LINEARLY ORDERABLE SPACES(1)

BY

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A topological space is *linearly orderable*(2) if it is homeomorphic to a linearly ordered space topologized with the interval topology. For example, certain subsets of the real line R are linearly orderable, whereas other subsets of R are not. Examples of linearly orderable subsets of R are:

(a) $$[0,1) \cup \{2\}$$

which is homeomorphic to the linearly ordered space $(0,1] \cup \{2\}$ topologized with the interval topology,

(b) $$\{-1\} \cup \bigcup_{n=1}^{\infty} \left\{\frac{1}{n}\right\}$$

which is homeomorphic to the integers. Examples of nonlinearly orderable subsets of R are:

(a) $(0,1) \cup \{2\}$,

(b) $$[-1,0] \cup \bigcup_{n=0}^{\infty} \left\{\frac{1}{3n+3}\right\} \cup \bigcup_{n=0}^{\infty} \left(\frac{1}{3n+2}, \frac{1}{3n+1}\right).$$

In a previous note we showed that every subset of the real line which contains no interval is linearly orderable (I. L. Lynn [2]).The purpose of this paper is to tackle the general problem for subsets of R. We do not achieve a complete characterization of the linearly orderable subsets of R. However, we do obtain the following three facts(3):

(1) *If no open subset of X is compact and X has only countably many components, then X is linearly orderable.*

(2) *If X is a union of intervals containing no isolated closed*(4) *interval, then X is linearly orderable.*

(3) *If X is a union of open or half-open intervals, then X is linearly orderable.*

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(1) This research was supported, in part, by the Office of Naval Research and is part of the author's doctoral dissertation.

(2) Our concept of linearly orderable topological space agrees with that of *ordered* topological space introduced by Eilenberg [1], if and only if the space is connected.

(3) Henceforth, X denotes a subspace of R.

(4) Henceforth, closure is in R.

These results are consequences of the following main theorem of this paper. Let $\eta(X)$ denote the set of end points of the open ends of those components of $R \sim X$ which are half-open intervals.

MAIN THEOREM. *If no open subset of X is compact and $\eta(X)^{-} \cap X$ is countable, then X is linearly orderable.*

Crucial for the completion of the proof of this main theorem is the following embedding lemma.

EMBEDDING LEMMA. *There is an order-preserving homeomorphism τ of X into $[0, 1]$ such that the closure (in R) of each component of the complement of the Cantor set is either the closure (in R) of a component of $\tau[X]$ or else disjoint rom $\tau[X]$ (equivalently, such that each component of $\tau[X]$ either consists of an inaccessible point of the Cantor set or else its interior (in R) is a component of the complement of the Cantor set).*

CONJECTURE. If X contains no compact, open set, then X is linearly orderable.

REMARK. The "compact, open" condition of the main theorem cannot be deleted because $(0, 1) \cup \{2\}$ is not linearly orderable and contains the compact, open set $\{2\}$.

1. **Preliminaries.** Recall, if Y is a set linearly ordered by a relation $<$, then a subbase for the interval topology $\mathscr{I}$ on Y, induced by $<$, consists of all sets of the form $\{y \text{ in } Y: y < a\}$ or $\{y \text{ in } Y: a < y\}$ for a in Y.

DEFINITION. *A linearly ordered topological space* $(Y, \mathscr{T}, <)$ is a set Y on which a topology $\mathscr{T}$ and a linear ordering $<$ have already been defined such that the *f*nterval topology generated by $<$ coincides with $\mathscr{T}$.

DEFINITION. *A linearly orderable topological space* $(Y, \mathscr{T})$ is a topological space for which a linear ordering $<$ can be defined such that the interval topology generated by $<$ coincides with $\mathscr{T}$.

LEMMA 1.1. *Let $(S, \mathscr{U}, <)$ be a linearly ordered space. If $(Y, \mathscr{T})$ is a subspace and $\mathscr{I}$ is the interval topology on Y, induced by $<$, then $\mathscr{I} \subset \mathscr{T}$.*

Note that the identity map on $(Y, \mathscr{T})$ onto $(Y, \mathscr{I}, <)$ is continuous. Hence Lemma 1.1 holds. Whence, if $\mathscr{T} \subset \mathscr{I}$, then $(Y, \mathscr{T}, <)$ is a linearly ordered space.

Before showing that the conclusion of the lemma cannot be strengthened, we ntroduce the following conventions. Henceforth we let R denote the real line with its usual topology and usual linear ordering $<$. The relative topology of a subspace is, whenever mentioned, denoted $\mathscr{T}$. The interval topology on a subset of R is henceforth assumed induced by $<$ and is, whenever mentioned, denoted $\mathscr{I}$.

EXAMPLE 1.2. Consider the subspace

$$X = \{-1\} \cup \bigcup_{n=1}^{\infty} \left\{\frac{1}{n}\right\}.$$

Clearly -1 is not a $\mathscr{T}$-limit point of X. But -1 is an $\mathscr{I}$-limit point on X. Therefore $\mathscr{I}$ is properly contained in $\mathscr{T}$. Whence $(X, \mathscr{T}, <)$ is not a linearly ordered space. But $(X, \mathscr{T})$, being homeomorphic to the linearly ordered subspace of integers, is linearly orderable.

EXAMPLE 1.3. The subspace $X = (0, 1) \cup \{2\}$ is not even linearly orderable. For a one-one, continuous map of the connected space $(0, 1)$ into a linearly ordered space Y is order-preserving or order-reversing. So a homeomorph of X in Y consists of a connected open interval G and an isolated point y. But, since G has neither a first point nor a last point, y is a limit point of G in the interval topology, induced by the linear ordering in Y, on $\{y\} \cup G$.

The following definition is crucial.

DEFINITION. We say, *X is not linearly ordered at a point e of X from below* (*above*), if e is the right-hand (left-hand) end point of a component of $R \sim X$ which is a half-open interval. In the contrary case, we say *X is linearly ordered at the point e of X from below* (*above*).

If X is not linearly ordered at a point e of X from below or above, we say *X is not linearly ordered at e*. In the contrary case, we say *X is linearly ordered at e*.

Observe that in Example 1.2, X is not linearly ordered at -1, because -1 is the end point of the component $(-1, 0]$ of $R \sim X$ which is a half-open interval. Similarly in Example 1.3, X is not linearly ordered at 2.

We next prove a lemma which characterizes linearly ordered subspaces of R and reduces our problem from a global to a local level.

LEMMA 1.4. *X is a linearly ordered space if and only if X is linearly ordered at each of its points.*

Proof. Suppose X is not a linearly ordered space. Then it follows from Lemma 1.1 that we may assume there is an e in X which is an $\mathscr{I}$-limit point of the set S of points of X above e, say, but not a $\mathscr{T}$-limit point. It follows that e is the left-hand end point of a component C of $R \sim X$ which is a half-open interval. For if C were an open interval, then the right-hand end point of C would be in X, whence e would not be an $\mathscr{I}$-limit point of X. Therefore X is not linearly ordered at e.

Conversely, suppose X is not linearly ordered at the point e of X. Then e is an end point of a component C of $R \sim X$ which is a half-open interval. Conequently e is an $\mathscr{I}$-limit point of the set S of points in X on the other side of C, for otherwise C would be an open interval. But e is not a $\mathscr{T}$-limit point of S. Thus by definition X is not a linearly ordered space.

COROLLARY 1.5. *Any open or closed subspace of R is a linearly ordered space.*

COROLLARY 1.6. *Any dense subspace of R is a linearly ordered space.*

A dense-in-itself subspace of R need not even be linearly orderable.

EXAMPLE 1.7. The space $(0, 1) \cup [2, 3]$ is not linearly orderable for reasons similar to those of Example 1.3.

DEFINITION. Let $\eta(X)$ be the set of points of X at which X is not linearly ordered.

Note that we seek topologically invariant conditions, whereas $\eta(X)$ depends upon the embedding of X in R.

EXAMPLE 1.8. Let X_1 be the set of right-hand end points of the components in $[0, 1]$ of the complement of the Cantor set. If x is in X_1, then x is the right-hand end point of a component in $R \sim X_1$ which is a half-open interval. Whence X_1 is not linearly ordered at x. Consequently $\eta(X_1) = X_1$, so X_1 is not linearly ordered at *any* of its points and thus at infinitely many points. Therefore by Lemma 1.4, X_1 is not a linearly ordered space. But no open subset of X_1 is compact and $\eta(X_1)^- \cap X_1$ is countable, so by our main theorem X_1 is linearly orderable.

DEFINITION 1.9. A point of the Cantor set is an *accessible* (*inaccessible*) point if it is (is not) an end point of a component of the complement of the Cantor set.

EXAMPLE 1.10. Let X_2 be the union of X_1 of Example 1.8 and the set of inaccessible points of the Cantor set. Then exactly as in the space X_1, X_2 is not linearly ordered at each accessible point of X_2. Each inaccessible point is a limit point of the set $\eta(X_2)$ of points at which X_2 is not linearly ordered. Thus $\eta(X_2)^- \cap X_2$ is uncountable. But X_2 is linearly orderable because X_2 contains no interval (I. L. Lynn [2]).

2. **The case $\eta(X)$ is finite.** Our purpose is to establish a topological characterization of any linearly orderable subspace X of R for which $\eta(X)$ is finite. We will obtain as a corollary: If no open subset of X is compact and $\eta(X)$ is finite, then X is linearly orderable. This will conclude the proof of the first half of our main theorem.

DEFINITION 2.1. We will say that X is an *interval space of two-sided limit points* if X contains no zero-dimensional component, and any end point in a component C of X is in $(X \sim C)^-$.

Note that this definition involves topological concepts and not those of order, since $X \subset R$. But it follows from the definition that each component of X is an interval, and each point of X is a two-sided limit point of X.

Any disjoint collection of open or half-open intervals in R, such that no half-open interval is isolated, is an interval space of two-sided limit points.

THEOREM 2.2. *If X is linearly orderable, then X is not the union of nonempty separated sets Y and Z such that Y is compact and Z is an interval space of two-sided limit points. The converse holds if $\eta(X)$ is finite.*

We will first show that our condition is necessary. We will then show that it is not sufficient.

Proof of necessity in the theorem. Suppose X has the above-described separation $Y \cup Z$. Let g be a homeomorphism of X into a linearly ordered space S. Select a point $g(z)$ in $g[Z]$. Without loss of generality we may assume that some point of $g[Y]$ precedes $g(z)$. Therefore, since $g[Y]$ is compact, there is a greatest point $g(y)$ of $g[Y]$ which precedes $g(z)$.

Now $\{g(y)\}$ is separated from $g[Z]$. But $g[Z]$ contains no least point which is greater than $g(y)$, because Z is an interval space of two-sided limit points. It follows that $g(y)$ is a limit point of $g[Z]$ in the interval topology, induced by the linear ordering in S, on $g[X]$. Therefore $g[X]$ is not a linearly ordered space. This concludes the proof.

The following example shows that, without some additional restriction, the converse is false.

EXAMPLE 2.3. Set $I=[-1,0]$, $P=\{1/(3n+3): n=0,1,\cdots\}$, and for $n=0,1,\cdots$, set $G_n=(1/(3n+2),1/(3n+1))$.

Set $X=I\cup P\cup\bigcup_{n=0}^{\infty}G_n$.

Suppose X has the above-described separation $Y\cup Z$.

Observe first that since Y and Z are separated and Y is compact, we must have $\bigcup_{n=0}^{\infty} G_n\subset Z$. Observe next that since Z contains no zero-dimensional component, we must have $P\subset Y$. It follows that 0 is a limit point of both Y and Z. Therefore Y and Z are not separated.

Consequently X cannot have the above-described separation.

We will now show X is not linearly orderable.

Let g be a homeomorphism of X into a linearly ordered space S. For $n=0,1,\cdots$, choose a point $g(z_n)$ of $g[G_n]$. The sequences $\{g(1/(3n+3))\}$, of isolated points of $g[X]$, and $\{g(z_n)\}$ converge to $g(0)$. But the component $g[I]$ is an interval, containing $g(0)$ as an end point. So $g(0)$ is the maximum or minimum of $g[I]$.

Without loss of generality we assume that $g(0)$ is the maximum of $g[I]$. It follows that for all but at most finitely many natural numbers n, $g(z_n)$ and $g(1/(3n+3))$ lie above $g(0)$. Fix an integer s such that $g(1/(3s+3))$ lies above $g(0)$. Let $g(z_m)$ be the greatest point in $\{g(z_n)\}$ below $g(1/(3s+3))$. Let $g(1/(3r+3))$ be the least point in $\{g(1/(3n+3))\}$ above $g(z_m)$.

