GENERALIZED COMMUTING PROPERTIES OF MEASURE-PRESERVING TRANSFORMATIONS(1)

BY

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1. **Introduction.** Let $(X, \mathfrak{B}, m)$ be a measure space where X is a set of elements, $\mathfrak{B}$ a σ-field of measurable subsets of X, and m a countably additive measure defined on $\mathfrak{B}$. A measure-preserving transformation T of the measure space $(X, \mathfrak{B}, m)$ is almost everywhere a one-to-one mapping of X into itself such that if $B \in \mathfrak{B}$ then TB and $T^{-1}B \in \mathfrak{B}$ with $m(TB) = m(T^{-1}B) = m(B)$. Let $\boldsymbol{G}$ be the group of all measure-preserving transformations of X onto itself with I denoting the identity transformation on X. Associated with a measure-preserving transformation T is a sequence $\boldsymbol{C}_n(T)$, $n = 0, 1, 2, \cdots$, of subfamilies of $\boldsymbol{G}$ defined inductively as follows:

$$\boldsymbol{C}_0(T) = \{S : S \in \boldsymbol{G},\ S = I \text{ a.e.}\},$$
$$\boldsymbol{C}_n(T) = \{S : S \in \boldsymbol{G},\ STS^{-1}T^{-1} \in \boldsymbol{C}_{n-1}(T)\}.$$

In view of the observation that $\boldsymbol{C}_1(T)$ is the set of measure-preserving transformations which commute with T almost everywhere $\boldsymbol{C}_n(T)$ will be called the *nth class of generalized T-commuting transformations*. It is clear that

$$\boldsymbol{C}_n(T) \subseteq \boldsymbol{C}_{n+1}(T), \qquad n = 0, 1, 2, \cdots.$$

If there exists an integer N such that $\boldsymbol{C}_N(T) = \boldsymbol{C}_{N+1}(T)$ then $\boldsymbol{C}_n(T) = \boldsymbol{C}_{n+1}(T)$ for all $n \geqq N$; and in this case we define $N(T) = \min\{N : \boldsymbol{C}_N(T) = \boldsymbol{C}_{N+1}(T)\}$, otherwise we set $N(T) = \infty$. We call $N(T)$ the *generalized commuting order of T*. If $RT_1R^{-1} = T_2$ a.e., where $R, T_1, T_2 \in \boldsymbol{G}$ then it is clear that $N(T_1) = N(T_2)$, i.e., $N(T)$ is a conjugacy invariant or spatial isomorphism invariant for measure-preserving transformations. It is of course not to be expected that the converse holds.

2. **Pure point spectrum.** In order to avoid pathologies which can arise with too liberal a choice of measure space we impose the mild restriction that $(X, \mathfrak{B}, m)$ be a Lebesgue space [8] with $m(X) = 1$. A measure-perserving transformation T is said to be ergodic if $TB = B$, $B \in \mathfrak{B}$ implies

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$m(B) = 0$ or $m(X - B) = 0$. A measure-preserving transformation T induces a unitary operation U_T on $L^2(X, \mathfrak{B}, m)$ defined by $U_T: f(x) \to f(Tx)$. The proper values and the proper functions of this unitary operator are also called proper values and proper functions of the measure-preserving transformation; furthermore, it is clear that if a complex number c is a proper value of T then $|c| = 1$. The proper value theorem [3, p. 34] relates the concept of ergodicity of a measure-preserving transformatiom with its proper values. A measure-preserving transformation T on a space of finite measure is ergodic if and only if the number 1 is a simple proper value of T; moreover, if a measure-preserving transformation T is ergodic the absolute value of every proper function of T is constant a.e., every proper value is simple, and the set of all proper values of T is a subgroup of the circle group. A measure-preserving transformation is said to have pure point spectrum if $L^2(X, \mathfrak{B}, m)$ can be spanned by its proper functions. In connection with generalized commuting properties of measure-preserving transformations the following theorem can be proved.

THEOREM. *If a measure-preserving transformation T and all of its powers are ergodic and T has pure point spectrum then its commuting order is two. Furthermore, $C_0(T)$, $C_1(T)$ and $C_2(T)$ are subgroups of G.*

We shall discuss this theorem and its proof in the light of the representation theorem for ergodic measure-preserving transformations with pure point spectrum on Lebesgue spaces. P. R. Halmos and J. von Neumann [4] showed such a transformation could be considered as measure theoretically the same as a rotation $T_a: x \to ax$ which is Haar measure-preserving on some compact separable Abelian group(2). They also proved that a rotation $T_a: x \to ax$ on a compact separable Abelian group X is ergodic if and only if $\{a^n: n = 0, \pm 1, \pm 2, \cdots\}$ is dense in X. Such groups admitting ergodic rotations are called monothetic and are discussed in [2] and [5]. The above theorem can now be stated in the following more detailed form.

