A DIFFERENCE PROPERTY FOR POLYNOMIALS AND EXPONENTIAL POLYNOMIALS ON ABELIAN LOCALLY COMPACT GROUPS

BY

F. W. CARROLL

1. Introduction. Let $f$ be a complex-valued function on $E^m$, Euclidean $m$-space, which has the property that for each $h \in E^m$, the function $\Delta_h f: \Delta_h f(x) = f(x + h) - f(x)$ is continuous (or is a polynomial, or an exponential polynomial). Then $f$ itself need not be continuous (or a polynomial, or an exponential polynomial), for there exist nonmeasurable additive functions on $E^m$, that is, nonmeasurable solutions $\Gamma$ of the functional equation $\Gamma(x + y) = \Gamma(x) + \Gamma(y)$. However, de Bruijn [1], [2] (for $E^1$) and Kemperman [7], [8] (for $E^m$) showed that, among many others, the classes of continuous functions, polynomials, and certain classes of trigonometric and exponential polynomials have the property that if $\Delta_h f$ is in the class for each $h \in E^m$, then there exists an additive function $\Gamma$ on $E^m$ such that $f - \Gamma$ is in the class.

Let $G$ be an abelian topological group, and let $\Omega$ be a class of complex-valued functions on $G$ which contains the constant functions, and such that $f, g \in \Omega$ implies $f - g \in \Omega$ and $f_h \in \Omega$ for each $h \in G$, where $f_h(x) = f(x + h)$. The class $\Omega$ is said to have the difference property if the following implication holds: let $f$ be a complex-valued function on $G$ such that $\Delta_h f \in \Omega$ for each $h \in G$. Then there is an additive function $\Gamma$ on $G$ such that $f - \Gamma \in \Omega$.

Except where the contrary is explicitly stated, $G$ will denote an abelian locally compact group. The product of $m$ copies of the reals will be denoted by $E^m$, and the group of integers by $C$. All functions considered are complex-valued. It is known [3] that the class of continuous functions on $G$ has the difference property. The principal results of this paper are Theorems 1 and 2, which give necessary and sufficient conditions on $G$ in order that the class of polynomials on $G$ (as defined, for instance, in [6]) and the class of exponential polynomials on $G$ have the difference property.

2. The difference property for polynomials. A function $P$ on $G$ is a polynomial of degree $N$ $(N < \infty)$, provided

(P1) for each $(a, b) \in G \times G$, the function

$$P_{ab}: P_{ab}(\lambda) = P(a + \lambda b) \quad (\lambda \in C)$$

is a polynomial on $C$. 

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147
(P2) if \( N_{a,b} \) denotes the degree of \( P_{a,b} \), then \( N = \max \{ N_{a,b} : (a,b) \in G \times G \} \), and

(P3) \( P \) is continuous on \( G \).

An element \( b \in G \) is said to be compact if the closure of the subgroup generated by \( b \) is compact. The set \( B \) of all compact elements is a closed subgroup [11]. Since \( G = E^m + G' \), where \( G' \) contains a compact open subgroup [12], it follows that \( G/B = E^m + G_1 \), where \( G_1 \) is discrete. If a function \( P \) has properties (P1) and (P3) on \( G \), if \( a \in G \) and \( b \in B \), then the set of polynomial values \( \{ P_{ab}(\lambda) : \lambda \in C \} \) is bounded, since \( \{a + \lambda b : \lambda \in C \} \) has compact closure. Thus \( P_{ab}(\lambda) = P(a) \), so that \( P \) is constant on cosets of \( B \), and is essentially a function on \( G/B \).

A set \( H \subseteq G \) is a Hamel basis for \( G \) provided that for no finite subset \( \{b_0, b_1, \ldots, b_k\} \) of \( H \) do there exist integers \( N \not= 0 \), \( n_1, \ldots, n_k \) such that \( Nb_0 = n_1b_1 + \cdots + n_kb_k \), and provided that \( H \) is maximal with respect to this property. It follows from the Hausdorff maximal principle that every abelian group has a (possibly empty) Hamel basis. If \( x \in G \), then there exist uniquely determined elements \( b_1, \ldots, b_k \) in \( H \) and integers \( N \not= 0, n_1, \ldots, n_k \) such that

\[
(2.1) \quad Nx = n_1b_1 + \cdots + n_kb_k.
\]

If (2.1) holds, we shall write

\[
x \sim (n_1/N)b_1 + \cdots + (n_k/N)b_k = r_1(x)b_1 + \cdots + r_k(x)b_k.
\]

In this way there is associated with each \( b \in H \) a rational-valued function \( r_a \) on \( G \), and \( r_a \) is easily seen to be additive.

