SOME RESULTS IN THE LOCATION OF THE ZEROS OF LINEAR COMBINATIONS OF POLYNOMIALS

BY

ZALMAN RUBINSTEIN(1)

We study here the location of the zeros of linear combinations of polynomials of the form \( f(z) - \lambda g(z) \), where \( f(z) \) and \( g(z) \) are arbitrary polynomials with complex coefficients and \( \lambda \) is a complex number. It is known [3] that this question is closely connected with the study of the zeros of polynomials of the form \( (z - \alpha)^n - \lambda (z - \beta)^{n/2} \), which indeed is the main object of this paper.

We start with a particular case.

THEOREM 1. Let the polynomials \( f(z) = z^n + \cdots \), and \( g(z) = z^{r} + \cdots \), \( n = 2r \), have zeros in the circles \( |z - a| \leq r_1 \) and \( |z - b| \leq r_2 \), respectively, then all the zeros of the polynomial

\[
(1) \quad f(z) - \lambda g(z)
\]

are in the union of the \( n \) circles

\[
(2) \quad |z - a - \frac{1}{2} \lambda^{2/n} + \lambda^{1/n} \left( a - b + \frac{1}{4} \lambda^{2/n} \right)^{1/2} | \leq (r_1 + r_2)^{1/2} |\lambda|^{1/n} + r_1,
\]

where \( \lambda^{1/n} \) assumes all the \( n \)th roots of \( \lambda \).

Proof. The equation \( f(z) - \lambda g(z) = 0 \) can be replaced by Grace’s theorem [3] by the equation \( (z - \alpha)^n - \lambda (z - \beta)^{n/2} = 0 \), where \( |\alpha - a| \leq r_1 \), and \( |\beta - b| \leq r_2 \).

Solving for \( z \) we obtain

\[
z = \alpha + \frac{1}{2} \lambda^{2/n} \pm \lambda^{1/n} \left[ (\alpha - \beta) + \frac{1}{4} \lambda^{2/n} \right]^{1/2}.
\]

Denoting generically the region \( |z - c| \leq R \) by \( C(c, R) \) we have

\[
\alpha - \beta \in C(a - b, r_1 + r_2),
\]

\[
\left( \alpha - \beta + \frac{1}{4} \lambda^{2/n} \right)^{1/2} \in C \left( \pm \left( a - b + \frac{1}{4} \lambda^{2/n} \right), (r_1 + r_2)^{1/2} \right);
\]

hence

Received by the editors August 21, 1963 and, in revised form, March 11, 1964.

(1) This work was supported by the Air Force Office of Scientific Research.
\[
z \in C \left( a + \frac{1}{2} \lambda^{2/n} \pm \lambda^{1/n} \left( a - b + \frac{1}{4} \lambda^{2/n} \right)^{1/2}, \frac{r_1 + r_2}{1/2} \lambda^{2/n} + r_1 \right).
\]

(2) follows since, by assumption, \( n \) is an even number.

The result is sharp for \( \lambda = 0 \), and for \( a = b \).

For the general case we have

**Theorem 2\(^{(2)}\).** Let \( f(z) = z^n + \cdots, g(z) = z' + \cdots, n > r \), have zeros in the circles \( |z - a| \leq r_1 \) and \( |z - b| \leq r_2 \), respectively. Then all the zeros of the polynomial \( f(z) - \lambda g(z) \) are in the circle

\[
|z - a| \leq r_1 + d,
\]

where \( d \) is the positive root of the equation

\[
d_n^{1/r} - Md - N = 0
\]

with

\[
M = |\lambda|^{1/r}, \quad N = |\lambda|^{1/r}(|a - b| + r_1 + r_2).
\]

**Proof.** Consider the equation

\[
(z - a)^n = \lambda(z - \beta)^r, \quad |a - \alpha| \leq r_1, \quad |b - \beta| \leq r_2.
\]

For \( z_0 \) satisfying \( (z_0 - a)^n = \lambda(z_0 - \beta)^r \), \((z_0 - \alpha)^{n/r - 1} = \lambda^{1/r}((z_0 - \beta)/(z_0 - \alpha))\). Let \( d_1 \) be a positive number satisfying

\[
d_1^{n/r} - Md_1 - N > 0.
\]

