GEOMETRY OF THE ZEROS OF THE SUMS OF LINEAR FRACTIONS

BY

J. L. WALSH

The object of the present paper is to study the zeros of functions of the form

\[ f(z) = \sum_{k=1}^{n} \frac{a_k}{z-a_k} - \sum_{k=1}^{n} \frac{b_k}{z-b_k}, \quad a_k > 0, \quad b_k > 0, \]

where the \(a_k\) and \(b_k\) have various geometric configurations as their loci. We investigate also functions of this form where the \(a_k\) and \(b_k\) are nonreal.

The appropriateness of this study arises from the facts that (i) Lagrange's interpolation formula for a polynomial with prescribed real values in real points \(a_k\) and \(b_k\) has a factor of precisely form (1), and a similar remark holds for nonreal values and nonreal points; (ii) Riemann sums for a Cauchy integral are of these same forms, in the respective real and nonreal cases; (iii) the logarithmic derivative of a rational function is of form (1), which enables us to study the location of the critical points. Our main theorems (Theorems 1 and 2) refer respectively to the real and nonreal cases just mentioned, where the locus of the \(a_k\) and \(b_k\) is a line segment with the \(a_k\) and \(b_k\) real, or a circular disk with the \(a_k\) and \(b_k\) not necessarily real.

Theorem 2 is a special case of a much more general theorem due to Marden [3], but is proved in detail here particularly because of the applications (i) and (ii), not mentioned by Marden. Namely, the present methods apply also to the case (Theorems 3 and 4) where the locus of the \(a_k, b_k,\) etc., is a circumference rather than a disk, a case not included in Marden's treatment yet important precisely for the study of a Cauchy or Cauchy-Stieltjes integral.

As is frequently done [2] in the study of zeros of such functions as (1), we interpret the conjugate of \(f(z)\) as the force at \(z\) due to repelling particles at the \(a_k\) and attracting particles at the \(b_k\), where each particle repels with a force equal to its mass \(a_k\) or \(-b_k\) times the inverse distance; the original problem of finding the zeros of \(f(z)\) is equivalent to the problem of finding the positions of equilibrium in this field of force.

**Theorem 1.** Let the conditions of (1) be satisfied, with \(A = \sum a_k > B = \sum b_k\), and let all \(a_k\) and \(b_k\) lie on the interval \(-1 \leq z \leq +1\). Then all nonreal zeros...
of \( f(z) \) lie in the closed interior of the ellipse
\[
Bx^2 + Ay^2 = \frac{AB}{A-B}.
\]

All real zeros of \( f(z) \) lie in the interval
\[
|x| \leq \frac{A+B}{A-B}.
\]

Indeed the sets mentioned constitute the locus of the zeros of \( f(z) \) for all \( f(z) \) satisfying the hypothesis.

In the field of force already introduced, if a point \( z_0 \) is considered, it is frequently convenient (cf. [2]) to replace \( n \) positive (or negative) particles \( \alpha_k \) (or \( \beta_k \)) by a single equivalent particle whose mass is the sum of the masses of the original particles and which exerts the same force at \( z_0 \). If the particle \( \alpha_k \) is inverted in the unit circle whose center is \( z_0 \), the corresponding force at \( z_0 \) is represented by the vector from the inverse of \( \alpha_k \) to the point \( z_0 \) multiplied by the mass of the particle; the total force at \( z_0 \) due to all the particles \( \alpha_k \) is represented by the vector (weighted by the total mass) from the center of gravity of the weighted inverses to \( z_0 \); the equivalent particle of the \( \alpha_k \) is located at the inverse of this center of gravity. We often have occasion to use the fact that if a number of initial points of vectors with common terminal point and various weights are given, their center of gravity lies in their convex hull; this center of gravity is the initial point of the vector resultant, weighted by the sum of weights of the given vectors.