THEOREM. *Let T_a and all its powers be ergodic on a compact separable Abelian group X; then $N(T_a) = 2$. Furthermore, $C_0(T_a)$, $C_1(T_a)$ and $C_2(T_a)$ are subgroups of G which can be characterized as follows:* (i) *$C_0(T_a)$ is, as defined, the family of measure-preserving transformations almost everywhere equal to the identity;* (ii) *$C_1(T_a)$ is the family of measure-preserving transformations almost everywhere equal to rotations on the group;* (iii) *$C_2(T_a)$ is the family of measure-preserving transformations almost everywhere equal to rotations composed with continuous automorphisms of the group X.*

(2) The standard facts concerning topological groups used in this work are contained in [6] and [7]. By separable we mean satisfying the second axiom of countability.

We shall establish the theorem in a sequence of propositions the first of which is to be found in [4] with only an indication of its proof.

Proposition 1. *Let T_a be an ergodic rotation on a compact separable Abelian group X. If $S \in C_1(T_a)$ i.e. $ST_a = T_aS$ a.e., there exists $b \in X$ such that $S = T_b$ a.e.*

Proof. Let $x^* \in X^*$ the character group of X. Since S commutes with T_a a.e., we have

$$x^*(ST_ax) = x^*(T_aSx) = x^*(a \cdot Sx) = x^*(a)x^*(Sx) \quad \text{a.e.}$$

Thus $x^*S(\cdot)$ is a proper function of T_a associated with the proper value $x^*(a)$. But $x^*(\cdot)$ also is a proper function T_a associated with the proper value $x^*(a)$. The proper value theorem asserts that all proper values of an ergodic measure-preserving transformation are simple. Hence for each $x^* \in X^*$

$$x^*(Sx) = \phi(x^*)x^*(x) \quad \text{a.e.,}$$

where $\phi(x^*)$ is a constant of absolute value 1 which depends only on the character x^*. This relation yields

$$x_1^*x_2^*(Sx) = \phi(x_1^*x_2^*)x_1^*x_2^*(x) \quad \text{a.e.,}$$

$$x_1^*x_2^*(Sx) = x_1^*(Sx)x_2^*(Sx) = \phi(x_1^*)\phi(x_2^*)x_1^*(x)x_2^*(x) \quad \text{a.e.}$$

for $x_1^*, x_2^* \in X^*$ which in turn imply

$$\phi(x_1^*x_2^*) = \phi(x_1^*)\phi(x_2^*),$$

$$\phi(x^{*-1}) = \overline{\phi(x^*)}$$

for $x_1^*, x_2^*, x^* \in X^*$. In other words ϕ is a homomorphism of the character group X^* into the multiplicative group of complex numbers of absolute value 1. The character group of a compact group is discrete so ϕ is automatically continuous. Thus ϕ is a character on X^*, i.e., $\phi \in X^{**}$. By Pontrjagin's duality theorem, there exists an element $b \in X$ such that $\phi(x^*) = x^*(b)$. Thus $x^*(Sx) = x^*(b)x^*(x) = x^*(bx)$ a.e. Let N_{x^*} be the zero measure set of points x where $x^*(Sx) \neq x^*(bx)$. Since X is separable X^* is countable so that $N = \bigcup_{x^* \in X^*} N_{x^*}$ is also of measure zero. Therefore for all x^* we have $x^*(Sx) = x^*(bx), x \in X - N$. From the Peter-Weyl theorem there are sufficiently many characters to distinguish the elements of X; consequently $Sx = bx, x \in X - N$.

The commutativity of X states that if $S = T_b$ a.e. for some $b \in X$, then $S \in C_1(T_a)$. This fact along with Proposition 1 completely characterizes the first class of generalized commuting transformations of T_a.