For \( |z_0 - a| \geq d_1 \), \((z_0 - \beta)/(z_0 - \alpha)\) belongs to the circle \( |z - 1| \leq |\alpha - \beta|/d_1 \); hence

\[
|\lambda^{1/r} \frac{z_0 - \beta}{z_0 - \alpha}| \leq |\lambda|^{1/r} \left( 1 + \frac{|\alpha - \beta|}{d_1} \right),
\]

but

\[
|z_0 - \alpha|^{n/r - 1} \geq d_1^{n/r - 1} > |\lambda|^{1/r} \left( 1 + \frac{|\alpha - \beta|}{d_1} \right),
\]

for all \( \alpha, \beta \) such that \( |\alpha - a| \leq r_1 \), and \( |\beta - b| \leq r_2 \). We get a contradiction, which proves that \( |z_0 - \alpha| < d_1 \).

It is worthwhile to remark that if \( M + N > 1 \) an estimate for the positive zero \( d \) is the expression

\[
\frac{(n - r)(M + N)^{n/r} + rN}{(n - r)(M + N) + rN} \leq (M + N)^{n/r - r}.
\]

For \( M + N < 1 \) a bound for the same is \((n - r + rN)/(n - rM) \leq 1\).

\(^{(2)}\) Theorem 2 was proved independently and by a different method by Mishael Zedek [5].
Different estimates can be obtained by means of estimates similar to those used in the proof of Theorem 2, which are sharp for \( \lambda = 0 \) or asymptotically for \( \lambda \to \infty \). We indicate some of them which are of a relatively simple form.

**Theorem 3.** Let \( f(z) \) and \( g(z) \) be as in Theorem 2. All the zeros of the polynomial \( f(z) - \lambda g(z) \) are in each of the following regions:

\[
|z| \leq \frac{|a| - r_1}{d(|a| - r_1) - 1} \left[ (|b| + r_2)d + 1 \right],
\]

where \( r > n, \ d = |\lambda|^{1/r}(r_1 + |a|)^{-n/r}, \) and \( d(|a| - r_1) - 1 > 0. \)

\[
|z - b| \leq r_2 + 2 \max \left[ |\lambda|^{-1/r-n}, (|a - b| + r_1 + r_2)^{n/r}|\lambda|^{1/r-n} \right],
\]

where \( r = nk, \ k \geq 2. \)

\[
|z - \delta_kb - \frac{\delta_k b}{\delta_k - 1}| \leq m + \frac{\delta_k |b| (r_2 + 1)}{|\delta_k - 1|}, \quad k = 1, \ldots, n,
\]

where \( n > r, \ w_k^n = \lambda, \ \delta_k^* = \lambda/(1 - \lambda), \ k = 1, \ldots, n; \)

\[
m = \max_{1 \leq k \leq n} \frac{1}{|1 - w_k|}(|a - w_kb| + r_1 + |w_k|r_2).
\]

**Proof of (4).** Let

\[
F_1(z) = (z - \alpha)^n - \lambda (z - \beta)^r,
\]

\[
G(z) = z^r F_1 \left( \frac{1}{z} \right) = z^{-r}(1 - za)^n - \lambda (1 - \beta z)^r;
\]

hence \( G(z) \) can also be written in the form:

\[
G(z) = (-\alpha)^n (z - \gamma)^r - \lambda (1 - \beta z)^r,
\]

where \( \gamma \) ranges over a circle including 0 and the points \( 1/\alpha \). If \( G(z_0) = 0 \), then \( z_0 = (\delta + \gamma)/(1 + \delta \beta) \), where \( \delta = \lambda^{1/r}(-a)^{-n/r}. \) Any zero of \( F_1(z) \) is thus of the form \( (1 + \delta \beta)/(\delta + \gamma) \). Let \( C(a, b) \) denote the circle \( |z - a| \leq b. \) If \( \alpha \in C(a, r_1) \), then

\[
\frac{1}{\alpha} \in C \left( \frac{\overline{\alpha}}{|a|^2 - r_1^2}, \frac{r_1}{|a|^2 - r_1^2} \right)
\]

and

\[
\gamma \in C \left( \frac{e^{-i\phi}}{2(|a| - r_1)}, \frac{1}{2(|a| - r_1)} \right), \quad \phi = \arg a.
\]

Thus
\[ |\gamma| \leq (|a| - r_1)^{-1} < |\lambda|^{1/r} (r_1 + |a|)^{-n/r} \leq |\delta| \]

by our assumption \( d(|a| - r_1) - 1 > 0 \).

