STRICTLY SINGULAR OPERATORS AND THEIR CONJUGATES(1)

BY

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In [7] Kato introduces the concept of a strictly singular operator, a generalization of the concept of a compact operator. He proves that for X and Y B-spaces the strictly singular operators form a closed subspace in the space of all bounded linear operators from X to Y and that the product of a strictly singular operator with a bounded operator is strictly singular, so when $X = Y$ they form a two-sided ideal. He further shows that the Riesz-Schauder theorem holds for the spectrum of a strictly singular operator on a B-space.

It is also shown in [7] that if X and Y are Hilbert spaces then all the strictly singular operators from X to Y are compact. In the general case, however, the strictly singular operators not only fail to be compact but strict singularity is not preserved under conjugation. In [4], Goldberg and Thorp give an example where T is strictly singular but T' (the conjugate operator) is not and we present below an example where T' is strictly singular but T is not. One of our main purposes is to relate the strict singularity of T to that of T' under certain conditions on the domain and range spaces.

In the following, X and Y denote normed linear spaces unless otherwise specified; also, an operator, map, etc. always means a bounded linear operator.

1. In this section we give examples of pairs of spaces (X, Y) for which every map from X to Y is strictly singular and an example in which T' is strictly singular but T is not.

DEFINITION 1.1. A map from X to Y is said to be *strictly singular* if whenever the restriction of T to a subspace M of X has a continuous inverse, M is finite dimensional.

If the words "a subspace M" are replaced in the definition by the words "a closed subspace M", the new definition is equivalent to the original.

As is well-known [3, p. 515, problem 30], a compact operator from X to Y maps weakly convergent sequences onto norm convergent sequences. The converse is true if X is reflexive. The converse need not be true when Y is reflexive. To see this let T be the identity map injecting l into l^2. Then T is obviously not compact.

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Yet if x_n converges weakly to x then x_n converges to x in norm in l [3, p. 296]. But then Tx_n converges in norm to Tx by the continuity of T. Even though the T above is not compact it is shown in [4] that this T is strictly singular. This suggests the following:

THEOREM 1.2. *Let T map from a B-space X into a reflexive space Y. If T maps weakly convergent sequences into norm convergent sequences, then T is strictly singular.*

Proof. Suppose that the restriction T_0 of T to some closed subspace M of X is an isomorphism. Since M is isomorphic to a subspace of the reflexive space Y, M is reflexive. Therefore the closed unit sphere $S(M)$ of M is weakly sequentially compact. The hypothesis implies that $T_0S(M)$ is norm sequentially compact and since T_0 has a bounded inverse, $S(M)$ is norm sequentially compact. Hence M is finite dimensional.

A possible converse to Theorem 1.2, that every strictly singular operator maps weakly convergent sequences into norm convergent sequences, is false. In [4] an example is given where T maps a reflexive space X into a reflexive space Y and is strictly singular but not compact. If T mapped weakly convergent sequences into norm convergent sequences it would be compact, since X is reflexive.

The T in Theorem 1.2 is weakly compact because it maps into a reflexive space. There are many spaces X for which T weakly compact with domain X implies that T maps weakly convergent sequences into norm convergent sequences. For example [3, pp. 494, 497, 508 and 511]:

(1) $C(S)$, S a compact Hausdorff space.

(2) $B(S)$, S a set.

(3) $L_\infty(S,\Sigma,\mu)$ and $L(S,\Sigma,\mu)$, (S,Σ,μ) a positive measure space.

(4) $rca(S)$ and $ba(S)$, which are isometrically isomorphic to the conjugate spaces of, respectively, $C(S)$ and $B(S)$.

For the last class of spaces in our list we recall that a closed subspace M of X is *complemented* (in X) if X can be written as the direct sum $X = M \oplus N$, where N is closed subspace of X.

(5) Any complemented subspace M of the spaces (1)–(4) above also has the property that a map from it to a reflexive space is strictly singular.

