ON THE EXTREME EIGENVALUES
OF TOEPLITZ OPERATORS OF THE HANKEL TYPE II

BY
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1. Introduction. Let $\nu$ be an arbitrary but fixed positive number and set

$$
\mu(x) = x^{2\nu+1}[2^{\nu+1/2}\Gamma(\nu+3/2)]^{-1}, \quad 0 \leq x < \infty,
$$

and

$$
C_\nu = [2^{-1/2}\Gamma(\nu+1/2)]^{-1}.
$$

We define

$$
J(x) = C_\nu^{-1}x^{1/2-\nu}J_{-1/2}(x), \quad 0 \leq x < \infty,
$$

where $J_{-1/2}$ is the Bessel function of the first kind of order $\nu - 1/2$. Let $\Lambda \subset [0, \infty)$ be a Borel measurable set and denote by $L_{2,\mu}(\Lambda)$ the Hilbert space of functions defined on $\Lambda$ with inner product $\int fg^* d\mu$ where $g^*$ is the complex conjugate of $g$.

Let $\Omega \subset [0, \infty)$ be a set of positive but finite measure $d\mu$ and $F$ a real bounded function in $L_{1,\mu}([0, \infty))$. We define the operator $B_A$ on $L_{2,\mu}(A\Omega)$, $A > 0$, by

$$
B_A f(x) = \int_{A\Omega} \rho(x,y)f(y) d\mu(y),
$$

where

$$
\rho(x,y) = \int_0^\infty F(t)J(xt)J(yt) d\mu(t).
$$

Here $L_{1,\mu}([0, \infty)) = L_{1,\mu}$ is the space of all functions defined on $[0, \infty)$ such that $\int |f| d\mu < \infty$.

Under various conditions on $F$ we derive asymptotic formulae for the $k$th largest eigenvalue of $B_A$ as $A \to \infty$. Our considerations fall into three cases. $F$ will always be a bounded real function in $L_{1,\mu}$ that has a unique maximum at $\xi_0$, $0 \leq \xi_0 < \infty$ and is such that $\lim \sup F(\xi) < F(\xi_0)$ as $\xi \to \infty$. The three cases are differentiated by the character and position of the maximum and the character of the set $\Omega$. They are

I. $\xi_0 = 0; \quad F(\xi) \sim F(0) - \sigma \xi^\omega$ as $\xi \to 0^+$, $\omega > 0$, $\sigma > 0$; $\Omega$ as positive and finite measure $d\mu$.

II. $\xi_0 = 0; \quad F(\xi) \sim F(0) - L(\xi) \xi^\omega$ as $\xi \to 0^+$, $\omega > 0$; $L(\xi)$ is slowly oscil-
lating as $\xi \to 0^+$; $\Omega = [0, a]$.

III. $\xi_0 \neq 0$ and $F(\xi) \sim F(\xi_0) - \sigma_1 L(\xi - \xi_0) |\xi - \xi_0|^\omega$ as $\xi \to \xi_0^+$, $F(\xi) \sim F(\xi_0) - \sigma_2 L(\xi - \xi_0) |\xi - \xi_0|^\omega$ as $\xi \to \xi_0^-$; $\omega > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$; $\Omega = [0, a]$; $L(\xi)$ is even and slowly oscillating as $\xi \to 0$.

As a representative result consider Case I. If $\lambda(A, k)$ is the $k$th largest eigenvalue of $B_A$, then there exists an operator depending only upon $\sigma, \nu$, and $\omega$ such that if $0 < \mu(1) \leq \mu(2) \leq \cdots$ are its positive eigenvalues

$$\lambda(A, k) = F(0) - \sigma A^{-\omega} \mu(k) + o(A^{-\omega})$$

as $A \to \infty$.


The idea behind the technique is to reformulate the problem so that we can consider a related sequence of operators on a fixed Hilbert space. We then show that this sequence converges suitably to an operator with known eigenvalues and use a perturbation theorem.

The results of this paper constituted part of my doctoral dissertation at Washington University in St. Louis. I wish to express appreciation to Professor I. I. Hirschman for suggesting this problem and to express my thanks for his direction of my career as a graduate student.

2. Preliminaries. In this section we introduce the necessary information concerning the Hankel transform and Bessel function.

If $x, y$, and $z$ are non-negative real numbers, set $\Delta(x, y, z)$ equal to the area of a triangle with sides $x, y$, and $z$ if such exists and zero otherwise. Let

$$D(x, y, z) = \frac{2^{x-2} \Gamma(\nu + 1/2)^2 |\Delta(x, y, z)|^{2\nu-2}}{\Gamma(1/2) \Gamma(\nu)(xyz)^{2\nu-1}}.$$

If we define convolution by

$$f \ast g \cdot (x) = \int_0^x \int_0^y f(y)g(z) D(x, y, z) d\mu(y)d\mu(z)$$

then $L_{1,\nu}$ is a Banach algebra. $D(x, y, z)$ satisfies

$$\int_0^x D(x, y, z) d\mu(x) = 1.$$
The Hankel transform is defined on $L_{1,u}$ by

$$f^\sim(x) = \int_0^\infty J(xt)f(t)d\mu(t), \quad 0 \leq x < \infty.$$  

For $f$ and $g$ in $L_{1,u}$ we have

$$(f * g)^\sim = f^\sim g^\sim.$$  

The Hankel transform on $L_{2,u}([0, \infty)) = L_{2,u}$ is defined by

$$f^\sim(x) = \int_0^\infty J(xt)f(t)d\mu(t),$$  

where the partial integrals converge in the norm of $L_{2,u}$. This is a unitary mapping of $L_{2,u}$ onto $L_{2,u}$ and in this case

$$(f^\sim)^\sim = f.$$  

These results are all well known and can be found in I. I. Hirschman [2]. We now list some formulas.