Now $\{g(1/(3r+3))\}$ is separated from $g[G_m]$. But $g(1/(3r+3))$ is a limit point of the open interval $g[G_m]$ in the interval topology, induced by the linear ordering in S, on $g[X]$. Therefore $g[X]$ is not a linearly ordered space.

So the converse is false.

We now attack the proof of sufficiency in the theorem.

LEMMA 2.4. *If a homeomorphism f of X into R is order-preserving or order-reversing, then x is in $\eta(X)$ if and only if $f(X)$ is in $\eta(f[X])$.*

Proof. Obvious.

LEMMA 2.5. *Let a homeomorphism g of X be the identity on $X\sim[p,q]$ and order-preserving (reversing) on $X\cap[p,q]$ into $[p',q']\subset[p,q]\cup(R\sim X)$. Then x is in $\eta(X)$ if and only if $g(x)$ is in $\eta(g[X])$, except possibly for these six points:*

(1) $\alpha = \sup\{x \text{ in } X : x \leqq p\}$,
(2) $\beta = \inf\{x \text{ in } X : p \leqq x\}$,
(3) $\gamma = \sup\{x \text{ in } X : x \leqq q\}$,
(4) $\delta = \inf\{x \text{ in } X : q \leqq x\}$,
(5) $\alpha' = \sup\{g(x) \text{ in } g[X] : g(x) \leqq p'\}$,
(6) $\delta' = \inf\{g(x) \text{ in } g[X] : q' \leqq g(x)\}$.

Proof. By the preceding lemma x is in $\eta(X \cap [p,q])$ if and only if $g(x)$ is in $\eta(g[X \cap [p,q]])$. Clearly x is in $\eta(X \sim [p,q])$ if and only if $g(x)$ is in $\eta(g[X \sim [p,q]])$.

It follows that for points of X distinct from the above six points, x is in $\eta(X)$ if and only if $g(x)$ is in $\eta(g[X])$.

This concludes the proof of the lemma.

LEMMA 2.6. *If $\eta(X)$ is finite, there is a homeomorphism g of X into R such that $\eta(g[X])$ contains at most one point.*

Proof. By Lemma 2.4, we may assume that $X \subset [0,1]$. Suppose that $\eta(X)$ contains exactly $n \geqq 2$ points, and $a < b$ are two of these points. It suffices to show that $\eta(g[X])$ contains at most $n-1$ points for some homeomorphism g.

Observe first if X is not linearly ordered at the point x of X from both below and above, then x is an isolated point of X. Therefore there is a homeomorphism f of X which is the identity on $X \sim \{x\}$ and has $f(x)$ as the minimum of $f[X]$. It follows that without loss of generality we need only prove our lemma valid for those cases in which X is not linearly ordered at x from only one side. We therefore have only four cases to consider.

Case 1. X is not linearly ordered at a and b from below, but X is linearly ordered at a and b from above.

Because X is not linearly ordered at b from below, b is the right-hand end point of a component C in $R \sim X$ which is a half-open interval. Let q denote the left-hand end point of C. Then q is in C and is a limit point of X from below q only.

Define a homeomorphism g of X to be the identity on $X \sim [a,q]$ and order-reversing on $X \cap [a,q]$ into $[a,q]$. Then $g(a)$ is the immediate predecessor in $g[X]$ of $g(b) = b$. It is now easily verified by Lemma 2.5 that $\eta(g[X])$ contains exactly $n-2$ points.

Case 2. X is not linearly ordered at a and b from above, but X is linearly ordered at a and b from below.

It follows from Lemma 2.4 that an order-reversing homeomorphism of X transposes Case 2 into Case 1.

Case 3. X is not linearly ordered at a from above and at b from below, but X is linearly ordered at a from below and at b from above.

Because X is not linearly ordered at a from above, a is the left-hand end point of a component C in $R \sim X$ which is a half-open interval. Let p denote the right-hand end point of C. Then p is in C and is a limit point of X from above p only. Similarly b is the right-hand end point of a component C' in $R \sim X$ which is a half-open interval. Let q denote the left-hand end point of C'. Then q is in C' and is a limit point of X from below q only, and also $p < q$.

Define a homeomorphism g of X to be the identity on $X \sim [p,q]$ and order-preserving on $X \cap [p,q]$ into $[-2,-1]$. It is easily verified by Lemma 2.5 that if the point $\delta' = \inf X$ is in X, then $\eta(g[X])$ contains exactly $n-1$ points. If on the other hand δ' is not in X, then it is similarly easily seen that $\eta(g[X])$ contains exactly $n-2$ points.

Case 4. X is not linearly ordered at a from below and at b from above, but X is linearly ordered at a from above and at b from below.

Define a homeomorphism g of X to be the identity on $X \sim [a,b]$ and order-preserving on $X \cap [a,b]$ into $[-2,-1]$. The proof is now similar to Case 3.

The proof of the lemma is concluded.

LEMMA 2.7. *Let a be in X, where $X \subset [0,1]$. Set $Z = \{x \text{ in } X : x < a\}$ and $Y = X \sim Z$. Let Y and Z be nonempty and separated, each point of Z be a two-sided limit point of X, and $C' = \{p\}$ be a (zero-dimensional) component of Z. Then there is a homeomorphism g of X into R such that $g(p)$ is the minimum of $g[Z]$, each point of $g[Z] \sim \{g(p)\}$ is a two-sided limit point of $g[X]$, and each point of $g[Y]$ is below $g(p)$.*

Proof. X is zero-dimensional at p. Therefore p has a neighborhood base of open and closed sets. These sets have empty boundary. Consequently we can choose a strictly increasing sequence $\{s_m\}$, $m = 1,2,3,\cdots$, of points of $R \sim X$ converging to p such that $s_1 < \inf X$. We can also choose a strictly decreasing sequence $\{t_m\}$ in $R \sim X$ converging to p such that $t_2 < a \leqq \sup X < t_1$.

For $m = 1,2,3,\cdots$, define open intervals S_m, T_m, U_m, and V_m as follows: $S_m = (s_m, s_{m+1})$, $T_m = (t_{m+1}, t_m)$, $U_m = (1/(2m+1), 1/2m)$, and $V_m = (1/2m, 1/(2m-1))$. Now for each natural number m we define f_{2m} to be an order-preserving homeomorphism on S_m onto U_m, and we define f_{2m-1} to be an order-preserving homeomorphism on T_m onto V_m.

Set $A = \{p\} \cup \bigcup_{m=1}^{\infty}(S_m \cup T_m)$, and $B = \{0\} \cup \bigcup_{m=1}^{\infty}(U_m \cup V_m)$.

The following function $\bar{f}$ is now defined on A onto B. For x in A, let

$$\bar{f}(x) = \begin{cases} f_{2m}(x), & \text{if } x \text{ is in } S_m, \\ f_{2m-1}(x), & \text{if } x \text{ is in } T_m, \\ 0, & \text{if } x = p. \end{cases}$$

We will show that $\bar{f}$ is a homeomorphism of A onto B. Now if m is a natural number, then the restriction of $\bar{f}$ to S_m or T_m is a homeomorphism onto U_m or V_m respectively. Moreover $\bar{f}$ is a one-one function on A onto B. Therefore we need only show that $\bar{f}$ is continuous at the point p of A and that $\bar{f}^{-1}$ is continuous at the point 0 of B.

We will show first that $\bar{f}$ is continuous at p. Let $\{x_r\}$ be a strictly increasing sequence in A converging to p. The strictly increasing sequence $\{s_m\}$ also converges to p. So there is an integer $N_{(m)}$ such that $s_m < x_r < p$ if $r > N_{(m)}$, whence x_r is in $\bigcup_{n=m}^{\infty} S_n$. So $\bar{f}(x_r)$ is in $\bigcup_{n=m}^{\infty} U_n$. Therefore $\bar{f}(x_r)$ is within $1/2m$ of 0. So the sequence $\{\bar{f}(x_r)\}$ converges to $\bar{f}(p)$. Thus $\bar{f}$ is continuous at p from below. An analogous demonstration shows $\bar{f}$ is continuous at p from above. Whence $\bar{f}$ is continuous at p. It follows that $\bar{f}$ is continuous on A.

We will show next that $\bar{f}^{-1}$ is continuous at 0. Let $\{\bar{f}(x_r)\}$ be a strictly decreasing sequence in B converging to 0. The strictly increasing sequence $\{s_m\}$ and the strictly decreasing sequence $\{t_m\}$ both converge to p. So there is an integer $N_{(m)}$ such that $s_{N_{(m)}}$ and $t_{N_{(m)}}$ are within $1/m$ of p. Thus each point of $\bigcup_{n=N_{(m)}}^{\infty}(T_n \cup S_n)$ is within $1/m$ of p. There is an integer $M(N_{(m)})$ such that $\bar{f}(x_r)$ is within $1/(2N_{(m)}-1)$ of 0 if $r > M(N_{(m)})$, whence $\bar{f}(x_r)$ is in $\bigcup_{n=N_{(m)}}^{\infty}(U_n \cup V_n)$. So x_r is in $\bigcup_{n=N_{(m)}}^{\infty}(S_n \cup T_n)$. Thus x_r is within $1/m$ of p. Therefore the sequence $\{x_r\}$ converges to p. So $\bar{f}^{-1}$ is continuous at $\bar{f}(p) = 0$ from above. But 0 is the minimum of B. Thus $\bar{f}^{-1}$ is continuous at 0. It follows that $\bar{f}^{-1}$ is continuous on B.

Consequently $\bar{f}$ is a homeomorphism on A onto B.

Obviously our given space X is contained in A.

Let f denote the restriction of $\bar{f}$ to X. Then f is a homeomorphism of X into $[0,1]$, and $f(p) = 0$ is the minimum of $f[X]$.

We will show next that each point of $f[Z] \sim \{f(p)\}$ is a two-sided limit point of $f[X]$.

Let z be a point of $Z \sim \{p\}$. Then there is an integer n such that z is in $S_n \cup T_n$. Suppose z is in the open interval S_n. Now each point of Z is a two-sided limit point of X. So z is two-sided limit point of $X \cap S_n$. But f is an order-preserving homeomorphism on $X \cap S_n$ into U_n. Therefore $f(z)$ is a two-sided limit point of $f[X] \cap U_n$.

A similar argument prevails if z is in T_n.

Therefore each point of $f[Z] \sim \{f(p)\}$ is a two-sided limit point of $f[X]$.

Now by definition of Y and Z we have a is the minimum of Y, and a is above each point of Z, and Y and Z are separated. Moreover we chose the points t_2 and t_1 such that $t_2 < a \leqq \sup Y < t_1$. But f is an order-preserving homeomorphism on $X \cap T_1$ into $V_1 = (\frac{1}{2}, 1)$. Therefore $f(a)$ is the minimum of $f[Y]$, and $f(a)$ is above each point of $f[Z]$, and $f[Y]$ and $f[Z]$ are separated.

Define a homeomorphism h of $f[X]$ to be the identity on $f[X] \sim f[Y]$ and order-preserving on $f[Y]$ into the interval $[-2, -1]$.

Set $g = h \circ f$. Then g is the required homeomorphism of X.

This concludes the proof of the lemma.

Proof of sufficiency in the theorem. By Lemma 2.4, we may assume that $X \subset [0,1]$. Now $\eta(X)$ is finite. Whence by Lemma 2.6 there is a homeomorphism g of X into R such that $\eta(g[X])$ consists of at most one point. But the rest of our hypothesis is topological. Consequently we also assume that $\eta(X)$ contains at most one point.

If $\eta(X) = \varnothing$, by Lemma 1.4 X is linearly ordered.

Consequently we assume $\eta(X) = \{a\}$.

Let us suppose that: X is not linearly ordered at a from below, but X is linearly ordered at a from above.

Set $Z = \{x \text{ in } X : x < a\}$ and $Y = X \sim Z$. (Z is not compact.)

Now a is the right-hand end point of a component in $R \sim X$ which is a half-open interval. Therefore X is the union of the nonempty separated sets Y and Z. It follows from our hypothesis that Y is not compact or Z is not an interval space of two-sided limit points.

Suppose first that Y is not compact. Then, since Y is bounded, there is a point q in $Y^- \sim Y$.

Set $y = \sup Y$. Then $a < q \leqq y \leqq 1$.