To see that (5) holds, let T map M to a reflexive space Y and let P be a continuous projection of X onto M. Then by Theorem 1.2, $T \circ P$ is strictly singular and thus T is strictly singular.

We now show that T' may be strictly singular even though T is not.

EXAMPLE 1.3. Let T be an isometric isomorphism of l^2 into $C[0,1]$ (or $B(S)$, S an infinite set). Since T is an isomorphism it is not strictly singular, but T' is a map from $(C[0,1])'$ (or $B(S)'$) into a reflexive space and is thus strictly singular.

Suppose we have a map T from X to Y. Let $Z = X \oplus Y$ be normed by $\|(x,y)\| = \max(\|x\|, \|y\|)$. Define $L: Z \to Z$ by $L((x,y)) = (0, Tx)$. This device

is due to Phillips and is used in the unpublished note referred to in [4]. The map L is strictly singular (compact) iff T is strictly singular (compact) and the same is true of the conjugate maps; this is best seen by considering L as the composition of the natural map from Z to X with T with the natural map from Y to Z. The properties of the conjugate follow from the equation $L'((x',y')) = (T'y',0)$. Hence, using the example given in [4] and the example above we can construct endomorphisms in which strict singularity is not inherited. Also, using the example given in [4] of T mapping a reflexive space X to a reflexive space Y which is strictly singular but not compact we can construct as above an endomorphism on a reflexive space which is strictly singular but is not compact.

2. In this section we discuss the connection between the strict singularity of T and that of T'.

DEFINITION 2.1. A normed linear space X is *subprojective* if, given any closed infinite dimensional subspace M of X, there exists a closed infinite dimensional subspace N contained in M and a continuous projection of X onto N.

THEOREM 2.2. *Let X be a B-space and Y be subprojective. If $T' : Y' \to X'$ is strictly singular then so is $T : X \to Y$.*

Proof. Suppose that T is not strictly singular. Then there is an infinite dimensional closed subspace M of X such that T restricted to M has a bounded inverse. Since Y is subprojective, there is a closed infinite dimensional subspace K of TM and a continuous projection P of Y onto K. Note that K can be written as TN, where N is a closed infinite dimensional subspace of M. Since P maps Y onto K, P' is a 1-1 continuous map of K' onto an infinite dimensional subspace R of Y' [12, p. 237]. We will show that T' restricted to R has a bounded inverse and is therefore not strictly singular. For any r in R,

$$\| T'r \| = \sup_{0 \neq x \in X} \frac{\| T'r(x) \|}{\| x \|} \geqq \sup_{0 \neq x \in N} \frac{\| r(Tx) \|}{\| x \|}.$$

Since T restricted to M has a bounded inverse, there exists a positive number a such that $\| Tx \| \geqq a \| x \|$ for all x in $N \subseteq M$. In addition, since r is a member of $P'K'$ there is a k' in K' such that $P'k' = r$. Combining this information,

$$\| T'r \| \geqq \sup_{0 \neq y \in K} \frac{\| P'k'(y) \|}{\| y \|} a = \sup_{0 \neq y \in K} \frac{\| k'(Py) \|}{\| y \|} a = \| k' \| a.$$

Finally, because $\| P' \| \, \| k' \| \geqq \| P'k' \| = \| r \|$, $\| T'r \| \geqq (a/\| P' \|) \| r \|$. Thus T' has a bounded inverse on the subspace R of Y'.

COROLLARY 2.3. *Suppose X is reflexive and X' is subprojective. Then if $T : X \to Y$ is strictly singular so is T'.*

Proof. In this case $T'' = J_Y \circ T \circ J_X^{-1}$, where J_X and J_Y are the canonical

embeddings of, respectively, X onto X'' and Y into Y''. Then if T is strictly singular so is T'' and by Theorem 2.2 so is T'.

COROLLARY 2.4. *If X is a Hilbert space and if $T: X \to Y$ is strictly singular, so is T'. If Y is a Hilbert space and if $T': Y' \to X'$ is strictly singular, so is T.*

In the example of T strictly singular but T' not, in [4], the range space is a Hilbert space. In our example of T' strictly singular but T not, the domain is a Hilbert space.

