TOPOLOGICAL ENTROPY

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Introduction. The purpose of this work is to introduce the notion of entropy as an invariant for continuous mappings.

1. Definitions and general properties. Let $X$ be a compact topological space.

Definition 1. For any open cover $\mathcal{U}$ of $X$ let $N(\mathcal{U})$ denote the number of sets in a subcover of minimal cardinality. A subcover of a cover is minimal if no other subcover contains fewer members. Since $X$ is compact and $\mathcal{U}$ is an open cover, there always exists a finite subcover. To conform with prior work in ergodic theory we call $H(\mathcal{U}) = \log N(\mathcal{U})$ the entropy of $\mathcal{U}$.

Definition 2. For any two covers $\mathcal{U}, \mathcal{V}$, $\mathcal{U} \vee \mathcal{V} = \{ A \cap B | A \in \mathcal{U}, B \in \mathcal{V} \}$ defines their join.

Definition 3. A cover $\mathcal{V}$ is said to be a refinement of a cover $\mathcal{U}$, $\mathcal{U} < \mathcal{V}$, if every member of $\mathcal{V}$ is a subset of some member of $\mathcal{U}$.

We have the following basic properties.

Property 00. The operation $\vee$ is commutative and associative.

Property 0. The relation $<$ is a reflexive partial ordering (1) on the family of open covers of $X$.

Property 1. $\mathcal{U} < \mathcal{V}, \mathcal{V} < \mathcal{W} \Rightarrow \mathcal{U} \vee \mathcal{V} < \mathcal{U} \vee \mathcal{W}$.

Proof. Consider $A' \cap B' \in \mathcal{U} \vee \mathcal{V}$ where $A' \in \mathcal{U}$ and $B' \in \mathcal{V}$. By hypothesis there exists $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that $A' \subseteq A, B' \subseteq B$. Thus $A' \cap B' \subseteq A \cap B$ where $A \cap B \in \mathcal{U} \vee \mathcal{V}$.

Remark. With the proper substitutions of $\mathcal{U}, \mathcal{V}$ and the cover $\{X\}$ in the statement above we obtain $\mathcal{U} < \mathcal{U} \vee \mathcal{V}$ and $\mathcal{V} < \mathcal{U} \vee \mathcal{V}$ which reveals that the family of open covers is a directed set with respect to the relation $<$.

Property 2. $\mathcal{U} < \mathcal{V} \Rightarrow N(\mathcal{U}) \leq N(\mathcal{V}), H(\mathcal{U}) \leq H(\mathcal{V})$.

Proof. Let $\{B_1, \ldots, B_{N(\mathcal{V})}\}$ be a minimal subcover of $\mathcal{V}$. Since $\mathcal{U} < \mathcal{V}$ there exists a subcover $\{A_1, \ldots, A_{N(\mathcal{V})}\}$ of $\mathcal{U}$. Therefore $N(\mathcal{U}) \leq N(\mathcal{V})$ and also $H(\mathcal{U}) \leq H(\mathcal{V})$.

Property 3. $\mathcal{U} < \mathcal{V} \Rightarrow N(\mathcal{U} \vee \mathcal{V}) = N(\mathcal{V}), H(\mathcal{U} \vee \mathcal{V}) = H(\mathcal{V})$.

Proof. It follows from Property 1 that $\mathcal{V} < \mathcal{U} \vee \mathcal{V}$ so that $N(\mathcal{V}) \leq N(\mathcal{U} \vee \mathcal{V})$. On the other hand $\mathcal{U} > \mathcal{U} \vee \mathcal{V}$ which is a consequence of the hypothesis. Thus $N(\mathcal{U} \vee \mathcal{V}) \leq N(\mathcal{V})$.

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**Property 4.** \( N(\mathfrak{A} \cup \mathfrak{B}) \leq N(\mathfrak{A}) \cdot N(\mathfrak{B}) \) and \( H(\mathfrak{A} \cup \mathfrak{B}) \leq H(\mathfrak{A}) + H(\mathfrak{B}) \).

**Proof.** Let \( \{ A_1, \ldots, A_{n(\mathfrak{A})} \} \) be a minimal subcover of \( \mathfrak{A} \) and \( \{ B_1, \ldots, B_{n(\mathfrak{B})} \} \) be a minimal subcover of \( \mathfrak{B} \). Then \( \{ A_i \cap B_j \mid i = 1, \ldots, n(\mathfrak{A}), j = 1, \ldots, n(\mathfrak{B}) \} \) is a subcover of \( \mathfrak{A} \cup \mathfrak{B} \). Consequently \( N(\mathfrak{A} \cup \mathfrak{B}) \leq N(\mathfrak{A}) \cdot N(\mathfrak{B}) \).

Let \( \varphi \) be a continuous mapping of \( X \) into itself. Let \( \mathfrak{A} \) be an open cover of \( X \) then from continuity, the family of \( \varphi^{-1}\mathfrak{A} = \{ \varphi^{-1}A \mid A \in \mathfrak{A} \} \) is again an open cover.

**Property 5.** \( \mathfrak{A} \subset \mathfrak{B} \Rightarrow \varphi^{-1}\mathfrak{A} \subset \varphi^{-1}\mathfrak{B} \).

**Property 6.** \( \varphi^{-1}(\mathfrak{A} \cup \mathfrak{B}) = \varphi^{-1}\mathfrak{A} \cup \varphi^{-1}\mathfrak{B} \).