Define a homeomorphism g of X to be the identity on $X \sim [a,q]$ and order-preserving (respectively reversing) on $X \cap [a,q]$ into $[2,3]$ if y is (respectively is not) in X.

It follows that if y is in X, then $g(y) = y$ is the immediate predecessor in $g[X]$ of $g(a)$. Whence it is easily verified from Lemma 2.5 that $\eta(g[X]) = \varnothing$. If on the other hand y is not in X, then it is again easily seen that $\eta(g[X]) = \varnothing$. Thus in either case it follows from Lemma 1.4 that $g[X]$ is a linearly ordered space.

Therefore if Y is not compact, then X is linearly orderable.

Suppose next that Y is compact and Z contains an isolated or one-sided limit point p'.

Since p' is in Z, $p' < a$. Whence X is linearly ordered at p'.

It follows from the above that p' is the minimum of X or has an immediate predecessor or successor in X.

Without loss of generality we assume that p' is the minimum of X or has an immediate predecessor q' in X.

Define a homeomorphism g of X to be the identity on Z and order-preserving on Y into $[-2,-1]$ (respectively $[(2q'+p')/3, (2p'+q')/3]$) if p' is the minimum (respectively has an immediate predecessor q') in X. It is an immediate consequence of Lemma 2.5 that $\eta(g[X]) = \varnothing$ in either case, because Y is compact. Whence by Lemma 1.4 $g[X]$ is a linearly ordered space.

Therefore if Y is compact and Z contains an isolated or one-sided limit point, then X is linearly orderable.

Suppose finally that Y is compact and each point of Z is a two-sided limit point. It follows that Z contains a zero-dimensional component $C' = \{p\}$. Therefore there is a homeomorphism g of X satisfying Lemma 2.7. It is now easily verified that $\eta(g[X]) = \varnothing$ because Y is compact. Whence by Lemma 1.4 $g[X]$ is a linearly ordered space.

Therefore if Y is compact and each point of Z is a two-sided limit point, then X is linearly orderable.

In summary we have shown, if $\eta(X) = \{a\}$ and X is not linearly ordered at a from below, but X is linearly ordered at a from above, then the condition in the theorem is sufficient.

The remaining possibilities, satisfying X is not linearly ordered at a, can be transposed into the preceding situation (or a linearly ordered space) by the obvious homeomorphisms. Therefore X is linearly orderable, since the condition in the theorem is topological.

This concludes the proof of the theorem.

Corollary 2.8. *If no open subset of X is compact and $\eta(X)$ is finite, then X is linearly orderable.*

Proof. If X is not linearly orderable and $\eta(X)$ is finite, then by the theorem, X contains a (nonempty, proper) compact, open set Y.

Example 2.9. The Cantor set is the complement in $[0,1]$ of open intervals G_n, $n = 1,2,3,\cdots$.

Set $Z = \bigcup_{n=1}^{\infty} G_n^-$.

Then Z contains no zero-dimensional component, since a component of Z is G_n^-.

Since the Cantor set is nowhere dense, Z is dense on $[0,1]$. In addition neither 0 nor 1 is in Z. It follows that any end point in a component C of Z is in $(Z \sim C)^-$.

Set $Y = [2,3]$ and $X = Y \cup Z$. By Theorem 2.2, X is not linearly orderable.

The preceding example shows that spaces of closed intervals are generally not linearly orderable.

3. The main theorem.

We first prove the following lemma which ensures, later on, that a sequence of homeomorphisms which will be constructed are uniformly convergent.

Embedding Lemma 3.1. *There is an order-preserving homeomorphism τ of X into $[0,1]$ such that the closure in R of each component of the complement of the Cantor set is either the closure in R of a component of $\tau[X]$ or else disjoint from $\tau[X]$.*

Proof. We may assume $X \subset (0,1)$. If C is a component of X, let $[a,b]$ be the (possibly degenerate) smallest closed interval containing C. The following function f is now defined on X into $(0,1)$. For each component C of X, if $x \in C$, let

$$f(x) = \begin{cases} x, & \text{if } a,b \in C, \\ (a+x)/2, & \text{if } a \in C,\ b \notin C, \\ (x+b)/2, & \text{if } a \notin C,\ b \in C, \\ (a+x+b)/3, & \text{if } a,b \notin C. \end{cases}$$

We will show that f is continuous on X. Since $f|C$ is a homeomorphism for each $C \subset X$, we need only show that f is continuous at each end point of C contained in C. Suppose first that the left-hand end point a of C is contained in C. Suppose further that we have a strictly increasing sequence $\{x_n\}$ of points of X converging to a. Now either an end point or midpoint of each component is a fixed point of f. Consequently there is a strictly increasing sequence $\{y_n\}$ of fixed points of f converging to a. It follows that, since f is strictly increasing and since $f(a) = a$, the strictly increasing sequence $\{f(x_n)\}$ converges to $f(a)$. So f is continuous at a from below a. Now suppose that $\{x_n\}$ is a strictly decreasing sequence converging to a. If $a = b$, that is, the component containing a consists of a point, then a similar argument prevails. If $a < b$, then the result is clear. Thus f is also continuous at a from above a. Whence f is continuous at a. By means of a similar argument, when then right-hand end point b of C is contained in C, f is continuous at b. So f is continuous.

An argument similar to the above shows that f^{-1} is also continuous. Therefore f is an order-preserving homeomorphism.

Let Z denote the set of end points of the components of $f[X]$.

We will now show that $R \sim Z$ is dense in R.

Suppose that Z contains an open interval J. If $J \subset f[X]$, then $J \cap Z = \varnothing$. Consequently J contains a point y of $R \sim f[X]$. Whence y is in Z and in $R \sim f[X]$. So y is an end point of an interval in $f[X]$. But y is an interior point of J. Thus J contains an interval in $f[X]$. Whence Z contains an interval in $f[X]$. This is a contradiction. It follows that $R \sim Z$ is dense in R.

Since $R \sim Z$ is dense in R, we can choose a countable set D contained in $R \sim Z$ that is dense in R. It follows that, ordered with respect to the usual ordering in R, D is an η-set([5]).

Now it follows directly from the standard construction of the Cantor set that the components in $[0, 1]$ of the complement of the Cantor set form an η-set, when ordered with respect to their position.

From this it follows that the set S consisting of the right-hand end points of these components is also an η-set, where the order is induced by that in R.

Since D and S are each η-sets, there is a similarity mapping g of D onto S. Let I denote the set of inaccessible points of the Cantor set. We will now define an extension $\bar{g}$ of g such that $\bar{g}$ is on R onto $I \cup S$, $\bar{g}$ is strictly increasing, and $\bar{g}^{-1}$ is continuous.

For r in $R \sim D$, set

$$\alpha = \sup\{g(d) : d \text{ is in } D \text{ and } d < r\}$$

and

$$\beta = \inf\{g(d) : d \text{ is in } D \text{ and } r < d\}.$$

([5]) An η-set Q is a countable linearly ordered set, with neither a first nor a last element, such that between any two elements of Q lie infinitely many elements of Q. An η-set is order-isomorphic with the rationals.

Because D is dense on R and g is strictly increasing, $d < r < d'$ implies $g(d) < \alpha \leqq \beta \leqq g(d')$. If $\alpha < \beta$, then it follows that the open interval (α,β) is a component of the complement of the Cantor set. Thus there is a d in D such that $\beta = g(d)$, which is impossible. So $\alpha = \beta$. Clearly β is not a left-hand end point of a component of the complement of the Cantor set. Therefore β is in I. We now set $\bar{g}(r) = \beta$ for each r in $R \sim D$, and $\bar{g}(d) = g(d)$ for each d in D. Then $\bar{g}$ is a strictly increasing function on R to $I \cup S$, because g is strictly increasing on a dense subset of R. Whence $\bar{g}$ is onto $I \cup S$ because R is connected and each point of I is a two-sided limit point of S. Finally $\bar{g}^{-1}$ is continuous because it is monotone and R is connected.

Now since $\bar{g}$ maps D onto S, $\bar{g}$ maps $R \sim D$ onto I. Moreover $Z \subset R \sim D$, because $D \subset R \sim Z$. Therefore $\bar{g}$ maps Z into I.

We will now show that $\bar{g}|Z$ is a homeomorphism of Z into I.

Let $\{z_n\}$ be a strictly increasing sequence of points of Z converging to the point z of Z. If the strictly increasing sequence $\{\bar{g}(z_n)\}$ does not converge to $\bar{g}(z)$, then, because $\bar{g}(z)$ is an inaccessible point of the Cantor set, there is a d in D such that $\bar{g}(z_n) < \bar{g}(d) < \bar{g}(z)$ for all n. Whence $z_n < d < z$ for all n. This is a contradiction. So $\bar{g}|Z$ is continuous at z from below z. A similar argument shows that $\bar{g}|Z$ is continuous at z from above z. Whence $\bar{g}|Z$ is continuous at z. Therefore $\bar{g}|Z$ is continuous. Since we have already shown above that $\bar{g}^{-1}$ is continuous, it follows that $\bar{g}|Z$ is a homeomorphism of Z into I.

We now make correspond to each component $C \subset f[X]$ an (possibly degenerate) interval $C' \subset [0,1]$ such that: (1) z is an end point of C if and only if $\bar{g}(z)$ is an end point of C', (2) z is in C if and only if $\bar{g}(z)$ is in C'.

Because of the above correspondence, the fact that Z is the set of end points of the components of $f[X]$, and finally that $\bar{g}|Z$ is an order-preserving homeomorphism of Z into I, it clearly follows that there is an order-preserving homeomorphism of $f[X]$ onto $E = \bigcup_{C \subset f[X]} C'$.

We will now show that the one-dimensional components of E together with the components in $[0,1]$ of the complement of the Cantor set disjoint from E form an η-set, when ordered with respect to their position in $[0,1]$.

Let $\{K_n\}_{n=1,2,\dots}$ be an enumeration of the above-described components. Now suppose that K_s precedes K_r and $s \neq r$. We will show that between K_s and K_r there is a K_m and $m \neq s, r$. Observe that, because $\bar{g}|Z$ is a homeomorphism of Z into I, end points of components of E are inaccessible points of the Cantor set. Thus the intersection of E with any interval either is a nowhere dense subset of the inaccessible points of the Cantor set or else contains an interval. Thus if K_s or K_r is disjoint from E, there is such a K_m. Suppose on the other hand that K_s and K_r are contained in E. Then the distance from K_s to K_r is positive because of the nature of the mapping f and the fact that $\bar{g}$ is strictly increasing. Whence again there is such a K_m. From this it follows that the $\{K_n\}$ form a dense linear order. Consequently it only remains for us to show that the $\{K_n\}$ has neither a first nor a last element.

Now Z is the set of all the end points of the components of $f[X]$, whence $\bar{g}[Z]$ is the set of all the end points of the components of E. Moreover $Z \subset (0,1) \subset R$, and $\bar{g}[R] = I \cup S \subset (0,1)$. Thus since $\bar{g}$ is order-preserving, we have $0 < \bar{g}(0) < \bar{g}(z) < \bar{g}(1) < 1$ for any point z in Z. So $\bar{g}(0)$ and $\bar{g}(1)$ are respectively the lower and upper bounds of the set of end points of the components of E, and hence of E. Therefore there are neighborhoods of 0 and 1 disjoint from E. It follows that the $\{K_n\}$ has neither a first nor a last element.

Consequently the $\{K_n\}$ is an η-set.

The $\{K_n^0\}$ is dense on $[0,1]$. For since the Cantor set is nowhere dense, its complement is dense on $[0,1]$. But if L is a component of the complement of the Cantor set, then L is either disjoint from E or contained in E. Whence L is contained in K_n^0 for some n. Thus the $\{K_n^0\}$containing a dense subset of $[0,1]$ is itself dense in $[0,1]$.

We now pause briefly in our proof in order to make a remark. The following technique, which will conclude the proof of the embedding lemma, will be referred to again. Consequently we will prefix what follows by the number 3.2.

3.2. Let $\{L_n\}_{n=1,2,\ldots}$ be an enumeration of the components of the complement in $[0,1]$ of the Cantor set. Then the $\{L_n\}$ ordered by their position in $[0,1]$ is an η-set. We have shown above that the $\{K_n\}$ is also an η-set. Hence there is a similarity mapping h of the $\{L_n\}$ onto the $\{K_n\}$. By change of notation we may assume $h(L_n) = K_n$ for each n. Now for each natural number n we let h_n be an order-preserving homeomorphism of L_n onto K_n^0.

