UNIFORMIZATION OF SYMMETRIC RIEMANN SURFACES BY SCHOTTKY GROUPS

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1. Introduction. A Riemann surface $S$ is called symmetric if there exists an anti-conformal map $\phi$ of $S$ onto itself such that $\phi^2 = $ identity. We say that $\phi$ is a symmetry on $S$.

The classical "retrospection theorem" asserts the existence of representations of closed Riemann surfaces of genus $g$ by "Schottky groups," groups generated by Möbius transformations $A_1, \cdots, A_g$ such that $A_i$ maps the exterior of $\Gamma_i$ into the interior of $\Gamma'_i$, where $\Gamma_1, \Gamma'_1, \cdots, \Gamma_g, \Gamma'_g$ are disjoint Jordan curves bounding a $2g$-times connected domain, a standard fundamental domain for the group.

We will show that a closed symmetric Riemann surface of genus $g$ can be represented by a Schottky group which has a standard fundamental domain which exhibits the symmetry. This result is contained in Theorems I, II and III of §§4-6. The proof does not use the classical theorem.

As a corollary, in §7, we obtain a new proof of the Koebe theorem: every $n$-times connected planar domain can be conformally mapped onto a plane domain exterior to $n$ disjoint circles.

Techniques from the theory of quasiconformal mappings are used to obtain these results.

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2. Quasiconformal mappings. We recall [1], [3] that a homeomorphism $w(z)$ of a plane domain $\Delta$ onto another plane domain $\tilde{\Delta}$ is said to be quasiconformal if it has generalized derivatives satisfying, at each point $z \in \Delta$, a Beltrami equation $w_z = \mu(z)w_\bar{z}$ with $\mu(z) \in M_\Delta$, where $\mu(z) \in \bar{M}_\Delta$ if it is defined and measurable in $\Delta$ and ess.sup$|\mu(z)| \leq k < 1$ for $z \in \Delta$.

For a given $\mu(z) \in M_C$ ($C$ the complex plane) there exists a unique quasiconformal mapping $w^*(z)$ of $C$ onto itself satisfying $w_z = \mu(z)w_\bar{z}$ and normalized by the conditions $w(0) = 0$ and $w(1) = 1$.

If $\mu(z) \in M_C$ is compatible with the Möbius transformation $A(z)$:

$$\mu \circ A = (A_\mu / \overline{A_\mu}) \mu$$

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or with the anti-Möbius transformation \( B(z) = (az + b) / (cz + d) \):

\[
\mu \circ B = (B_\mu \overline{B_\mu})^\mu
\]

then \( A^* = \omega^* \circ A \circ (\omega^*)^{-1} \) and \( B^* = \omega^* \circ B \circ (\omega^*)^{-1} \) are Möbius and anti-Möbius transformations respectively. If \( G \) is a Schottky group generated by the transformations \( \{ A_j \} \), then the transformations \( \{ A_j^* \} \) generate a Schottky group \( G^* \).

We need, also, the following lemmas:

**Lemma 1.** Let \( M(z) \) and \( N(z) \) be two anti-Möbius transformations. If \( \mu(z) \) is compatible with \( M(z) \) and \( M \circ N(z) \) then it is compatible with \( N(z) \).

The proof is by calculation.

**Lemma 2.** If \( \mu(z) \in M_C \) and is compatible with the anti-Möbius transformation \( R(z) \), reflection in \( C_{a,\rho} \) (the circle with radius \( \rho \) and center at \( z = a \)), that is

\[
R(z) = a + \rho^2/z - a
\]

then \( w^*(C_{a,\rho}) \) is a circle (with center at \( w^*(a) \)) and

\[
R^*(w^*(z)) = w^*(a) + \frac{\lambda^2}{w^*(z) - w^*(a)},
\]

i.e., reflection in \( w(C_{a,\rho}) \).

**Proof.** Let \( \psi(z) = w^*(z) + \rho^2/w^*(R(z)) - w^*(a) \). Then it is easily shown that \( \psi(z) - w^*(a) \) and \( w^*(z) - w^*(a) \) both satisfy the same Beltrami equation. They are both 0 when \( z = a \) and \( \infty \) when \( z = \infty \). It follows by uniqueness that one is a multiple of the other. Then \( \frac{w^*(z) - w^*(a)}{w^*(R(z)) - w^*(a)} = \text{constant} \).