(2)  

$$|J(x)| \leq 1,$$

(3)  

$$J(0) = 1,$$

(4)  

$$\frac{d}{dz} (z^{-s}J_s(z)) = - z^{-s}J_{s+1}(z),$$

(5)  

$$\Delta J(xy) = - x^2 J(xy),$$

where

$$\Delta = \left( \frac{d}{dy} \right)^2 + 2\nu \frac{d}{y \ dy}.$$  


$$H_{\nu}^{(1)}(z) = (\pi z/2)^{-1/2} \exp\left[ i(z - \nu\pi/2 - \pi/4) \right]$$

(6)  

$$\cdot \left[ \sum_{m=0}^{M-1} (\nu, m) (-2iz)^{-m} + O(|z|^{-M}) \right]$$

as $|z| \to \infty$ and $-\pi < \arg z < 2\pi$.

$$H_{\nu}^{(2)}(z) = (\pi z/2)^{-1/2} \exp\left[ -i(z - \nu\pi/2 - \pi/4) \right]$$

(7)  

$$\cdot \left[ \sum_{m=0}^{M-1} (\nu, m) (2iz)^{-m} + O(|z|^{-M}) \right]$$

as $|z| \to \infty$ and $-2\pi < \arg z < \pi$.  

\[ J_r(z) = (\pi/2)^{-1/2} \left\{ \cos(z - \nu \pi/2 - \pi/4) \right\} \\
\sum_{m=0}^{m-1} (-1)^m (\nu, 2m) (2z)^{-2m-1} + O(|z|^{-2M}) \\
- \sin(z - \nu \pi/2 - \pi/4) \left\{ \sum_{m=0}^{M-1} (-1)^m (\nu, 2m + 1) (2z)^{-2m-1} + O(|z|^{-2M-1}) \right\} \]

as \(|z| \to \infty\) and \(-\pi < \arg z < \pi\). Here \((\nu, m)\) is Hankel’s symbol

\[ (\nu, m) = \Gamma(1/2 + \nu + m)[m! \Gamma(1/2 + \nu - m)]^{-1}. \]

(2), (3) and (5) can be found in I. I. Hirschman [2], (4) in Watson [10, 47]; (6), (7) and (8) in Erdélyi [1, p. 85].

3. Perturbation theory. Let \(0 \leq S\) be a self-adjoint operator on a separable Hilbert space \(\mathcal{H}\) and \(F\) a bounded operator on \(\mathcal{H}\). We define

\[ \mathcal{S} = \{ f | Ff \in \mathcal{D}(S^{1/2}) \}. \]

Here \(S^{1/2}\) is the unique positive square root of \(S\) and \(\mathcal{D}(S^{1/2})\) is the domain of \(S^{1/2}\). Let \(\mathcal{M}\) be the closure of \(\mathcal{S}\) in \(\mathcal{H}\). Then \(\mathcal{M}\) is a closed subspace of \(\mathcal{H}\) and is itself a Hilbert space.

**Theorem 3a.** With the above definitions, there exists a self-adjoint operator \(S_F\) on the Hilbert space \(\mathcal{M}\) with the properties

1. \(\mathcal{D}(S_F) \subseteq \mathcal{S}\),
2. \((S_F f|g) = (S^{1/2} F f|S^{1/2} F g)\) for all \(f \in \mathcal{D}(S_F)\) and \(g \in \mathcal{S}\).

**Proof.** See F. Riesz and B. Sz.-Nagy [8, p. 326].

Let \(A\) be a closed linear operator on \(\mathcal{H}\), not necessarily densely defined.

**Definition 3b.** A subset \(\mathcal{L}\) of \(\mathcal{D}(A)\) is said to be a core for \(A\) if \(|(f, g)| g = Af, f \in \mathcal{L}| \subset \mathcal{H} \times \mathcal{H}\) is dense in \(|(f, g)| g = Af, f \in \mathcal{D}(A)|\); that is, if \(\mathcal{L} \times A \mathcal{L}\) is dense in the graph of \(A\).

**Definition 3c.** Let \(A_n, n = 1, 2, \ldots, \) and \(A\) be closed linear operators in \(\mathcal{H}\) and let \(\mathcal{L} = \{ f| A_n f \to Af \text{ as } n \to \infty \}\). If \(\mathcal{L} = \mathcal{D}(A)\) we say that \(A\) is the strong limit of the \(A_n\)'s. If \(\mathcal{L}\) is a core for \(A\) we say that \(A\) is the closure of the strong limit of the \(A_n\)'s.

We will use “\(\rightarrow\)" for strong convergence in a Hilbert space and “\(\rightharpoonup\)" for weak convergence.

**Theorem 3d.** Suppose the following conditions are satisfied:

(i) \(0 \leq S\) is a self-adjoint operator in \(\mathcal{H}\); 
(ii) \(F\) is a bounded operator in \(\mathcal{H}\); 
(iii) \(0 \leq S_n\) is a self-adjoint operator in \(\mathcal{H}\), \(n = 1, 2, 3, \ldots, F_n\) is a bounded operator in \(\mathcal{H}\), \(n = 1, 2, 3, \ldots\) such that \(\mathcal{D}(F_n) \subset \mathcal{D}(S_n), n = 1, 2, \ldots\);
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(iv) $F$ is the strong limit of $F_n$ as $n \to \infty$;
(v) $S_n^{1/2}$ is the closure of the strong limit of $S_n^{1/2}$ as $n \to \infty$;
(vi) $S_n^{1/2} F$ is the closure of the strong limit of $S_n^{1/2} F_n$ as $n \to \infty$.