With the hypothesis of Theorem 1 we first choose \( \text{Im}(z_0) > 0 \). For fixed \( z_0 \) the particle \( \alpha_0 \) equivalent to the given \( \alpha_k \) lies in the circular segment \( S(z_0) \) bounded by the interval \(-1 \leq x \leq 1\) and by an arc of the circle through \(-1, +1, \) and \( z_0 \) whose endpoints are \( z = +1 \) and \(-1\); the arc lies in the closed half-plane \( \text{Im}(z) \leq 0 \). This remark follows from the fact that the inverse in the unit circle whose center is \( z_0 \) of the interval \(-1 \leq x \leq 1\) is an arc of a circle through \( z_0 \); the convex hull of this arc is a certain segment of a circle whose inverse is \( S(z_0) \). Moreover, \( S(z_0) \) is the actual locus of the equivalent particle when all possible choices of the \( \alpha_k \) are considered, not restricted in total number or in respective (positive) masses.

The locus of the particle \( \beta_0 \) equivalent to the \( \beta_k \) is also \( S(z_0) \), and \( z_0 \) is a position of equilibrium if and only if \( \alpha_0 \) and \( \beta_0 \) (in their respective loci) are collinear, with \( |z_0 - \alpha_0|/|z_0 - \beta_0| = A/B \). If \( \alpha_0' \) and \( \beta_0' \) are any two positions of \( \alpha_0 \) and \( \beta_0 \) collinear with \( z_0 \) and in their proper loci, say with \( \beta_0' \) on the interval \(-1 \leq x \leq 1\) and \( \alpha_0' \) on the circular arc partially bounding \( S(z_0) \), then the ratio \( |z_0 - \alpha_0|/|z_0 - \beta_0| \) can be increased by rotating the line \( z_0 \alpha_0 \beta_0 \) about \( z_0 \), and by sliding \( \beta_0 \) from \( \beta_0' \) along the interval and sliding \( \alpha_0 \) from \( \alpha_0' \) along the circular arc in the sense so as algebraically to decrease the
ordinate of $a_0$; this increase of the ratio is always possible as long as the abscissa of $a_0$ is not zero. The maximum of the ratio occurs when $\beta_0$ is on the interval, say $\beta_0 = \beta'_0$, and when the abscissa of $a_0$ is zero, say $a_0 = a'_0$. However, the ratio can take on all values and only values between unity and this maximum inclusive, for suitable choices of $a_0$ and $\beta_0$ in their proper loci and collinear with $z_0, |z_0 - a_0| \geq |z_0 - \beta_0|$. Thus $z_0$ can be a position of equilibrium if and only if we have

\[
\frac{|z_0 - a'_0|}{|z_0 - \beta'_0|} \geq \frac{A}{B}.
\]

If we set $z_0 = x_0 + iy_0$, and note that the center of the circle an arc of which bounds $S(z_0)$ in part has the ordinate $b = (x_0^2 + y_0^2 - 1)/(2y_0)$, inequality (4) can be rewritten as

\[
y_0 + (1 + b^2)^{1/2} - b \geq \frac{A}{B},
\]

which is equivalent to

\[
Bx_0^2 + Ay_0^2 \leq \frac{AB}{A - B};
\]

this inequality is valid for both $y_0 > 0$ and $y_0 < 0$, so the proof of the first part of Theorem 1 is complete. It may be noted that the foci of the ellipse (2) are $z = \pm 1$ and $-1$, and its eccentricity is $[(A - B)/A]^{1/2}$; thus the ellipse corresponding to an arbitrary interval as assigned locus of the $a_k$ and $\beta_k$ is found at once.

If $z_0 = x_0 + iy_0$ is real and given, say $z_0 > 1$, the maximum of the first member of (4) is $(z_0 + 1)/(z_0 - 1)$, and $z_0$ can be a zero of $f(x)$ if and only if we have

\[
\frac{z_0 + 1}{z_0 - 1} \geq \frac{A}{B},
\]

so $z_0$ is a zero of some $f(z)$ if and only if we have

\[
z_0 \geq \frac{A + B}{A - B}.
\]

A similar discussion applies if we have $z_0 < -1$.