**Theorem 1.** Let \( G \) be an abelian locally compact group, and let \( B \) be the group of compact elements of \( G \). A necessary and sufficient condition in order that the class of polynomials on \( G \) have the difference property is that \( G/B = E^m + G_1 \), where \( G_1 \) has a finite Hamel basis.

The following example shows the necessity of the condition.

**Example 1.** Let \( G/B = G_1 + E^m \) where \( G_1 \) is discrete and has an infinite Hamel basis. Then there exists a continuous function on \( G \) which is not a polynomial, but each of whose differences is a polynomial. It suffices to show that such a function \( f \) exists on \( G_1 \), for then the function

\[
f_1 : f_1(x) = f(P_2x),
\]

where \( P_1 \) and \( P_2 \) are the natural mappings from \( G \) to \( G/B \) and from \( G/B \) to \( G_1 \) respectively, is an extension of \( f \) which has the required properties on \( G \). Let \( b_1, b_2, \ldots \) be a countably infinite subset of the Hamel basis of \( G_1 \), let \( r_1, r_2, \ldots \) be the corresponding rational-valued functions, and let

\[
f(x) = \sum_{i=1}^{\infty} (r_i(x))' \quad (x \in G_1).
\]
If \( h \) is an arbitrary fixed element of \( G_1 \), there is an integer \( N = N(h) \) such that \( \nu > N \) implies \( r_\nu(h) = 0 \). Thus, from the additivity of the \( r_\nu \),

\[
f(x + h) - f(x) = \sum_{\nu=1}^{N} \left[ (r_\nu(x) + r_\nu(h)) - (r_\nu(x)) \right],
\]
a polynomial.

It may be noticed that the function \( f \) in Example 1 satisfies (P1). In fact, the following is true: if \( G \) is an arbitrary abelian group, and if \( P \) is a function on \( G \) such that, for each \((a, b) \in G \times G\), the function given by \( \Delta_\lambda P(a + \lambda b) \) \((\lambda \in \mathbb{C})\) is a polynomial on \( \mathbb{C} \), then \( P \) has property (P1). For, given \((a, b)\), it is easy to construct a polynomial \( Q_{a,b} \) on \( \mathbb{C} \) such that

\[
Q_{a,b}(\lambda + 1) - Q_{a,b}(\lambda) = \Delta_\lambda P(a + \lambda b) \quad (\lambda \in \mathbb{C}),
\]

\[
Q_{a,b}(0) = P(a).
\]

Then \( P(a + \lambda b) = Q_{a,b}(\lambda) \).

Example 1 shows that, in general, the degree \( N_{a,b} \) of \( P_{a,b} \) is not a bounded function on \( G \times G \). Even on \( \mathbb{C} + \mathbb{C} \), there exists a function satisfying (P1) with \( N_{a,b} \) unbounded [4].

**Proof of Theorem 1. Sufficiency.** Let \( f \) be a function on \( G \) such that \( \Delta_\lambda f \) is a polynomial for each \( h \in G \). Then \( f \) may be taken to be a continuous function, since otherwise there is an additive function \( \Gamma_1 \) on \( G \) such that \( f - \Gamma_1 \) is continuous, and the differences of \( f - \Gamma_1 \) will be polynomials. Also, \( f \) has property (P1) and therefore can (and will) be considered simply as a function on \( G/B \).

Clearly, \( G/B \) contains a dense subgroup \( G' \) which has a finite Hamel basis, viz., \( G' = R^n + G_1 \), where \( R \) denotes the subgroup of \( E \) consisting of the rational numbers. If \( H = \{h_1, \ldots, h_p\} \) is a Hamel basis for \( G' \), and if \( G'' \) is the group generated by \( H \), then \( G'' \) is isomorphic to \( C^p \). The isomorphism \( \phi \) can be chosen so that \( \phi h_i = \varepsilon_i \) has \( \delta_{ij} \) as its \( j \)th coordinate. Let polynomials \( f_i \) \((i = 1, \ldots, p)\) be defined on \( C^p \) by

\[
f_i(x) = \Delta_{\varepsilon_i} f(x) \quad (x \in G'').
\]