3. Subprojective spaces.

LEMMA 3.1. *Closed subspaces of subprojective spaces are subprojective. Any space which is isomorphic to a subprojective space is subprojective.*

Proof. The proof follows directly from the definition.

We shall now consider some specific spaces.

THEOREM 3.2. *The spaces l^p $(1 \leqq p < \infty)$ and c_0 are subprojective.*

Proof. By Lemma 2, p. 214 of [8], l^p is subprojective. Theorem 3, p. 217 of [8] states that if M is a subspace of c_0 which is isomorphic to c_0, then M is complemented in c_0. Then Theorem 1, p. 194 of [1] completes the proof that c_0 (which is isomorphic to c) is subprojective.

COROLLARY 3.3. *The spaces $l^p(S)$ $(1 \leqq p < \infty)$ and $c_0(S)$ are subprojective.*

Proof. These spaces are defined in [2]. The proof follows from Theorem 3.2 using the fact that every separable subspace is contained in a complemented subspace which is isomorphic to, respectively, l^p or c_0.

The space $L_p(S, \Sigma, \mu)$, in the special case where S is $[0,1]$, Σ is the Lebesgue measurable subsets of S, and μ is Lebesgue measure, is denoted by L_p.

THEOREM 3.4. *The spaces L_p $(2 \leqq p < \infty)$ are subprojective and the spaces L_p $(1 < p < 2)$ are not subprojective.*

Proof. For the case $2 \leqq p < \infty$, the subprojectivity of L_p follows directly from Collollary 1, p. 167 and Corollary 2, p. 168 in [6]. Now we consider $1 < p < 2$. If $1 < p \neq q < 2$, then L_p contains a subspace M isomorphic to l^q [5]. If L_p is subprojective, then M contains an infinite dimensional subspace N complemented in L_p. Since N is complemented in M it is isomorphic to l^q [8, Theorem 1, p. 213] and since N is complemented in L_p it is either isomorphic to l^2 or contains a subspace isomorphic to l^p [6, Corollary 3, p. 168]; but neither of the last two alternatives can hold together with the first, since the spaces $l^{p'}$ and $l^{q'}$ $(1 < p' \neq q' > 1)$ have incomparable linear dimension [1, Theorem 7, p. 205].

THEOREM 3.5. *Let X be any of the spaces* (1)–(5) *given in* §1. *If X contains an infinite dimensional reflexive subspace then X is not subprojective.*

Proof. Any projection of X into an infinite dimensional reflexive subspace is strictly singular by Theorem 1.2 and thus has finite dimensional range. The conclusion of the theorem follows.

COROLLARY 3.6. *The spaces $C[0,1]$, L_1 and $B(S)$, for S an infinite set, are not subprojective.*

Proof. There is an isomorphism of l^2 into $C[0,1]$ [1, Theorem 9, p. 185], and thus of l^2 into $B(S)$. Also, by a remark on p. 168 of [6], L_1 contains a subspace isomorphic to l^2.

Let X be a B-space. We write $X \in P$ and say that X is in P if X is complemented in each B-space which contains it. In order that X be in P it is necessary and sufficient that X be isomorphic to a complemented subspace of a $B(S)$. (This was pointed out to me by Henry Cohen and Spencer Dickson.) The necessity is clear. The sufficiency follows from Theorems 2 and 3 of [11] and the fact that $B(S)$ is in P [2, p. 94].

THEOREM 3.7. *Let X be an infinite dimensional space which is in P. Then X is not subprojective.*

Proof. Assume that $X \in P$ and that X is subprojective. Then there is a separable subspace N of X complemented in X. By Theorems 2 and 3, pp. 218–219 of [11], $N \in P$. But Theorem 6, p. 221 of [8] tells us that no separable infinite dimensional space can be in P.