On the other hand, an arbitrary point of $-1 \leq x \leq 1$ belongs to the locus; for instance $z = 0$ is a zero of the particular function

\[
f(z) = \frac{A}{z - a_0} - \frac{B}{z - \beta_0},
\]

provided we have merely $A\beta_0 = B\alpha_0$, so $\alpha_0$ and $\beta_0$ can be chosen positive and as small as desired. Theorem 1 is established.
Under the hypothesis of Theorem 1 except that now we take $A = B$, the locus of zeros of the totality of the functions $f(z)$ consists nontrivially of the entire plane. Indeed, let $z_0$ be a given nonreal point of the plane and let $z_1$ be an interior point of the circular segment $S(z_0)$ already defined. We set

$$f(z) = \frac{1}{z - (z_1 + \delta)} + \frac{1}{z - (z_1 - \delta)} - \frac{2}{z - (z_1 + \epsilon)},$$

where $|\delta|$ and $|\epsilon|$ ($> 0$) are chosen so small that $z_1 \pm \delta$ and $z_1 + \epsilon$ lie within a circle interior to $S(z_0)$. The function $f(z)$ vanishes when $z = z_1 + \delta^2/\epsilon$, and $\epsilon$ and $\delta$ can be so chosen that this number is $z_0$. This discussion does not apply if $z_0$ is real, but in that case a slight modification of the discussion of (5) does apply, and shows that $z_0$ belongs to the locus of zeros of all $f(z)$.

As an application of Theorem 1 we formulate

**Corollary 1.** Let $r(z)$ be a rational function of $z$ whose finite zeros and poles lie on the segment $-1 \leq z \leq +1$, of respective total orders $A$ and $B$ or $B$ and $A$, $A > B$. Then all finite nonreal critical points of $r(z)$ lie in the closed interior of the ellipse (2), and all finite real critical points lie in the closed interval (3).

The logarithmic derivative of $r(z)$ is of form (1), where the $\alpha_k$ are the zeros of $r(z)$ and the $\beta_k$ are the poles, each enumerated a number of times according to its multiplicity, and where all $a_k$ and $b_k$ are unity. The corollary follows from Theorem 1.

As a second application we have

**Corollary 2.** Let $f(z)$ be defined by the Stieltjes integral

$$f(z) = \int_{-1}^{1} \frac{d\sigma(t)}{t - z}, \quad -1 \leq t \leq 1,$$

where the total positive variation of $\sigma(t)$ on $-1 \leq t \leq 1$ is $A$ and the total negative variation is $-B$, $A > B$. Then all finite nonreal zeros of $f(z)$ lie in the closed interior of the ellipse (2), and all real zeros lie in the closed interval (3).

The proof of Corollary 2 follows by considering the partial sums approximating the Stieltjes integral, and by Theorem 1. If the total negative variation of $\sigma(t)$ is greater than the total positive variation it suffices to consider the zeros of $-f(z)$.

In the proof of theorems such as Theorem 1 on the geometry of zeros of functions, two methods of proof are frequently used: (i) study of the loci of particles equivalent to various categories of particles; (ii) study of the total forces due to various categories of particles. We have just employed method (i), and now proceed to use method (ii) in a different problem.

**Theorem 2 (Marden).** Let the function $f(z)$ be of the form
\[ f(z) = \sum_{k=1}^{m} \frac{a_k}{z - \alpha_k} - \sum_{k=1}^{n} \frac{b_k}{z - \beta_k} + \sum_{k=1}^{p} \frac{ic_k}{z - \gamma_k} - \sum_{k=1}^{q} \frac{id_k}{z - \delta_k}, \]

where all the \(a_k, b_k, c_k,\) and \(d_k\) are non-negative. We set \(A = \sum a_k, B = \sum b_k, C = \sum c_k, D = \sum d_k,\) and suppose \((A - B)^2 + (C - D)^2 \neq 0.\) If \(T: |z| \leq 1\) is the simultaneous locus of the points \(\alpha_k, \beta_k, \gamma_k,\) and \(\delta_k,\) for all \(a_k, b_k, c_k, d_k\) satisfying the conditions given, then the locus of the zeros of \(f(z)\) is the disk \((7)\)

\[
|z| \leq \frac{A + B + C + D}{(A - B)^2 + (C - D)^2}^{1/2}.
\]