We now define the following function p on $\bigcup_n L_n$ onto $\bigcup_n K_n^0$: for z in L_n, $p(z) = h_n(z)$. Now the extension $\bar{p}$ of p, defined by letting $\bar{p}(x) = \lim_{z \to x} p(z)$ for each x in $[0,1]$, is a continuous map of $[0,1]$ onto $[0,1]$. For since p is monotone its one-sided limits exist for each x in $(\bigcup_n L_n)^- = [0,1]$. If at any point these one-sided limits were distinct, then there would be a jump in the range of p. But we have shown above that the $\bigcup_n K_n^0$ is dense in $[0,1]$.

Observe that $\bar{p}$ is strictly increasing, for p is strictly increasing on a dense subset of $[0,1]$.

Thus $\bar{p}$ is a one-one continuous map on the compact set $[0,1]$, and hence is a homeomorphism.

We will now show that the embedding of X according to our lemma is accomplished.

First of all since f, $\bar{g}$ and $\bar{p}$ are order-preserving homeomorphisms, it is easily seen that there is an order-preserving homeomorphism of X onto $\bar{p}[E]$. Consequently we need only show that the components of $\bar{p}[E]$ are embedded according to our lemma.

Now $\{L_n\}$ is our enumeration of the components in $[0,1]$ of the complement of the Cantor set, and $L_n = \bar{p}[K_n^0]$ for each n. But for each n, K_n is either a component of the complement of the Cantor set disjoint from E or else a one-dimensional component of E. So if K_n is a one-dimensional component of E it clearly follows that the closure in R of L_n is the closure in R of a component of $\bar{p}[E]$. Suppose on the other hand that K_n is a component of the complement of the Cantor set disjoint from E. Since the end points of components of E are inaccessible points of the Cantor set whereas the end points of K_n are accessible points of the Cantor set, the closure in R of K_n is disjoint from E. It follows that the closure in R of L_n is disjoint from $\bar{p}[E]$.

We have shown that the closure in R of each component of the complement of the Cantor set is either the closure in R of a component of $\bar{p}[E]$ or else disjoint from $\bar{p}[E]$. Setting $\bar{p}[E] = \tau[X]$, this concludes the proof of our embedding lemma.

The next lemma gives us a useful equivalent formulation of the embedding lemma. In addition it gives a clearer insight into the manner in which components of $\tau[X]$ are embedded with respect to the Cantor set and its complement.

LEMMA 3.3. *The following conditions on a subset X of $[0,1]$ are equivalent:*

A. *The closure in R of each component of the complement of the Cantor set is either the closure in R of a component of X or else disjoint from X.*

B. *Each component of X either consists of an inaccessible point of the Cantor set or else its interior in R is a component of the complement of the Cantor set.*

Proof. Assume A. Suppose that a component C of X does not consist of an inaccessible point of the Cantor set. It follows that there is a component C' of the complement of the Cantor set whose closure in R meets C. But then it follows from A that the closure in R of C' is the closure in R of C. Therefore it follows that the interior in R of C is C'.

Assume B. Suppose that the closure in R of a component C' of the complement of the Cantor set meets a component C of X. It therefore follows from B that the interior in R of C is C', because the Cantor set is dense in itself. Thus the closure in R of C' is the closure in R of C.

This concludes the proof of the lemma.

Proof of the main theorem. If $\eta(X)$ is finite, then by Corollary 2.8, X is linearly orderable.

Suppose that $\eta(X)$ is infinite.

The proof consists of three parts.

In part I we construct a sequence of homeomorphisms $\{h_u\}$ of X into $[0,1]$ such that for each u:

(1) There are precisely $2u$ points in $\eta(X)$ whose images are not in $\eta(h_u[X])$.

(2) For any point x of X distinct from these $2u$ points, x is in $\eta(X)$ if and only if $h_u(x)$ is in $\eta(h_u[X])$.

In part II we show that the pointwise limit, $\lim_u h_u = h$, exists and is a homeomorphism of X into $[0,1]$.

In part III we show that the topology $\mathscr{T}$ of the limit space $h[X]$ coincides with the interval topology $\mathscr{I}$, that is $\eta(h[X]) = \varnothing$.

Part I-1 of the proof. Let τ be an order-preserving homeomorphism of X into $[0,1]$. Then x is in $\eta(X)$ if and only if $\tau(x)$ is in $\eta(\tau[X])$. Moreover, since X contains no compact, open set and τ is a homeomorphism, $\tau[X]$ contains no compact, open set. Therefore, because of the embedding lemma, we may assume that $X \subset [0,1]$ and that the closure in R of each component of the complement of the Cantor set is either the closure in R of a component of X or else disjoint from X.

Let $\{e_k\}$ be an enumeration of $\eta(X)^- \cap X$ and let $e_{n(k)}$ be the kth point of $\eta(X)$ in this enumeration. For $k = 1,2,3,\cdots$, let J_k denote that component of X containing $e_{n(k)}$ as an end point. Because X contains no compact, open set, the correspondence between $e_{n(k)}$ and J_k is one-one.

Set $R \sim [\bigcup_k J_k]^- = \bigcup_k G_k$, the $\{G_k\}$ being a disjoint collection of possibly empty (the $\{G_k\}$ might be a finite collection of nonempty open intervals), open intervals. Set $V_k = G_k \cap X$ for $k = 1,2,3,\cdots$.

We will proceed by induction as follows. For $t = 1,2,\cdots$, we will let $\Omega(t)$ denote ten assertions (i),$\cdots$,(x). Then, under the assumption that $\Omega(k)$ is valid for $k \leqq t$, we will prove ten lemmas. We will then show, under the assumption that $\Omega(t)$ is valid for $t \leqq u$, that $\Omega(u+1)$ is valid.

For each positive integer t, let $\Omega(t)$ denote the following ten assertions (i),$\cdots$,(x):

(i) Distinct positive integers $m(1),\cdots,m(t), j(1),\cdots,j(t)$, have been determined so that the integer $m(t)$ is the smallest positive integer different from $m(1),\cdots, m(t-1), j(1),\cdots,j(t-1)$. The integer $j(t)$ will be specified in (iv).

(The purpose of $j(t)$ is to associate $e_{n(j(t))}$ with $e_{n(m(t))}$. The "bad" points will be transformed into "good" points in pairs, exactly as we did in the preceding section.)

(ii) Let h_0 be the identity map on X.

A homeomorphism h_t of X into $[0,1]$ has been determined so that $h_t[X]$ is linearly ordered at $h_t(e_{n(m(1))}),\cdots,h_t(e_{n(m(t))}), h_t(e_{n(j(1))}),\cdots,h_t(e_{n(j(t))})$. For all other $h_t(x)$ in $h_t[X]$, $h_t[X]$ is linearly ordered at $h_t(x)$ if and only if X is linearly ordered at x.

(iii) Half-open intervals K_t and L_t satisfying conditions to be specified in (v) and (vii) respectively have been determined.

(We will map K_t, containing $h_{t-1}(e_{n(j(t))})$ as an end point, onto L_t. This will transform the "bad" points $h_{t-1}(e_{n(j(t))})$ and $h_{t-1}(e_{n(m(t))})$ of $h_{t-1}[X]$ into "good" points in the next space.)

(iv) Specification of $j(t)$.

Let B_t be the set of positive integers distinct from $\{m(1),\cdots,m(t),j(1),\cdots,j(t-1)\}$. Let $A_t = \{k: k \in B_t \text{ and } h_{t-1}[J_k] \subset L_i \Leftrightarrow h_{t-1}[J_{m(t)}] \subset L_i, \text{ for } i = 1,\cdots,t-1\}$.

(iv-1) In case $A_t = \varnothing$, let

$$r_t = d\left(h_{t-1}[J_{m(t)}], \bigcup_{k \in B_t} h_{t-1}[J_k]\right),$$

where d is the usual distance function between two sets. Then $j(t)$ is selected from B_t so that

$$d(h_{t-1}[J_{m(t)}], h_{t-1}[J_{j(t)}]) < r_t + \frac{1}{t} .$$

(iv-2) In case $A_t \neq \varnothing$, let

$$r_t = d\left(h_{t-1}[J_{m(t)}], \bigcup_{k \in A} h_{t-1}[J_k]\right),$$

and $F_t = \{h_{t-1}(e_k) : h_{t-1}(e_k) \in L_i \Leftrightarrow h_{t-1}(e_{n(m(t))}) \in L_i$, for $i = 1, \cdots, t-1$, and for $k < n(m(t))\}$. Then $j(t)$ is selected from A_t so that $d(h_{t-1}[J_{m(t)}], h_{t-1}[J_{j(t)}]) < r_t + 1/t$ and $h_{t-1}[J_{j(t)}]$ is on the same side of each two-sided $\mathscr{T}$-limit point of $\eta(h_{t-1}[X])$ that is in F_t as $h_{t-1}[J_{m(t)}]$ is.

(The latter restriction on the selection of $j(t)$ in (iv-2) will guarantee that if x is a two-sided $\mathscr{T}$-limit point of $\eta(X)$, then $h(x)$ is a two-sided $\mathscr{T}$-limit point of $h[X]$. Whence $h[X]$ is linearly ordered at $h(x)$).

(v) Specification of K_t.

Observe that $h_{t-1}[J_k]$ is that component of $h_{t-1}[X]$ containing $h_{t-1}(e_{n(k)})$ as an end point. Denote the other end point of $h_{t-1}[J_k]$ by $c_{t-1,k}$.

Then K_t is a half-open interval containing $h_{t-1}[J_{j(t)}]$ and having $h_{t-1}(e_{n(j(t))})$ as one end point and having as its other end a point p_t with the following nine properties.

(v-1) If $h_{t-1}[J_{j(t)}]$ is a half-open interval, then $p_t = c_{t-1,j(t)}$.

(v-2) $p_t \in h_{t-1}[X]^- \sim h_{t-1}[X]$.

(v-3) If $h_{t-1}[X]$ is not linearly ordered at $h_{t-1}(e_{n(j(t))})$ from below (respectively above), then $p_t > c_{t-1,j(t)}$ (respectively $p_t < c_{t-1,j(t)}$). (Because X contains no compact, open set, it is impossible to have both situations occur simultaneously.)

(v-4) If $c_{t-1,j(t)}$ is contained in an open interval disjoint from

$$\eta(h_{t-1}[X]) \sim \{h_{t-1}(e_{n(j(t))})\},$$

then p_t is in this interval.

(Conditions (v-1) and (v-4) are used to show that the sequence $\{r_u\}$ converges to 0 as u approaches infinity and that h^{-1} is continuous. They make the distinction between those points $c_{0,j(t)}$ of X which are or are not limit points of $\eta(X)$.)

(v-5) $d(p_t, h_{t-1}[J_{j(t)}]) < 1/t$.

(v-6) For $i = 1, \cdots, t$, the subset of $h_{t-1}[X]$ between p_t and $c_{t-1,j(t)}$ is either disjoint from each $h_{t-1}[V_i]$, or else contained in one and only one $h_{t-1}[V_i]$.

(v-7) For $i = 1, \cdots, n(m(t)), n(j(1)), \cdots, n(j(t))$, the point $h_{t-1}(e_i)$ is not between, and distinct from, p_t and $c_{t-1,j(t)}$.

(Conditions (v-6) and (v-7) ensure that the sequence $\{h_u(x)\}$ is eventually constant for each x in X, and hence that h exists and is one-one.)

(v-8) For $i = 1, \cdots, t-1$, the point p_t is a limit point of L_i if and only if $h_{t-1}[J_{j(t)}] \subset L_i$.

(v-9) For $i = 1, \cdots, t-1$, the half-open interval K_i is separated from K_t.

(vi) The closure in R of each component of the complement of the Cantor set is either the closure in R of a component of $h_t[X]$ or else disjoint from $h_t[X]$.

(Conditions (v-9) and (vi) are used to show that the sequence $\{h_u\}$ is uniformly convergent, and hence that h is continuous.)

(vii) Specification of the half-open interval L_t.

(vii-1) $L_t \subset \bigcap_{i=0}^{i=t-1} R \sim h_i[X]$.

(vii-2) The end points of L_t are in the Cantor set.

(vii-3) The set of components of the complement of the Cantor set which lie in L_t is either of the same, or reverse, order-type, when ordered by their position, as the set of those in K_t.

(It will turn out that the end points p_t and $h_{t-1}(e_{n(j(t))})$ of K_t are in the Cantor set.)