Now

\[ z_0 \in C \left( \frac{\beta d^2 - \gamma}{d^2 - |\gamma|^2}, \frac{d |\beta | - 1}{d^2 - |\gamma|^2} \right), \]

where \( d = |\lambda|^{1/r} (r_1 + |a|)^{-n/r} \). Taking into account the inequalities \(|\gamma| \leq (|a| - r_1)^{-1}, |\beta| \leq |b| + r_2 \), we arrive at (4) after a short calculation.

**Proof of (5).** From \( F_1(z) = (z - \alpha)^n - \lambda(z - \beta)^r \) it follows that

\[ z F_1 \left( \frac{1}{z} + \beta \right) = -\lambda + z^{-n} \left[ 1 + (\beta - \alpha)z \right]^n. \]

If \( F_1(\zeta) = 0 \), then

\[ z^r \zeta + z^{r-1} - \mu = 0, \]

with \( \zeta = 1/z + \beta, \mu = (\lambda)^{1/n}, \gamma = \alpha - \beta \).

The left-hand side of (7) can be written in the form

\[ \left( \frac{\gamma}{\mu} z^k + 1 \right) (z^{k-1} - \mu) - \frac{\gamma}{\mu} z^{2k-1}. \]

It follows by Szegö's Theorem [3, p. 60] that

\[ |z| \geq \frac{1}{2} \text{Min} \left[ |\mu|^{1/k}, |\gamma|^{-1/k}, |\mu|^{1/k-1} \right] \]

and \(|\zeta - \beta| \leq 2 \text{Min} \left[ |\lambda|^{1/r-n}, |\lambda|^{1/r} |\beta - \alpha|^{-n/r} \right]^{-1}\), (5) follows easily.

It is worthwhile to remark that by the same manipulation we can also obtain a lower bound for the zeros of \( f(z) - \lambda g(z) \) namely writing

\[ -\gamma z^k + z^{k-1} - \mu z + \frac{\mu}{\gamma} = \left( -\mu z + \frac{\mu}{\gamma} \right) \left( z^{k-1} \frac{\gamma}{\mu} + 1 \right). \]

It follows by the same theorem due to Szegö that all the zeros of \( -\gamma z^k + z^{k-1} - \mu z \) are in \(|z| \leq 2 \text{Max} \left[ 1/|\gamma|, (\mu/\gamma)^{1/k-1} \right] \). The final estimate is \(|\zeta - \beta| \geq 2 \text{Max} \left[ |\alpha - \beta|^{-1}, |\lambda|^{1/nk} |\alpha - \beta|^{-k} \right]^{-1}\). To obtain a meaningful result it is necessary to suppose that \( \text{Min} |\alpha - \beta| > 0 \); then

\[ |\zeta - b| \geq 2 \text{Max} \left[ |a - b| - (r_1 + r_2) \right]^{-1}, \]

\[ |\lambda|^{1/r} \left[ (|a - b| - (r_1 + r_2))^{-1/k} \right]^{-1} - r_2. \]
Proof of (6). Write \( F_i(z) = f_i(z) - \lambda g_i(z), \ f_i(z) = (z - \alpha)^n - \lambda(z - \beta)^n, \ g_i(z) = (z - \beta)^r - (z - \beta)^n. \)

The zeros of \( f_i(z) \) are in the union of the circles

\[
C \left( \frac{a - w_k b}{1 - w_k}, \frac{r_1 + |w_k| r_2}{|1 - w_k|} \right)
\]

(see, e.g., [3, p. 57]); hence in \( C(0, r) \).

The zeros of \( g_i(z) \) are in \( C(b, r_2 + 1) \). Since \( f_i(z) \) and \( g_i(z) \) are both of degree \( n \) we can use the result in [3] to obtain (6).