Then, clearly,

\[
(2.2) \quad \Delta_{\varepsilon_i} f_j = \Delta_{\varepsilon_j} f_i \quad (i, j = 1, \ldots, p).
\]

But (2.2) implies that there exists a polynomial \( g^* \) on \( C^p \) such that \( g^*(x + \varepsilon_i) - g^*(x) = f_i(x) \) \((i = 1, \ldots, p)\). Explicitly, \( g^* = g_p \), where \( g_k \) is given on \( C^k \) \((k = 1, 2, \ldots)\) by

\[
g_0 = 0,
\]

\[
g_k(n_1, \ldots, n_{k-1}, n_k) = g_{k-1}(n_1, \ldots, n_{k-1})
\]

\[
+ \sum_{m=0}^{M} c_{m,k}(n_1, \ldots, n_{k-1}) (B_{m+1}(n_k) - B_{m+1})/(m+1).
\]
Here, the $c_{m,k}$ are the polynomials obtained from

$$f_k(n_1, \ldots, n_{k-1}, n_k, 0, \ldots, 0) = \sum_{m=0}^{M} c_{m,k}(n_1, \ldots, n_{k-1})(n_k)^m,$$

while $B_m(x)$ and $B_n$ are the $m$th Bernoulli polynomial and number respectively. This assertion can be proved by induction on $p$. (Note that the well-known identity

$$\frac{(B_{m+1}(x + 1) - B_{m+1}(x))/(m + 1) = x^n}$$

implies that the sum (2.3) is

$$\left(-\sum_{j=n_k}^{n_k-1} + \sum_{j=0}^{n_k-1}\right) f_k(n_1, \ldots, n_{k-1}, j),$$

the first (respectively, second) sum in (2.4) being empty if $n_k \geq 0$ (resp., if $n_k \leq 0$).)

Also, $g^*$ has a unique extension as a polynomial $g^{**}$ on all of $R^p$. Let $g$ denote the function on $G'$ given as follows: if

$$x \sim \sum_{i=1}^{p} r_i h_i, \text{ then } g(x) = g^{**}(r_1, \ldots, r_p).$$

Clearly, $g$ satisfies (P1) and is of bounded degree. Since $\Delta_h(f - g)(x) = 0$ for each $x \in G''$ ($i = 1, \ldots, p$), it follows that $f - g$ is constant on $G''$. If $x$ is any point in $G'$, then there exists a positive integer $K$ such that $\mu K x \in G''$ for all $\mu \in C$. Then the polynomial in $x$ given by $f(\lambda x) - g(\lambda x)$, being constant for $\lambda = \mu K$ ($\mu \in C$), is a constant on $C$, so that $f$ is a polynomial on $G'$. If $N - 1$ is its degree, then $\Delta^N f(a)$ vanishes for all $(a, b) \in G' \times G'$, and therefore, by continuity, for all $(a, b) \in G \times G$. But this implies that $f$ is a polynomial of degree $N - 1$ on $G$:

$$f(a + \lambda b) = \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{\mu} (-1)^{\mu} \binom{\lambda}{\mu} \binom{\mu}{\nu} f(a + \nu b) \quad (\lambda \in C; a, b \in G).$$

3. The difference property for exponential polynomials. A function $z$ on $G$ is a generalized character if $z$ is a continuous homomorphism from $G$ to the multiplicative group of nonzero complex numbers. A function $e$ on $G$ is an exponential polynomial if

$$e = \sum_{i=1}^{n} P_i z_i,$$

where each $P_i$ is a polynomial and each $z_i$ is a generalized character. If each $P_i$ is a constant, and each $z_i$ is an ordinary character, then $e$ is said to be a trigonometric polynomial.
Theorem 2. Let $G$ be an abelian locally compact group. A necessary and sufficient condition in order that the class of exponential polynomials on $G$ have the difference property is that $G$ be compactly generated.

Let $f$ be a function on $G$ such that

$$\Delta_h f = \sum_{\alpha \in A} P^\alpha_h z_{\alpha} \quad (h \in G)$$

where $\{z_{\alpha} : \alpha \in A\}$ is the set of all generalized characters on $G$, each $P^\alpha_h$ is a polynomial on $G$, and, for each fixed $h \in G$,

$$P^\alpha_h = 0 \quad \text{for} \ \alpha \neq \alpha_1(h), \ldots, \alpha_k(h).$$

Since the class of continuous functions on $G$ has the difference property, $f$ may be taken to be continuous.

