0 \) because the form is positive definite. We let \( I(E \otimes E) \) denote the elements of \( E \otimes E \) invariant under \( G \) and \( I_H(E \otimes E) \) the elements invariant under \( H \). Under the map \( f \to \sum (\partial f/\partial x_i) \otimes x_i \) of \( S_2 \to E_1 \otimes E_1 \) the Killing form maps into \( au + bv \in I(E_1 \otimes E_1) \) where \( u, v \in I_H(E_1 \otimes E_1) \) are the invariants of Lemma 3. Since \( I(E \otimes E) \subseteq I_H(E \otimes E) \), Lemma 3 shows that \( \dim(I(E_p \otimes E_p)) \leq 2 \). Suppose \( \dim(I(E_p \otimes E_p)) = 2 \) for some \( p = 1, \ldots, n - 1 \). We prove that \( \dim(I(E_{n-1} \otimes E_{n-1})) = 2 \). If \( p = n - 1 \) there is nothing to prove so assume \( p < n - 1 \). From Lemma 3 we see that \( u^p \in I(E_p \otimes E_p) \) and \( u^{p-1} v \in I(E_p \otimes E_p) \). Since \( v^2 = 0 \) it follows that both \( au^p v = u^{p-1} v (au + bv) \) and \( au^{p+1} + bu^p v = u^p (au + bv) \) are in \( I(E_{p+1} \otimes E_{p+1}) \). Since \( p < n - 1 \) we
have \( u^p v \neq 0 \) and since \( a \neq 0 \) it follows that \( \dim I(E_{p+1} \otimes E_{p+1}) = 2 \). We conclude by induction that \( \dim I(E_{n-1} \otimes E_{n-1}) = 2 \). Let \( Z \) be the 1-dimensional \( G \)-module defined by the homomorphism \( \gamma \to \det \gamma, \gamma \in G \). Then \( E_{n-1} \cong E_1 \otimes Z \) as \( G \)-modules. Since \( \det \gamma = \pm 1 \), \( Z \otimes Z \cong C \) is the trivial \( G \)-module. Then \( E_{n-1} \otimes E_{n-1} \cong E_1 \otimes E_1 \) as \( G \)-modules so that \( \dim I(E_1 \otimes E_1) = 2 \) which contradicts the irreducibility of \( E_1 \). Thus \( \dim I(E_p \otimes E_p) = 1 \) for all \( p \) and hence \( E_p \) is irreducible for all \( p \).

Let \( \theta : S \to S/F \) be the natural map and extend \( \theta \) to a map, denoted again \( \theta \), of \( S \otimes E \to S/F \otimes E \) by letting it be the identity on \( E \). Then \( \theta \) is a homomorphism of \( G \)-modules and of \( C \)-algebras. Since \( \theta \) is a homomorphism of \( G \)-modules we certainly have \( \theta I(S \otimes E) \subseteq I(S/F \otimes E) \). Actually we have \( \theta I(S \otimes E) = I(S/F \otimes E) \) because, using complete reducibility of the representations of \( G \), we may choose a graded \( G \)-module \( T \) such that \( S = F \oplus T \) and then \( \theta : I(T \otimes E) \to I(S/F \otimes E) \) is an isomorphism. Set \( z_i = \theta(df_i) \). Since \( I(S \otimes E) \) is generated over \( I(S) \) by the \( df_i \) and the unit element, and since every element \( s \in I(S) \) may be written as \( s = s_0 + s_1 \) with \( s_0 \in C \) and \( s_1 \in F \), it follows that the \( z_i \) together with the unit element generate \( I(S/F \otimes E) \) as algebra over \( C \). Thus the \( \binom{n}{p} \) elements \( z_{i_1} \cdots z_{i_p} \) generate \( I(S/F \otimes E_p) \) as vector space over \( C \). In fact they form a basis for \( I(S/F \otimes E_p) \) because Lemma 2 shows that \( \dim I(S/F \otimes E_p) = \dim E_p = \binom{n}{p} \). Thus the \( z_{i_1} \cdots z_{i_p} \) are linearly independent over \( C \).