We continue to interpret the conjugate of \(f(z)\) as defining a field of force in the \(z\)-plane. If \(z_0\) is a zero of \(f(z),\) then \(\omega z_0\) with \(|\omega| = 1\) is a zero of \(f(z)\) with the original \(\alpha_k, \beta_k, \gamma_k, \delta_k\) replaced by \(\omega \alpha_k, \omega \beta_k, \omega \gamma_k, \omega \delta_k,\) so it is sufficient for us to study \(z_0 = a,\) real; we take \(a > 1,\) and then we make a translation of the plane so that \(\Gamma\) becomes \(\Gamma_1: |z + a| \leq 1\) and \(z_0\) becomes \(z_1 = 0.\) The inverse of \(\Gamma_1\) in the unit circle whose center is \(z_1\) is

\[
|z + \frac{a}{a^2 - 1}| \leq \frac{1}{a^2 - 1},
\]

and the force exerted at \(z_1\) due to all the particles \(\alpha_k\) is represented by a vector with initial point \(z_1\) and terminal point in the disk

\[
C_1: \left| z - \frac{aA}{a^2 - 1} \right| \leq \frac{A}{a^2 - 1};
\]

in fact \(C_1\) is the locus of the terminal points of such vectors for all possible choices of the \(a_k\) and \(\alpha_k,\) with \(A\) fixed; compare [2, p. 13]. The “disk” \(C_1\) is represented by the formula given even if \(A = 0.\)

Likewise, the locus of the terminal points of the vectors with initial points in \(z_1\) and representing the force at \(z_1\) for all possible choices of the \(b_k\) and \(\beta_k\) with \(B\) fixed is the disk

\[
C_2: \left| z + \frac{AB}{a^2 - 1} \right| \leq \frac{B}{a^2 - 1}.
\]

The locus of the terminal points of the vectors with initial points in \(z_1\) representing the total force at \(z_1\) due to the particles at the \(\alpha_k\) and \(\beta_k\) is the disk which is the “sum” of \(C_1\) and \(C_2:\)

\[
(7) \quad \left| z - \frac{a(A - B)}{a^2 - 1} \right| \leq \frac{A + B}{a^2 - 1},
\]

in the sense that if \(C_1\) and \(C_2\) are the loci of \(z_1\) and \(z_2\) then \((7)\) is the locus of \(z_1 + z_2.\)

By a similar method, and with the note that the conjugate of \(f(z)\) defines
the forces, it follows that the locus of the terminal points of the vectors with initial points in $z_1$, representing the total force at $z_1$ due to the particles at the $\gamma_k$ and $\delta_k$ is the disk

$$\left| z + \frac{ia(C-D)}{a^2-1} \right| \leq \frac{C+D}{a^2-1}. \tag{8}$$

For the two sets of forces we have vectors with initial points in $z_1 (=0)$ and terminal points whose loci are the respective disks (7) and (8). The total resultant force is represented by a vector whose initial point is $z_1$ and the locus of whose terminal points lies in the disk

$$\left| z - \frac{a(A-B) - ia(C-D)}{a^2-1} \right| \leq \frac{A+B+C+D}{a^2-1}. \tag{9}$$

A necessary and sufficient condition that $z_1$ be a possible position of equilibrium is that the total force may be zero, or that $z_1 (=0)$ should lie in the disk (9), namely

$$\left| \frac{a(A-B) - ia(C-D)}{a^2-1} \right| \leq \frac{A+B+C+D}{a^2-1}, \quad a = |z_0| > 1,$n

which is essentially (6). The second member of (6) is greater than unity unless three of the four numbers $A, B, C, D$ are zero.