Distinct generalized characters are linearly independent over the ring of polynomials on $G$ (Lemma 3.1, below). Hence, $f$ will be an exponential polynomial if and only if there exist polynomials $\{Q^\alpha : \alpha \in A\}$ such that

$$\Delta_h (Q^\alpha z_{\alpha}) = P^\alpha_h z_{\alpha} \quad (h \in G, \alpha \in A),$$

and

$$Q^\alpha = 0 \quad \text{for} \ \alpha \neq \alpha_1, \ldots, \alpha_k,$$

for then it is clear that

$$f = \sum_{\alpha \in A} Q^\alpha z_{\alpha}.$$

The proof of the sufficiency portion of Theorem 3 consists in constructing polynomials $Q^\alpha$ satisfying (3.3), and showing that (3.4) also holds. If $G$ is not compactly generated, however, then it is not true that (3.2) and (3.3) imply (3.4), even when $\{|\Delta_h f: h \in G\}$ are all trigonometric polynomials. This is shown in Theorem 3, from which the necessity of the condition in Theorem 2 will follow.

Lemma 3.1. Let $G$ be an arbitrary group, let $z_1, \ldots, z_n$ be distinct homomorphisms of $G$ into the multiplicative group of nonzero complex numbers, and let $P_1, \ldots, P_n$ be complex functions satisfying (P1). If

$$P_1 z_1 + \cdots + P_n z_n = 0,$$

then $P_1 = \cdots = P_n = 0$.

Proof. First, consider the special case $G = C$. Since $z_1, \ldots, z_n$ are distinct, the complex numbers $z_1(1), \ldots, z_n(1)$ are necessarily distinct. If not all $P_j$ are identically zero, then, reordering if necessary, it may be assumed that for some integer $p$, $1 \leq p \leq n$, $
\[ z_j(1)/z_1(1) = e^{i\beta_j} \neq 1 \quad (\beta_j \text{ real}, \; 2 \leq j \leq p), \]
\[ |z_j(1)| > |z_1(1)| \quad (p + 1 \leq j \leq n), \]
\[ P_j(\lambda) = c_j\lambda^m + O(\lambda^{m-1}) \; \text{as} \; \lambda \to \infty \quad (1 \leq j \leq p; \; c_j \neq 0). \]

It follows upon division of the terms of (3.5) by \( \lambda^m z_1(\lambda) \) that
\[ (3.6) \quad c_1 = -\sum_{j=2}^p c_j e^{i\beta_j} + O(\lambda^{-1}) \; \text{as} \; \lambda \to \infty. \]

Thus, taking \( \lambda = 1, 2, \ldots, N \) in (3.6), adding, and dividing by \( N \), it is seen that
\[ (3.7) \quad c_1 = -\left(1/N\right) \sum_{j=2}^p c_j e^{i\beta_j} (e^{i\beta_j N} - 1)/(e^{i\beta_j} - 1) + O(N^{-1}\log N). \]

Letting \( N \) approach infinity in (3.7), it follows that \( c_1 = 0 \), a contradiction. In the general case, we prove by induction on \( n \) that \( P_1(x) = 0 \). This is obvious for \( n = 1 \), since \( z_1 \) is never zero. Now let \( n > 1 \), and suppose the result holds for all \( p < n \). Choose \( x_0 \in G \) such that \( z_1(x_0) \neq z_2(x_0) \); this is possible since the \( z \)'s are distinct. Suppose that \( z_1(x_0) = z_2(x_0) = \cdots = z_p(x_0) \) for some integer \( p, \; 1 \leq p < n \), while \( z_j(x_0) \neq z_1(x_0) \) for \( p < j \leq n \). Then
\[ \sum_{j=1}^n P_j(x + \lambda x_0) z_j(x + \lambda x_0) = \left[ \sum_{j=1}^p P_j(x + \lambda x_0) z_j(x) \right] z_1(\lambda x_0) \]
\[ + \sum_{j=p+1}^n P_j(x + \lambda x_0) z_j(x + \lambda x_0). \]

For each fixed \( x \), this expression can be considered as an exponential polynomial on \( C \), and \( z_1(\lambda) = z_1(\lambda x_0) \) is distinct from the other generalized characters. Its coefficient is therefore zero for each \( \lambda \in C \). Hence, for each \( x \in G \)
\[ \sum_{j=1}^p P_j(x) z_j(x) = 0, \]
so that \( P_1(x) \equiv 0 \), by the induction assumption.