(vii-4) The end point of L_t in L_t is an inaccessible point of the Cantor set if $J^0_{j(t)} = \varnothing$, and is the end point of a component of the complement of the Cantor set which lies in L_t otherwise.

(vii-5) Length $L_t < 1/t$.

(vii-6): (a) $d(L_t, h_{t-1}[J_{m(t)}]) < 1/t$.

(b) $d(L_t, h_{t-1}[X]) > 0$.

(vii-7) If $h_{t-1}[X]$ is not linearly ordered at $h_{t-1}(e_{n(m(t))})$ from above (respectively below), then L_t has a minimum (respectively maximum) which is the immediate successor (respectively predecessor) of $h_{t-1}(e_{n(m(t))})$ (with respect to the space $h_{t-1}[X]$).

(vii-8) For $i = 1, \cdots, t$, either $L_t \subset L_i$ or else $d(L_t, L_i) > 0$.

(vii-9) For $i = 1, \cdots, t$, we have $d(L_t, K_i) > 0$.

(vii-10) For $i = 1, \cdots, t-1$, if $h_{t-1}[J_m(t)] \subset L_i$, then $L_t \subset L_i$.

(This property, intimately tied to the definition of A_t in (iv), and to (v-8), ultimately ensures that h^{-1} is continuous.)

(viii) A homeomorphism f_t of K_t onto L_t has been determined so that each component of the complement of the Cantor set which lies in K_t is mapped onto a component of the complement of the Cantor set which lies in L_t.

(ix) A homeomorphism g_t of $h_{t-1}[X]$ into $[0,1]$ has been determined so that:

(ix-1) g_t is the identity on $h_{t-1}[X] \sim K_t$.

(ix-2) g_t coincides with f_t on $h_{t-1}[X] \cap K$.

(x) $h_t = g_t \circ h_{t-1}$.

This completes the statement of $\Omega(t)$.

Part I-2 of the proof. We proceed to prove the ten lemmas.

LEMMA 3.4. *Assume $\Omega(k)$ is valid for all $k \leqq t$. Any point of $h_t[X]$ is in one and only one of $h_{t-1}[X] \sim K_t$ or L_t.*

Proof. Let $h_t(x)$ be a point of $h_t[X]$. Now $h_t = g_t \circ h_{t-1}$ (assertion (x)).

Suppose first that $h_{t-1}(x)$ is in $h_{t-1}[X] \sim K_t$. Since g_t is the identity on $h_{t-1}[X] \sim K_t$ (condition (ix-1)), it follows that $h_t(x)$ is in $h_{t-1}[X] \sim K_t$.

Suppose next that $h_{t-1}(x)$ is in $h_{t-1}[X] \cap K_t$. Because f_t is a homeomorphism of K_t onto L_t (assertion (viii)) and g_t coincides with f_t on $h_{t-1}[X] \cap K_t$ (condition (ix-2)), it follows that $h_t(x)$ is in L_t.

Consequently $h_t(x)$ is in $h_{t-1}[X] \sim K_t$ or in L_t. Therefore, since $d(L_t, h_{t-1}[X]) > 0$ (condition (vii-6)-(b)) this concludes the proof of the lemma.

LEMMA 3.5. *Assume that for all $k \leqq t$, $\Omega(k)$ holds. The end point of the half-open interval L_t that belongs to L_t is $h_t(e_{n(j(t))})$.*

Proof. Since f_t is a homeomorphism of K_t onto L_t (assertion (viii)), f_t maps the end point of K_t in K_t onto the end point of L_t in L_t. Now the end point of K_t in K_t is $h_{t-1}(e_{n(j(t))})$(assertion (v)) and g_t coincides with f_t on $h_{t-1}[X] \cap K_t$ (condition (ix-2)). Therefore, since $h_t = g_t \circ h_{t-1}$, it follows that $h_t(e_{n(j(t))})$ is the end point of L_t that is in L_t.

LEMMA 3.6. *Assume that $\Omega(k)$ holds for all $k \leqq t$. The point $h_{t-1}(e_{n(m(t))})$ is in $h_{t-1}[X] \sim K_t$.*

Proof. By assertion (i), $m(t) \neq j(t)$. It follows that the components $h_{t-1}[J_{m(t)}]$ and $h_{t-1}[J_{j(t)}]$ of $h_{t-1}[X]$ are distinct. Therefore $h_{t-1}(e_{n(m(t))}) \neq c_{t-1,j(t)}$ because these points are end points of distinct components, and $h_{t-1}(e_{n(m(t))})$ is in $h_{t-1}[J_{m(t)}]$. Whence it follows from condition (v-7) and the fact that $h_{t-1}(e_{n(m(t))})$ is not in $h_{t-1}[J_{j(t)}]$, that $h_{t-1}(e_{n(m(t))})$ is not in K_t. This concludes the proof of the lemma.

LEMMA 3.7. *If for each $k \leqq t$, $\Omega(k)$ is valid, then $h_{t-1}(e_{n(m(t))}) = h_t(e_{n(m(t))})$.*

Proof. This is a direct consequence of the preceding lemma and assertions (ix-1) and (x).

LEMMA 3.8. *Assume $\Omega(k)$ is valid for all $k \leqq t$. The point $h_t(e_{n(j(t))})$ is the immediate predecessor or the immediate successor of $h_t(e_{n(m(t))})$ in the space $h_t[X]$.*

Proof. Since X contains no compact, open set, it follows from assertion (ii) that $h_{t-1}[X]$ is not linearly ordered at $h_{t-1}(e_{n(m(t))})$ from one and only one side. Therefore, from condition (vii-7) and Lemmas 3.5 and 3.7, the open interval between $h_t(e_{n(m(t))})$ and $h_t(e_{n(j(t))})$ lies in $R \sim L_t$ and in $R \sim h_{t-1}[X]$. Thus, because it is a consequence of Lemma 3.4 that $h_t[X]$ is disjoint from

$$(R \sim L_t) \cap (R \sim h_{t-1}[X]),$$

the proof of the lemma is concluded.

LEMMA 3.9. *Assume that for all $k \leqq t, \Omega(k)$ is valid. For $i = 1, \cdots, k \leqq t$, the following two propositions $P_{k,i}$ and $Q_{k,i}$ hold.*

(1) $P_{k,i}$. *$h_k(e_{n(m(i))})$ is the immediate predecessor or the immediate successor of $h_k(e_{n(j(i))})$ in $h_k[X]$.*

(2) $Q_{k\,i}$. *For* $q = i, i+1, \cdots, k$, *we have*

$$h_k(e_{n(m(i))}) = h_{q-1}(e_{n(m(i))})$$

and

$$h_k(e_{n(j(i))}) = h_q(e_{n(j(i))}).$$

Proof. The proof is by induction on k.

Let s be a positive integer $\leqq t$ such that the lemma is valid for all $k < s$.

Now $P_{s,s}$ and $Q_{s\,s}$ hold by Lemmas 3.8 and 3.7. Therefore, since in case $s = 1$ the proof is complete, we assume that $s > 1$.

We now show that the lemma is valid for $k = s$ by induction downward on i.

Let v be a positive integer such that $P_{s,i}$ and $Q_{s,i}$ are valid for $v < i \leqq s$.

We must show that $P_{s,v}$ and $Q_{s,v}$ are valid.

We first show that $P_{s,v}$ is valid.

By reasoning similar to the demonstration of Lemma 3.7, we easily see that $h_{s-1}(e_{n(m(v))}) = h_s(e_{n(m(v))})$ and $h_{s-1}(e_{n(j(v))}) = h_s(e_{n(j(v))})$. Therefore, since $P_{s-1,v}$ is valid, it follows that the open interval U between $h_s(e_{n(m(v))})$ and $h_s(e_{n(j(v))})$ lies in $R \sim h_{s-1}[X]$. It follows from condition (vii-7) that U also lies in $R \sim L_s$, because $d(L_s\; h_{s-1}[X]) > 0$ (condition (vii-6)-(b)) and $h_{s-1}[X]$ is not linearly ordered at $h_{s-1}(e_{n(m(s))})$ from one and only one side. Therefore we derive from Lemma 3.4 that $h_s[X]$ is disjoint from U. Whence it follows that $P_{s,v}$ is valid.

We next show that $Q_{s,v}$ is valid.

Now $Q_{s-1\,v}$ is valid. So for $q = v, v+1, \cdots, s-1$, we have $h_{s-1}(e_{n(m(v))}) = h_{q-1}(e_{n(m(v))})$ and $h_{s-1}(e_{n(j(v))}) = h_q(e_{n(j(v))})$. Since from the above we also have $h_{s-1}(e_{n(m(v))}) = h_s(e_{n(m(v))})$ and $h_{s-1}(e_{n(j(v))}) = h_s(e_{n(j(v))})$, it follows that $Q_{s,v}$ is true.

Consequently the lemma holds for $k = s$.

This concludes the proof of the lemma.

LEMMA 3.10. *Assume that $\Omega(k)$ is valid for all $k \leqq t$. The sets K_t and $h_{t-1}[X] \sim K_t$ are separated.*

Proof. By assertion (v) the half-open interval K_t contains $h_{t-1}(e_{n(j(t))})$ as one end point and has p_t as its other end point. We see from condition (v-2) that p_t is not in $h_{t-1}[X]$. Thus because $h_{t-1}[X]$ is not linearly ordered at $h_{t-1}(e_{n(j(t))})$ from one and only one side, it follows from (v-3) that K_t and $h_{t-1}[X] \sim K_t$ are separated.

LEMMA 3.11. *Assume that for each $k \leqq t$, $\Omega(k)$ holds. Any point of X is a two-sided $\mathscr{T}$-limit point of X if and only if its corresponding image in $h_t[X]$ is a two-sided $\mathscr{T}$-limit point of $h_t[X]$.*

Proof. It follows from condition (ix-1) that g_t is order-preserving on $h_{t-1}[X] \sim K_t$. Moreover, since f_t is a homeomorphism of K_t onto L_t, it follows from condition (ix-2) that g_t is order-preserving or order-reversing on $h_{t-1}[X] \cap K_t$. Therefore, utilizing the preceding lemma, it is easily seen that our desired conclusion is obtained, since $d(L_t, h_{t-1}[X]) > 0$.

LEMMA 3.12. *For all $k \leqq t$, assume that $\Omega(k)$ is valid. Let J be a component of X. If $h_t[J] \not\subset L_t$, then:*
(1) *$h_t[J]$ and K_t are separated,*
(2) *g_t is the identity on $h_{t-1}[J]$,*
(3) *$d(h_t[J], L_t) > 0$.*

Proof. Since J is a component of X, it follows from Lemma 3.10 that $h_{t-1}[J]$ is contained in one and only one of $h_{t-1}[X] \cap K_t$ or $h_{t-1}[X] \sim K_t$. Therefore, if $h_t[J] \not\subset L_t$ then $h_{t-1}[J] \subset h_{t-1}[X] \sim K_t$ and g_t is the identity on $h_{t-1}[J]$. Since $d(L_t, h_{t-1}[X]) > 0$, the proof of the lemma is concluded.

LEMMA 3.13. *Let $\Omega(k)$ be valid for all $k \leqq t$. Let J be a component of X. For $i = 1, \cdots, t$:*
(1) *$h_t[J]$ and K_i are separated, and*
(2) *if $h_t[J] \not\subset L_i$, then $d(h_t[J], L_i) > 0$.*

Proof. If $h_t[J] \not\subset L_i$ for each i, then we derive the conclusion of the lemma by the preceding lemma.

In the contrary case, select the largest integer v such that $h_t[J] \subset L_v$. Then, for $i = v+1, v+2, \cdots, t$, we derive the conclusion of the lemma, as above. If $i \leqq v$, then $d(L_v, K_i) > 0$ (condition (vii-9)), and it follows from condition (vii-8) that if $h_t[J] \not\subset L_i$, then $d(L_v, L_i) > 0$. Whence the conclusion of the lemma follows.

This completes Part I-2 of the proof.

Part I-3 of the proof. Assume that $\Omega(t)$ is valid for $t \leqq u$. We will select the integer $j(u+1)$ and the half-open intervals K_{u+1} and L_{u+1} so that $\Omega(u+1)$ is valid.

Set $m(u+1)$ equal to the smallest positive integer different from $m(1), \cdots, m(u)$, $j(1), \cdots, j(u)$.

We now select the positive integer $j(u+1)$.

Define B_{u+1} and A_{u+1} as in assertion (iv).

If $A_{u+1} = \varnothing$, define r_{u+1} as in condition (iv-1). Then $j(u+1)$ is simply selected from B_{u+1} as in (iv-1).