For \( z' \) on \( C_{a,\rho} \), \( R(z') = z' \) and \( |w^*(z') - w^*(a)| = \lambda \) where \( \lambda \) is a positive constant. Hence \( w^*(C_{a,\rho}) \) is a circle with center \( w^*(a) \) and radius \( \lambda \).

**Remark.** If, in Lemma 2, \( C_{a,\rho} \) is the unit circle, then \( R^* = w^* \circ R \circ (w^*)^{-1} \) is again reflection in the unit circle. If, in addition, \( \mu \) is compatible with the Möbius transformations \( A \) and \( B \) and \( A = R \circ B \circ R \), then \( A^* = R^* \circ B^* \circ R^* \). This follows by a simple calculation.

**Lemma 3.** If \( \mu(z) \in M_C \) and is compatible with the anti-Möbius transformation

\[
Q(z) = a - \rho^2/z - a
\]

which is reflection in the circle \( C_{a,\rho} \) of radius \( \rho \) and center \( a \), followed by a rotation about \( a \) by the angle \( \pi \), then \( w^*(C_{a,\rho}) \) is a "quasicircle" (i.e., if \( w_1 \) is on \( w^*(C_{a,\rho}) \) then the line through \( w_1 \) and the "center" \( b = w^*(a) \) intersects \( w^*(C_{a,\rho}) \) in a point \( w_2 \) such that \( (w_2 - b)(w_1 - b) = \text{negative constant} \). The anti-Möbius transformation \( Q^*(z) \) maps the exterior of \( w^*(C_{a,\rho}) \) into its interior in such a way that a point on \( w^*(C_{a,\rho}) \) is mapped into its "diametrically opposed" point.
Proof. As in Lemma 2, one can show that

\[ [w^*(z) - w^*(a)][w^*(Q(z)) - w^*(a)] = \text{constant} = c. \]

Setting \( z = a + \rho \) and then \( z = a - \rho \) we see that \( c \) is real. Suppose \( c > 0 \).

Then for \( z_0 \) such that \( w^*(z_0) = w^*(a) + \sqrt{c} \), \( w^*(Q(z_0)) = w^*(a) + \sqrt{c} \) also, so that (since \( w^* \) is a homeomorphism) \( z_0 = Q(z_0) \). But \( Q(z) \) is fixed point free. Hence \( c = -\lambda^2 \) and

\[ Q^*(w^*(z)) = w^*(Q(z)) = w^*(a) - \lambda^2/w^*(z) - w^*(a). \]

Suppose now that \( w_0 \) is on \( w^*(C_{a,p}) \). Let \( w_0 = w^*(a) + \rho e^{i\theta} \). Then \( Q^*(w_0) = w^*(a) - \lambda^2/w_0 - w^*(a) = w^*(a) - (\lambda^2/\rho)e^{i\theta} \). But then \( Q^*(w_0) \), which is on \( w^*(C_{a,p}) \), is diametrically opposed to \( w_0 \). Hence \( w^*(C_{a,p}) \) is a quasicircle.

We recall also [3] that if a homeomorphic map \( f \) of a compact Riemann surface \( S \) onto another compact Riemann surface \( S' \) is given, there exists a quasiconformal map \( \tilde{f} \) of \( S \) onto \( S' \) (i.e., a map which is quasiconformal in terms of local parameters) which is homotopic to \( f \). If, in addition, \( S \) and \( S' \) admit anti-conformal involutions \( \phi \) and \( \phi' \) and if \( f \circ \phi = \phi' \circ f \), then \( \tilde{f} \) may be chosen so as to satisfy the relation \( \tilde{f} \circ \phi = \phi' \circ \tilde{f} \). If \( \mu(z) \in M_\infty \) and \( \mu(z) \) is compatible with the generators \( \{A_i\} \) of a Schottky group \( G \), then, denoting by \( L \) and \( L' \) the set of limit points of \( G \) and \( G' \) respectively, \( w^*: C \rightarrow C \) induces a quasiconformal map of \( (C - L)/G \) onto \( (C - L')/G' \). Furthermore, if \( S, S' \) and \( S'' \) are three Riemann surfaces and \( f^* \) and \( h^* \) quasiconformal maps; \( f^*: S \rightarrow S' \) and \( h^*: S \rightarrow S'' \) both satisfying (in terms of local coordinates) the same Beltrami equation on \( S \), then \( h^* \circ (f^*)^{-1} \) is a conformal map of \( S' \) onto \( S'' \).