**Property 7.** \( N(\mathfrak{A}) \leq N(\varphi^{-1}\mathfrak{A}) \).

**Proof.** Let \( \{ A_1, \ldots, A_{n(\mathfrak{A})} \} \) be a minimal subcover of \( \mathfrak{A} \). Since \( \{ \varphi^{-1}A_1, \ldots, \varphi^{-1}A_{n(\mathfrak{A})} \} \) is a cover, possibly not minimal, we have \( N(\varphi^{-1}\mathfrak{A}) \leq N(\mathfrak{A}) \).

**Remark.** When \( \varphi \) is onto then \( N(\mathfrak{A}) = N(\varphi^{-1}\mathfrak{A}) \).

**Property 8.**

\[
\lim_{n \to \infty} \left( \prod_{k=0}^{n-1} \varphi^{-k}\mathfrak{A} \right)/n = \lim_{n \to \infty} H(\mathfrak{A} \cup \varphi^{-1}\mathfrak{A} \cup \cdots \cup \varphi^{-(n+1)}\mathfrak{A})/n
\]

exists and is finite.

**Proof.**

\[
H(\mathfrak{A} \cup \cdots \cup \varphi^{-m-n+1}\mathfrak{A}) = H(\mathfrak{A} \cup \cdots \cup \varphi^{-m+1}\mathfrak{A} \cup \varphi^{-m}(\mathfrak{A} \cup \cdots \cup \varphi^{-1}\mathfrak{A}))
\leq H(\mathfrak{A} \cup \cdots \cup \varphi^{-m+1}\mathfrak{A}) + H(\varphi^{-m}(\mathfrak{A} \cup \cdots \cup \varphi^{-1}\mathfrak{A}))
\leq H(\mathfrak{A} \cup \cdots \cup \varphi^{-m+1}\mathfrak{A}) + H(\mathfrak{A} \cup \cdots \cup \varphi^{-n+1}\mathfrak{A}).
\]

The first equality follows from Property 6; the next inequality from Property 4; and the final inequality from Property 7.

Letting \( H_n = H(\mathfrak{A} \cup \cdots \cup \varphi^{-n+1}\mathfrak{A}) \) we have \( H_{m+n} \leq H_m + H_n \) and \( H_n \geq 0 \) for all positive integers \( m, n \). It is a standard exercise in analysis to prove that \( \lim_{n \to \infty} H_n/n \) exists and is finite.

**Definition.** The entropy \( h(\varphi, \mathfrak{A}) \) of a mapping \( \varphi \) with respect to a cover \( \mathfrak{A} \) is defined as \( \lim_{n \to \infty} H(\mathfrak{A} \cup \varphi^{-1}\mathfrak{A} \cup \cdots \cup \varphi^{-n+1}\mathfrak{A})/n \).

**Property 9.** \( h(\varphi, \mathfrak{A}) \leq H(\mathfrak{A}) \).

**Proof.** This follows from Properties 4 and 7.

**Property 10.** \( \mathfrak{A} \subset \mathfrak{B} \Rightarrow h(\varphi, \mathfrak{A}) \leq h(\varphi, \mathfrak{B}) \).

**Proof.** This follows from Properties 1, 2 and 5.

**Property 11.** If \( \varphi \) is a homeomorphism then \( h(\varphi, \mathfrak{A}) = h(\varphi^{-1}, \mathfrak{A}) \).

**Proof.**

\[
H(\mathfrak{A} \cup \cdots \cup \varphi^{-n+1}\mathfrak{A}) = H(\varphi^{-n}(\mathfrak{A} \cup \cdots \cup \varphi^{-1}\mathfrak{A}))
= H(\mathfrak{A} \cup \varphi^{-1}\mathfrak{A} \cup \cdots \cup \varphi^{-(n-1)}\mathfrak{A})
= H(\mathfrak{A} \cup (\varphi^{-1})^{-1}\mathfrak{A} \cup \cdots \cup (\varphi^{-1})^{-n+1}\mathfrak{A}).
\]

**Definition.** The entropy \( h(\varphi) \) of a mapping \( \varphi \) is defined as the sup \( h(\varphi, \mathfrak{A}) \) where the supremum is taken over all open covers \( \mathfrak{A} \). (Considering \( \{ h(\varphi, \mathfrak{A}) \mid \mathfrak{A} \} \)
as a net, \( h(\varphi) = \lim_{n \to \infty} h(\varphi, \mathcal{A}_n) \).

**Definition.** A sequence \( \{\mathcal{A}_n\} \) of open covers is refining if
\( (1) \mathcal{A}_n \in \mathcal{A}_{n+1}. \)
\( (2) \) For every open cover \( \mathcal{B} \) there exists \( \mathcal{A}_n \) such that \( \mathcal{B} \in \mathcal{A}_n. \)
A refining sequence of covers when it exists simplifies the computation of entropy as the next property reveals.

**Property 12.** If \( \{\mathcal{A}_n\} \) is a refining sequence of covers
\[ h(\varphi) = \lim_{n \to \infty} h(\varphi, \mathcal{A}_n). \]

**Proof.** This property follows from Property 10 above.

**2. General theorems.**

**Theorem 1.** Entropy is an invariant in the sense that \( h(\psi \varphi^{-1}) = h(\varphi) \) where \( \varphi \) is a continuous mapping of \( X \) into itself and \( \psi \) is a homeomorphism of \( X \) onto some \( X' \).

**Proof.**
\[
h(\psi \varphi^{-1}, \mathcal{A}) = \lim_{n \to \infty} H(\psi \mathcal{A} \cup \psi^{-1} \mathcal{A} \cup \cdots \cup \psi^{-n+1} \mathcal{A} \cup \mathcal{A})/n
= \lim_{n \to \infty} H(\mathcal{A} \cup \varphi^{-1} \mathcal{A} \cup \cdots \cup \varphi^{-n+1} \mathcal{A})/n
= h(\varphi, \mathcal{A}).
\]
As \( \mathcal{A} \) ranges over all open covers of \( X \), \( \psi \mathcal{A} \) ranges over all open covers of \( X' \) since \( \psi \) is a homeomorphism; hence \( h(\psi \varphi^{-1}) = h(\varphi) \).