Set $S_n,F = F_n^* S_n F_n$, where $F_n^*$ is the adjoint of $F_n$ and let $S_n,F = \int_0^\infty \lambda d\psi_n(\lambda)$ be the spectral resolution of $S_n,F$ on $\mathcal{H}$. Here we assume $\psi_n(\lambda) = \psi_n(\lambda^+)$ for $0 \leq \lambda < \infty$ and $n = 1, 2, \cdots$ and that $\psi_n(0^-) = 0$.

Let

$$S_F = \int_0^\infty \lambda d\psi(\lambda)$$

be spectral resolution of $S_F$ on $\mathcal{H}$ and again we assume that $\psi(\lambda) = \psi(\lambda^+)$ for $0 \leq \lambda < \infty$ and $\psi(0^-) = 0$. Then for every $f \in \mathcal{H}$ and $\lambda$ not in the point spectrum of $S_F$

$$\psi_n(\lambda) f \to \psi(\lambda) f,$$

where "$\to$" is in $\mathcal{H}$.

Proof. See I. I. Hirschman [3].

We note that if $\mathcal{H}_1$ is a subspace of $\mathcal{H}$ and if $E$ is a projection in $\mathcal{H}_1$ considered as a Hilbert space, then $E$ can be considered as a projection in $\mathcal{H}$, namely, as the projection of $\mathcal{H}$ on $E \mathcal{H}_1$. It is in this sense that the above convergence is in $\mathcal{H}$. This convention will be used throughout.

Theorem 3e. Suppose that $0 \leq R_n$, $n = 1, 2, 3, \cdots$ are bounded self-adjoint operators defined on subspaces $\mathcal{N}_n$ of a Hilbert space $\mathcal{H}$. Let $0 < R$ be a self-adjoint operator defined on a subspace $\mathcal{N}$ of $\mathcal{H}$. Let

$$R_n = \int_0^\infty \lambda dE_n(\lambda),$$

$$R = \int_0^\infty \lambda dE(\lambda)$$

be the spectral resolutions of $R_n$ on $\mathcal{N}_n$ and of $R$ on $\mathcal{N}$.

Suppose further that:

(a) $E_n(\lambda) \to E(\lambda)$ as $n \to \infty$ for all $\lambda > 0$ and not in the point spectrum of $R$. Here "$\to$" is in $\mathcal{H}$;

(b) there is a number $m > 0$ such that if $f_n \in \mathcal{N}_n$, $\|f_n\| = 1$ and $(R_n f_n | f_n) \leq m_1 < m$ for $n \in p_1$ then $p_1$ contains a subsequence $p_2$, where $f_n \to f$ as $n \to \infty$ in $p_2$ and $f \neq 0$. Here "$\to$" is in $\mathcal{H}$ and $p_k$, $k = 1, 2$, denote subsequences of the natural numbers.

Then

$$\dim E(\lambda) < \infty, \quad 0 \leq \lambda < m,$$

$$\dim E_n(\lambda) \to \dim E(\lambda) \quad \text{as} \quad n \to \infty \quad \text{for} \quad 0 \leq \lambda < m$$

and $\lambda$ not in the point spectrum of $R$. 

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Proof. See I. I. Hirschman [3].
In order to apply Theorem 3e we will need the following result.

THEOREM 3f. F \textbf{M} reduces S_F.

Proof. Let \( f \in F \mathcal{S} \). Clearly, if \( g \in \mathcal{S} \), then \( Fg \in \mathcal{S} \). Then writing \( g = Fg + (I - F)g \) we see that \((I - F)g \in \mathcal{S}\). Thus \((I - F) \mathcal{S} \subset \mathcal{S}\) and as \( \mathcal{S} \) is dense in \( \mathcal{M} \), \((I - F) \mathcal{S}\) is dense in \((I - F) \mathcal{M}\). Also \( F \mathcal{S} \) is dense in \( F \mathcal{M} \).

Let \( h \in (I - F) \mathcal{S} \). Then by (2) we have

\[
(S_F|h) = (S^{1/2} F|S^{1/2} Fh) = 0
\]

and hence \( S_F f \in F \mathcal{M} \).

4. Definitions and preliminaries—Case I. We shall assume that \( F(\xi) \) satisfies the following conditions:

(i) \( F(\xi) \) is a bounded real function in \( L_{1,u} \).

(ii) \( F(0) = M \) is the unique maximum of \( F \) and \( \lim \sup F(\xi) < M \) as \( \xi \to \infty \).

(iii) \( F(\xi) \sim M - \sigma \xi^\omega \) as \( \xi \to 0^+ \), \( \omega > 0 \), \( \sigma > 0 \).

We also assume that \( \Omega \subseteq [0, \infty) \) has positive but finite measure \( d\mu \).

We will set up an apparatus, part of which may seem superfluous for this case, but which sets a pattern for the later cases where it will be necessary.

We define four Hilbert spaces \( \mathcal{H}, \mathcal{H}^-, \mathcal{L}, \mathcal{L}^- \), all equal to \( L_{2,u} \). We will use the variable \( x \) for functions in \( \mathcal{H} \), \( \xi \) for \( \mathcal{H}^- \), \( t \) for \( \mathcal{L}^- \), and \( y \) for \( \mathcal{L} \).

This will be the convention throughout this paper. For Case I and Case II a function denoted by \( f^- \) will always be in \( \mathcal{H}^- \) or \( \mathcal{L}^- \) and will always be the Hankel transform of a function in \( \mathcal{H} \) or \( \mathcal{L} \), respectively.