3. Symmetric surfaces. If a symmetry \( \phi \) on \( S \) leaves fixed a point of \( S \), then it leaves fixed a closed, analytic, Jordan curve through the point, which we call a transition curve.

If the \( \tau \geq 0 \) transition curves separate \( S \), a symmetric Riemann surface of genus \( g \), into two disjoint surfaces (orthosymmetry), we say that \( S \) is symmetric of type \((g, + \tau)\) with respect to the symmetry \( \phi \); otherwise (diassymmetry) it is of type \((g, - \tau)\). In the former case \( S/\phi \) is an orientable surface with \( \tau \) holes and \((g - \tau + 1)/2 \) handles. In the latter case \( S/\phi \) is a nonorientable surface with \( \tau \) holes. Since, topologically, on a nonorientable surface a handle can be replaced by two cross caps, \( S/\phi \) is homeomorphic to a surface with \( \tau \) holes and, say, \( k \) cross caps. It is easily seen that \( k = g - \tau + 1 \). From these remarks we observe that if \( S \) is symmetric of type \((g, \epsilon \tau)\), \( \epsilon = \pm 1 \), then:

\[ \text{if } \epsilon = +1, \text{ then } g - \tau + 1 \text{ is even and } 0 \leq g - \tau + 1 \leq g, \]

or

\[ \text{if } \epsilon = -1, \text{ then } 0 \leq \tau \leq g. \]

4. Orthosymmetric surfaces. Given \( \epsilon = \pm 1 \) and integers \( g > 0 \) and \( \tau \geq 0 \) satisfying (1), we construct a "standard model of type \((g, \epsilon \tau)\)." We as-
sume at first that \( \epsilon = +1 \) and hence \( \tau > 0 \). Let \( C_s \) \((1 \leq s \leq \tau - 1)\) and \( H_r \) \((1 < r < g - \tau + 1)\) be \( g \) disjoint circles exterior to the unit circle \( C_0 \), and with centers \((a_s, a_r)\) on the real axis. Denote by \( R_s(z) \) reflection in the circle \( C_s \). Let \( A_s(z) \) be a Möbius transformation which maps the exterior of \( H_r \) onto the interior of \( H_{r+1}, r = 1, 3, \ldots, g - \tau \).

Reflect the circles \( C_1, \ldots, C_{\tau - 1} \) and \( H_1, \ldots, H_{\tau - 1} \) in \( C_0 \), obtaining circles \( C'_{\tau - 1}, \ldots, C'_{1} \) and \( H_{\tau - 1}, \ldots, H_1 \). The exterior of these \( 2g \) circles we denote by \( F \) and note that \( F \) is a standard fundamental domain of the Schottky group \( G \) generated by the \( g \) Möbius transformations

\[
A_1, A_3, \ldots, A_{g-\tau}, A_1', A_3', \ldots, A_{\tau - 1}', R_0 \circ R_1, \ldots, R_0 \circ R_{\tau - 1}
\]

where \( A_s'(z) = R_0 \circ A_s \circ R_0(z) \). We observe that \( R_0 \circ R_s(z) \) (reflection in \( C_s \) followed by reflection in \( C_0 \)) maps \( C_s \) onto \( C'_s \) in such a way that points on \( C_s \) and \( C'_s \) which are symmetrically situated with respect to \( C_0 \), are identified by \( R_0 \circ R_s(z) \) and hence by the group \( G \).

Denoting by \( \pi \) the canonical mapping of \((C - L)\) onto \((C - L)/G\), we see that the surface \( F/G = (C - L)/G \) is symmetric of type \((g, + \tau)\) with respect to the symmetry \( R, R^\circ \pi = \pi^\circ R_0 \). We call it the standard model of type \((g, + \tau)\). If it is identified under \( R \), the resulting surface \((F/G)/R = F/[G, R_0] \) has, by computing the Euler characteristic, \( \tau \) holes and 

\[(g - \tau + 1)/2 \] handles.

Given a symmetric surface \( S \) of type \((g, + \tau)\), there exists, therefore, a homeomorphism \( f: (F/G)/R = ((C - L)/G)/R \to S/\phi \). We extend \( f \) to a map of \( F/G \) onto \( S \) by the requirement \( f^\circ R = \phi^\circ f \). The homeomorphism \( f \) can be deformed into a quasiconformal map, which we again denote by \( f \), of \( F/G \) onto \( S \) satisfying the same requirement.