If $A_{u+1} \neq \varnothing$, define r_{u+1} and F_{u+1} as in (iv-2). Then since F_{u+1} is a finite set, it is easily seen by means of Lemma 3.13-(2) that we can select an integer $j(u+1)$ from A_{u+1} as in (iv-2).

Proof that assertion (v) **is valid for** $t = u+1$. Observe that $h_u[J_k]$ is that component of $h_u[X]$ containing $h_u(e_{n(k)})$ as an end point. Denote the other end point of $h_u[J_k]$ by $c_{u,k}$.

We proceed to select K_{u+1}.

Now X contains no compact, open set. Therefore it follows from assertion (ii) that $h_u[X]$ is not linearly ordered at $h_u(e_{n(j(u+1))})$ from one and only one side. In addition it is easily verified that each neighborhood of $c_{u,j(u+1)}$ contains a point of $h_u[X]^- \sim h_u[X]$. Whence we can select p_{u+1} so that it satisfies properties (v-1), $\cdots$, (v-5).

Consider property (v-6).

Suppose first that for $t = 1, \cdots, u+1$, the sets $h_u[V_t]$ and $h_u[J_{j(u+1)}]$ are separated. Then we also select p_{u+1} so that for $t = 1, \cdots, u+1$, no point of $h_u[V_t]$ is between p_{u+1} and $c_{u,j(u+1)}$.

Suppose next that for some t, $t = 1, \cdots, u+1$, the sets $h_u[V_t]$ and $h_u[J_{j(u+1)}]$ are not separated. Then it follows from the definition of V_t that there is a neighborhood of $c_{0,j(u+1)}$ meeting X only in points of V_t and $J_{j(u+1)}$. Whence there is a neighborhood of $h_u(c_{0,j(u+1)}) = c_{u,j(u+1)}$ meeting $h_u[X]$ solely in points of $h_u[V_t]$ and $h_u[J_{j(u+1)}]$. So in this case we also select p_{u+1} so that the subset of $h_u[X]$ between $c_{u,j(u+1)}$ and p_{u+1} is contained in $h_u[V_t]$.

Consequently we can select p_{u+1} so that it satisfies properties (v-1), $\cdots$, (v-6).

Since property (v-7) refers to a finite point set, it follows that we can select p_{u+1} so that it satisfies properties (v-1), $\cdots$, (v-7).

Consider properties (v-8) and (v-9).

If $h_u[J_{j(u+1)}] \subset L_t$ for some t, $t = 1, \cdots, u$, it is easily seen from Lemmas 3.5 and 3.9-(2) that if $c_{u,j(u+1)}$ is an end point of L_t, then it is the end point of the open end of L_t. Therefore it follows from Lemma 3.13 that we can select p_{u+1} so that it also satisfies these last two conditions.

Thus we select K_{u+1} satisfying the required properties.

Proof that assertion (vii) is valid for $t = u+1$. By assertion (ii), $h_u[X] \subset [0,1]$. Therefore, since p_{u+1} is in $h_u[X]^- \sim h_u[X]$ and $h_u(e_{n(j(u+1))})$ is the end point of a component of $h_u[X]$, it follows from assertion (vi) that these end points of K_{u+1} are in the Cantor set. Consequently the set of components of the complement of the Cantor set which lie in K_{u+1} is one of the following order-types, when ordered by their position: $1, \eta$, $1+\eta, \eta+1$, or $1+\eta+1$.

Suppose first that for $t = 1, \cdots, u$, the point $h_u(e_{n(m(u+1))})$ is not in L_t. Then from Lemma 3.12-(2) we derive that h_t is the identity on $e_{n(m(u+1))}$ for each t.

Because X contains no compact, open set, it follows that $e_{n(m(u+1))}$ is the end point of a unique component C in $R \sim X$ which is a half-open interval. Therefore it follows from the embedding of X and Lemma 3.3 that either $e_{n(m(u+1))}$ is an inaccessible point of the Cantor set or else the interior in R of $J_{m(u+1)}$ is a component of the complement of the Cantor set. The other end point of C, being in $X^- \sim X$, is in the Cantor set. Consequently the set of components of the complement of the Cantor set which lie in C is one of the following order-types, when ordered by their position: η, $1+\eta$, or $\eta+1$ (the last two order-types imply that $e_{n(m(u+1))}$ is the sup C or inf C, respectively).

We now show that $C \subset \bigcap_{i=0}^{i=u} R \sim h_i[X]$.

Let s be a positive integer such that $C \subset \bigcap_{i=0}^{i=t} R \sim h_i[X]$ for $t < s \leqq u$.

From the above, h_{s-1} is the identity on $e_{n(m(u+1))}$. Moreover $h_{s-1}(e_{n(m(s))})$ and $h_{s-1}(e_{n(m(u+1))})$ are distinct points of $\eta(h_{s-1}[X])$. Therefore from condition (vii-7) and the definition of C, it follows that $C \subset R \sim L_s$, because $C \subset R \sim h_{s-1}[X]$. Whence from Lemma 3.4 we conclude that $C \subset R \sim h_s[X]$.

It follows that $C \subset \bigcap_{i=0}^{i=u} R \sim h_i[X]$.

We proceed to select L_{u+1} in C.

Since X contains no compact, open set, $h_u[X]$ is not linearly ordered at $h_u(e_{n(m+1))})$ from one and only one side.

By selection, $m(u+1) \neq j(u+1)$, and we have selected K_{u+1}. It follows that the proofs of Lemmas 3.6 and 3.10 hold for $t = u+1$. Therefore $d(h_u(e_{n(m(u+1))}), K_{u+1}) > 0$.

By supposition, $h_u(e_{n(m(u+1))})$ is not in L_t, for $t = 1, \cdots, u$. Therefore it follows from Lemma 3.13 that $d(h_u(e_{n(m(u+1))}), K_t) > 0$ and $d(h_u(e_{n(m(u+1))}), L_t) > 0$ for each t.

It is now easily verified from the above that, for this case, C contains an interval L_{u+1} satisfying the required ten properties.

In the contrary case, select the largest integer $v \leqq u$ such that $h_u(e_{n(m(u+1))})$ is in L_v. Then by an argument similar to the above it follows from the definition of v that $h_v(e_{n(m(u+1))}) = h_t(e_{n(m(u+1))})$ for $t = v, v+1, \cdots, u$.

It follows from Lemma 3.5 that $h_v(e_{n(m(u+1))})$ is in L_v^0. Therefore, because X contains no compact, open set, $h_v(e_{n(m(u+1))})$ is the end point of a unique component C in $L_v \cap R \sim h_v[X]$ which is a half-open interval. Consequently we conclude from assertion (vi) and Lemma 3.3 that either $h_v(e_{n(m(u+1))})$ is an inaccessible point of the Cantor set or else the interior in R of $h_v[J_{m(u+1)}]$ is a component of the complement of the Cantor set. The other end point of C, being in $h_v[X]^- \sim h_v[X]$, is in the Cantor set. Whence the set of components of the complement of the Cantor set which lie in C is of order-type η, $1 + \eta$, or $\eta + 1$, when ordered by their position.

An argument similar to the above shows that $C \subset \bigcap_{i=v}^{i=u} R \sim h_i[X]$. Whence, since $C \subset L_v$ and

$$L_v \subset \bigcap_{i=0}^{i=v-1} R \sim h_i[X]$$

(condition vii-1), it follows that

$$C \subset \bigcap_{i=0}^{i=u} R \sim h_i[X].$$

We proceed to select L_{u+1} in C.

As above, $h_u[X]$ is not linearly ordered at $h_u(e_{n(m(u+1))})$ from one and only one side, and $d(h_u(e_{n(m(u+1))}), K_{u+1}) > 0$. Also, as above, $d(h_u(e_{n(m(u+1))}), K_t) > 0$ and $d(h_u(e_{n(m(u+1))}), L_t) > 0$ for $t = v+1, v+2, \cdots, u$.

Moreover it follows from Lemma 3.13-(1) that $d(h_u(e_{n(m(u+1))}), K_t) > 0$ for $t = 1, \cdots, v$. Finally, $L_v \subset L_t$ or $d(L_v, L_t) > 0$ for $t \leqq v$ (condition (vii-8)).

It is now easily verified from the above that C contains an interval L_{u+1} satisfying the required ten properties.

Thus we select L_{u+1} satisfying the required properties.

Assertion (iii)) is now valid for $t = u + 1$, because K_{u+1} and L_{u+1} are selected.

The homeomorphism f_{u+1} of K_{u+1} onto L_{u+1} is defined by a technique similar to the one prefixed by 3.2 in the embedding lemma. Hence assertion (viii).

By assertion (ii), $h_u[X] \subset [0,1]$. By construction, $L_{u+1} \subset [0,1]$ (see condition (vii-2)).

Define the following function g_{u+1} on $h_u[X]$ into $[0,1]$:

(1) g_{u+1} is the identity on $h_u[X] \sim K_{u+1}$.

(2) g_{u+1} coincides with f_{u+1} on $h_u[X] \cap K_{u+1}$.

Because K_{u+1} and L_{u+1} have been selected, it follows that the proof of Lemma 3.10 holds for $t = u + 1$, and $d(L_{u+1}, h_u[X]) > 0$. Therefore the domains (respectively, ranges) of g_{u+1} corresponding to (1) and (2) above are separated. It follows that g_{u+1} is a homeomorphism.

Setting $h_{u+1} = g_{u+1} \circ h_u$, (x) is valid for $t = u + 1$.

Proof that assertion (ii) is valid for $t = u + 1$. Since all assertions other than (ii) and (vi) hold for $t = u + 1$, it follows that the proof of Lemma 3.8 holds for $t = u + 1$. Whence, because X contains no compact, open set, $h_{u+1}[X]$ is linearly ordered at $h_{u+1}(e_{n(m(u+1))})$ and $h_{u+1}(e_{n(j(u+1))})$. Therefore, because p_{u+1} is in $h_u[X]^- \sim h_u[X]$, it is easily verified by Lemma 2.5 that for all other $h_{u+1}(x)$ in $h_{u+1}[X]$, $h_{u+1}(x)$ is in $\eta(h_{u+1}[X])$ if and only if $h_u(x)$ is in $\eta(h_u[X])$.

It follows that assertion (ii) is valid for $t = u + 1$.

Proof that assertion (vi) is valid for $t = u + 1$. Let C be a component of X. Suppose first that C consists of a point x. It follows from assertion (vi) and Lemma 3.3 that $h_u(x)$ is an inaccessible point of the Cantor set. Therefore, in case $h_u(x)$ is in $h_u[X] \sim K_{u+1}$, then, since $h_{u+1}(x) = h_u(x)$, $h_{u+1}(x)$ is an inaccessible point of the Cantor set. In the contrary case, recall that the end points of K_{u+1} and of L_{u+1} lie in the Cantor set, and the set of components of the complement of the Cantor set in L_{u+1} is either of the same, or reverse, order-type as the set of those in K_{u+1}. It therefore follows from conditions (vii-4), (viii) and (ix-2) that $h_{u+1}(x)$ is an inaccessible point of the Cantor set.

Suppose next that C is an interval. Then because $h_u[X] \sim K_{u+1}$ and $h_u[X] \cap K_{u+1}$ are separated, an argument similar to the above yields that the interior in R of $h_{u+1}[C]$ is a component of the complement of the Cantor set.

It follows from Lemma 3.3 that assertion (vi) is valid for $t = u + 1$.

This completes part I of the proof. Namely for all u, integers $m(u)$ and $j(u)$ and half-open intervals K_u and L_u have been selected so that assertions (i) through (x) are satisfied.

Part II-1 of the proof. We proceed to prove several lemmas.

LEMMA 3.14. *The sets* $h_u[J_{m(u+1)}]$, $u = 0, 1, \cdots$, *are disjoint.*

Proof. Let $s < v$ be non-negative integers.

Suppose that $h_s[J_{m(s+1)}] \neq h_v[J_{m(s+1)}]$. Select the largest integer $w < v$ such that $h_w[J_{m(s+1)}] \neq h_v[J_{m(s+1)}]$. It follows that $h_v[J_{m(s+1)}] = h_{w+1}[J_{m(s+1)}] \subset L_{w+1}$. Now $L_{w+1} \subset \bigcap_{i=0}^{i=w} R \sim h_i[X]$. So in particular $h_v[J_{m(s+1)}] \subset R \sim h_s[X]$. But now, since it follows from Lemma 3.9-(2) that $h_v(e_{n(m(s+1))}) = h_s(e_{n(m(s+1))})$, we have a contradiction. Therefore $h_s[J_{m(s+1)}] = h_v[J_{m(s+1)}]$.