4. **Superprojective spaces.** The codimension of a subspace N of X ($\operatorname{cod}(N)$) is the dimension of the vector space X/N. The codimension of N, as well as the dimension $\dim(N)$, will be a non-negative integer or the symbol ∞.

DEFINITION 4.1. A space X is *superprojective* if, given any closed subspace N with infinite codimension, there exists a closed subspace M containing N where M has infinite codimension and is the image of a continuous projection of X.

For a subset M of X, $M^{\perp} = \{x' \text{ in } X' : x'(x) = 0 \text{ for all } x \text{ in } M\}$. For a subset M of X', ${}^{\perp}M = \{x \text{ in } X : x'(x) = 0 \text{ for all } x' \text{ in } M\}$. We need the following well-known results:

(1) If M is a closed subspace of X, then $X'/M^{\perp}$ is isomorphic to M' and $M^{\perp}$ is isomorphic to $(X/M)'$.

(2) If $X = M \oplus N$, M and N closed subspaces, then $X' = M^{\perp} \oplus N^{\perp}$.

(3) If M is a closed subspace of X, then ${}^{\perp}(M^{\perp}) = M$.

Let X be a B-space and M a subspace of X'. Then M is closed in the relativized weak* topology of X' iff $({}^{\perp}M)^{\perp} = M$ [12, p. 232]. Directly from this theorem we see that:

(4) Let X be a B-space and M a reflexive subspace of X'; then $({}^{\perp}M)^{\perp} = M$.

LEMMA 4.2. *A space isomorphic to a superprojective space is superprojective.*

Proof. Argue from the definition.

THEOREM 4.3. *Let X be one of the spaces* (1)–(5) *in* §1. *Then if X can be mapped onto an infinite dimensional reflexive space, X is not superprojective.*

Proof. Let T map X onto the infinite dimensional reflexive space Y and denote the null manifold of T by $N(T)$. Then $X/N(T)$ is infinite dimensional and reflexive. Assume that X is superprojective. Then there is a closed subspace M of X which contains $N(T)$, has infinite codimension and is complemented in X. Say $X = M \oplus R$. Now the quotient of $X/N(T)$ by $M/N(T)$ is reflexive, since $X/N(T)$ is reflexive, and is isomorphic to X/M. But then by Theorem 1.2, the projection of X onto R is strictly singular and so R is finite dimensional; a contradiction.

COROLLARY 4.4. *The spaces L_1 and $l(S)$ (for S an infinite set) are not superprojective.*

Proof. The space L_1 can be mapped onto any separable B-space [1, p. 245]. The same holds for l and thus for $l(S)$.

THEOREM 4.5. *Let X be a subprojective B-space. Let N be a closed subspace of X' with infinite codimension. Then if N is reflexive (or more generally, if* $\operatorname{cod}({}^{\perp}N)^{\perp} = \infty$) *there is a complemented subspace of X' with infinite codimension which contains N.*

Proof. Note that $\dim({}^{\perp}N) = \operatorname{cod}({}^{\perp}N)^{\perp} = \infty$. So by the subprojectivity of X there exists an infinite dimensional subspace M which is complemented in X and is contained in ${}^{\perp}N$. Then $X' = M^{\perp} \oplus R$ and we have $M^{\perp} \supseteq ({}^{\perp}N)^{\perp} \supseteq N$ and $\operatorname{cod}(M^{\perp}) = \dim(X'/M^{\perp}) = \dim M' = \infty$.

We note that all that we needed was that $({}^{\perp}N)^{\perp}$ have infinite codimension whenever the codimension of N was infinite. To see that this is not true in general consider Jc_0, the canonical image of c_0 in m. The space m/Jc_0 cannot be finite dimensional for then m would be separable. Yet $({}^{\perp}(Jc_0))^{\perp} = m$.