The map \( f \) defines in \( F \) a function \( \mu(z) = f^*_z/f^*_z \) where \( f^*_z(z) = \xi^\circ f^\circ z^{-1} \) (\( z \) and \( \xi \) being local coordinates near \( p_0 \) on \( F/G \) and near \( f(p_0) \) on \( S \) respectively). Due to the above requirement, \( \mu(z) \) is compatible with \( R_0(z) \). We extend \( \mu \) to \( C \) by requiring that it be compatible with \( G \) and observe that \( \mu \in M_C \). Let \( w^\rho(z) \) be the (unique) quasiconformal map of \( C \) onto itself satisfying \( w^\rho = \mu(z)w_z \) with \( w(0) = 0 \) and \( w(1) = 1 \). Denote by \( \tilde{w}^\rho \) the induced map of \((C - L)/G\) onto \((C - L^\rho)/G^\rho \). It is easily seen that \( F^\rho = w^\rho(F) \) is a standard fundamental domain of the Schottky group \( G^\rho \). But then \( h = f^\circ (\tilde{w}^\rho)^{-1} \) is a conformal map of \((C - L^\rho)/G^\rho \) onto \( S \). The Schottky group \( G^\rho \), which has the fundamental domain \( F^\rho \), therefore represents the symmetric surface \( S \).

We examine now the fundamental domain \( F^\rho \). By the remark following Lemma 2,

(a) \( F^\rho \) is symmetric with respect to reflection in the unit circle and

\[
h \circ R^\rho = f^\circ (\tilde{w}^\rho)^{-1} \circ R^\rho = f^\circ R^\circ (\tilde{w}^\rho)^{-1} = \phi^\circ f^\circ (\tilde{w}^\rho)^{-1} = \phi^\circ h
\]

so that the symmetry \( \phi \) in \( S \) is represented by reflection in the unit circle.
(b) \( A_r^w(w) \) and \( A'^r w(w) \) map the exterior of the symmetrically situated Jordan curves \( H_r^r \) and \( H'^r_r \) onto the interior of the symmetrically situated Jordan curves \( H_{r+1}^r \) and \( H'^{r+1}_r \) respectively \((r = 1, 3, \ldots, g - r)\). Here, we denote by \( \Gamma^w \), the image of a curve \( \Gamma \) under \( w^r \). Furthermore

\[
A^r_w(w) = R_0^w \circ A^r \circ R_0^w(w).
\]

(c) \( C^r_t \) and \( C'^r_t \) are circles (by Lemmas 1 and 2) and the Möbius transformation \( (R_0^w \circ R_0^t)^w = R_0^w \circ R_0^t \) maps the exterior of \( C^r_t \) onto the interior of \( C'^r_t \) in such a way that two symmetrically situated points on \( C^r_t \) and \( C'^r_t \) are identified under the group \( G^t \) (specifically, by the element \( R_0^w \circ R_0^t \)). As a result, the points on \( P^t \) which lie on these circles are left fixed (as is the unit circle \( C^t_0 \)) by the symmetry: reflection in \( C^t_0 \). We summarize these results in

**Theorem I.** A symmetric surface \( S \), of type \((g, + t)\) with respect to a symmetry \( \phi \), can be represented by a Schottky group which has a fundamental domain symmetric with respect to reflection in the unit circle \( C \), and bounded by (i) \( t - 1 \) identified pairs of symmetrically situated circles \( \Gamma_1, \Gamma'_1, \ldots, \Gamma_{r-1}, \Gamma'_{r-1} \) and (ii) \((g - t + 1)/2\) identified pairs of Jordan curves in the exterior of \( C \) and \((g - t + 1)/2\) symmetrically situated identified pairs of Jordan curves in the interior of \( C \). The symmetry \( \phi \) on \( S \) is represented by reflection in \( C \) and the \( t \) transition curves on \( S \) by \( C \) and the \( t - 1 \) pairs of circles \( \Gamma_1, \Gamma'_1, \ldots, \Gamma_{r-1}, \Gamma'_{r-1} \).