The Killing form induces a natural isomorphism \( E_p \cong E_p^* \) of \( G \)-modules and hence a natural isomorphism \( S/F \otimes E_p \cong \text{Hom}(E_p, S/F) \) of \( G \)-modules. Under this isomorphism the invariants \( I(S/F \otimes E_p) \) correspond to \( \text{Hom}_G(E_p, S/F) \). For \( (i) \in \Omega \), let \( \phi_{(i)} \) be the image in \( \text{Hom}_G(E_p, S/F) \) of \( z_{i_1} \cdots z_{i_p} \). The linear independence of the \( z_{i_1} \cdots z_{i_p} \) implies linear independence of the \( \phi_{(i)} \). Now if \( M \) is an irreducible \( G \)-module and \( \phi_1, \cdots, \phi_q \) are linearly independent \( G \)-module homomorphisms of \( M \) into a \( G \)-module \( N \) then the sum \( \sum_{l} \phi_{(i)}(M) \) is direct. This follows at once from Schur's lemma by induction on the number of summands. In the case at hand this means that the sum \( \sum_{(i) \in \Omega} \phi_{(i)}(E_p) \) is direct. Since the number of summands is \( \binom{n}{p} = \dim E_p \), the sum \( \sum_{(i) \in \Omega} \phi_{(i)}(E_p) \) is the isotypic component of \( S/F \) of type \( E_p \). Thus \( E_p \) occurs as an irreducible constituent of \( (S/F)_q \) for precisely the values \( q = m_{i_1} + \cdots + m_{i_p}, i_1 < \cdots < i_p \). From the definition of \( \phi_{(i)} \) as the image of \( \theta(df_{i_1} \cdots df_{i_p}) \) it follows that a basis for \( \phi_{(i)}(E_p) \) is given by the \( \binom{n}{p} \) minors of \( J \) which involve \( f_{i_1}, \cdots, f_{i_p} \), reduced mod \( F \). This completes the proof of Theorem 2.

For the corollaries we argue as follows:

(2a) The linear independence over \( C \) of the minors of \( J \) after reduction mod \( F \) amounts to the linear independence of the elements \( \theta(df_{i_1} \cdots df_{i_p}) \) over \( C \). This we have shown.

(2b) The proof of the theorem shows that \( I(S/F \otimes E) \) is generated as algebra over \( C \) by the \( z_j = \theta(df_j) \) and the unit element. Since \( z_j \bar{z}_j = -z_j \bar{z}_j, I(S/F \otimes E) \) is a homomorphic image of an exterior algebra on \( n \) generators. But
dim I(S/F ⊗ E) = dim E = 2^n so I(S/F ⊗ E) is in fact an exterior algebra on the
z_i = \theta(df_i).

(2c) The proof of the theorem shows that I(E_p ⊗ E_p) = Cw^p for all p = 1, \ldots, n
where w = au + bv may be identified with the Killing form. Suppose we have an
isomorphism E_p \simeq E_q of G-modules. Then \binom{p}{q} = dim E_p = dim E_q = \binom{n}{q}
so q = p or q = n – p. Suppose q = n – p. Let \chi_p be the character of E_p and let
y \in G be a reflection. Then \chi_p(y) is the pth elementary symmetric function of the
eigenvalues 1, 1, \ldots, 1, -1 so that \chi_p(y) = \binom{n-1}{p} – \binom{n-1}{q-1}. Now \chi_p(y) = \chi_{n-p}(y)
shows n – p = p. Thus in any case q = p. It follows that I(E_p ⊗ E_q) = 0 for q ≠ p and
hence I(E ⊗ E) = \sum_p I(E_p ⊗ E_p) is generated over \mathbb{C} by the unit element and an
element w = au + bv which satisfies w^{n+1} = 0 and which may be identified with
the Killing form.

(2d) From Theorem 2 or directly from (5.1) we conclude that I(S/F ⊗ E_p)(t) = \sigma_p(t^{m_1}, \ldots, t^{m_n}). Thus one has a homogeneous isomorphism

(5.2) I(S/F ⊗ E_1) \simeq I(S/F ⊗ E_{n-1})

of graded vector spaces if and only if there exists an integer k such that
\sum_i \sigma_1(t^{m_1}, \ldots, t^{m_n}) = \sigma_{n-1}(t^{m_1}, \ldots, t^{m_n}). Comparing coefficients on both sides shows
that this condition is equivalent to the existence of an integer k such that
k + m_1 + \ldots + m_{n-i+1} = m_1 + \ldots + m_n. This is equivalent in turn to the statement that
m_1 + m_{n-i+1} is independent of i, the double duality. The same kind of coefficient
comparison shows that double duality implies the isomorphisms

I(S/F ⊗ E_p) \simeq I(S/F ⊗ E_{n-p}).