Proposition 2. *Let T_a be an ergodic rotation on a compact separable Abelian group X. If $S \in C_2(T_a)$, then there exists an element $b \in X$ and a* *continuous automorphism K of X such that $S = T_b K$ a.e.*

Proof. Essentially we are characterizing those measure-preserving transformations which supply spatial isomorphisms between ergodic rotations. By hypothesis $ST_aS^{-1}T_a^{-1} \in C_1(T_a)$. Proposition 1 assures the existence of an element $c \in X$ such that $ST_aS^{-1}T_a^{-1} = T_c$ a.e. or $ST_aS^{-1} = T_d$ a.e., where $d = ac$. For each $x^* \in X^*$ the function $x^*S(\cdot)$ is a proper function of T_a associated with the proper value $x^*(d)$ because $x^*(ST_ax) = x^*(T_dSx)$ $= x^*(d \cdot Sx) = x^*(d)x^*(Sx)$ a.e. Next we shall show that for each $x^* \in X^*$ there exists a character denoted by K^*x^* such that $K^*x^*(a) = x^*(d)$, for assuming the contrary there would exist $y^* \in X^*$ such that $y^*(d) \neq x^*(a)$ for any $x^* \in X^*$. As is well known, proper functions associated with different proper values are orthogonal in $L^2(X)$; so $y^*S(\cdot)$ having $y^*(d)$ as a proper value would be orthogonal to every $x^* \in X^*$. Since X^* is a complete orthomormal system in $L^2(X)$ a contradiction would immediately arise. For each $x^* \in X^*$ the character K^*x^* is uniquely defined for suppose L^*x^* were another satisfying $L^*x^*(a) = x^*(d)$. We would have $L^*x^*(a)$ $= K^*x^*(a)$; that is the two characters would agree on the generating element a which means that they must agree everywhere. The above relation between K^*x^* and x^* then reveals that $K^*x^*(\cdot)$ and $x^*S(\cdot)$ are proper functions of T_a both having the same proper value $x^*(d)$. Since T_a is ergodic, the proper value theorem asserts that $x^*(d)$ is simple. Therefore for each $x^* \in X^*$ there exists a complex number $\phi(x^*)$ depending on x^* such that

$$\phi(x^*)K^*x^*(x) = x^*(Sx) \quad \text{a.e.} \tag{*}$$

In addition it is clear that $|\phi(x^*)| = 1$. The set

$$\{x^*S : x^* \in X^*\} = \{\phi(x^*)K^*x^* : x^* \in X^*\}$$

is the image of X^* under the unitary operator U_s; consequently, since X^* is complete in $L^2(X)$, so is $\{K^*x^* : x^* \in X^*\}$. Thus we can conclude that the mapping $K^* : x^* \to K^*x^*$ maps X^* onto itself. From (*) we obtain

$$\begin{aligned}\phi(x_1^*x_2^*)K^*x_1^*x_2^*(x) &= x_1^*x_2^*(Sx) = x_1^*(Sx)x_2^*(Sx)\\ &= \phi(x_1^*)K^*x_1^*(x)\phi(x_2^*)K^*x_2^*(x) \quad \text{a.e.}\end{aligned}$$

for $x_1^*, x_2^* \in X^*$. Since characters are continuous we obtain

$$\phi(x_1^*x_2^*)K^*x_1^*x_2^*(x) = \phi(x_1^*)\phi(x_2^*)K^*x_1^*(x)K^*x_2^*(x)$$

everywhere. In particular for $x = e$, the unit element of the group, the equation yields

$$\phi(x_1^*x_2^*) = \phi(x_1^*)\phi(x_2^*)$$

and consequently

$$K^*x_1^*x_2^* = K^*x_1^*K^*x_2^*$$

for $x_1^*, x_2^* \in X^*$. In addition it follows that

$$\phi(x^{*-1}) = \overline{\phi(x^*)}$$

and

$$K^*x^{*-1} = (K^*x^*)^{-1}$$

for $x^* \in X^*$. Thus ϕ is a homomorphism of X^* into the complex numbers of absolute value 1 and K^* is an endomorphism of X^*. That K^* is one-to-one and thus an automorphism of X^* results also from (*); for suppose $K^*x^* = e^*$, the identity character. Then

$$x^*(Sx) = \phi(x^*)K^*x^*(x) = \phi(x^*)e^*(x) = \text{constant} \quad \text{a.e.}$$

Since S is a measure-preserving transformation of X onto itself, $x^*(x) = \text{constant}$ a.e., and by continuity $x^*(x) = \text{constant}$ everywhere on X. But the only constant character is e^*; therefore $x^* = e^*$. The continuity of ϕ and K^* is automatically provided by X^* being discrete. Thus ϕ is a character on X^* and the duality theorem of Pontrjagin asserts the existence of an element $b \in X$ such that $\phi(x^*) = x^*(b)$ for $x^* \in X^*$. Furthermore, the continuous automorphism K^* induces a continuous automorphism K^{**} on the character group X^{**} of X^* defined by $K^{**}x^{**}(x^*) = x^{**}(K^*x^*)$. The duality theorem is invoked once again to guarantee the existence of a continuous automorphism K of X satisfying $K^*x^*(x) = x^*(Kx)$ for all $x^* \in X^*$. Then from (*) we have

$$x^*(Sx) = x^*(b)x^*(Kx) = x^*(b \cdot Kx) = x^*(T_bKx) \quad \text{a.e.}$$

for all $x^* \in X^*$. From the countability of X^* and the Peter-Weyl theorem we conclude as before that $S = T_bK$ a.e.