5. Diasymmetric surfaces with fixed points. We now extend the results of §4 to symmetric surfaces for which \( t = 0 \) but \( \epsilon = -1 \).

To obtain the standard model of type \((g, - t)\) we construct \( g \) pairs of circles:

\[
C_1, C'_1, \ldots, C_{r-1}, C'_{r-1}, K_1, K'_1, \ldots, K_{g-t+1}, K'_{g-t+1}
\]

symmetrically situated with respect to the unit circle \( C_0 \) as in §4. If we let \( Q_i(z) \) be reflection in \( K_i \), followed by rotation about the center \( b_i \) of \( K_i \) by the angle \( \pi \), and define \( R_{g-t}(z) \) as in §4 we find that \( F \), the exterior of the \( 2g \) circles, is a fundamental domain of the Schottky group

\[
G = \{ R_0^w \circ Q_1, \ldots, R_0^w \circ Q_{g-t+1}, R_0^w \circ R_{g-t+1}, \ldots, R_0^w \circ R_{r-1} \}.
\]

Again, under \( R_0^w \circ R_{g-t} \), symmetrical points on \( C_s \) and \( C'_s \) are identified and \( R_0^w \circ Q_t \) maps each point \( P \) of \( K_t \) onto the point of \( K'_t \) which is diametrically opposed to \( R_0^w(P) \), the reflection of \( P \) in \( C_0 \).

\( F/G \), the standard model of type \((g, - t)\) has genus \( g \), is symmetric with respect to \( R(z) \), and has \( t \) transition curves. \((F/G)/R \) has \( t \) holes and \( g - t + 1 \) holes with diametrically opposed points identified (i.e., cross caps). Then, if \( S \) is a symmetric surface of type \((g, - t)\), \( t \neq 0 \), there exists a homeomorphism \( f: F/G \to S \) satisfying \( f \circ R = \phi \circ f \).
The procedure of §4 can be repeated to obtain a Schottky group $G^*$ (with a fundamental domain $F^*$) which represents $S$. $F^*$ has the properties (a) and (c) of §4 and, in addition, by Lemmas 1 and 3, (b') $K^*_i$ and $K'^*_i$ are quasicircles and the Möbius transformation $(R_0 \circ Q)\*^*(z)$ maps the exterior of $K^*_i$ onto the interior of $K'^*_i$ in such a way that each point $P$ on one quasicircle is identified with the point $P'$ on the other quasicircle which is diametrically opposed to the reflection $R_0^*(P)$ of $P$. As a result, a point on $F^*$ which lies on one of the quasicircles is identified, under the group $\{G^*, R_0^*\}$, with its diametrically opposed point on the quasicircle.

We state

**Theorem II.** A symmetric Riemann surface $S$ of type $(g, -\tau)$, $\tau \neq 0$, with respect to a symmetry $\phi$, can be represented by a Schottky group which has a standard fundamental domain symmetric with respect to reflection in the unit circle $C$, and bounded by (i) $\tau - 1$ identified pairs of symmetrically situated circles $\Gamma_1, \Gamma'_1, \ldots, \Gamma_{\tau-1}, \Gamma'_{\tau-1}$ and (ii) $g - \tau + 1$ identified pairs of symmetrically situated quasicircles $\Lambda_1, \Lambda'_1, \ldots, \Lambda_{g-\tau+1}, \Lambda'_{g-\tau+1}$. The symmetry $\phi$ on $S$ is represented by reflection in $C$; the $\tau$ transition curves on $S$ by $C$ and the $\tau - 1$ pairs of circles $\Gamma, \Gamma'$. The pairs of quasicircles $\Lambda_i, \Lambda'_i$ represent (when identified under reflection) $g - \tau + 1$ cross caps on $S/\phi$.

6. Fixed point free diasymmetric surfaces. To extend the results of §§4 and 5 to symmetric surfaces of type $(g, 0)$ we use a different representation of the symmetry. This is clearly necessary, since the reflection $R_0(z)$ always leaves fixed the points on the unit circle.