We define unitary maps \( \phi \) of \( \mathcal{H} \) onto \( \mathcal{H}^- \) and \( \psi \) of \( \mathcal{L} \) onto \( \mathcal{L}^- \) by

\[
\phi f \cdot (\xi) = f^-(\xi)
\]

and

\[
\psi f \cdot (t) = f^-(t).
\]

We define maps \( \chi_A \) of \( \mathcal{H}^- \) onto \( \mathcal{H}^- \) and \( \chi_A^* \) of \( \mathcal{L}^- \) onto \( \mathcal{L}^- \) by

\[
\chi_A f \cdot (t) = f(tA^{-1})A^{-r-1/2}
\]

and

\[
\chi_A^* f \cdot (\xi) = f(A\xi)A^{r+1/2}.
\]

It is evident that \( \chi_A, \chi_A^* \) are unitary and that \( \chi_A^* \) is the adjoint of \( \chi_A \).

Define the projection \( E_A \) in \( \mathcal{H} \) by

\[
E_A f(x) = C_{A\Omega}(x)f(x),
\]

where \( C_{A\Omega} \) is the characteristic function of \( A\Omega \).

\( E_A \) is unitarily equivalent to the projection \( E_A^- \) in \( \mathcal{H}^- \) and \( F_A^- \) in \( \mathcal{L}^- \) defined by
We define the operator $T^*$ in $L^\infty$ by
\[
T^* f \cdot (\xi) = (\lambda - F(\xi)) f(\xi).
\]
$T^*$ is unitarily equivalent to the operators $T$ in $L^\infty$ and $T_\lambda^*$ in $L^\infty$ defined by
\[
T = \phi^{-1} T^* \phi,
\]
\[
T_\lambda^* = \chi_\lambda T^* \chi_\lambda^*.
\]
We define the projection $F$ in $L^\infty$ by
\[
F f \cdot (y) = C_{\psi}(y) f(y).
\]
$F$ is unitarily equivalent to the projection $F^*$ in $L^\infty$ defined by
\[
F^* = \psi F \psi^{-1}.
\]
Finally we define the operators $S_{T_\lambda^*}$ and $S^*$ on $L^\infty$ by
\[
S_{T_\lambda^*} f \cdot (t) = \sigma_\lambda^{-1} A^{*} T_\lambda^* f \cdot (t),
\]
\[
S^* f \cdot (t) = \sigma^{-1} t f(t).
\]
If $\lambda(1) \geq \lambda(2) \geq \cdots$ are the positive eigenvalues of $A_\psi$ then $M - \lambda(1) \leq M - \lambda(2) \leq \cdots$ are the eigenvalues of
\[
E_\lambda T f \big|_{E_\lambda^{+}} = \left( M - \lambda(1) \right) A_\psi^{-1} f \big|_{E_\lambda^{+}},
\]
where these symbols are to read "$E_\lambda T f$ restricted to $E_\lambda^{+}$," etc. The eigenvalues of
\[
F_{\psi} S_{T_\lambda^*} F_{\psi} \big|_{F_{\psi}^{+}}
\]
are $(M - \lambda(1)) A_\psi^{-1} \leq (M - \lambda(2)) A_\psi^{-1} \leq \cdots$. In the following sections we will show that $F_{\psi} S_{T_\lambda^*} F_{\psi} \big|_{F_{\psi}^{+}}$ converges to $S_{F^*}$ of Theorem 3a as $A \to \infty$ so that if $0 < \mu(1) \leq \mu(2) \leq \cdots$, $\lim_{k} \mu(k) = \infty$, are the positive eigenvalues of $S_{F^*} \big|_{F^*}$ then using the perturbation theorem we will have
\[
(M - \lambda(A,k)) A^{-1} = \mu(k) + o(1)
\]
as $A \to \infty$ for $k = 1, 2, \cdots$, or equivalently
\[
\lambda(A,k) = M - \mu(k) A^{-1} + o(A^{-1}).
\]
Recall that $S_{F^*}$ is a self-adjoint operator defined on a subspace $M^*$ of $L^\infty$ and $\mathcal{D}(S_{F^*}^*) \subseteq \mathcal{D}^*$, where
\[
\mathcal{D}^* = \{ f : F^* f \in \mathcal{D}((S^*)^{1/2}) \} = \mathcal{D}((S^*)^{1/2} F^*).
\]
Recall also that by Theorem 3f, $F^* M^*$ reduces $S_{F^*}$ so that $S_{F^*} \big|_{F^*}$ is
an operator in $F^\sim \mathcal{M}^\sim$. Since this is the operator that we will eventually be interested in it is important that we determine the nature of $F^\sim \mathcal{M}^\sim$.

It is clear that if $\Omega$ is an interval then $\mathcal{S}^\sim$ is dense in $\mathcal{L}^\sim$ and thus $F^\sim \mathcal{M}^\sim = F^\sim \mathcal{L}^\sim$. Suppose that $\Omega$ is a nowhere dense set of positive measure and that $\omega > 1$. Let $f \in F^\sim \mathcal{S}^\sim$. Then $t^{-1/2}f \in L_{2,\omega}$ and $f$ has its support on $\Omega$. We claim that $f$ is continuous. Indeed, for $0 < y_0 < \infty$ and $\epsilon > 0$ such that $1 + \epsilon = \omega$,

\[ |f(y) - f(y_0)|^2 \leq \int_0^\infty |f^\sim (t)|^2 t^{1+\epsilon} d\mu(t) \int_0^\infty |J_{-1/2}(yt)y^{1/2-\epsilon} - J_{-1/2}(y_0 t)y_0^{1/2-\epsilon}|^2 C_{-1} t^{-\epsilon} dt. \]

Since $J_{-1/2}(t) = O(t^{-1/2})$ as $t \to \infty$, see (7) §2,

\[ \int_0^\infty |J_{-1/2}(yt)y^{1/2-\epsilon} - J_{-1/2}(y_0 t)y_0^{1/2-\epsilon}|^2 t^{-\epsilon} dt \to 0 \]

as $y \to y_0$.

But $f$ vanishes on the complement of $\Omega$, an everywhere dense set, and hence vanishes identically. Thus we see that $F^\sim \mathcal{S}^\sim$ contains only the zero function.