Observe that

$$(T_bK)T_a(T_bK)^{-1}T_a^{-1} = T_{a^{-1}\cdot K(a)}$$

for any $b \in X$ and any continuous automorphism K on X. Thus if $S = T_bK$ a.e. then $S \in C_2(T_a)$. This fact along with Proposition 2 completely characterizes $C_2(T_a)$.

LEMMA. *If T_a is an ergodic rotation on a compact separable Abelian group X which is spatially isomorphic to a rotation T_d composed with a continuous automorphism K, i.e., there exists a measure-preserving transformation S of X onto itself such that $ST_aS^{-1} = T_dK$ a.e., then all characters $x^* \in X^*$ have finite orbits under K.*

Proof. For fixed $x^* \in X^*$ consider the Fourier expansion

$$x^*(Sx) \sim \sum_{\gamma \in \Gamma} c_\gamma x_\gamma^*(x),$$

where the characters are indexed with a countable set Γ with the coefficients c_γ square summable. The automorphism K maps characters into characters, and consequently we shall denote by $x^*_{K^{-1}\gamma}$ the character defined by $x^*_{K^{-1}\gamma}(x) = x^*_\gamma(Kx)$. Since

$$x^*(T_aSx) = x^*(ST_dKx) \quad \text{a.e.},$$
$$x^*(a) \cdot x^*(Sx) = x^*(ST_dKx) \quad \text{a.e.}$$

we obtain from the Fourier expansion

$$\sum_{\gamma\in\Gamma} x^*(a)c_\gamma x^*_\gamma(x) \sim \sum_{\gamma\in\Gamma} c_\gamma x^*_\gamma(d)x^*_\gamma(Kx) \sim \sum_{\gamma\in\Gamma} c_\gamma x^*_\gamma(d)x^*_{K^{-1}\gamma}(x).$$

Reindexing the series we have

$$\sum_{\gamma\in\Gamma} x^*(a)c_\gamma x^*_\gamma(x) \sim \sum_{\gamma\in\Gamma} c_{K\gamma}x^*_{K\gamma}(d)x^*_\gamma(x).$$

Therefore,

$$x^*(a)c_\gamma = c_{K\gamma}x^*_{K\gamma}(d),$$

whence,

$$|c_\gamma| = |c_{K\gamma}|.$$

Suppose $x^*_{\gamma_0}$ has an infinite orbit under K, i.e., $x^*_{\gamma_0}, x^*_{K\gamma_0}, \cdots, x_{K^n\gamma_0}, \cdots$ were all distinct. Since $|c_{\gamma_0}| = |c_{K\gamma_0}| = \cdots = |c_{K^n\gamma_0}| = \cdots$ and the coefficients $c_\gamma, \gamma \in \Gamma$, are square summable we would have $c_{\gamma_0} = 0$. Thus in the Fourier expansion of x^*S if the coefficent $c_\gamma \neq 0$ then there exists an integer n depending upon γ such that $x^*_{K^n\gamma} = x^*_\gamma$. Because $\{x^*S : x^* \in X^*\}$ is complete in $L^2(X)$ and every x^*S, $x^* \in X^*$ can be expanded in terms of characters whose orbits under K are finite, the set of characters whose orbits are finite under K is also complete in $L^2(X)$. This combined with the fact that characters are orthogonal to each other shows that all characters must have finite orbits under K.