THEOREM 4.6. *Let X be a superprojective B-space and N be a closed infinite dimensional subspace of X'. If N is reflexive (more generally, if N is closed in the relativized weak* topology) then there exists an infinite dimensional complemented subspace contained in N.*

Proof. We note that $\operatorname{cod}({}^{\perp}N) = \dim(X/{}^{\perp}N)' = \dim({}^{\perp}N)^{\perp}$ which is infinite. Then there is a complemented subspace M of infinite codimension which contains ${}^{\perp}N$. Then $M^{\perp}$ is complemented in X', $M^{\perp} \subseteq ({}^{\perp}N)^{\perp} = N$, and $\dim(M^{\perp}) = \dim(X/M)' = \infty$.

COROLLARY 4.7. *Let X be reflexive. Then X is superprojective (subprojective) iff X' is subprojective (superprojective).*

COROLLARY 4.8. *The spaces $l^p(S)$ $(1<p<\infty)$ are superprojective. The spaces L_p are superprojective for $1 < p \leqq 2$ and are not superprojective for $2 < p < \infty$.*

If we consider the spaces c_0 and l we see that X subprojective does not necessarily imply that X' is superprojective. The other various implications are open in the general case, except where counterexamples are provided by the L_p spaces.

Let S be an infinite set. Since $B(S)$ is isomorphic to $(l(S))'$ we see that every infinite dimensional reflexive subspace of $B(S)$ is contained in a complemented subspace of infinite codimension. A. Pelczynski has sent me the following result: The space $B(S)$, for S an infinite set, is not superprojective. Pelczynski shows that a compact Hausdorff space K is not dispersed (see [9]) iff there is a continuous map of $C(K)$ onto l^2. In this case, by Theorem 4.3, $C(K)$ is not superprojective. He then shows that in the representation of $B(S)$ as a $C(K)$, K is not dispersed.

We obtain a mapping theorem by arguing as in Theorem 4.3. The corollary to this theorem is the dual of [1, Theorem 1, p. 194] for the spaces l^p.

THEOREM 4.9. *Let X be a superprojective B-space and let there be a map of X onto an infinite dimensional B-space Y. Then there is a map of Y onto an infinite dimensional complemented subspace of X.*

COROLLARY 4.10. *If there is a map of l^p $(1<p<\infty)$ onto an infinite dimensional B-space Y then there is a map of Y onto l^p.*

Proof. Any infinite dimensional complemented subspace of l^p is isomorphic to l^p [8, Theorem 2, p. 213].

THEOREM 4.11. *Let X be reflexive, Y a superprojective B-space and T a map from X to Y'. Then if T' is strictly singular so is T.*

Proof. Assume that T is not strictly singular; then T is an isomorphism when restricted to an infinite dimensional closed subspace M of X. Then M is reflexive; so TM is reflexive and by Theorem 4.6, it contains an infinite dimensional complemented subspace. As in Theorem 2.2, this is enough to show that T' is not strictly singular.

We remark that in the above corollary we do not know that Y is subprojective; if it were then the corollary would be a special case of Theorem 2.2.

5. **Spaces which contain no infinite dimensional reflexive subspaces.** The following from [4] motivates this section. If a B-space X contains no infinite dimensional reflexive subspaces and Y is reflexive, then any map from X to Y or Y to X is strictly singular.

DEFINITION 5.1. If X contains no infinite dimensional reflexive subspaces then X is *very irreflexive.*

THEOREM 5.2. *Let X be a B-space. There is no map of X onto an infinite dimensional reflexive space iff X' is very irreflexive.*

Proof. If T maps M onto Y, Y reflexive and infinite dimensional, then T' is an isomorphism of Y' with a subspace of X'. Conversely, let N be an infinite dimensional reflexive subspace of X'. Then $(X/{}^{\perp}N)'$ is isomorphic to $({}^{\perp}N)^{\perp} = N$ and is thus reflexive. There is a natural map of X onto the reflexive infinite dimensional space $X/{}^{\perp}N$.

COROLLARY 5.3. *Let X denote one of the spaces* (1)–(5) *of* §1. *Then if X is superprojective, X' is very irreflexive.*

Proof. Theorem 4.3.