At first it might seem possible that given an integer $n$ an $\Omega$ could be found such that the dimension of $F^\sim \mathcal{M}^\sim$ would be $n$. This is not the case. We will show that if $F^\sim \mathcal{M}^\sim$ contains a function other than the zero function, its dimension is infinite.

**Lemma 4a.** Let $W(x)$ be positive, nondecreasing and such that $W(x + y) \leq W(x)W(y)$. Let $f(y)$ be such that $\int_0^\infty |f(y)|^2 W(x) d\mu(x) < \infty$ and let $g(y)$ be such that $\int_0^\infty |g(y)| W(x)^{1/2} d\mu(x) < \infty$. Then

\[ \int_0^\infty |f * g \cdot (y)|^2 W(y) d\mu(y) < \infty. \]

**Proof.** We have, using Schwarz's inequality,

\[ |f * g \cdot (y)|^2 \leq C_1 \int_0^\infty \int_0^\infty |f(z)|^2 |g(x)| W(x)^{-1/2} D(x, y, z) d\mu(z) d\mu(x), \]

where

\[ C_1 = \int_0^\infty |g(x)| W(x)^{1/2} d\mu(x). \]

Since $D(x, y, z) = 0$ for $y \geq x + z$ we may assume $y < x + z$ and hence $W(y) \leq W(x + z) \leq W(x)W(z)$. Thus, using (1) §2 we get
\begin{align*}
\int_0^\infty |f \ast g \cdot (y)|^2 W(y) \, d\mu(y) \\
\leq C_1 \int_0^\infty \int_0^\infty \int_0^\infty |f(z)|^2 W(z)|g(x)| W(x)^{1/2} D(x, y, z) \, d\mu(z) \, d\mu(x) \, d\mu(y) \\
\leq C_2 \int_0^\infty |f(z)|^2 W(z) \, d\mu(z) < \infty.
\end{align*}

Now suppose there exists an \( f \in F^\wedge, f \neq 0 \). Let \( \Omega_1 \subseteq \Omega \) be the support of \( f \) and let \( I \) be an interval such that the \( d\mu \) measure of \( I \cap \Omega_1 \) is positive. Let \( h(y) \) be an infinitely differentiable, positive function that vanishes off \( I \) and set \( W(t) = (1 + t)^{-1} \). Since \( F(fh) = fh \) we have \( f^\wedge \ast h^\wedge \in F^\wedge \). Applying Lemma 4a we get that \( f^\wedge \ast h^\wedge \in F^\wedge \).

Let \( I_n, n = 1, 2, \ldots \), be a sequence of disjoint intervals such that the \( d\mu \) measure of \( I_n \cap \Omega_1 \) is positive. The corresponding sequence of functions \( f^\wedge \ast h_n^\wedge, n = 1, 2, \ldots \) will be independent and hence \( F^\wedge \) has infinite dimension.

In general \( F^\wedge \neq F^\wedge \wedge \). For example, let \( \Omega \) be the disjoint union of a nowhere dense set of positive measure and a finite interval.

If \( F^\wedge = \{ 0 \}, S_f^\wedge \) has no eigenvalues, but by convention we will say it has infinitely many all equal to plus infinity.

5. Convergence of operators—Case I.

**Lemma 5a.** Suppose \( L, L_n \) are multiplier transformations on \( \mathcal{L} \)

\[ L u \cdot (t) = u(t) h(t), \]
\[ L_n u \cdot (t) = u(t) h_n(t). \]

Then if

(i) \[ \lim_{n \to \infty} h_n(t) = h(t), \]

(ii) \[ |h_n(t)| \leq K |h(t)|, \]

\( L \) is the strong limit of the \( L_n \).

**Proof.** Routine.

**Theorem 5b.** \( (S^\wedge)^{1/2} \) is the strong limit of \( (S_A^\wedge)^{1/2} \).

**Proof.** We have

\[ S_A^\wedge f \cdot (t) = s_A(t) f(t), \]

where

\[ s_A(t) = \sigma^{-1} A^{-1}(M - F(t A^{-1})). \]
By condition (iii) on $F$

\[(1) \quad s_A(t) \to t^\gamma \quad \text{as } A \to \infty.\]

Now $M - F(\xi) = \sigma t^\gamma \epsilon(\xi)$ for $0 \leq \xi \leq 1$, where $\epsilon(\xi)$ is bounded and $\epsilon(\xi) \to 1$ as $\xi \to 0^+$. Using this for $0 \leq t \leq A$ and the fact that $F$ is bounded for $t > A$, we obtain

\[(2) \quad s_A(t) \leq C t^\gamma\]

for all $A$ and $t$. (1) and (2) are precisely the conditions for Lemma 5a and the theorem is proved.

We now compute $F_A^\gamma$. From the definition

\[F_A^\gamma = \chi_A \phi E_A \phi^{-1} \chi_A^\gamma.\]

A straightforward computation shows that $F_A^\gamma = F^\gamma$ for all $A$. Since $(S_A^\gamma)^{1/2}$ converges strongly to $(S^\gamma)^{1/2}$, it is immediate that $(S^\gamma)^{1/2} F^\gamma$ is the strong limit of $(S_A^\gamma)^{1/2} F^\gamma$.

6. The asymptotic formula—Case I. Let $S_F^\gamma$ be constructed from $F^\gamma$ and $S^\gamma$ as in Theorem 3a. Then

\[\mathcal{D}(S_F^\gamma) \subseteq \mathcal{D}^\gamma = \{ f | F^\gamma f \in \mathcal{D} ((S^\gamma)^{1/2}) \}\]

and $S_F^\gamma$ is a self-adjoint operator on $\mathcal{M}^\gamma = \text{closure of } \mathcal{D}^\gamma$.