**Lemma 3.2.** Let \( G \) be an abelian topological group, let \( z \neq 1 \) be a generalized character on \( G \), and let \( \{P_k : h \in G\} \) be a collection of polynomials on \( G \) such that
\[ (3.8) \quad P_h(x + h') z(h') - P_h(x) = P_k(x + h) z(h) - P_k(x), \]
for all \( h, h', x \) in \( G \). Then there exists a polynomial \( Q \) on \( G \) such that
\[ (3.9) \quad \Delta_{\lambda}(Qz) = P_k z \quad (h \in G). \]

**Proof.** Let \( h \in G \) be such that \( |z(h)| < 1 \), or, if there is no \( h \) with this property, i.e., if \( |z| \equiv 1 \), let \( z(h) \neq 1 \). Consider the expression
If \(|z(h)| < 1\), then the \(r\) in the summation may be replaced by 1, and the \(\lim\) omitted. Since \(P_h\) is a polynomial,

\[
(3.11) \quad P_h(x + nh) = \sum_{k=0}^{N} c_k(x)n^k \quad (x \in G),
\]

where each \(c_k(x)\) is a polynomial on \(G\), obtainable explicitly from setting \(n = 0, 1, \ldots, N\) in (3.11) and solving by Cramer's rule. The sum in (3.10) is given for \(0 < r < 1\) by

\[
- \sum_{k=0}^{N} c_k(x) \sum_{n=0}^{\infty} n^k(z(h)r)^n = - \sum_{k=0}^{N} c_k(x) \left\{ \begin{array}{c} y \\ dy \end{array} \right\}^k \left( \frac{1}{1-y} \right) \bigg|_{y=r(z(h))}.
\]

Hence the limit in (3.10) exists and yields a polynomial on \(G\); let this polynomial be denoted by \(Q\). Let \(h'\) and \(x\) be arbitrary elements of \(G\). Then

\[
(3.12) \quad Q(x + h')z(h') - Q(x) = \lim_{r \to 1-} \left\{ - \sum_{n=0}^{\infty} (z(h)r)^n \left[ P_h(x + h' + nh)z(h') - P_h(x + nh) \right] \right\}.
\]

From (3.8), the right-hand side of (3.12) is

\[
\lim_{r \to 1-} \left\{ - \sum_{n=0}^{\infty} (z(h)r)^n \left[ P_h(x + (n+1)h)z(h) - P_h(x + nh) \right] \right\}
\]

\[
(3.13) \quad = P_h(x) - \lim_{r \to 1-} (1 - r) \sum_{n=1}^{\infty} P_h(x + nh)z(h)^n r^{n-1} = P_h(x),
\]

so that (3.9) follows from (3.12) and (3.13).

**Proof of Theorem 2.** Sufficiency. Let \(f\) be a function (which may and will be assumed to be continuous) such that (3.1) and (3.2) hold. From (3.1), Lemma 3.1, and the identity

\[
\Delta_h \Delta_h f = \Delta_h \Delta_h f,
\]

it follows that (3.8) holds for each generalized character \(z\). For each \(z\) except \(z_0 = 1\), it follows from Lemma 3.2 that there exists a polynomial \(Q^i\) satisfying (3.9); in particular, if \(P^i_h = 0\) for all \(h\), then \(Q^i = 0\). But only finitely many \(Q^i\) can be nonzero. For suppose that \(Q^i \not\equiv 0\) \((i = 1, 2, \ldots)\), with \(P^i_h\) and \(z_i\) the corresponding polynomials and generalized characters. For each \(h \in G\), (3.2) shows that there exists an integer \(i(h)\) such that \(P^i_h = 0\) for \(i \geq i(h)\). Hence, from (3.3),

\[
Q^i(x + h)z_i(h) - Q^i(x) = 0 \quad (i \geq i(h), x \in G),
\]

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whence it follows that \( z_i(h) = 1 \) for \( i \geq i(h) \). Let

\[ H_j = \{ h : h \in G, z_i(h) = 1 \text{ for all } i \geq j \}. \]