**Theorem 2.** \( h(\varphi^k) = kh(\varphi) \) for \( k \) a positive integer.

**Proof.**
\[
h(\varphi^k) = h(\varphi^k, \mathcal{A} \cup \varphi^{-1} \mathcal{A} \cup \cdots \cup \varphi^{-k+1} \mathcal{A})
= k \lim_{n \to \infty} H(\mathcal{A} \cup \varphi^{-1} \mathcal{A} \cup \cdots \cup \varphi^{-k+1} \mathcal{A} \cup \varphi^{-k} \mathcal{A} \cup \cdots \cup \varphi^{-2k+1} \mathcal{A} \cup \cdots \cup \varphi^{-(n-1)k} \mathcal{A} \cup \cdots \cup \varphi^{-nk+1} \mathcal{A})/nk
= kh(\varphi, \mathcal{A})
\]
for any open cover \( \mathcal{A} \). Thus \( h(\varphi^k) \geq kh(\varphi) \). On the other hand, since
\[
\mathcal{A} \cup (\varphi^k)^{-1} \mathcal{A} \cup \cdots \cup (\varphi^{k-1})^{-n+1} \mathcal{A} < \mathcal{A} \cup \varphi^{-1} \mathcal{A} \cup \cdots \cup \varphi^{-nk+1} \mathcal{A},
\]
\[
h(\varphi, \mathcal{A}) = \lim_{n \to \infty} H(\mathcal{A} \cup \varphi^{-1} \mathcal{A} \cup \cdots \cup \varphi^{-nk+1} \mathcal{A})/nk
\geq \lim_{n \to \infty} H(\mathcal{A} \cup (\varphi^k)^{-1} \mathcal{A} \cup \cdots \cup (\varphi^k)^{-n+1} \mathcal{A})/nk
= h(\varphi^k, \mathcal{A})/k,
\]
for any open cover \( \mathcal{A} \); thus \( kh(\varphi) \geq h(\varphi^k) \).
Corollary. If \( \varphi \) is a homeomorphism then \( h(\varphi^k) = |k|h(\varphi) \) for any integer \( k \).

Theorem 3. Let \( X \) and \( Y \) be two compact topological spaces. Let \( \varphi_1 \) be a continuous mapping of \( X \) into itself and \( \varphi_2 \) a continuous mapping of \( Y \) into itself. Then

\[
h(\varphi_1 \times \varphi_2) = h(\varphi_1) + h(\varphi_2)
\]

where \( \varphi_1 \times \varphi_2 \) is the continuous mapping of \( X \times Y \) into itself defined by \( \varphi_1 \times \varphi_2 : (x,y) \mapsto (\varphi_1 x, \varphi_2 y) \).

Proof. Open covers of \( X \times Y \) of the form \( \mathcal{A} \times \mathcal{B} = \{ A \times B \mid A \in \mathcal{A}, B \in \mathcal{B} \} \) have the property that \( N_{X \times Y}(\mathcal{A} \times \mathcal{B}) = N_X(\mathcal{A}) \cdot N_Y(\mathcal{B}) \) and \( (\mathcal{A} \times \mathcal{B}) \vee (\mathcal{A}' \times \mathcal{B}') = (\mathcal{A} \vee \mathcal{A}') \times (\mathcal{B} \vee \mathcal{B}') \) where \( \mathcal{A}, \mathcal{A}' \) are covers of \( X \) and \( \mathcal{B}, \mathcal{B}' \) are covers of \( Y \). Although a cover itself usually indicates which space it covers, subscripts on \( N \) signifying the set being covered can be employed as above in order to reduce ambiguity. Consequently,

\[
h(\varphi_1 \times \varphi_2, \mathcal{A} \times \mathcal{B}) = h(\varphi_1, \mathcal{A}) + h(\varphi_2, \mathcal{B})
\]

which implies \( h(\varphi_1 \times \varphi_2) \geq h(\varphi_1) + h(\varphi_2) \). To establish the reverse inequality we need only show that for an arbitrary cover \( \mathcal{C} \) of \( X \times Y \) there exists a refinement of the form \( \mathcal{A} \times \mathcal{B} \) where \( \mathcal{A} \) is a cover of \( X \) and \( \mathcal{B} \) is a cover of \( Y \). Since every open subset of \( X \times Y \) is a union of rectangles \( A \times B \), an open subset of \( X \), \( B \) open subset of \( Y \), we can obtain a refinement of \( \mathcal{C} \) consisting only of open rectangles and from this choose a minimal subcover \( \mathcal{C}' \); i.e., \( \mathcal{C}' = \{ A'_1 \times B'_1, \ldots, A'_{N(\mathcal{C})} \times B'_{N(\mathcal{C})} \} \) and \( \mathcal{C} \subseteq \mathcal{C}' \). Let \( \mathcal{A}' = \{ A'_1, \ldots, A'_{N(\mathcal{C})} \} \) and \( \mathcal{B}' = \{ B'_1, \ldots, B'_{N(\mathcal{C})} \} \). Let \( A_x \) be the intersection of all sets from \( \mathcal{A}' \) which contain the element \( x \in X \) and \( B_y \) be the intersection of all sets from \( \mathcal{B}' \) which contain the element \( y \in Y \). These newly defined sets are of course open, and we can choose a finite number of points \( x_1, \ldots, x_m \) in \( X \) and \( y_1, \ldots, y_n \) in \( Y \) such that \( \mathcal{A} = \{ A_{x_1}, \ldots, A_{x_m} \} \) and \( \mathcal{B} = \{ B_{y_1}, \ldots, B_{y_n} \} \) are covers of \( X \) and \( Y \) respectively. Consider any set \( A_{x_i} \times B_{y_j} \subseteq \mathcal{A} \times \mathcal{B} \). Since \( \mathcal{C}' \) is a cover of \( X \times Y \), \( (x_i, y_j) \in A'_k \times B'_k \) for some integer \( k \) between 1 and \( N(\mathcal{C}') \). Because \( x_i \in A'_{x_i} \) and \( y_j \in B'_{y_j} \) it follows that \( A_{x_i} \subseteq A'_{x_i} \) and \( B_{y_j} \subseteq B'_{y_j} \); that is \( A_{x_i} \times B_{y_j} \subseteq A'_{x_i} \times B'_{y_j} \) which implies the desired result \( \mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{A} \times \mathcal{B} \).