If $z_0$ is a zero of $f(z)$ for a particular choice of the $\alpha_k$ etc., and if $0 < \rho < 1$, the point $\rho z_0$ is a zero of $f(z)$ with the $a_k, b_k, c_k, d_k$ unchanged and the $\alpha_k, \beta_k, \gamma_k, \delta_k$ multiplied by $\rho$. Moreover $z_0 = 0$ is a zero of $f(z)$ with suitably chosen $\alpha_k, \beta_k, \gamma_k, \delta_k$ small in modulus, and this completes the proof of Theorem 2.

**Corollary 1.** If $f(z)$ in Theorem 2 is of the form

$$f(z) = \sum_{k=1}^{m} \frac{a_k}{z - \alpha_k} + \sum_{k=1}^{n} \frac{ic_k}{z - \gamma_k}, \quad A + C \neq 0,$n

the locus of its zeros is the disk

$$|z| \leq \frac{A+C}{(A^2+C^2)^{1/2}}.$$n

**Corollary 2.** If $f(z)$ in Theorem 2 is of the form

$$f(z) = \sum_{k=1}^{m} \frac{a_k}{z - \alpha_k} - \sum_{k=1}^{n} \frac{b_k}{z - \beta_k}, \quad A > B,$n

the locus of its zeros is the disk

$$|z| \leq \frac{A+B}{A-B}.$n

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If $r(z)$ is a rational function not identically constant, and if the exact degrees of its numerator and denominator are $A$ and $B$, its logarithmic derivative is of form (10); the conclusion of Corollary 2 is essentially that all zeros of the derivative lie in the disk (11), a result [1] proved by the present author in 1918.

If $f(z)$ is multiplied by $\omega$ with $|\omega| = 1$, the zeros of the new function $f_1(z)$ ($= \omega f(z)$) are unchanged, yet the second member of (6) is not unchanged by such an arbitrary transformation; indeed the denominator in the second member of (6) is precisely $|A - B + iC - iD|$, which is invariant, but the numerator is not invariant. This seeming paradox is resolved if we consider for instance the special case $B = C = D = 0$. The original theorem refers to the zeros of $\sum a_k(z - \alpha_k)^{-1}, a_k > 0$ whereas if we set $\omega = \cos \theta + i\sin \theta, 0 < \theta < \pi/2$; the new function $f_1(z)$ is to be written $\sum a_k(\cos \theta + i\sin \theta) \cdot (z - \alpha_k)^{-1}$, $\sum a_k = A$, which is quite different from the function $\sum a'_k(z - \alpha_k)^{-1} + \sum c'_k(z - \beta_k)^{-1}$ for all $a'_k, c'_k$ having prescribed sums $\sum a'_k = A \cos \theta, \sum c'_k = A \sin \theta$.

As a consequence of the facts just discussed, we formulate the following remark. In the application of Theorem 2 we may replace $f(z)$ by $\omega f(z)$, where $\omega$ is a constant of modulus unity; this change may modify the second member of (6). In particular if $f(z)$ can be written so as to contain one or more terms of the form

$$z^{-\lambda_k}$$

where $\arg \lambda_k$ is independent of $k$, then as far as those terms are concerned it is favorable to choose $\arg \omega = -\arg \lambda_k$.

Corollary 2 to Theorem 1 has an analogue here, concerning the integral

$$\phi(z) \equiv \int_\gamma \frac{d\alpha(t)}{t - z}.$$

If $\gamma$ lies in the closed interior of the unit circle, if $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ where $\alpha_1(t)$ and $\alpha_2(t)$ are real, and if $A$ and $-B$, and $C$ and $-D$, are the respective total positive and negative variations of $\alpha_1(t)$ and $\alpha_2(t)$ on $\gamma$, and if $(A - B)^2 + (C - D)^2 \neq 0$, then all zeros of the approximating sums of $\phi(z)$ (which approach $\phi(z)$) lie in the closed interior of a variable disk that approaches (6), so by Hurwitz’s theorem all zeros of $\phi(z)$ lie in the closed interior of (6).