\( H_j \) is a closed subgroup of \( G \), \( H_{j+1} \supset H_j \) and \( G = \bigcup H_j \), so that at least one \( H_j \) is of positive Haar measure, and thus open \([5]\). Therefore \( H_k \) is open for all \( k \geq j \). Let \( A \) be a compact neighborhood of \( 0 \) which generates \( G \). Then \( \bigcup \{ H_k : k \geq j \} \) covers \( G \), so that \( A \) is covered by some \( H_N \), whence \( H_N = G \). Therefore \( z_i = 1 \) for all \( i \geq N \), contradicting the distinctness of the \( z_i \). Since \( Q^a = 0 \) for all but finitely many \( a \), the function given by \( g = \sum Q^a z_a \) (the summation taken for all \( a \) such that \( z_a \neq 1 \)) is an exponential polynomial, and, for each \( h, \Delta_h(f - g) \) is clearly a polynomial on \( G \). But the function \( f - g \) is continuous, and the class of polynomials on \( G \) has the difference property from Theorem 1, since \( G \) is compactly generated only if \( G_i \) is finitely generated. Hence, \( f - g \) is a polynomial on \( G \).

In the proof just given, use was made of the fact that a compactly generated group \( G \) is not the countable union of a strictly increasing sequence of closed subgroups. Conversely,

**Lemma 3.3.** If the locally compact abelian group \( G \) is not compactly generated, there is a sequence \( \{ H_j \} \) of closed subgroups of \( G \), such that \( H_j \subset H_{j+1} \) (strictly) and \( \bigcup H_j = G \).

**Proof.** There is a compact subgroup \( G' \) of \( G \) such that \( G/G' = E^p + G_2 \), with \( G_2 \) discrete. Since \( G \) is not compactly generated, it follows that \( G_2 \) is not finitely generated. It is known \([9]\) that

\[ G_2 = \bigcup_{n=1}^{\infty} S_n, \]

where each \( S_n \) is a direct sum of cyclic groups, and \( S_n \subset S_{n+1} \). If the inclusion is proper for infinitely many \( n \), the choice of the \( H_j \) is clear, and the lemma follows. Otherwise, \( G_2 \) is itself a direct sum of infinitely many cyclic groups:

\[ G_2 = \sum_{a} A_a. \]

Let \( \{ A_{a_1}, A_{a_2}, \ldots \} \) be a countably infinite subset of \( \{ A_a \} \), and let

\[ H_j = E^p + \sum A_a : a \neq a_{j+1}, a_{j+2}, \ldots. \]

Then \( H_j \subset H_{j+1} \) properly, and their union is \( G \).

**Theorem 3.** Let \( G \) be an abelian locally compact group. The class of trigonometric polynomials on \( G \) has the difference property if and only if \( G \) is compactly generated.

**Proof.** The sufficiency is clearly a corollary of the sufficiency proof of Theorem 2. If \( G \) is not compactly generated, let \( \{ H_j \} \) be the sequence given
by Lemma 3.3, and for each j let \( z_j \) be a character identically 1 on \( H_j \) but not identically 1 on \( G \); such characters exist \([11]\). Let \( \sum a_j \) be a convergent infinite series of positive numbers, and let \( f \) be defined by

\[
(3.14) \quad f = \sum_{j=1}^{n} a_j z_j.
\]

If \( h \in G \) is given, there exists an integer \( k = k(h) \) such that \( h \in H_{k+1} \). Then

\[
f(x + h) - f(x) = \sum_{j=1}^{k} a_j (z_j(h) - 1)z_j(x),
\]

a trigonometric polynomial.

Suppose now that \( f \), given by (3.14) is also given by

\[
(3.15) \quad f = \sum_{j=1}^{n} P_j z_{a_j} + \Gamma,
\]

with polynomials \( P_j \), generalized characters \( z_{a_j} \), and an additive function \( \Gamma \). Let \( z_j \) be a character appearing in (3.14) but not in (3.15), and let \( h \in G \) be chosen such that \( z_j(h) \neq 1 \). Then \( z_j \) appears in the expression for the exponential polynomial \( A_h f \) obtained from (3.14) but not in that obtained from (3.15). This contradicts Lemma 3.1. Thus the necessity portions of both Theorem 2 and of Theorem 3 are established.

**References**

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