In the next theorem we apply the elementary lemma.

Lemma. Suppose \( \{ a_n \} \) and \( \{ b_n \} \) are two sequences of real numbers not less than 1 such that \( \lim_{n \to \infty} (\log a_n)/n = a \) and \( \lim_{n \to \infty} (\log b_n)/n = b \) exist. Then \( \lim_{n \to \infty} \log(a_n + b_n)/n = \max\{a,b\} \).

Proof. For any \( c > a, b \) there exists an integer \( n_0 \) such that \( \log a_n < nc \) and \( \log b_n < nc \) whenever \( n \geq n_0 \). Thus \( \log (a_n + b_n) < nc + \log 2 \) for \( n \geq n_0 \). Consequently

\[
a, b \leq \liminf_{n \to \infty} (\log(a_n + b_n))/n \leq \limsup_{n \to \infty} (\log(a_n + b_n))/n < c.
\]
Therefore \( \lim_{n \to \infty} \frac{(\log(a_n + b_n))}{n} = \max\{a, b\} \).

**Theorem 4.** Let \( X_1 \) and \( X_2 \) be two closed subsets of \( X \) such that \( X = X_1 \cup X_2 \) and \( \varphi X_1 \subseteq X_1, \varphi X_2 \subseteq X_2 \) for a continuous mapping \( \varphi \) of \( X \) into itself. Then

\[
\varphi = \max \{ h(\varphi_1), h(\varphi_2) \},
\]

where \( \varphi_1 \) and \( \varphi_2 \) are the restrictions of \( \varphi \) to \( X_1 \) and \( X_2 \) respectively.

**Proof.** Let \( i = 1 \) or \( 2 \). For any open cover \( \mathcal{A} \) of \( X \) the family \( (\mathcal{A})_i = \{ A \cap X_i \mid A \in \mathcal{A} \} \) defines an open cover of \( X_i \) open in the subspace topology of \( X_i \). Employing subscripts on \( N \) to indicate the space whose cover is being counted we have \( N_i((\mathcal{A})_i) \leq N(\mathcal{A}) \). For open covers \( \mathcal{A} \) and \( \mathcal{B} \) of \( X \) we also have \( (\mathcal{A} \cup \mathcal{B})_i = (\mathcal{A})_i \cup (\mathcal{B})_i \). Furthermore \( \varphi_i^{-1}(\mathcal{A})_i = (\varphi_i^{-1}\mathcal{A})_i \) because of the invariance of \( X_i \). Let \( \mathcal{A}_i \) be an arbitrary open cover of \( X_i \), open in the subspace topology of \( X_i \). There exists an open cover \( \mathcal{A} \) of \( X \) such that \( (\mathcal{A})_i = \mathcal{A}_i \); namely, \( \mathcal{A} = \{ A \cup (X - X_i) \mid A \in \mathcal{A}_i \} \).

\[
N_i\left( \bigvee_{k=0}^{n-1} \varphi_i^{-k}\mathcal{A}\right) = N_i\left( \bigvee_{k=0}^{n-1} (\varphi_i^{-k}\mathcal{A})_i \right) = N_i\left( \left( \bigvee_{k=0}^{n-1} \varphi_i^{-k}\mathcal{A}\right)_i \right) \leq N\left( \bigvee_{k=0}^{n-1} \varphi_i^{-k}\mathcal{A}\right).
\]

Thus \( h(\varphi_i, \mathcal{A}) \leq h(\varphi, \mathcal{A}) \). Hence \( h(\varphi_i) \leq h(\varphi) \). On the other hand for any open cover \( \mathcal{A} \) of \( X \) we have

\[
N\left( \bigvee_{k=0}^{n-1} \varphi^{-k}\mathcal{A}\right) \leq N_1\left( \bigvee_{k=0}^{n-1} (\varphi^{-k}\mathcal{A})_1 \right) + N_2\left( \bigvee_{k=0}^{n-1} (\varphi^{-k}\mathcal{A})_2 \right)
\]

and as before

\[
N_i\left( \bigvee_{k=0}^{n-1} \varphi^{-k}\mathcal{A}\right)_i = N_i\left( \bigvee_{k=0}^{n-1} \varphi^{-k}(\mathcal{A})_i \right), \quad i = 1, 2;
\]

whereupon

\[
\log N\left( \bigvee_{k=0}^{n-1} \varphi^{-k}\mathcal{A}\right) \leq \log \left[ N_1\left( \bigvee_{k=0}^{n-1} \varphi^{-k}(\mathcal{A})_1 \right) + N_2\left( \bigvee_{k=0}^{n-1} \varphi^{-k}(\mathcal{A})_2 \right) \right].
\]