Proposition 3. *Let T_a and all of its powers $T_{a^n} = T_{a^n}$, $n = 0, 1, 2, \cdots$, be ergodic rotations on a compact separable Abelian group X. Then $C_3(T_a) = C_2(T_a)$.*

Proof. Let $S \in C_3(T_a)$. From Proposition 2 we have $ST_aS^{-1}T_a^{-1} = T_bK$ a.e., where K is a continuous automorphism of X and T_b is some rotation on X. Thus $ST_aS^{-1} = T_dK$, where $d = K(a)b$. Consider the function $x^*S(\cdot)$, where x^* is some character of X. By the preceding lemma there exists an integer n depending on x^* such that $x^*(K^nx) = x^*(x)$; consequently,

$$x^*(ST_a^nx) = x^*((T_dK)^nSx) = x^*(d_n \cdot K^nSx) = x^*(d_n)x^*(Sx) \quad \text{a.e.},$$

where $d_n = d \cdot Kd \cdots K^{n-1}d$. The function $x^*S(\cdot)$ is a proper function of

T_a^n which by hypothesis is ergodic. Invoking the proper value theorem we have that x^*S is almost everywhere equal to a constant $\phi(x^*)$ multiplied by some character, denoted by L^*x^*, i.e.,

$$x^*(Sx) = \phi(x^*)L^*x^*(x) \quad \text{a.e.}$$

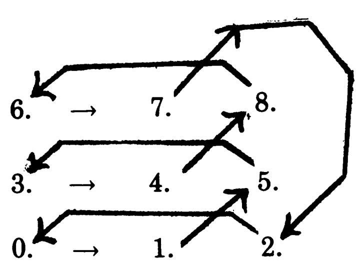
for $x^* \in X^*$. But this is the previous relation (*). From this and the orthogonality of characters follows that L^*x^* is uniquely defined. Then proceeding in exactly the same manner as Proposition 2 we show that there exists $c \in X$ and a continuous automorphism L of X such that $S = T_cL$ a.e., in other words $S \in C_2(T_a)$.

COUNTEREXAMPLE. In order to demonstrate the essential role in Proposition 3 played by the hypothesis that all powers of T_a be ergodic consider the finite cyclic group $G_9 = \{0,1,2,3,4,5,6,7,8\}$ the integers with addition modulo 9. The rotation $T_a: x \to x + a$ (modulo 9) is ergodic if and only if a is a generating element of G_9, i.e., if and only if a is relatively prime to 9. Furthermore, since 9 is not a prime number every ergodic rotation has a power which is not ergodic. We can enumerate all of the automorphisms of G_9 by K_k, where k is an integer less than and relatively prime to 9. The automorphism K_k is completely determined by the relation $K_k(1) = k$. The possible choices for k are $k = 1,2,4,5,7$ and 8. We denote by I the identity automorphism and by J the involution automorphism, $J: x \to -x$ (modulo 9). Since $K^n(1) = k^n$ (modulo 9), $2^6 \equiv 4^3 \equiv 5^6 \equiv 7^3 \equiv 8^2 \equiv 1$ (modulo 9) and $5^3 \equiv 4$ (modulo 9), we have the following relations:

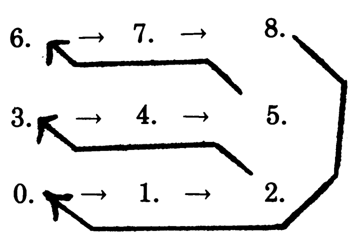
$$K_1 = (K_2)^6 = (K_4)^3 = (K_5)^6 = (K_7)^3 = (K_8)^2 = I,$$
$$(K_5)^3 = K_4, K_3 = J.$$

Consider the rotation $T_1: x \to x + 1$ (modulo 9) and the permutation T_1K_4 on G_9. Their behavior on G_9 is described in the diagrams below:

T_1K_4



T_1



The transformations T_1 and T_1K_4 are spatially isomorphic; the relation $S^{-1}T_1S = T_1K_4$ is satisfied by a permutation S whose behavior on G_9 is determined by the following diagram:

$$\begin{tikzcd}
6. \arrow[r] & 7. \arrow[r] & 8. \arrow[ll, bend right, "S"'] \\
3. \arrow[r] & 4. \arrow[r] & 5. \arrow[ll, bend right] \\
0. \arrow[r] & 1. \arrow[r] & 2. \arrow[ll, bend right]
\end{tikzcd}$$

Thus $S^3 = I$. Next we shall show that S is not of the form T_aK, where T_a is some rotation on G_9 and K is one of the automorphisms of G_9 listed above. Suppose $S = T_aK$. Then $(T_aK)^3 = I$. From this we obtain

$$T_bK^3 = I,$$

where $b \equiv K^2(a) + K(a) + a$ (modulo 9). Thus K^{-3} is an automorphism that is also a rotation: the only such automorphism is $K^{-3} = K^3 = I$. This reduces the possibilities for K to those automorphisms whose cube is the identity, i.e., $K = K_4$ or $K = K_7$. Suppose $S = T_aK_4$. We get

$$2 = S(1) = T_aK_4(1) = 4 + a \pmod 9$$

consequently, $a = 7$. On the other hand

$$0 = S(2) = T_7K_4(2) = 8 + 7 = 6 \pmod 9$$

which is a contradiction. The remaining possibility is $S = T_aK_7$. Here, however, we get

$$2 = S(1) = T_aK_7(1) = a + 7 \pmod 9$$

so that $a = 4$. On the other hand

$$0 = S(2) = T_7K_7(2) = 7 + 14 = 3 \pmod 9$$

which is again impossible. Therefore S is not in $C_3(T_1)$.