It follows that $h_s[J_{m(s+1)}] \cap h_v[J_{m(v+1)}] = \varnothing$.

LEMMA 3.15. *The sequence* $\{r_u\}$ *converges to* 0 *as u approaches infinity.*

Proof. Suppose the contrary. Then there is an $\varepsilon > 0$ and a subsequence $\{r_{u(k)} + 1\}$ such that $r_{u(k)+1} > \varepsilon$ for $k = 1, 2, \cdots$. Whence the sequence $\{h_{u(k)}[J_{m(u(k)+1)}]\}$, being a bounded, infinite and (by the preceding lemma) disjoint collection, contains a subsequence which converges to a point x in $[0,1]$. Therefore by a change of notation we may without loss of generality assume that the sequence $\{h_{u(k)}[J_{m(u(k)+1)}]\}$ converges to x as k approaches infinity. Thus there is a fixed integer k such that $1/u(k) < \varepsilon$ and $h_{u(k+s)}[J_{m(u(k+s)+1)}]$ is contained in the $\varepsilon/2$ neighborhood of x for $s = 0, 1, \cdots$.

Suppose first that $h_{u(k)}[J_{m(u(k+s)+1)}] = h_{u(k+s)}[J_{m(u(k+s)+1)}]$ for $s = 0, 1, \cdots$. Then (since our collection is infinite) we select a pair of non-negative integers $p < q$ such that for $i = 1, \cdots, u(k)$, we have $h_{u(k)}[J_{m(u(k+p)+1)}]$ is in L_i if and only if $h_{u(k)}[J_{m(u(k+q)+1)}]$ is in L_i.

Now suppose that $h_{u(k+p)}[J_{m(u(k+q)+1)}] \neq h_{u(k+q)}[J_{m(u(k+q)+1)}]$. Select the largest integer $v < u(k+q)$ such that $h_v[J_{m(u(k+q)+1)}] \neq h_{u(k+q)}[J_{m(u(k+q)+1)}]$. It follows that $h_{u(k+q)}[J_{m(u(k+q)+1)}] \subset L_{v+1}$. Now $L_{v+1} \subset \bigcap_{j=0}^{j=v} R \sim h_j[X]$. So in particular $h_{u(k+q)}[J_{m(u(k+q)+1)}] \subset R \sim h_{u(k)}[X]$. But, since $h_{u(k+q)}[J_{m(u(k+q)+1)}] = h_{u(k)}[J_{m(u(k+q)+1)}]$, we now have a contradiction.

It follows that $h_{u(k+p)}[J_{m(u(k+q)+1)}] = h_{u(k)}[J_{m(u(k+q)+1)}]$.

Now for $i = u(k) + 1, u(k) + 2, \cdots, u(k+p)$, we have $L_i \subset \bigcap_{j=0}^{j=i-1} R \sim h_j[X]$. So in particular $L_i \subset R \sim h_{u(k)}[X]$ for each such i.

Consequently it follows from the above that $h_{u(k+p)}[J_{m(u(k+p)+1)}]$ is in L_i if and only if $h_{u(k+p)}[J_{m(u(k+q)+1)}]$ is in L_i, for $i = 1, 2, \cdots, u(k+p)$ (since $h_{u(k+p)}[J_{m(u(k+p)+1)}] = h_{u(k)}[J_{m(u(k+p)+1)}]$). Whence $m(u(k+q)+1)$ is in $A_{u(k+p)+1}$. It follows that in this case $r_{u(k+p)+1} < \varepsilon$, which is a contradiction.

In the contrary case, select a non-negative integer p such that $h_{u(k)}[J_{m(u(k+p)+1)}] \neq h_{u(k+p)}[J_{m(u(k+p)+1)}]$.

Choose the largest integer $v < u(k+p)$ such that

$$h_v[J_{m(u(k+p)+1)}] \neq h_{u(k+p)}[J_{m(u(k+p)+1)}].$$

(Recall that for each $u, u = 0, 1, \cdots$, there are precisely two points $h_u(e_{n(m(u+1))})$ and $h_u(e_{n(j(u+1))})$ such that $h_u[X]$ is not linearly ordered at these points, whereas $h_{u+1}[X]$ is linearly ordered at $h_{u+1}(e_{n(m(u+1))})$ and at $h_{u+1}(e_{n(j(u+1))})$. Moreover for any point distinct from the above-named points, $h_{u+1}[X]$ is linearly ordered at $h_{u+1}(x)$ if and only if $h_u[X]$ is linearly ordered at $h_u(x)$. Furthermore, it follows from conditions (v-1) and (v-4) on the point p_{u+1} that the transformation g_{u+1} moves either precisely one point at which $h_u[X]$ is not linearly ordered, namely the point $h_u(e_{n(j(u+1))})$ in $h_u[J_{j(u+1)}]$, or else infinitely many points at which $h_u[X]$ is not linearly ordered.)

It follows that g_{v+1} moves infinitely many points at which $h_v[X]$ is not linearly ordered. Therefore the end point $c_{v,j(v+1)} = h_v(c_{0,j(v+1)})$ in $h_v[J_{j(v+1)}]$ is a limit point of points (in $h_v[X]$) at which $h_v[X]$ is not linearly ordered. Whence we conclude that the end point $h_{u(k+p)}(c_{0\ j(v+1)})$ in $h_{u(k+p)}[J_{j(v+1)}]$ is a limit point of corresponding points (in $h_{u(k+p)}[X]$) at which $h_{u(k+p)}[X]$ is not linearly ordered.

As in the proof of Lemma 3.14 we see that $h_{v+1}[J_{j(v+1)}] = h_{u(k+p)}[J_{j(v+1)}]$. It follows that $h_{u(k+p)}[J_{j(v+1)}] \subset L_{v+1}$.

By Lemmas 3.5 and 3.9-(2), the end point of L_{v+1} in L_{v+1} is $h_{v+1}(e_{n(j(v+1))}) = h_{u(k+p)}(e_{n(j(v+1))})$. It follows that $h_{u(k+p)}(c_{0,j(v+1)})$ is either an interior point of L_{v+1} or equals $h_{u(k+p)}(e_{n(j(v+1))})$.

It follows from Lemmas 3.6 and 3.9-(2) that $h_{u(k+p)}(e_{n(m(v+1))})$ is not in L_{v+1}. By Lemma 3.9-(1), $h_{u(k+p)}(e_{n(j(v+1))})$ is the immediate predecessor or the immediate successor of $h_{u(k+p)}(e_{n(m(v+1))})$ with respect to the space $h_{u(k+p)}[X]$.

As in the previous case, we have $L_i \subset R \sim h_{v+1}[X]$ for $i = v+2, v+3, \cdots, u(k+p)$. Also, $h_{u(k+p)}[J_{m(u(k+p)+1)}] = h_{v+1}[J_{m(u(k+p)+1)}]$ and is contained in L_{v+1}.

It follows from the above that $h_{u(k+p)}[J_{m(u(k+p)+1)}]$ is in L_i if and only if $h_{u(k+p)}[J_{j(v+1)}]$ is in L_i, for $i = 1,2,\cdots,u(k+p)$. Moreover, for each such i, it follows from Lemma 3.13-(2) that if $h_{u(k+p)}(c_{0,j(v+1)})$ is not in L_i, then $d(h_{u(k+p)}(c_{0,j(v+1)}), L_i) > 0$. Therefore we also conclude from the above that there is an arbitrarily large integer j in $A_{u(k+p)+1}$.

We now have $h_{u(k+p)}[J_j]$ and $h_{u(k+p)}[J_{m(u(k+p)+1)}]$ are contained in L_{v+1}, and length $L_{v+1} < 1/(v+1) < 1/u(k) < \varepsilon$. Whence again $r_{u(k+p)+1} < \varepsilon$, which is impossible.

This concludes the proof of the lemma.

LEMMA 3.16. *As u approaches infinity, $d(K_{u+1}, L_{u+1})$ converges to* 0.

Proof. From the preceding lemma it follows that as u approaches infinity $d(h_u[J_{j(u+1)}], h_u[J_{m(u+1)}])$ and hence, $d(K_{u+1}, h_u[J_{m(u+1)}])$, converges to 0.

It is clear that $d(h_u[J_{m(u+1)}], L_{u+1})$ converges to 0, and it follows from Lemma 3.14 that length $h_u[J_{m(u+1)}]$ converges to 0, as u approaches infinity.

The conclusion of the lemma follows.

This completes part II-1 of the proof.

Part II-2 of the proof. We will show that the pointwise limit, $\lim_u h_u = h$, exists and is a continuous function of X into $[0,1]$.

Recall that for $u = 0,1,\cdots$, the end points of K_{u+1} and of L_{u+1} are in the Cantor set.

(We make the following observation: Let the components of the complement of the Cantor set be ordered according to their position. Then if a component C of the complement of the Cantor set has length $1/3^u$, it follows that there are two other components of the complement of the Cantor set, one above C and one below C, each of which is at a distance $1/3^u$ from C and each of which has length $> 1/3^u$.)

Let $\varepsilon > 0$ be given.

Choose a natural number v such that $1/3^v < \varepsilon$. Then there is an integer N_v such that $u \geqq N_v$ implies $d(K_{u+1}, L_{u+1}) < 1/3^v$, length $L_{u+1} < 1/3^v$ and length $K_{u+1} < 1/3^v$ (the $\{K_{u+1}\}$ is a bounded, disjoint, infinite collection). Therefore, utilizing the above observation, it is easily verified that for $u \geqq N_v$, $d(h_u, h) \leqq 1/3^v < \varepsilon$.

It follows that the sequence of homeomorphisms $\{h_u\}$ of X into $[0,1]$ is uniformly convergent. Whence h exists and is a continuous function of X into $[0,1]$.

Part II-3 of the proof. We will show that h is one-one on X into $[0, 1]$.

LEMMA 3.17. *If for some non-negative integer w, the point x is in $J_{m(w+1)}$, then $h_w(x) = h_u(x)$ for $u = w, w+1, \cdots$.*

Proof. Suppose the contrary.

Select the least integer $v > w$ such that $h_w(x) \neq h_v(x)$. It follows that $h_v[J_{m(w+1)}] \subset L_v \subset \bigcap_{i=0}^{i=v-1} R \sim h_i[X]$. Whence $h_v[J_{m(w+1)}] \subset R \sim h_w[X]$, contradicting the result $h_v(e_{n(m(w+1))}) = h_w(e_{n(m(w+1))})$ of Lemma 3.9-(2).

LEMMA 3.18. *If for some non-negative integer w, the point x is in $J_{j(w+1)}$, then $h_{w+1}(x) = h_u(x)$ for $u = w+1, w+2, \cdots$.*

The proof is similar to the preceding proof.

(Recall that $\{e_k\}$ is an enumeration of $\eta(X)^- \cap X$, and that $e_{n(k)}$ is the kth point of $\eta(X)$ in this enumeration.)

LEMMA 3.19. *If for some positive integer r, the point e_r is not in $\bigcup_k J_k$, and w is the least integer such that $r < n(m(w+1))$, then $h_w(e_r) = h_u(e_r)$ for $u = w, w+1, \cdots$.*

Proof. It is easily seen that if the conclusion were false, this would contradict condition (v-7).

(Recall that G_k is a maximal open interval in $R \sim [\bigcup_k J_k]^-$ and that $V_k = G_k \cap X$.)

LEMMA 3.20. *Suppose that for some positive integer w, the point x is in V_w.*

(1) *If V_w is separated from $J_{j(u+1)}$ for each non-negative integer u, then $h_w(x) = h_u(x)$ for $u = w-1, w, \cdots$.*

(2) *In the contrary case, since at most two non-negative integers $v \leqq r$ exist such that V_w is not separated from $J_{j(v+1)}$ and $J_{j(r+1}$, we have:*

(a) *If* $w \leqq r+1$, *then* $h_{r+1}(x) = h_u(x)$ *for* $u = r+1, r+2, \cdots$.

(b) *If* $w > r+1$, *then* $h_{w-1}(x) = h_u(x)$ *for* $u = w-1, w, \cdots$.

Proof of (1). For $u = 0, 1, \cdots, h_u[V_w]$ and $h_u[J_{j(u+1)}]$ are separated. Therefore, since X contains no compact, open set, it is easily seen that the conclusion of the lemma follows from condition (v-6).