The remarks just made concerning $\phi(z)$ suggest the study of the hypothesis of Theorem 2 except that now the $\alpha_k$ and $\beta_k$ are required to lie on the unit circumference $\gamma$. If the locus of positive particles $\alpha_k$ is $\gamma$, and if $z_0$ lies exterior to $\gamma$, the locus of the equivalent particle is the closed interior of $\gamma$, as becomes obvious at once by inversion in the unit circle whose center.
is \( z_0 \). If the locus of these \( \alpha_k \) is \( \gamma \) and \( z_0 \) lies interior to \( \gamma \), the locus of the equivalent particle is the closed exterior of \( \gamma \) including the point at infinity. If \( z_0 \) lies interior to \( \gamma \), we may consider the equivalent particles for each category of particles to lie at infinity, whence \( f(z_0) = 0 \). We have

**Theorem 3.** Let the hypothesis of Theorem 2 be modified so that all particles \( \alpha_k, \beta_k, \gamma_k, \delta_k \) lie on \( \gamma \); \( |z| = 1 \). As far as concerns points \( z \) not on \( \gamma \), the locus of the zeros \( z \) of \( f(z) \) is the disk (6).

This proof of Theorem 3 involves essentially applying the method of proof of Theorem 2, but not applying Theorem 2 itself.

To study the points \( z \) on \( \gamma \), we consider (as in the proof of Theorem 2) the actual forces at \( z_0 \) due to the various categories of particles, and the locus of the terminal points of the vectors representing these forces, when the initial points lie in \( z_0 \). We omit the assumption \((A - B)^2 + (C - D)^2 \neq 0\). As before, let us choose \( z_0 \) positive and then translate the plane, so that \( \gamma \) becomes \( |z + 1| = 1 \) and \( z_0 \) becomes \( z_1 = 0 \). The inverse of \( \gamma \) in the unit circle whose center is \( z_1 \) is the line (better, the finite points of the line) \( x = -1/2 \), and the locus of the terminal points of all vectors each corresponding to a set of particles \( \alpha_k \) is the line \( x = A/2 \) unless \( A = 0 \). The locus of the terminal points of all vectors each corresponding to a set of particles \( \beta_k \) is the line \( x = -B/2 \) unless \( B = 0 \), and for the composition of a pair of these vectors we have as locus the line \( x = (A - B)/2 \); however, it is to be noted that all vectors are null vectors if we have \( A = B = 0 \). The loci for the vectors corresponding to the \( \gamma_k \) and \( \delta_k \) are respectively the lines \( y = -C/2 \) and \( y = D/2 \) unless \( C = 0 \) or \( D = 0 \); for the composition of a pair of these vectors we have as locus the line \( x = -C/2 \), except if \( C = D = 0 \). The locus for the negatives of these last mentioned vectors having their initial points in \( z_1 \) is \( y = (C - D)/2 \), which always intersects the line \( x = (A - B)/2 \); the total sum of all vectors is null for a suitable configuration depending on given \( A, B, C, D \), with the exceptions noted.

**Theorem 4.** With the hypothesis of Theorem 3, the locus of the zeros of \( f(z) \) contains the entire circumference \( \gamma \) provided we have \( A + B \neq 0 \) and \( C + D \neq 0 \). The locus contains the entire circumference also if \( A = B, C + D = 0 \), or if \( A + B = 0, C = D \). The locus contains no point of \( \gamma \) if \( A \neq B, C + D = 0 \) or if \( A + B = 0, C \neq D \).

The case \( A = B = C = D = 0 \) is of course trivial.

It is of interest to indicate how Theorems 1 and 2 apply to the study of zeros of restricted infrapolynomials; for these methods, compare [4], [5]. The category of restricted infrapolynomials on a set \( E \) as used here includes the category of similarly restricted polynomials of least norm on \( E \), where
norm is in the sense of least weighted pth powers ($p > 0$) or in the sense of (weighted) Tchebycheff.