For later use we remark that (5.2) is equivalent to the existence of a homogeneous
isomorphism

(5.3) I(S ⊗ E_1) \simeq I(S ⊗ E_{n-1})

of graded vector spaces.

6. Let M, N be G-modules. We say that M and N are skew isomorphic if there
exists a 1-1 \mathbb{C}-linear map \theta of M onto N such that \theta x = (\det y)x for all
x \in M and all y \in G. We call \theta a skew isomorphism between M and N. Since
\det y = \pm 1 the relation of skew isomorphism is symmetric. Again we let Z denote
the 1-dimensional G-module defined by the homomorphism y \rightarrow \det y and let z be a
generator of Z. If we set \tilde{M} = M \otimes Z we see that x \rightarrow x \otimes z, x \in M, defines a skew
isomorphism between M and \tilde{M}. Since Z \otimes Z \simeq \mathbb{C}, \tilde{M} \gamma and M are isomorphic as
G-modules.

From Lemma 1 we conclude that \tau'(S/F)(t^{-1}, y) = \det y(S/F)(t, y) where
r = \sum_i m_i. Thus \tau_{r-q}(y) = (\det y)\tau_q(y) for all y \in G and all q = 0, \ldots, r so that
(S/F)_q and (S/F)_{r-q} are skew isomorphic. If M is an irreducible G-module and N
is any G-module, then M is irreducible and the multiplicity of M in N is equal to the multiplicity of M in N. Hence the

**Theorem.** If M is an irreducible G-module, then the multiplicity of M in (S/F)_q is equal to the multiplicity of M in (S/F)_r where r = \sum m_i. Thus in the notation of Theorem 1, with suitable ordering, q(M) = r - q(M).

We say that x \in M is a skew invariant if yx = (det y)x for all y \in G. Let \hat{I}(M) denote the subspace of skew invariant elements of M. Then the map x \mapsto x \otimes z defines a natural isomorphism \hat{I}(M) \cong I(\hat{M}) of vector spaces. It follows that we have an isomorphism

\[ (6.1) \hat{I}(S \otimes M) \cong I(S \otimes \hat{M}) \]

of graded vector spaces which is homogeneous of degree zero. From Lemma 2 we see that \hat{I}(S \otimes M) is free over I(S) of rank equal to dim M.

Since E_n = E_1 is an isomorphism of G-modules, the isomorphism (5.3) equivalent to the double duality becomes

\[ (6.2) \hat{I}(S \otimes E_1) \cong I(S \otimes E_1). \]

Now I(S \otimes E_1) is generated freely over I(S) by the df_i. If we can construct the homogeneous derivation \hat{d} of Theorem 3 then \hat{I}(S \otimes E_1) is generated freely over I(S) by the \hat{d}f_i and the homogeneous isomorphism (6.2) is defined by df_i \mapsto \hat{d}f_i. Thus in this formulation the double duality is equivalent to the existence of the map \hat{d}.

In connection with (6.2) it is worth noting that the homogeneous isomorphism \hat{I}(S \otimes E_0) \cong I(S \otimes E_0) amounts to the familiar fact that every skew invariant polynomial may be written as an invariant polynomial multiplied by det J.

7. Let q = m_1 + \cdots + m_{n-1}. We have a sequence of natural maps

\[ S_q \otimes E_{n-1} \xrightarrow{\psi_1} S_q \otimes E_1 \xrightarrow{\psi_2} \text{Hom}(E_1, S_q) \xrightarrow{\psi_3} \text{Hom}(S_1, S_q) \]

where \psi_1 is a skew isomorphism of G-modules induced by the natural duality in the Grassman algebra, where \psi_2 is an isomorphism of G-modules induced by the natural isomorphism of vector spaces, and \psi_3 is the isomorphism of G-modules induced by the identification of E_1 with S_1. The composite map

\[ \psi : S_q \otimes E_{n-1} \to \text{Hom}(S_1, S_q) \]