It is shown in [4] that $c_0(S)$ and $l(S)$ are very irreflexive. We offer a possibly stronger result which yields this property for $l(S)$.

THEOREM 5.4. *Suppose that X has the property that strong and weak convergence of sequences are the same. Then X is very irreflexive.*

Proof. Let M be a reflexive subspace of X. Then $S(M)$ is weakly sequentially compact and thus norm sequentially compact and so M is finite dimensional.

COROLLARY 5.5. *Let (S,Σ,μ) be a positive measure space in which every point has nonzero measure. Then $L(S,\Sigma,\mu)$ is very irreflexive, or equivalently, $l(S)$ is very irreflexive.*

Proof. Corollary 13, p. 295 of [3]. Note that there is an isometry between such $L(S,\Sigma,\mu)$ and $l(S)$.

6. **Ideals of strictly singular operators.** We denote the ring of all maps on a B-space X by $B(X)$. The closed ideal of all strictly singular operators is denoted by S.

DEFINITION 6.1. An infinite dimensional Banach space X is an *h-space* if each closed infinite dimensional subspace of X contains a complemented subspace isomorphic to X.

An h-space is separable. As we have seen, c_0 and $l^p (1 \leqq p < \infty)$ are h-spaces. Any B-space isomorphic to an h-space is an h-space; in particular, c is an h-space. We now generalize a result [10, pp. 24–25] from Hilbert spaces to h-spaces.

THEOREM 6.2. *Let X be an h-space. Then S is the greatest ideal in $B(X)$.*

Proof. We will show that any ideal which contains elements not in S must coincide with $B(X)$. Let T be a member of $B(X)$ which is not in S.Then there is a closed infinite dimensional subspace N such that the restriction of T to this subspace, T_1, is an isomorphism. The subspace $T_1 N$ contains, by hypothesis, a subspace R which is isomorphic to X and complemented in X. We let $A : R \to X$ be an isomorphism of R onto X and $P : X \to R$ be a continuous projection of X onto R. Further, because T_1 has a bounded inverse, $T_1^{-1}R$ is isomorphic to X; let $C : X \to T^{-1}R$ be an isomorphism of X onto $T_1^{-1}R$. Now note that C and $A \circ P$

are in $B(X)$. Thus any ideal containing T must contain $(A \circ P) \circ T \circ C$, a 1-1 map of X onto X, which, by the interior mapping principle, has a bounded inverse. Thus any ideal containing T contains the identity map and is therefore $B(X)$. So every proper ideal in $B(X)$ is contained in S.

We thank the referee for pointing out the paper *Normally solvable operators and related ideals* by Feldman, Gohberg and Markus (Izv. Moldav. Filial AN SSSR **10(76)**, 51–69. (Russian)) in which it is shown that the compact operators are the only closed two-sided ideal in (BX) for $X = c$ and $l^p (1 \leqq p < \infty)$.

We note that h-spaces have another property in connection with strictly singular operators. If X is an h-space, Y a B-space, and all the maps from X to Y are strictly singular, then so are all the maps from Y to X. This property follows directly.

7. **Another case in which T' strictly singular implies T is strictly singular.** Example 1.3 suggests that restrictions on the domain space are of no avail if we want T' strictly singular to imply that T is strictly singular; this will be true if we limit ourselves to generalizations of the projection properties of Hilbert space as we have done in this paper. We can, however, get the desired theorem with a strong restriction on the domain space and a minor condition on the range space.

THEOREM 7.1. *Let X be isomorphic to $c_0(S)$ and Y be a separable B-space. Then if T' mapping Y' to X' is strictly singular so is T.*

Proof. It suffices to consider $X = c_0(S)$ and S an infinite set. If T is not strictly singular, using the fact that c_0 is an h-space, we can find a subspace M on which T is an isomorphism and have M isomorphic to c_0. Then TM is isomorphic to c_0. Since Y is separable, TM is complemented in Y [8, Theorem 4, p. 217]. Now we argue as in Theorem 2.2.

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