**Theorem 5.** Let the two disjoint point sets $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\beta_1, \beta_2, \ldots, \beta_n$ consist of the distinct points indicated and lie on $-1 \leq z \leq 1$. Let the real polynomial $P(z) = N z^{n+1} + \ldots$ have the coefficient $N$ prescribed, and also the (real) values $P(\alpha_j)$, and be a thus restricted infrapolynomial (i.e., have no restricted underpolynomial) on $E: |\beta_j|$. Set $\sum^n_j [P(\alpha_j)/\omega'(\alpha_j)] = N_0$, where $\omega(z) = \prod^n_j(z - \alpha_j) \cdot \prod^n_j(z - \beta_j)$. Let $A$ and $-B$ be the sum of the positive and negative numbers respectively among $P(\alpha_j)/\omega'(\alpha_j), N - N_0$; we suppose $A > B$. Then all nonreal zeros of $P(z)$ lie in the closed interior of the ellipse (2), and all real zeros lie in the interval (3).

The polynomial $P(z)$ can be expressed by the Lagrange formula

$$P(z) = \omega(z) \sum^m_1 \frac{P(\alpha_j)}{\omega'(\alpha_j)(z - \alpha_j)} + \omega(z) \sum^n_1 \frac{P(\beta_j)}{\omega'(\beta_j)(z - \beta_j)};$$

here the coefficients $P(\alpha_j)$ are prescribed, and the coefficients $P(\beta_j)$ are not prescribed, but are subject to the condition

$$\sum^n_1 P(\beta_j)/\omega'(\beta_j) = N - N_0.$$

It is then clear that for $P(z)$ thus restricted to be an infrapolynomial on $E$ the condition

$$\text{sg}[P(\beta_j)/\omega'(\beta_j)] = \text{sg}(N - N_0), \quad j = 1, 2, \ldots, n,$$

is necessary and sufficient. Indeed, if (15) is satisfied, there exists no restricted underpolynomial $Q(z)$ of $P(z)$ on $E$, for there exists no set of values $Q(\beta_j)$ with

$$\sum^n_1 Q(\beta_j)/\omega'(\beta_j) = N - N_0$$

such that

$$|Q(\beta_j)| < |P(\beta_j)| \quad \text{if} \quad P(\beta_j) \neq 0$$

and

$$Q(\beta_j) = 0 \quad \text{if} \quad P(\beta_j) = 0.$$

Conversely, if $P(z)$ is a restricted infrapolynomial and if we have both (14) and $\sum^n_1 |P(\beta_j)/\omega'(\beta_j)| > |N - N_0|$, then we can set

$$\frac{Q(\beta_j)}{\omega'(\beta_j)} = \frac{|P(\beta_j)/\omega'(\beta_j)| \cdot |N - N_0|}{\sum^n_1 |P(\beta_j)/\omega'(\beta_j)|}.$$
whence (16), (17), and (18) are valid and $Q(z)$ is an underpolynomial of $P(z)$ on $E$.

By virtue of (13) with (14) and (15), Theorem 5 now follows from Theorem 1. If a given polynomial $P(z)$ satisfies all the requirements of Theorem 5 except that now $A < B$, we need merely reverse the signs of the $P(\alpha_j)$ and of $N$ to apply Theorem 5 as stated. But we draw no conclusion if $A = B$, namely if $N = 0$.

A similar application of Theorem 2, still by use of equations (13), (14), and (15), yields

**Theorem 6.** Let the two disjoint point sets $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\beta_1, \beta_2, \ldots, \beta_n$ consist of the distinct points indicated, and lie in the disk $\Gamma: |z| \leq 1$. Let the polynomial $P(z) = N z^{m+n-1} + \cdots$ have the coefficient $N$ prescribed, and also the values $P(\alpha_j)$, and be a thus restricted infrapolynomial on $E$: $|\beta_j|$. Let $N_0$ and $\omega(z)$ be as defined in Theorem 5. Let $A, B, C, D$ be respectively the sum of the positive numbers among $\Re[S], \Re[-S], \Re[-iS], \Re[iS]$, where $S$ is the set $\{P(\alpha_j)/\omega'(\alpha_j), N - N_0\}$, and where we suppose $(A - B)^2 + (C - D)^2 \neq 0$. Then all zeros of $P(z)$ lie in the disk (6).

In connection with Theorem 6, the remark following the proof of Theorem 2 is significant.

**References**


Harvard University,
Cambridge, Massachusetts