Let

\[S_F^\gamma = \int_0^{\infty} \lambda d\psi^\gamma(\lambda)\]

be the spectral resolution of $S_F^\gamma$ on $\mathcal{M}^\gamma$ and let

\[S_{A,F}^\gamma = \int_0^{\infty} \lambda d\psi_{A,F}^\gamma(\lambda)\]

be the spectral resolution of $F^\gamma S^\gamma F = S_{A,F}^\gamma$. By Theorem 3d, we have

\[(1) \quad \psi_{A,F}^\gamma(\lambda) \to \psi^\gamma(\lambda)\]

for $0 \leq \lambda < \infty$, and $\lambda$ not in the point spectrum of $S_F^\gamma$.

Let $R^\gamma$ be $S_F^\gamma$ restricted to $F^\gamma \mathcal{M}^\gamma$ and $R_{A,F}^\gamma$ be $S_{A,F}^\gamma$ restricted to $F^\gamma \mathcal{M}^\gamma$. It is easy to show that $R^\gamma > 0$ and $R_{A,F}^\gamma > 0$. Then $R^\gamma$ has the spectral resolution on $F^\gamma \mathcal{M}^\gamma$,

\[R^\gamma = \int_0^{\infty} \lambda dG^\gamma(\lambda),\]

where $G^\gamma(\lambda) = \psi^\gamma(\lambda) - \psi^\gamma(0)$. For $0 \leq \lambda < \infty$ and $G^\gamma(0) = 0$, and $R_{A,F}^\gamma$ has the spectral resolution on $F^\gamma \mathcal{M}^\gamma$. 

\[ R^\sim = \int_0^\infty \lambda \, dG^\sim (\lambda), \]

where \( G^\sim (\lambda) = \psi^\sim (\lambda) - \psi^\sim (0) \) for \( 0 \leq \lambda < \infty \) and \( G^\sim (0) = 0 \). Since \( \psi^\sim (0) = I - F^\sim \) and \( \psi^\sim (0) = I - F^\sim \), where \( I \) is the identity operator, it follows from (1) that

(2) \[ G^\sim (\lambda) \to G^\sim (\lambda) \]

for all \( \lambda, 0 \leq \lambda < \infty, \lambda \) not in the point spectrum of \( R^\sim \).

**Lemma 6a.** If \( f^n \in F^\sim \), \( \| f^n \| = 1 \), \( n = 1, 2, 3, \ldots \), and \( f^n \to f^\sim \) as \( n \to \infty \), then \( f^n(t) \to f^\sim(t) \) uniformly on any compact set \( Z \).

**Proof.** \( f^n \in F^\sim \) implies \( F^\sim f^n = f^n \); that is,

(2) \[ f^n(t) = \int_0^\infty f^n(y) [J(yt) C_0(y)] \, d\mu(y). \]

Now \( f^n \to f^\sim \) implies \( f \to f \) and since \( J(yt) C_0(y) \in L^2 \), we have \( f^n(t) \to f^\sim(t) \) for each fixed \( t \) as \( n \to \infty \).

Now

\[ f^n(t) - f^n(s) = \int_0^\infty f^n(y) [J(yt) - J(yt)] C_0(y) \, d\mu(y) \]

and by Schwarz's inequality, we get

\[ |f^n(t) - f^n(s)| \leq \int_0^\infty [J(yt) - J(yt)]^2 C_0(y) \, d\mu(y). \]

This implies that \( \{f^n(t)\} \) is an equicontinuous set of functions and hence \( f^n(t) \to f^\sim(t) \) uniformly on any compact set.

We denote by \( p \) a subsequence of the natural numbers and let \( A(1) < A(2) \)

\[ \ldots, \quad \text{where} \quad A(k) \to \infty \text{ as } k \to \infty. \]

**Lemma 6b.** With the above definitions let \( f_n \in F^\sim \), \( \| f_n \| = 1 \), \( (R^\sim f_n)/f_n \) \( \leq m < \infty \) for \( n \in p \). If \( f_n \to f \) as \( n \to \infty \) in \( p_1, p_1 \), a subsequence of \( p \), then \( f \neq 0 \).

**Proof.** We claim that given \( m_1 > 0 \) there exists numbers \( t_0 \) and \( A_0 \) such that \( s_A(t) > m_1 \) for \( t > t_0 \) and \( A > A_0 \). Since \( M - F(\xi) = s_A(\xi) \) for \( 0 \leq \xi \leq 1 \), where \( 0 < \epsilon(\xi) \to 1 \) as \( \xi \to 0^+ \), we have for \( 0 \leq t \leq 1 \) that \( \epsilon(tA^{-1}) > m_2 \) \( \to 0 \) and \( s_A(t) = \epsilon(tA^{-1}) > m_2 t \). For \( tA^{-1} > 1 \), \( s_A(t) = \sigma^{-1} \epsilon(M - F(tA^{-1})) > m_3 \) \( > 0 \) and \( s_A(t) = \sigma^{-1} \epsilon(M - F(tA^{-1})) > m_3 A^{-1} \), etc.

Now pick \( n_0 \) and \( t_0 \) so that \( s_{A(t)}(t) > 2m \) for \( n > n_0, t > t_0 \). Then for \( n > n_0 \)

\[ m \geq (R^\sim f_n)/f_n \geq 2m \int_0^\infty |f_n(t)|^2 \, d\mu(t) \]

and hence
\[ \int_0^{t_0} |f_n(t)|^2 \, d\mu(t) \geq 1/2. \]

But by Lemma 6a, \( f_n(t) \to f(t) \) uniformly on \([0, t_0]\) and hence \( \|f\|_2 \geq 1/2. \)

**Theorem 6c.** Let \( F(\xi) \) satisfy conditions (i), (ii), (iii), of §4 and let \( \Omega \) be a set of positive but finite measure \( d\mu \). Let \( \lambda(A, 1) \geq \lambda(A, 2) \geq \cdots \) be the positive eigenvalues of \( B_A \) and \( 0 < \mu(1) \leq \mu(2) \leq \cdots, \mu(k) \to \infty \) as \( k \to \infty \), be the positive eigenvalues of \( S_F^{-} \), where \( \mu(k) = + \infty, k = 1, 2, 3, \cdots \) if \( F^{-} \not\in \mathcal{M} \).