We conclude this discussion by proving some results about the location of part of the zeros of the polynomial \( (z - \alpha)^n - \lambda(z - \beta)^r \).

**Theorem 4.** At least \( n \) zeros of the polynomial \( (z - \alpha)^n - \lambda(z - \beta)^r \) are in the circle

\[
|z - \alpha| \leq \frac{n}{r - n} |\alpha - \beta| \quad \text{if} \quad n < r \leq 2n,
\]

\[
|z - \alpha| \leq |\alpha - \beta| \quad \text{if} \quad r \geq 2n,
\]

and at most \( n \) zeros of the above polynomial are in the circle

\[
|z - \alpha| \leq |\alpha - \beta| \quad \text{if} \quad n < r \leq 2n,
\]

\[
|z - \alpha| \leq \frac{n}{r - n} |\alpha - \beta| \quad \text{if} \quad r \geq 2n,
\]

for all complex \( \lambda \).

**Proof.** By a straightforward calculation one obtains that \( \text{Re}\left(\frac{(z - A)}{(z - B)}\right) > 0 \ (\ < 0) \) if and only if

\[
z \in C \left( \frac{A + B}{2}, \frac{|A - B|}{2} \right), \quad \left( z \in C \left( \frac{A + B}{2}, \frac{|A - B|}{2} \right) \right)
\]

for \( A \neq B \).

Now

\[
\frac{\partial}{\partial \theta} \arg \left[ \frac{(z - \alpha)^n}{\lambda(z - \beta)^r} \right]_{|z - \alpha| = r}
\]

\[= \text{Re} \left[ (z - \alpha) \left( \frac{n}{z - \alpha} - \frac{r}{z - \beta} \right) \right] = (n - r) \text{Re} \left[ \frac{z + r \alpha - n \beta}{z - \beta} \right]. \tag{8}\]

Since \( n < r \) it follows that (8) is positive if and only if
In this case
\[
\Delta \arg \left[ \frac{(z - \alpha)^n - \lambda(z - \beta)^r}{|z - \alpha|} \right] 
\]
\[
\Delta \arg \left[ \frac{(z - \alpha)^n}{-\lambda(z - \beta)^r} \right] + \Delta \arg \left[ (- \lambda)(z - \beta)^r \right] 
\]
\[
\Delta \arg \left[ \frac{(z - \alpha)^n}{-\lambda(z - \beta)^r} \right] + \Delta \arg \left[ -\lambda(z - \beta)^r \right] = 2\pi n. 
\]
Thus if
\[
C(\alpha, R) \subset C \left( \frac{r(\alpha + \beta) - 2n\beta}{2(r - n)}, \frac{r|\alpha - \beta|}{2(r - n)} \right),
\]
then the polynomial \((z - \alpha)^n - \lambda(z - \beta)^r\) has at most \(n\) zeros in the circle \(C(\alpha, R)\). It is easy to see that we can take
\[
R = \frac{r - |r - 2n|}{2(r - n)} |\alpha - \beta|. 
\]
This proves the second part of the theorem. Similarly
\[
\frac{\partial}{\partial \theta} \arg \left[ \frac{(z - \alpha)^n}{-\lambda(z - \beta)^r} \right] < 0 
\]
if and only if
\[
z \in C \left( \frac{r(\alpha + \beta) - 2n\beta}{2(r - n)}, \frac{r|\alpha - \beta|}{2(r - n)} \right),
\]
and we can set
\[
R = \frac{|\alpha - \beta|}{2(r - n)} (r + |r - 2n|). 
\]
It follows in particular that for \(r = 2n\), the circle \(|z - \alpha| \leq |\alpha - \beta|\) contains exactly \(n\) zeros of the polynomial \((z - \alpha)^n - \lambda(z - \beta)^r\).

The following theorem generalizes a result due to Biernacki and Jankowski [1, 2].