Now applying the lemma

\[
h(\varphi, \mathcal{A}) \leq \max \{ h(\varphi_1, (\mathcal{A})_1), h(\varphi_2, (\mathcal{A})_2) \}
\]

which yields upon taking suprema

\[
h(\varphi) \leq \max \{ h(\varphi_1), h(\varphi_2) \}.
\]

**Theorem 5.** Let \( \sim \) be an equivalence relation on a compact set \( X \). Let \( \varphi \) be a continuous mapping of \( X \) into itself such that \( \varphi x \sim \varphi y \) if \( x \sim y \). If \( \overline{\varphi} \) is the mapping of \( X/\sim \) into itself defined by \( \overline{\varphi}x = \pi\varphi \) where \( \pi \) is the projection of \( X \) onto \( X/\sim \) then

\[
h(\overline{\varphi}) \leq h(\varphi).
\]
Proof. Let $\mathcal{A}$ be an open cover of $X/\sim$. Then $\pi^{-1}\mathcal{A}$ is an open cover of $X$ and $N_X(\pi^{-1}\mathcal{A}) = N_{X/\sim}(\mathcal{A})$. Therefore $h(\varphi, \pi^{-1}\mathcal{A}) = h(\varphi, \mathcal{A})$ and hence 

$$h(\varphi) = \sup_{\mathcal{A}} h(\varphi, \mathcal{A}) \equiv \sup_{\mathcal{A}} h(\varphi, \pi^{-1}\mathcal{A}) = \sup_{\mathcal{A}} h(\varphi, \mathcal{A}) = h(\varphi).$$

3. Computation of entropy in metric spaces. Let $X$ be a compact metric space with metric $d$.

**Definition.** The diameter $d(\mathcal{A})$ of a cover $\mathcal{A}$ is defined by

$$d(\mathcal{A}) = \sup_{A \in \mathcal{A}} d(A)$$

where $d(A)$ is the diameter of the set $A$.

**Lebesgue's Covering Lemma.** For every open cover $\mathcal{A}$ of a compact metric space $X$ there exists $\epsilon > 0$ such that if $U$ is a set with $d(U) < \epsilon$ then $U$ is contained in one of the members of $\mathcal{A}$. The supremum of all such numbers $\epsilon$ is called the Lebesgue number of $\mathcal{A}$.

**Rephrasing of Lebesgue's Covering Lemma.** For open covers $\mathcal{A}$ and $\mathcal{B}$ of $X$, if $d(\mathcal{B}) < \text{Lebesgue number of } \mathcal{A}$ then $\mathcal{A} \supset \mathcal{B}$.

**Corollary.** If $\{\mathcal{A}_n\}$ is a sequence of open covers such that

1. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$,
2. $d(\mathcal{A}_n) \to 0$, as $n \to \infty$,

then $\{\mathcal{A}_n\}$ is a refining sequence.

**Remark.** This corollary assures the existence of refining sequences in metric spaces. For example, the sequence $\{\mathcal{A}_n\}$, where $\mathcal{A}_n$ is the set of all spheres of diameter less than $1/n$, is refining. In addition from any sequence $\{\mathcal{B}_n\}$ of covers satisfying condition (2) of the corollary we can construct $\mathcal{A}_n = \bigvee_{k=0}^{n} \mathcal{B}_k$ which satisfies both (1) and (2) and thus is refining.

**Example 1.** If $\varphi$ is an isometry of $X$ onto itself then $h(\varphi) = 0$.

**Proof.** Let $\mathcal{A}_p$ be the family of all open sets of diameter less than $1/p$. Such a family enjoys the property that $\mathcal{A}_p \vee \mathcal{A}_p = \mathcal{A}_p$. Since $\varphi$ is an isometry, $\varphi^{-1}\mathcal{A}_p = \mathcal{A}_p$. This implies $\mathcal{A}_p \vee \varphi^{-1}\mathcal{A}_p \vee \cdots \vee \varphi^{-n+1}\mathcal{A}_p$. Therefore $h(\varphi, \mathcal{A}_p) = 0$. According to the previous corollary $\{\mathcal{A}_p\}$ is a refining sequence so that we can conclude $h(\varphi) = 0$.

**Example 1a.** Let $(X, \varphi)$ be an equicontinuous compact dynamical system, then $h(\varphi) = 0$.

**Proof.** The metric $d'$ defined by $d'(x, y) = \sup_{-r < \varphi^n < r} d(\varphi^n x, \varphi^n y)$ is equivalent to $d$. With respect to this new metric, $\varphi$ is an isometry and the above statement applies.

**Example 1b.** Let $X$ be a compact separable topological group and
\[ \varphi : x \mapsto axb, a, b \subseteq X. \] Then \( h(\varphi) = 0. \)

**Proof.** \( X \) is metrizable, say with metric \( d \). The rotation \( \varphi \) is an isometry with respect to the metric \( d' \) defined by \( d'(x, y) = \sup_{u,v \in X} d(uxv, uyv) \) which is equivalent to \( d \).