Then

\[ \lambda(A, k) = M - \sigma A^{-}\mu(k) + o(A^{-}). \]

**Proof.** Lemma 6b and (2) are the hypotheses for Theorem 3e and hence

\[ \sigma^{-1} A^{-}(n) \left( M - \lambda(A(n), k) \right) = \mu(k) + o(1). \]

But this is equivalent to (3).

7. **Definitions and preliminaries—Case II.** We say that a function \( L \) is slowly oscillating as \( \xi \to 0^+ \) if for all \( \epsilon > 0 \)

\[ \xi^\epsilon L(\xi) \text{ is increasing in a neighborhood of } 0, \xi > 0, \]

\[ \xi^{-\epsilon} L(\xi) \text{ is decreasing in a neighborhood of } 0, \xi > 0. \]

We shall assume that \( F \) satisfies the following conditions:

(i) \( F \) is a bounded real-valued function in \( L_{1,\nu} \).

(ii) \( F \) has a unique maximum \( M \) at \( \xi = 0 \) and \( \lim \sup F(\xi) < M \) as \( \xi \to \infty \).

(iii) \( M - F(\xi) \sim \xi^\nu L(\xi) \) as \( \xi \to 0^+ \), where \( L(\xi) \) is positive, continuous on \( 0 < \xi < \infty \), \( L(\xi) = O(1) \) as \( \xi \to \infty \), bounded away from 0 as \( \xi \to \infty \) and is slowly oscillating as \( \xi \to 0^+ \).

Define the Hilbert spaces \( \mathcal{H}, \mathcal{H}^-, \mathcal{E}, \) and \( \mathcal{E}^- \) and the operators \( \phi, \psi, \chi_A, \chi_A^*, E_A, E_A^*, F_A^-, T^-A, T^-A^*, F^- \) as in §4.

In this case we assume that \( \Omega = [0, a] \) and without loss of generality that \( a = 1 \).

The operators \( S_A^- \) and \( S_A^- \) on \( \mathcal{E}^- \) are different than in §4 and are defined by

\[ S_A^- f \cdot (t) = A^{-}(L(A^{-1}))^{-1} T_A^- f \cdot (t), \]

\[ S_A^- f \cdot (t) = \iota f(t). \]

As in Case I, if \( \lambda(A, 1) \geq \lambda(A, 2) \geq \cdots \) are the positive eigenvalues of \( B_A \) then the eigenvalues of \( F_A^- S_A^- F_A^- \) restricted to \( F_A^- \mathcal{E}^- \) are \((M - \lambda(A, 1)) A^{-} \leq \cdots \mu(k) \to \infty \) as \( k \to \infty \) are the eigenvalues of \( S_F^{-} \) restricted to \( F^{-} \not\in \mathcal{M} \) the perturbation theorem will yield \((M - \lambda(A, k)) A^{-} \leq \mu(k) + o(1)\).

In the remainder of this paper all limits are taken as \( A \to \infty \) unless stated otherwise.
8. Convergence of \((S^*)^{1/2}\) to \((S^*)^{1/2}\) and \((S^{*})^{1/2}F^*\) to \((S^*)^{1/2}F^*\)—Case II. As in §5, \(F^* = F\) for all \(A\).

We now show that \((S^*)^{1/2}\) is the closure of the strong limit of \((S^{*})^{1/2}\) and that \((S^*)^{1/2}F^*\) is the closure of the strong limit of \((S^{*})^{1/2}F^*\).

We first state a well-known and easily verified result about slowly oscillating functions as a lemma.

**Lemma 8a.** If \(L(\xi)\) is slowly oscillating as \(\xi \to 0^+\), positive, continuous on \(0 < \xi < \infty\), \(L(\xi) = O(1)\) as \(\xi \to \infty\) and bounded away from 0 as \(\xi \to \infty\), then

\[
\lim L(\xi_1) [L(\xi_2)]^{-1} = 1
\]

as \(\xi_1 \to 0, \xi_2 \to 0\), where \(\xi_1, \xi_2\) satisfy \(0 < a < \xi_1, \xi_2^{-1} < b < \infty\); and for each \(\epsilon > 0\) there exists a constant \(C(\epsilon)\) such that

\[
L(\xi_1) [L(\xi_2)]^{-1} \leq C(\epsilon) \left[ (\xi_1, \xi_2^{-1}) + (\xi_1, \xi_2^{-1})^{-1} \right]
\]

for all \(0 < \xi_1, \xi_2 < \infty\).

Next, an easy computation shows that \(S^*f\cdot(t) = s_A(t)f(t)\), where

\[
s_A(t) = A^{\epsilon}[L(A^{-1})]^{-1} (M - F(tA^{-1})).
\]

**Lemma 8b.** Under the assumptions of §7 on \(F(\xi)\) we have

\[
\lim s_A(t) = r, \quad 0 \leq t < \infty;
\]

for any \(\epsilon > 0\) there exists a positive constant \(M(\epsilon)\) such that for all \(A > 0\)

\[
0 \leq s_A(t) \leq M(\epsilon) (t + t^{-1}) r;
\]

and for any \(m_1 > 0\), there exists \(A_0 > 0\) and \(t_0 > 0\) such that

\[
s_A(t) \geq m_1 \text{ for } t > t_0, A > A_0.
\]

**Proof.** By assumption, \(M - F(\xi) = \epsilon' L(\epsilon) \epsilon(\xi)\), where \(\epsilon(\xi)\) is bounded, \(\epsilon(\xi) \to 1\) as \(\xi \to 0^+\) and \(\epsilon(\xi) = O(\xi^{-\nu})\) as \(\xi \to \infty\), and thus

\[
s_A(t) = r L(tA^{-1}) [L(A^{-1})]^{-1} \epsilon(tA^{-1}).
\]

For any fixed \(t, 0 < t < \infty\), (1) of Lemma 8a immediately gives (3).