Let $K_0, \ldots, K_g$ be disjoint circles and let $Q_j(z), 0 \leq j \leq g$, be reflection in $K_j$ followed by rotation about the center of $K_j$ by the angle $\pi$. Denote by $K'_j$ the circle $Q_0(K_j)$, $1 \leq j \leq g$. Denoting by $F$ the exterior of the $2g$ circles $K_0, K'_1, \ldots, K'_g, K_g$ we see that $F$ is a standard fundamental domain of the Schottky group $G = \{Q_0 \circ Q_1, \ldots, Q_0 \circ Q_g\}$. $F/G$ is a symmetric surface (with respect to the symmetry $Q, Q \circ \pi = \pi \circ Q$). It has genus $g$ and no transition curves. Since, for a point $P$ on $K_0, Q_0(P)$ and $Q_0 \circ Q_0(P)$ are diametrically opposed points on $K'_0$. $(F/G)/Q = F/\{G, Q_0\}$ is a sphere with $g + 1$ cross caps. If $S$ is a symmetric surface of type $(g, 0)$ there exists a homeomorphism $f$ of $F/G$ onto $S$ such that $\phi \circ f = f \circ Q$.

As in §§4 and 5 we obtain a fundamental domain $F^*$ of a Schottky group $G^*$ which represents $S$. Furthermore,

(a) The symmetry $\phi$ on $S$ is represented by a symmetry $Q_0^*(w)$ which is of the form (see Lemma 3)

$$Q_0^*(w) = b - \lambda^2/w - b.$$

(b) Again by Lemma 3, $K^*_i$ and $K'^*_i$ are quasicircles and the Möbius transformation $(Q_0 \circ Q)\*^*^*(w)$ maps the exterior of $K^*_i$ onto the interior of $K'^*_i$ in such a way that each point $P$ on one quasicircle is identified with
that point \( P' \) on the other quasicircle which is diametrically opposed to the point \( Q_0(P) \). Therefore, points on \( F' \) which lie on the quasicircles are identified, under the group \( \{ G', Q_0 \} \) with their diametrically opposed points.

We assume without loss of generality that the “center” of \( K_0 \) is at the origin and that \( \lambda = 1 \), so that \( Q_0(w) = -\frac{1}{w} \). We can then state

**Theorem III.** A symmetric Riemann surface \( S \) of type \((g,0)\) with respect to a symmetry \( \phi \) can be represented by a Schottky group \( G \) having a fundamental domain bounded by \( g \) identified pairs of disjoint quasicircles \( \Lambda_1, \Lambda'_1, \ldots, \Lambda_g, \Lambda'_g \) which are symmetrically situated with respect to the symmetry \( Q(w) = -\frac{1}{w} \). The symmetry \( \phi \) on \( S \) is represented by the transformation \( Q(w) \). There is also a quasicircle \( \Lambda \) which is a closed Jordan curve and which contains in its interior the quasicircles \( \Lambda_1, \Lambda_2, \ldots, \Lambda_g \). \( Q(w) \) transforms \( \Lambda \) into itself in such a way that diametrically opposed points are identified, and transforms \( \Lambda_i \) into \( \Lambda'_i \) in such a way that a point \( P \) on \( \Lambda_i \) is identified with the point on \( \Lambda'_i \) which is diametrically opposed to the point on \( \Lambda'_i \) which is identified with \( P \) under the group \( G \). As a result, the quasicircle \( \Lambda \), together with the \( g \) pairs of quasicircles \( \Lambda_i, \Lambda'_i \) represent, when identified under \( Q \), \( g + 1 \) cross caps on \( S/\phi \).

7. **Mappings of multiply connected domains.** We recall that a multiply connected plane domain \( D \), bounded by \( n \) nondegenerate continua, can be mapped conformally onto a plane domain bounded by \( n \) closed analytic Jordan curves. This follows at once from the Riemann mapping theorem.

Given a domain \( D \) bounded by \( n \) closed analytic Jordan curves \( \gamma_1, \ldots, \gamma_n \), let \( S \) be the closed surface obtained by doubling \( D \) [2, pp. 118-119]. We observe that \( S \) is a Riemann surface of genus \( n - 1 \), orthosymmetric with respect to the symmetry \( \phi \) defined by the doubling process. Furthermore \( S/\phi = D \). Then by Theorem I, \( D \) is conformally equivalent to a region bounded by \( n \) disjoint circles. The above arguments give a new proof of the

**Koebe Theorem.** A multiply connected plane domain, bounded by \( n \) nondegenerate continua can be mapped conformally onto a plane domain bounded by \( n \) circles.

**Remark.** This theorem, which has been obtained as a corollary of Theorem I, can also be obtained directly from Lemma 2.

**References**


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