**Theorem 5.** Let \(P(z) = a_p z^p + a_{p-1} z^{p-1} + \cdots + a_0, Q(z) = b_q z^q + b_{q-1} z^{q-1} + \cdots + b_0, a_p b_q \neq 0, q > p, s \geq 1, t \geq 1\) have all their zeros in the circles \(|z| \leq R_1\) and \(|z| \leq R_2\), respectively. Let \(r = \min(s, t) \geq 1\). At least \(p\) zeros of the polynomial
\[
P(z) + \lambda Q(z) 
\]
are in the circle

\[ |z| \leq \text{Max} \left\{ \left( \frac{qR_1 + pR_2}{q - p} \right)^{1/r}, R_2 \right\}. \]

**Proof.**

\[ \text{Max} \frac{d}{d\theta} \arg \frac{P(z)}{Q(z)} \leq \text{Max} \frac{d}{d\theta} \arg P(z) - \text{Min} \frac{d}{d\theta} \arg Q(z). \]

For \( R > \text{Max}(R_1, R_2) \) we have:

\[ \text{Max} \frac{d}{d\theta} \arg \frac{P(z)}{Q(z)} \leq \text{Max} \sum_{k=1}^{p} \frac{z}{z - \alpha_k} - \text{Min} \sum_{k=1}^{q} \frac{z}{z - \beta_k} \]

\[ \leq p \text{Max} \frac{R}{R - |\alpha|} - q \frac{R}{R + |\beta|}, \]

where \( \alpha_k, \beta_k \) are the zeros of \( P(z) \) and \( Q(z) \), respectively, and the functions \( \alpha(z), \beta(z) \) satisfy \( |\alpha(z)| \leq R_1 / R^{-1}, \ |\beta(z)| \leq R_2 / R^{-1} \). This follows by a recent result due to Walsh \[4\]. If the \( m_k, \alpha_k, \) and \( z \) are given with \( m_k > 0, \ |\alpha_k| \leq A, \ |z| > A, \) and \( \sum_{l=1}^{n} m_k \alpha_k^l = 0 \) for \( l = 1, 2, \ldots, j \), then \( \alpha = \alpha(z) \) as defined by the equation

\[ \pi_{k=1}^{n} (z - \alpha_k)^n = (z - \alpha)^n \]

satisfies the inequality

\[ |\alpha(z)| \leq A^{j+1}/|z|^j. \]

Under the same conditions except that now \( |\alpha_k| \geq A, \ |z| < A, \) and \( \sum_{l=1}^{n} m_k \alpha_k^l = 0 \) and \( l = 1, 2, \ldots, j \), we have

\[ |\alpha(z)| \geq A^{j+1}/|z|^j. \]

In deriving (11) we also notice that

\[ \text{Re} \left( \frac{z}{z - \alpha} \right) \leq \left| \frac{z}{z - \alpha} \right| \leq \frac{R}{R - |\alpha|} \]

and

\[ \text{Re} \left( \frac{z}{z - \beta} \right) = \frac{R(R - r \cos(\theta - \phi))}{R^2 + r^2 - 2rR \cos(\theta - \phi)}, \]

with \( \beta = re^{i\theta}, \) \( z = Re^{i\phi}. \)

The last expression is an increasing function of \( \cos(\theta - \phi) \) and attains its minimum for \( \cos(\theta - \phi) = -1. \) Hence \( \text{Re}(z/(z - \beta)) \geq R/(R + |\beta|). \) It
follows now from (11) that
\[ \operatorname{Max} \frac{d}{d\theta} \arg \frac{P(z)}{Q(z)} \leq p \frac{R'}{R' - R_1'} - q \frac{R'}{R' + R_2'} < 0 \]
for \( R' > ((pR_2' + qR_1')/(q - p)) \). It is enough to set
\[ R = \operatorname{Max} \left( \frac{pR_2' + qR_1'}{q - p} \right)^{1/r}, R_2' \]
which implies \( R \geq \operatorname{Max}(R_1, R_2) \).
Now one proves similarly to what has been done in Theorem 4 that
\[ \Delta \arg (P + \lambda Q) \geq 2\pi p \]
which concludes the proof.
It is clear that
\[ R \leq R' = \operatorname{Max} \left[ \frac{pR_2 + qR_1}{q - p}, R_2 \right]. \]
The estimate \(|z| \leq R\) is due to Biernacki [1]. For large \( r \), \( R \) tends to \( \operatorname{Max}(R_1, R_2) \).

**Bibliography**