is a skew isomorphism of G-modules. Let \eta = \psi(df_1 \cdots df_{n-1}). Since df_1 \cdots df_{n-1} \in I(S_q \otimes E_{n-1}) we have \eta \in \hat{I}(\text{Hom}(S_1, S_q)) so that \eta is a skew homomorphism of S_1 into S_q. Since S_1 is irreducible, \eta must be injective, and from the definition of the map \psi we see that \eta x_i = u_i where u_i is the minor of order n - 1 of J obtained by deleting the derivatives of f_n and the derivatives with respect to x_i.
Our map \( \hat{d}: S \to S \otimes E_1 \) is homogeneous of degree \(-1\) and commutes with the action of \( G \). Let \( \hat{d} = d \circ \eta \). Then \( \hat{d} \) is a skew isomorphism of \( S_1 \) into \( S_{q-1} \otimes E_1 \). Since \( S \) is a polynomial ring we may extend \( \hat{d} \) to a derivation \( \hat{d}: S \to S \otimes E_1 \) of \( S \)-modules. By induction on the degree of a homogeneous element one sees that
\[
d_s y = (\det y) \gamma dy \quad \text{for all } \gamma \in G \quad \text{and all } s \in S.
\]
Thus \( \hat{d} \) maps \( I(S) \) into \( \hat{I}(S \otimes E_1) \).
Since \( \hat{d}x_i = du_i = \sum_k (\partial u_i / \partial x_k) dx_k \) we have
\[
(7.1) \quad \hat{d}f = \sum_i \sum_k \frac{\partial f}{\partial x_i} \frac{\partial u_i}{\partial x_k} dx_k.
\]
We claim that the elements \( \hat{d}f_1, \ldots, \hat{d}f_n \) are linearly independent over \( S \). If not, then we have a relation \( \sum_i s_i \hat{d}f_i = 0 \) where \( s_i \in S \) and where \( s_1 \), say, is not zero. Then multiplication by \( \hat{d}f_2 \cdots \hat{d}f_n \) shows that \( \hat{d}f_1 \cdots \hat{d}f_n = 0 \). On the other hand, computing directly from (7.1) shows that
\[
\hat{d}f_1 \cdots \hat{d}f_n = \det \left( \frac{\partial f_i}{\partial x_k} \right) \det \left( \frac{\partial u_i}{\partial x_k} \right) dx_1 \cdots dx_n
\]
which is not zero in view of our assumption about the algebraic independence of the \( u_i \). Thus the \( \hat{d}f_i \) are linearly independent over \( S \) and the sum \( P = \sum_i I(S) \hat{d}f_i \) is direct. Since \( G \) is a real group it has an invariant quadratic form \( f_1 \) and hence \( m_1 = 1 \). The degree of the map \( \hat{d} \) is thus \( q - 1 = m_1 + \cdots + m_{n-1} - 1 = m_2 + \cdots + m_{n-1} \) and the Poincaré series for the graded vector space \( P \) is thus
\[
P(t) = \frac{t^{m_2 + \cdots + m_{n-1}}(t^{m_1} + \cdots + t^{m_n})}{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})}.
\]
On the other hand, using (5.1), (6.1) and the double duality we see that
\[
P(t) = I(S \otimes E_n-1)(t) = \hat{I}(S \otimes E_1)(t). \quad \text{Since } P \subseteq \hat{I}(S \otimes E_1) \text{ is an inclusion of graded vector spaces we have } P = \hat{I}(S \otimes E_1). \text{ Thus } \hat{I}(S \otimes E_1) \text{ is freely generated over } I(S) \text{ by } \hat{d}f_1, \ldots, \hat{d}f_n \text{ and Theorem 3 is proved.}
\]
8. For the symmetric group on \( n \) letters \( x_1, \ldots, x_n \) we can give the following construction for the skew invariant differential 1-forms. Let \( \sigma_1, \ldots, \sigma_n \) be the elementary symmetric functions of \( x_1, \ldots, x_n \) and let
\[
\Delta(x_1, \ldots, x_{n-1}) = \prod_{1 \leq i < j \leq n-1} (x_i - x_j)
\]
be the fundamental skew invariant polynomial for the symmetric group on the letters \( x_1, \ldots, x_{n-1} \). Then a basis for the skew invariant differential 1-forms over the algebra of symmetric functions is given by the forms
\[
\omega_k = \sum_i (-1)^{i+1} \Delta(x_1, \ldots, \tilde{x}_i, \ldots, x_n) \frac{\partial \sigma_k}{\partial x_i} dx_i, \quad k = 1, \ldots, n,
\]
where \( \tilde{x}_i \) means that the letter \( x_i \) is to be omitted.
References


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