**Example 2.** Let \( X = \{ (x_1, x_2) | x_1 + x_2 = 1 \} \) be the unit circle. If \( \varphi \) is a homeomorphism of \( X \) onto itself then \( h(\varphi) = 0. \)

**Proof.** Let \( \mathcal{A}_p \) be a covering of \( X \) by intervals on \( X \) of arc length \( 1/p \). The covering \( \mathcal{A}_p \cup \varphi^{-1}\mathcal{A}_p \cup \cdots \cup \varphi^{-n+1}\mathcal{A}_p \) is a covering of \( X \) by intervals and \( N(\mathcal{A}_p \cup \cdots \cup \varphi^{-n+1}\mathcal{A}_p) \leq nN(\mathcal{A}_p) \). Thus \( h(\varphi, \mathcal{A}_p) = 0; \) and since \( \{ \mathcal{A}_p \} \) is refining, \( h(\varphi) = 0. \)

**Example 3.** Expressing the space of two-sided infinite sequences of zeros and ones by \( X = \prod_{i=1}^{\infty} X_i \), where \( X_i = \{0, 1\} \), the discrete topology on \( X_i \) with the discrete topology the space \( X \) is compact in the cartesian product topology by virtue of the Tychonoff theorem. Let \( (x)_i \) denote the \( i \)-th component of the sequence \( x \in X \). Then cartesian product topology on \( X \) is the same as that determined by the metric \( d \) where

\[
d(x, y) = \sum_{i=0}^{\infty} |(x)_i - (y)_i|/2^{|i|}.
\]

Consider the homeomorphism \( \varphi \) of \( X \) onto itself called the *shift* and defined by \( (\varphi x)_i = (x)_{i+1} \). Let \( \mathcal{A} = \{ \{ x \} | (x)_0 = 0 \}, \{ x \} | (x)_0 = 1 \} \) and

\[
\mathcal{A}_p = \bigvee_{k=-p}^{p} \varphi^k \mathcal{A}, \quad p = 0, 1, 2, \ldots.
\]

Since \( d(\mathcal{A}_p) \to 0 \), as \( n \to \infty \), the sequence \( \{ \mathcal{A}_p \} \) is refining. Next

\[
h(\varphi, \mathcal{A}) \leq h(\varphi, \mathcal{A}_p) = \lim_{n \to \infty} H \left( \bigvee_{k=-p}^{n-1} \varphi^{-k} \mathcal{A}_p \right)/n
\]

\[
= \lim_{n \to \infty} H \left( \bigvee_{k=-p}^{p} \varphi^k \mathcal{A} \cup \bigvee_{k=-p-1}^{p-1} \varphi^k \mathcal{A} \cup \cdots \cup \bigvee_{k=-p-n+1}^{p-n+1} \varphi^k \mathcal{A} \right)/n.
\]

From property (3) it then follows that

\[
h(\varphi, \mathcal{A}_p) = \lim_{n \to \infty} H \left( \bigvee_{k=-p-n+1}^{0} \varphi^k \mathcal{A} \right)/n \leq \lim_{n \to \infty} H \left( \bigvee_{k=-p-n+1}^{0} \varphi^k \mathcal{A} \right)/n
\]

\[
= \lim_{n \to \infty} H \left( \bigvee_{k=-p-n+1}^{0} \varphi^k \mathcal{A} \right)/n = h(\varphi, \mathcal{A}).
\]

Counting reveals that \( N(\bigvee_{k=0}^{n-1} \varphi^{-k} \mathcal{A}) = 2^n \) so that \( h(\varphi, \mathcal{A}) = \log 2 \). Thus \( h(\varphi, \mathcal{A}_p) = \log 2 \) which by the refining property of the sequence \( \{ \mathcal{A}_p \} \) implies that \( h(\varphi) = \log 2 \).

**Remark.** If \( X = \{ 0, 1, \ldots, N-1 \} \) above, then \( h(\varphi) = \log N. \) Furthermore
if \( X \), above is some compact Hausdorff space containing an infinite number of points, then \( h(\varphi) = \infty \).

**Example 4.** Let \( X \) be the two dimensional-torus, i.e., \( X = E^2/\sim \) where \( E^2 \) is the Euclidean plane and \( \sim \) is the equivalence relation which identifies two points in the plane if their corresponding coordinates differ by integers. A metric on \( X \) can be defined in terms of the metric on \( E^2 \) by taking the distance between two points of \( X \) to be the minimum distance between any representatives of these points in \( E^2 \). A continuous group automorphism \( \varphi \) of \( X \) has a representation \( \varphi: (x,y) \to (ax + by, cx + dy) \) (additions mod 1) where \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is a unimodular matrix \( A \), that is, a matrix of integers with determinant \( \pm 1 \). Suppose \( A \) has two linearly independent characteristic vectors \( \alpha, \beta \), associated with characteristic values \( \lambda, \mu \), where \( |\lambda| \geq 1 \). Then

\[
h(\varphi) = \log |\lambda|.
\]

**Proof.** Consider a covering of \( E^2 \) by all open parallelograms with sides parallel to \( \alpha \) and \( \beta \) and having length \( 1/p \). Each set is a representative of an equivalence class of sets under \( \sim \). Let \( \mathcal{A}_p \) be an open covering of \( X \) by these equivalence classes. If \( A \) is one of the above parallelograms then \( \varphi^{-n}A \) is equivalent to a parallelogram having sides of length \( |\lambda|^n/p \) and \( |\lambda|^{-n}/p \) which are again parallel to characteristic vectors. Considering one parallelogram equivalent to one of the sets of \( \mathcal{A}_p \), \( p > 1 \), it can be seen that it takes \( |\lambda|^{n-1} \) parallelograms to cover it which are equivalent to sets in \( \varphi^{-n+1}\mathcal{A}_p \). Thus

\[
p^2|\lambda|^{n-1} \leq N(\mathcal{A}_p \vee \varphi^{-1}\mathcal{A}_p \vee \cdots \vee \varphi^{-n+1}\mathcal{A}_p) \leq |\lambda|^{n-1}N(\mathcal{A}_p)
\]

from which follows \( h(\varphi, \mathcal{A}_p) = \log |\lambda| \), for \( p > 1 \). Since \( \{\mathcal{A}_p\}_{p=1,2,\cdots} \) is a refining sequence we have \( h(\varphi) = \log |\lambda| \).