3. **Anzai's skew product transformations.** Let X, Y be unit intervals with Borel measurability and Lebesgue measure and $Z = X \times Y$ be the unit square with the usual direct product measurability and measure. We shall consider the following skew product measure-preserving transformation defined on Z; let $T_{\gamma,\alpha}$ denote the measure-preserving transformation with α-function defined by $T_{\gamma,\alpha}: (x, y) \to (x + \gamma, y + \alpha(x))$ (additions modulo 1) where γ is an irrational number and $\alpha(\cdot)$ a real-valued measurable function defined on X. Conditions for ergodicity of $T_{\gamma,\alpha}$ along with proof that it is measure-preserving can be found in Anzai's paper [1]. Two other results from [1] upon which the subsequent work depends are the following: (Proper value criterion) The value $e^{2\pi i\lambda}$ is a proper value of $T_{\gamma,\alpha}$ if and only if there exists an integer p and a real-valued meas-

urable function $\theta(\cdot)$ on X such that $\lambda - p\alpha(x) = \theta(x+\gamma) - \theta(x)$ (modulo 1) a.e.; (Spatial isomorphism criterion) If S is a measure-preserving transformation such that $ST_{\gamma,\alpha}S^{-1} = T_{\gamma,\beta}$ a.e. where $T_{\gamma,\alpha}$ and $T_{\gamma,\beta}$ are ergodic skew product transformations with α-function and β-function, respectively, then S has either the form

$$\text{(i)} \qquad S:(x,y)\to(x+u,y+\theta(x))$$

(additions modulo 1) where u is a real number and $\theta(\cdot)$ is a real-valued measurable function on X such that

$$\beta(x+u) - \alpha(x) = \theta(x+\gamma) - \theta(x) \ \ (\text{modulo } 1) \quad \text{a.e.}$$

or

$$\text{(ii)} \qquad S:(x,y)\to(x+u,\theta(x)-y)$$

(additions modulo 1) where u and $\theta(\cdot)$ now satisfy

$$\beta(x+u) + \alpha(x) = \theta(x+\gamma) - \theta(x) \ \ (\text{modulo } 1) \quad \text{a.e.}$$

We shall restrict ourselves to the most tractable class of skew product transformation, namely, when $\alpha(x) = \nu x$ for some nonzero integer ν.

THEOREM. *The commuting order* $N(T_{\gamma,\nu}) = 2$. *Furthermore,* $C_0(T_{\gamma,\nu})$, $C_1(T_{\gamma,\nu})$ *and* $C_2(T_{\gamma,\nu})$ *are subgroups of* (X) *which can be characterized in the following manner*: (i) $C_0(T_{\gamma,\nu})$ *is as defined, the family of measure-preserving transformations almost everywhere equal to the identity*; (ii) $C_1(T_{\gamma,\nu})$ *is the family of measure-preserving transformations almost everywhere equal to some power of a* ν*th root of* $T_{\gamma,\nu}$ *composed with a transformation* R *of the form* $R:(x,y)\to(x,y+c)$ *where* c *is some constant*; (iii) $C_2(T_{\gamma,\nu})$ *is the family of measure-preserving transformations almost everywhere equal to one of the forms*

$$(x,y)\to(x+u,y+Kx+c),$$
$$(x,y)\to(-x+u,y+Kx+c),$$
$$(x,y)\to(x+u,Kx-y+c),$$
$$(x,y)\to(-x+u,Kx-y+c),$$

where K *is an arbitrary integer and* u *and* c *are arbitrary real numbers.*

This theorem will be established in a sequence of propositions.

PROPOSITION 1. *If* $S\in C_1(T_{\gamma,\nu})$, *i.e.,* $ST_{\gamma,\nu} = T_{\gamma,\nu}S$ *a.e., then* S *almost everywhere is of the form* $S(x,y)\to(x+n\gamma/\nu, y+nx+c)$ *where* c *is some real constant*; *in other words*

$$S = (T_{\gamma,\nu})^{n/\nu}R \quad \textit{a.e.},$$

where $(T_{\gamma,\nu})^{1/\nu}$ *is a* ν*th root of* T, *and* R *is a measure-preserving transformation* *defined by* $R:(x,y)\to(x,y+c)$ (*additions modulo* 1) *for some real constant* c.