The proof of (2) is similarly easily obtained.

It follows from Lemmas 3.17-3.20 that the sequence $\{h_u(x)\}$ is eventually constant for each x in X. Therefore h is one-one on X into $[0,1]$.

This concludes part II-3 of the proof.

Part II-4 of the proof. We will show that h^{-1} is continuous on $h[X]$ onto X.

Let $h(x)$ be a point of $h[X]$. It follows from the above that there is a least integer v such that $h(x) = h_{v+s}(x)$ for $s = 0, 1, \cdots$.

Suppose that h^{-1} is not continuous at $h(x)$. It follows that there is a sequence $\{h(x_k)\}$ in $h[X]$ and a $\delta > 0$ such that $\{h(x_k)\}$ converges to $h(x)$, whereas $d(h_v(x_k), h_v(x)) \geqq \delta$ for $k = 1, 2, \cdots$.

Now $L_{v+s} \subset \bigcap_{j=0}^{j=v+s-1} R \sim h_j[X]$ and so $L_{v+s} \subset R \sim h_v[X]$ for $s = 1, 2, \cdots$. Therefore $h_v(x)$ is not in L_{v+s} for $s = 1, 2, \cdots$. Whence from Lemma 3.13-(2) it follows that $d(L_{v+s}, h(x)) > 0$ for $s = 1, 2, \cdots$ (since $h(x) = h_{v+s}(x)$).

But the sequence $\{h_u\}$ converges uniformly to h. It follows that there is a (finite or infinite) sequence $v = u(0) < u(1) < \cdots < u(r) < \cdots$, such that for each positive and integral value of r we have $h_{u(r)}$ is the first transformation for which $h_{u(r-1)}(x_{r,k}) \neq h_{u(r)}(x_{r,k})$ for each point $x_{r,k}$ of a subsequence $\{x_{r,k}\}$ of the sequence $\{x_{r-1,k}\}$ (where $\{x_{0,k}\}$ is the sequence $\{x_k\}$). Whence it follows that for each such r, the point $h_{u(r)}(x_k)$ is in $L_{u(r)}$ for infinitely many k. Therefore, since $d(L_{v+s}, h(x)) > 0$ for $s = 1, 2, \cdots$, the sequence $u(0) < u(1) < \cdots < u(r) < \cdots$, is infinite because the sequence $\{h(x_k)\}$ converges to $h(x)$ and the sequence $\{h_u\}$ converges uniformly to h. So $d(L_{u(r)}, h(x))$ converges to 0 as r approaches infinity.

Suppose first that $h_{u(r)-1}[J_{m(u(r))}] \subset L_{u(r-1)}$ for $r = 2, 3, \cdots$. Then $L_{u(r)} \subset L_{u(r-1)}$ (condition (vii-10)) for $r = 2, 3, \cdots$. But $d(L_{u(1)}, h(x)) > 0$. This contradicts our assumption that the sequence $\{h(x_k)\}$ converges to $h(x)$.

In the contrary case, let r be the first natural number $\geqq 2$ such that $h_{u(r)-1}[J_{m(u(r))}] \not\subset L_{u(r-1)}$. Then as in the proof of Lemma 3.15 it is easily verified that because of condition (v-8), $h_{u(r)-1}[J_{m(u(r))}] \not\subset \bigcup_{i=1}^{i=u(r)-1} L_i$, since $h_{u(r)}$ moves a point of $L_{u(r-1)}$. Therefore it readily follows that

$$h_{u(r)-1}[J_t] \subset \bigcup_{i=1}^{i=u(r)-1} L_i$$

for $t > m(u(r))$ and $t \neq j(1), j(2), \cdots, j(u(r)-1)$, because $h_{u(r)}$ moves a point of $L_{u(r-1)}$. Thus $h_{u(r+1)-1}[J_{m(u(r+1))}] \subset L_{u(r)}$ since $h_{u(r+1)}$ moves a point of $L_{u(r)}$. Whence $L_{u(r+1)} \subset L_{u(r)}$. Similarly $L_{u(r+1+s)} \subset L_{u(r+s)}$ for $s = 1, 2, \cdots$. But

$d(L_{u(r)}, h(x)) > 0$. This contradicts our assumption that the sequence $\{h(x_k)\}$ converges to $h(x)$.

It follows that the sequence $\{x_k\}$ converges to x. Therefore h^{-1} is continuous on $h[X]$ onto X.

Since it follows from parts II-2, II-3 and the above that h is a homeomorphism of X into $[0,1]$, this concludes part II of the proof.

Part III of the proof. We will show that $\eta(h[X]) = \varnothing$.

LEMMA 3.21. *If for some positive integer w the point x of V_w is a two-sided $\mathscr{T}$-limit point of V_w, then $h(x)$ is a two-sided $\mathscr{T}$-limit point of $h[X]$.*

Proof. From the definition of V_w, the point x is in the open interval G_w disjoint from $X \sim V_w$. Therefore we conclude from Lemma 3.11 that $h_u(x)$ is a two-sided $\mathscr{T}$-limit point solely of $h_u[V_w]$ for $u = 0, 1, \cdots$. Whence the conclusion of the lemma follows from Lemma 3.20.

LEMMA 3.22. *If for some positive integers w and r, the point x is in $(V_w^- \sim V_w) \cap J_r$ and J_r is an interval, then $h(x)$ is a two-sided $\mathscr{T}$-limit point of $h[X]$.*

Proof. Obvious.

LEMMA 3.23. *If for some positive integer r, the point x of J_r is not an end point of J_r, then $h(x)$ is a two-sided $\mathscr{T}$-limit point of $h[X]$.*

Proof. Obvious.

LEMMA 3.24. *If for some positive integer r, the point x is in $(\eta(X)^- \sim \eta(X)) \cap J_r$ and J_r is an interval, then $h(x)$ is a two-sided $\mathscr{T}$-limit point of $h[X]$.*

Proof. Obvious.

LEMMA 3.25. *If for some positive integer w, the point x of X is in $(\eta(X)^- \sim \eta(X)) \cap (V_w^- \sim V_w)$, then $h(x)$ is a two-sided $\mathscr{T}$-limit point of $h[X]$.*

Proof. It follows from Lemma 3.11 that $h_{u(x)}$ is a two-sided $\mathscr{T}$-limit point of $h_u[X]$ for $u = 0,1,\cdots$. Whence it is easily seen from the definition of g_u for $u = 1,2,\cdots$, that $h_u(x)$ is a $\mathscr{T}$-limit point solely of $h_u[V_w]$ on one side of $h_u(x)$ and is a $\mathscr{T}$-limit point of $\eta(h_u[X])$ on the other side of $h_u(x)$. Therefore, since it follows from Lemmas 3.19 and 3.20 that $h(x)$ has the analogous property with respect to $h[V_w]$ and $h[\eta(X)]$, this concludes the proof of the lemma.

LEMMA 3.26. *If the point x of X is a two-sided $\mathscr{T}$-limit point of $\eta(X)$, then $h(x)$ is a two-sided $\mathscr{T}$-limit point of $h[X]$.*

Proof. Suppose the contrary.

It follows from Lemma 3.11 that $h_u(x)$ is a two-sided $\mathscr{T}$-limit point of $h_u[X]$ for $u = 0, 1, \cdots$. Whence it is easily seen from the definition of g_u for $u = 1,2,\cdots$, that $h_u(x)$ is a two-sided $\mathscr{T}$-limit point of $\eta(h_u[X])$.

Let v be the least integer such that $h(x) = h_{v+s}(x)$ for $s = 0,1,\cdots$. Then as in part II-4, $d(L_{v+s}, h(x)) > 0$ for $s = 1,2,\cdots$. Moreover for $u = 1,2,\cdots,v$, by Lemma 3.13-(2), if $h(x)$ is not in L_u, then $d(L_u, h(x)) > 0$. Therefore, since $h_v(x)$ is a two-sided $\mathscr{T}$-limit point of $h_v[X]$, either $h(x)$ is in the interior of L_i for some largest integer $i \leqq v$, or else $d(h(x), L_u) > 0$ for $u = 1,2,\cdots$.

It follows that for some integer $w \geqq v$, we selected $j(w+1)$ from A_{w+1} so that $h_w[J_{j(w+1)}]$ and $h_{w+1}[J_{j(w+1)}]$ are on opposite sides of the point $h_w(x)$ of F_{w+1}. This is impossible.

Therefore, since $h(x)$ is a two-sided $\mathscr{T}$-limit point of $h[\eta(X)]$, this concludes the proof of the lemma.

Lemma 3.27. *For $u = 1,2,\cdots$, the point $h(e_{n(j(u))})$ is the immediate predecessor or the immediate successor of $h(e_{n(m(u))})$.*

The conclusion follows from Lemma 3.9, 3.17 and 3.18.

Lemma 3.28. *If the point x of X has an immediate predecessor or an immediate successor y in X, then $h(x)$ is the immediate predecessor or the immediate successor of $h(y)$.*

Proof. For each non-negative integer u, the end point p_{u+1} of K_{u+1} is in $h_u[X]^- \sim h_u[X]$. Therefore for $u = 0,1,\cdots$, it follows from the definition of g_{u+1} that $h_u(x)$ is in K_{u+1} if and only if $h_u(y)$ is in K_{u+1}, and that $h_u(x)$ is the immediate predecessor or the immediate successor of $h_u(y)$.

Since the sequences $\{h_u(x)\}$ and $\{h_u(y)\}$ are eventually constant, the conclusion of the lemma follows.

Lemma 3.29. *If x is the minimum (respectively, maximum) of X, then $h(x)$ is the minimum (respectively, maximum) of $h[X]$.*

Proof. Since it follows from the choice of the end points of K_{u+1} and L_{u+1} for $u = 0,1,\cdots$, that $h_u(x)$ is never in K_{u+1} and that L_{u+1} is always above (respectively, below) $h_u(x)$, the conclusion of the lemma is easily obtained.

Lemma 3.30. $\eta(h[X]) = \varnothing$.

Proof. Suppose first that x is a two-sided $\mathscr{T}$-limit point of X.

If x is not a $\mathscr{T}$-limit point of $\eta(X)$, then it follows from Lemmas 3.21, 3.22 and 3.23 that $h[X]$ is linearly ordered at $h(x)$.

If x is a $\mathscr{T}$-limit point of $\eta(X)$, then it follows from Lemmas 3.24, 3.25 and 3.26 that $h[X]$ is linearly ordered at $h(x)$.

Suppose next that x is a one-sided $\mathscr{T}$-limit point of X.

Since X contains no compact, open set, it follows from Lemmas 3.27, 3.28 and 3.29 that $h[X]$ is linearly ordered at $h(x)$.

The limit space $h[X]$ is a linearly ordered space.

This concludes the proof of the main theorem.

REMARK 3.31. The homeomorphism h of X into $[0,1]$ has the following properties.

(1) $h[X]$ is a linearly ordered space.

(2) For each x in X, the point x is a two-sided $\mathscr{T}$-limit point of X if and only if $h(x)$ is a two-sided $\mathscr{T}$-limit point of $h[X]$.

(3) Any point of $J_{m(1)}$ is a fixed point of h.

(4) Since the sequence $\{h_u(x)\}$ is eventually constant for each x in X, each component of $h[X]$ either consists of an inaccessible point of the Cantor set, or else its interior in R is a component of the complement of the Cantor set.

REMARK 3.32. The homeomorphism $h \circ \tau$ of X into $[0,1]$ retains all the preceding properties of h except property (3).

COROLLARY 3.33. *If X contains no compact, open set and it has only countably many components, then X is linearly orderable.*

COROLLARY 3.34. *If X contains no isolated interval closed in R and its components are intervals, then X is linearly orderable.*

COROLLARY 3.35. *If X is a union of open or half-open intervals, then X is linearly orderable.*

EXAMPLE. 3.36. Let $X_3 = X_2 \cup [1,2)$, where X_2 is the space of Example 1.10. Let y be the maximum of a compact subset Y of X_3. Then $y \neq 2$. Thus y is a limit point of points of X_3 above y. Consequently Y is not open in X_3. Whence X_3 contains no compact, open set.

Thus X_3 is not zero-dimensional and contains no compact, open set. But $\eta(X_3)^- \cap X_3$ is uncountable. However it is easily verified, by the technique used in I. L. Lynn [2], that X_3 is linearly orderable.

CONJECTURE 3.37. If a subset X of R contains no compact, open set, then X is linearly orderable.

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