**Remark.** If \( X \) is an \( n \)-dimensional torus and \( \phi \) a continuous automorphism of \( X \) determined by an \( n \) by \( n \) unimodular matrix having real characteristic values \( \lambda_1, \cdots, \lambda_n \) and \( n \) linearly independent characteristic vectors, then a similar argument yields

\[
h(\varphi) = \sum_{|\lambda| \geq 1} \log |\lambda|.
\]

A curiosity based on the techniques of this work is the following.

**Theorem.** Let \( X \) be a compact metric space with an infinite number of points. Let \( \varphi \) be a continuous mapping of \( X \) into itself. For any open cover \( \mathcal{A} \) there exists a number \( \delta > 0 \) (depending on \( \mathcal{A} \) and \( \varphi \)) such that

\[
d(\varphi^{-1}\mathcal{A} \vee \varphi^{-2}\mathcal{A} \vee \cdots \vee \varphi^{-n}\mathcal{A}) \geq \delta > 0
\]

for all \( n \).

**Proof.** Suppose \( d(\varphi^{-1}\mathcal{A} \vee \cdots \vee \varphi^{-n}\mathcal{A}) \to 0 \), as \( n \to \infty \). There exists an integer \( N \) such that if \( n \geq N \) then \( d(\varphi^{-1}\mathcal{A} \vee \cdots \vee \varphi^{-n}\mathcal{A}) \) is less than the
Lebesgue number of $\mathfrak{A}$. Therefore $\mathfrak{A} < \varphi^{-1}\mathfrak{A} \vee \cdots \vee \varphi^{-n}\mathfrak{A}, n \geq N$. Thus 
\[ N(\mathfrak{A} \vee \varphi^{-1}\mathfrak{A} \vee \cdots \vee \varphi^{-n}\mathfrak{A}) = N(\varphi^{-1}\mathfrak{A} \vee \cdots \vee \varphi^{-n}\mathfrak{A}) = N(\mathfrak{A} \vee \cdots \vee \varphi^{-n+1}\mathfrak{A}). \]
By induction 
\[ N(\mathfrak{A} \vee \varphi^{-1}\mathfrak{A} \vee \cdots \vee \varphi^{-n}\mathfrak{A}) = N(\mathfrak{A} \vee \cdots \vee \varphi^{-N}\mathfrak{A}) \text{ for } n \geq N; \]
that is $N(\mathfrak{A} \vee \cdots \vee \varphi^{-n}\mathfrak{A})$ is bounded, say by the number $M$. Choose $M + 1$ distinct points $x_1, \ldots, x_{M+1}$ and let $n$ be so large that 
\[ d(\mathfrak{A} \vee \cdots \vee \varphi^{-n}\mathfrak{A}) < \min_{1 \leq i < j \leq M+1} d(x_i, x_j). \]
This is a contradiction because to cover $x_1, \ldots, x_{M+1}$ with sets whose diameters are smaller than $\min_{1 \leq i < j \leq M+1} d(x_i, x_j)$ requires at least $M + 1$ sets.

4. Background and unsolved problems. In ergodic theory the notion of entropy for measure-preserving transformations has been extensively studied by the Russian school. The measure-theoretic entropy is defined as follows \cite{4}. Let $(X, \mathcal{E}, \mu)$ be a measure space with $X$ a set of points, $\mathcal{E}$ a sigma-field of measurable subsets of $X$, and $\mu$ a countably additive measure on $\mathcal{E}$ with $\mu(X) = 1$. Let $\mathfrak{A} = \{A_1, \ldots, A_n\}$ be a finite measurable partition of $X$, i.e., $X = \bigcup_{i=1}^{n} A_i, A_i \in \mathfrak{A}, \mu(A_i \cap A_j) = 0, i \neq j$. The measure-theoretic entropy $H_\mu(\mathfrak{A})$ is defined by
\[
H_\mu(\mathfrak{A}) = -\sum_{i=1}^{n} \mu(A_i) \log \mu(A_i).
\]
Again $\mathfrak{A} \vee \mathfrak{B}$ denotes the common refinement of two measurable partitions $\mathfrak{A}$ and $\mathfrak{B}$ and we have $H_\mu(\mathfrak{A} \vee \mathfrak{B}) \leq H_\mu(\mathfrak{A}) + H_\mu(\mathfrak{B})$. If $\varphi$ is a measure-preserving transformation then $H_\mu(\varphi^{-1}\mathfrak{A}) = H_\mu(\mathfrak{A})$ and the limit $h_\mu(\varphi, \mathfrak{A}) = \lim_{n \to \infty} H_\mu(\mathfrak{A} \vee \cdots \vee \varphi^{-n+1}\mathfrak{A})/n$ exists for every finite measurable partition $\mathfrak{A}$.

The number $H_\mu(\mathfrak{A})$ perhaps appears somewhat mysterious. However, note that if $\mu(A_i) = 1/n, i = 1, \ldots, n$, then $H_\mu(\mathfrak{A}) = \log n$. The contents of this present work indicate that $H_\mu(\mathfrak{A})$ is merely a delicate method of counting the number of sets in a partition in such a manner that the measures of the sets are given their appropriate weight in the tally. The quantity $h_\mu(\varphi) = \sup h_\mu(\varphi, \mathfrak{A})$ where the supremum is taken over all finite measurable partitions $\mathfrak{A}$ is called the entropy of $\varphi$. This number is a spatial isomorphism invariant for measure-preserving transformations on $(X, \mathcal{E}, \mu)$.

The following theorem of J. G. Sinaí \cite{4} is used to compute entropies.