Proof. Applying Anzai's result to S we see that it must have the form:

(i) $$S:(x,y)\to(x+u,y+\theta(x)),$$

where $\nu u=\theta(x+\gamma)-\theta(x)$ (modulo 1) a.e. for some real-valued measurable function $\theta(\cdot)$ or

(ii) $$S:(x,y)\to(x+u,\theta(x)-y),$$

where $2\nu x+\nu u=\theta(x+\gamma)-\theta(x)$ (modulo 1) a.e.

Case (ii) can be immediately eliminated: for it follows that the function $e^{2\pi i\theta(\cdot)}$ is a "generalized proper value" (see [3, p. 57]) of the rotation T_γ: $x\to x+\gamma$ (modulo 1) on the circle, i.e.,

$$e^{2\pi i\theta(x+\gamma)}=e^{2\pi i\nu u}e^{2\pi i2\nu x}e^{2\pi i\theta(x)}\quad\text{a.e.}$$

but for T_γ the only "generalized proper values" are proper functions themselves, i.e.,

$$e^{2\pi i\theta(x)}=e^{2\pi inx+c}$$

for some integer n and real number c; therefore

$$2\nu x+\nu u=n(x+\gamma)-nx=n\gamma\ \text{(modulo 1)}\quad\text{a.e.}$$

which is impossible.

Suppose S satisfies case (i); then the function $e^{2\pi i\theta(\cdot)}$ is a proper function with $e^{2\pi in\gamma}$ the associated proper value of the ergodic rotation T_γ whose only proper values are $e^{2\pi in\gamma}$, each of which corresponds to a proper function which is almost everywhere equal to a constant times the function which takes the values $e^{2\pi inx}$. Therefore $e^{2\pi i\nu u}=e^{2\pi in\gamma}$ and $e^{2\pi i\theta(x)}=e^{2\pi i(nx+c)}$ a.e. for some n from which we see that S has the form

$$S:(x,y)\to(x+n\gamma/\nu,y+nx+c)$$

(additions modulo 1) where c is some real constant. Since they also commute with $T_{\gamma,\nu}$, the νth roots of $T_{\gamma,\nu}$ must be of the above form with $n=1$ from which we can conclude that

$$(T_{\gamma,\nu})^{1/\nu}:(x,y)\to(x+\gamma/\nu,y+x-(\nu-1)\gamma/2+p/\nu)$$

(additions modulo 1), where $p=0,1,2,\cdots,\nu-1$. The constant c then can be adjusted so that

$$S=(T_{\gamma,\nu})^{n/\nu}R\quad\text{a.e.},$$

where $R:(x,y)\to(x,y+c)$ (additions modulo 1).

It is easy to verify that transformations almost everywhere equal to one of the forms $(T_{\gamma,\nu})^{n/\nu}R$ commute with $T_{\gamma,\nu}$ almost everywhere. This fact along with Proposition 1 completely characterizes $C_1(T_{\gamma,\nu})$.

Proposition 2. *If $S \in C_2(T_{\gamma,\nu})$ then S almost everywhere satisfies one of the forms*

(i) $(x,y) \to (x+u, y+Kx+c)$,

(ii) $(x,y) \to (x+u, Kx-y+c)$,

(iii) $(x,y) \to (-x+u, Kx+y+c)$,

(iv) $(x,y) \to (-x+u, Kx-y+c)$,

where K is an arbitrary integer and u and c are arbitrary real numbers.

Proof. By Proposition 1

$$ST_{\gamma,\nu}S^{-1}T^{-1}_{\gamma,\nu} = (T_{\gamma,\nu})^{n/\nu}R \quad \text{a.e.},$$

where $R:(x,y) \to (x, y+c)$; whence

$$(*) \qquad ST_{\gamma,\nu}S^{-1} = (T_{\gamma,\nu})^{(n+\nu)/\nu}R \quad \text{a.e.}$$

The transformation on the right is the ergodic skew product transformation $T_{(n+\nu)\gamma/\nu,\beta}$ with β-function which satisfies $\beta(x) = (n+\nu)x + c'$ in which all the constants involved are lumped together in c'; whereupon we write (*) as $ST_{\gamma,\nu}S^{-1} = T_{(n+\nu)\gamma/\nu,\beta}$. Applying Anzai's criterion for proper values and the familiar argument of "generalized proper values" used before (Proposition 1) we see that the proper values of $T_{\gamma,\nu}$ are precisely the positive and negative integral powers of $e^{2\pi i\gamma}$ whereas the proper values of $T_{(n+\nu)\gamma/\nu,\beta}$ are precisely the positive and negative integral powers of $e^{2\pi i(n+\nu)\gamma/\nu}$. Since $T_{\gamma,\nu}$ is isomorphic to $T_{(n+\nu)\gamma/\nu,\beta}$, we have that $e^{2\pi i\gamma}$ is some power of $e^{2\pi i(n+\nu)\gamma/\nu}$ and vice versa. This can only happen if $(n+\nu)/\nu = \pm 1$. For the case $(n+\nu)/\nu = 1$ we can apply Anzai's spatial isomorphism criterion which yields that S has either the form

$$S:(x,y) \to (x+u, y+\theta(x))$$

(additions modulo 1), where u is a real number and $\theta(\cdot)$ is a real-valued measurable function such that

$$(n+\nu)(x+u) + c' - \nu x = \theta(x+\gamma) - \theta(x) \ \ (\text{modulo } 1) \quad \text{a.e.}$$

or

$$S:(x,y) \to (x+u, \theta(x) - y)$$

(additions modulo 1), where

$$(n+\nu)(x+u) + c' + \nu x = \theta(x+\gamma) - \theta(x) \ \ (\text{modulo } 1) \quad \text{a.e.}$$

In either case the argument of "generalized proper values" demonstrates that $\theta(x) = Kx + c$ a.e. for some integer K and real number c; consequently, S almost everywhere satisfies one of the forms (i) or (ii) in the statement of the proposition. For the case $(n+\nu)/\nu = -1$ we have that $n = -2\nu$ and $\beta(x) = -\nu x + c'$. If Q denotes the mapping $(x,y) \to (-x, -y)$ then

$$QT_{-\gamma,\beta}Q^{-1} = T_{\gamma,\beta'},$$

where $\beta'(x) = \nu x - c'$. From this and relation (*) we have

$$QST_{\gamma,\nu}S^{-1}Q^{-1} = T_{\gamma,\beta'} \quad \text{a.e.}$$

to which we apply the considerations of the previous case to obtain that QS satisfies almost everywhere one of the forms (i) and (ii) in the statement of the proposition. Hence S must satisfy almost everywhere one of the forms (iii) or (iv).

It is easy to check that all transformations almost everywhere equal to one of the four forms belong to $C_2(T_{\gamma,\nu})$. This fact along with Proposition 2 completely characterizes $C_2(T_{\gamma,\nu})$.

PROPOSITION 3. $C_3(T_{\gamma,\nu}) = C_2(T_{\gamma,\nu})$.

Proof. If $S_3 \in C_3(T_{\gamma,\nu})$ then $S_3T_{\gamma,\nu}S_3^{-1}T_{\gamma,\nu}^{-1} = S_2$ a.e., where S_2 has one of the four forms described in the previous proposition. Thus we can write

$$S_3T_{\gamma,\nu}S_3^{-1} = T \quad \text{a.e.,}$$

where T has one of the forms

(i) $(x,y) \to (x + u, y + Kx + c)$,

(ii) $(x,y) \to (x + u, Kx - y + c)$,

(iii) $(x,y) \to (-x + u, y + Kx + c)$,

(iv) $(x,y) \to (-x + u, Kx - y + c)$,

where K is an arbitrary integer and u and c are arbitrary real numbers.

For case (i) we show $u = \pm\gamma$ exactly as in the previous proposition; in addition $K \neq 0$ if T is to be isomorphic to $T_{\gamma,\nu}$. Furthermore we can apply the isomorphism criterion of Anzai in the same manner as in Propostion 2 to solve for S_3; whereupon it is clear that S_3 is one of the four forms of $C_2(T_{\gamma,\nu})$. Cases (ii), (iii) and (iv) can be reduced to case (i) by taking squares and considering $S_3T_{\gamma,\nu}^2S_3^{-1} = T^2$, and we solve for S_3 in the identical manner obtaining the same results.

4. **Some unsolved problems.** 1. What are $N(T_a)$ and $C_n(T_a)$ for ergodic rotations T_a whose powers are not necessarily ergodic?

2. What are $N(T)$ and $C_n(T)$, where T is the shift transformation on a two-sided infinite direct product of a measure space with itself consisting of a finite number of points?

3. Give examples for $N(T) = n$ for each integer n including $N(T) = \infty$.

4. In general is $C_n(T)$ a subgroup of G?

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