**Theorem.** If $\mathfrak{A}^*$ is a measurable partition such that $\mathcal{E}$ is the sigma-field generated by the family of sets $\bigcup_{n=1}^{\infty} \varphi^n\mathfrak{A}^*$, then 
\[ h_\mu(\varphi) = h_\mu(\varphi, \mathfrak{A}^*). \]

The material in this paper we patterned after another theorem of this sort due to Rohlin \cite{3}. 

**Theorem.** If $|\mathfrak{A}_n| n = 1, 2, \ldots$ is a sequence of partitions such that $\mathfrak{A}_n < \mathfrak{A}_{n+1}$ and the sigma-field generated by $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ is $\mathcal{E}$, then $h(\varphi) = \lim_{n \to \infty} h(\varphi, \mathfrak{A}_n)$. 

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Let us consider some examples.

1. Let \( X \) be compact separable Abelian group and \( \mu \) Haar measure. Suppose \( \varphi : x \to ax \) then \( h_\mu (\varphi) = 0 \) \[3\].

2. Let \( X \) be the \( n \)-dimensional torus and \( \mu \) Haar measure. Let \( \varphi \) be a continuous automorphism on \( X \) whose associated unimodular matrix has \( n \) real characteristic values \( \lambda_1, \ldots, \lambda_n \) and \( n \) linearly independent character vectors, then
   \[
   h_\mu (\varphi) = \sum_{|\lambda| \neq 1} \log |\lambda| \quad \text{[4].}
   \]

3. Let \( (X, \mathfrak{G}, \mu) = \prod_{i=1}^n (X_i, \mathfrak{G}_i, \mu_i) \), where \( X_i = \{0, 1\}, \mathfrak{G}_i = \{\varphi, \{0\}, \{1\}, X_i \} \) and \( \mu_i(\{0\}) = p, \mu_i(\{1\}) = 1 - p, 0 \leq p \leq 1, i = 0 \pm 1, \pm 2, \ldots \). Let \( \varphi \) be the shift transformation on \( (X, \mathfrak{G}, \mu) \), i.e., \( \varphi(x)_i = (x)_{i+1} \). Then
   \[
   h(\varphi) = -p \log p - (1 - p) \log (1 - p) \quad \text{[1].}
   \]
   If \( X_i \) is considered to be a compact topological group then \( \mu_i \) is Haar measure only when \( p = 1/2 \). In this case \( X \) is also a compact topological group with the direct product measure \( \mu \) being its Haar measure. The mapping \( \varphi \) is a continuous automorphism on \( X \) and \( h(\varphi) = \log 2 \) which coincides with the maximum of \( -p \log p - (1 - p) \cdot \log (1 - p) \) for \( 0 < p < 1 \).

We adopt the convention that \( h_\mu \) denotes measure-theoretic entropy with respect to a measure \( \mu \) while \( h \) denotes topological entropy. Examination of examples having both topological and measure-theoretic aspects leads to some conjectures.

**Conjecture 1.** Let \( X \) be a compact topological space and \( \mu \) a regular measure. If \( \varphi \) is a homeomorphism on \( X \) and also a measure-preserving transformation then \( h_\mu (\varphi) \leq h(\varphi) \).

**Conjecture 2.** Let \( X \) be a compact metric space and \( \varphi \) a homeomorphism on \( X \). A result of Kryloff and Bogoliouboff \[2\] states that there exists regular measures \( \mu \) with respect to which \( \varphi \) is measure preserving. Then
   \[
   h(\varphi) = \sup h_\mu (\varphi) \quad \text{where the supremum is taken over all such invariant regular measures. If this is true, is there something special about the measure or measures where this supremum is assumed?}
   \]

**Conjecture 3.** Let \( X \) be a compact separable group and \( \varphi \) a continuous automorphism on \( X \). Then
   \[
   h(\varphi) = h_\mu (\varphi),
   \]
   where \( \mu \) is Haar measure.

In other directions we conjecture

**Conjecture 4.** Suppose \( \varphi_t \) is a one parameter flow on a compact space \( X \). Then \( h(\varphi_t) = |t| h(\varphi) \). This is true for \( t \) rational. Is it true for all \( t \)?

**Conjecture 5.** Suppose \( X \) and \( Y \) are compact topological spaces and \( \varphi_x \) \( x \in X \) is a family of homeomorphisms on \( Y \) such that \( \varphi : (x, y) \to (x, \varphi_x(y)) \) is a continuous mapping of \( X \times Y \) onto itself. Is
From Theorem 4 this can be verified when $X$ is a finite set.

5. **Conclusion.** The notion of entropy has an abstract formulation which we have not dealt with here. It can be tailored to fit mappings on other mathematical structures. For example, let $G$ be an Abelian group; for a finite subgroup $\mathfrak{A}$ of $G$ let $N(\mathfrak{A})$ equal the order of $\mathfrak{A}$; let $\mathfrak{A} \vee \mathfrak{B}$ be the group generated by two finite subgroups $\mathfrak{A}$ and $\mathfrak{B}$ of $G$; let $\mathfrak{A} < \mathfrak{B}$ mean $\mathfrak{A}$ is a subgroup of $\mathfrak{B}$; and finally let $\varphi$ be an endomorphism of $G$. The basic properties now hold and we can define the entropy $h(\varphi, \mathfrak{A})$ of $\varphi$ with respect to a finite subgroup $\mathfrak{A}$. The entropy $h(\varphi)$ of $\varphi$ is then $\sup h(\varphi, \mathfrak{A})$ where supremum is taken over all finite subgroups $\mathfrak{A}$ of $G$; and analogies to the general theorems can be established.

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**References**