From (6) we see that there exists a constant \(C > 0\) such that

\[
s_A(t) \leq C r L(tA^{-1}) [L(A^{-1})]^{-1}
\]

for \(0 < t < \infty, A > A_0\), and using (2) of Lemma 8a we get (4).

Now, given \(\epsilon > 0\), there exists \(a(\epsilon) > 0\) such that for \(0 < tA^{-1} < a(\epsilon)\) and \(A^{-1} < a(\epsilon)\) we have

\[
\epsilon(tA^{-1}) \geq m_2, \quad L(tA^{-1}) [L(A^{-1})]^{-1} \geq (t + t^{-1})^{-1}.
\]

Hence for all sufficiently large \(A\) and \(0 < tA^{-1} < a(\epsilon)\),

\[
s_A(t) \geq m_2 r(t + t^{-1})^{-1}.
\]
Choose $m_3 > 0$ so that $M - F(\xi) > m_3$ for $\xi \geq a(\epsilon)$. Then 
\[ s_\lambda(t) \geq m_3 A^\omega \left[ L(A^{-1}) \right]^{-1} \]
for $tA^{-1} \geq a(\epsilon)$, etc.

**Theorem 8c.** $(S^*)^{1/2}$ is the closure of the strong limit of $(S_\lambda^*)^{1/2}$.

**Proof.** Let $f \in D((S^*)^{1/2})$ and $\epsilon > 0$ be given. We must find a 
\[ g \in D((S^*)^{1/2}) \]
such that $\|g - f\| < \epsilon$, $\|(S^*)^{1/2}g - (S^*)^{1/2}f\| < \epsilon$ and $(S^*)^{1/2}g \to (S^*)^{1/2}g$.

Let $f_\delta(t) = e^{-\lambda(t)}$. Then 
\[ \|f - f_\delta\|^2 = \int_0^\infty \left| 1 - e^{-\lambda(t)} \right|^2 |f(t)|^2 d\mu(t) \]
and hence by Lebesgue’s theorem, for $\delta$ sufficiently small 
\[ \|f - f_\delta\|^2 < \epsilon \]
(7)

We also have 
\[ \|(S^*)^{1/2}(f - f_\delta)\|^2 = \int_0^\infty \left| f(t) \right|^2 \left[ 1 - e^{-\lambda(t)} \right]^2 t^\omega d\mu(t) \]
and since $f \in D((S^*)^{1/2})$ we see that for $\delta$ sufficiently small 
\[ \|(S^*)^{1/2}(f - f_\delta)\| < \epsilon \]
(8)

Choose $\delta$ so that (7) and (8) are satisfied and set $g = f_\delta$. Then 
\[ \|(S^*)^{1/2}g - (S^*)^{1/2}g\|^2 = \int_0^\infty \left| f(t) \right|^2 e^{-2\lambda(t)} t^\omega - s_\lambda(t)^{1/2} |^2 d\mu(t) \]
and using (4) of Lemma 8b we see that 
\[ t^\omega - s_\lambda(t)^{1/2} \leq 2t^\omega \left[ 1 + M(\epsilon)(t^\omega + t^{-\omega}) \right]. \]
Now taking $\epsilon < \omega$, and using (3) of Lemma 8b in conjunction with Lebesgue’s theorem we get that 
\[ \|(S^*)^{1/2}g - (S^*)^{1/2}g\| \to 0. \]

Finally we must show that $(S^*)^{1/2}F^*$ is the closure of the strong limit of $(S^*)^{1/2}F^*$.

It is for this proof that we need $\Omega$ to be an interval. In Theorem 8c it is trivial that the approximating function $f_\delta$ is in $D((S^*)^{1/2})$. Here, given $f \in D((S^*)^{1/2}F^*)$ we must find an approximating function in 
\[ D((S^*)^{1/2}F^*) \]
and it is not clear that for an arbitrary $\Omega$ this is possible. Widom [14] gives conditions on $\Omega$ so that the approximation is possible, but it seems that the
most natural situation here is to take \( \Omega = [0, a] \) and without loss of generality, \([0, 1]\).

**Theorem 8d.** \( (S^*)^{1/2} F^* \) is the closure of the strong limit of \( (S^*)^{1/2} F^* \).

**Proof.** Let \( f^* \in \mathcal{D} ((S^*)^{1/2} F^*) \). It is clearly sufficient to consider two cases: (i) \( F^* f^* = f^* \) and (ii) \( F^* f^* = 0 \).

Case (i). Let \( g_\lambda \) be an even, nonnegative, infinitely differentiable function defined on \(-\infty < y < \infty\), vanishing off \([-\lambda, \lambda]\), and such that

\[
\int_0^\infty g_\lambda(y) \, d\mu(y) = 1
\]

for \( \lambda > 0 \) and

\[
\int_0^\infty g_\lambda(y) \, d\mu(y) \to 0
\]
as \( \lambda \to 0 \) for all \( \delta > 0 \). Clearly

\[
|g_\lambda(t)| \leq 1
\]

and

\[
g_\lambda(t) \to 1
\]
as \( \lambda \to 0 \).

Let \( \mathcal{H} \) be the set of all functions, \( f^* (t) \), such that \( F^* f^* = f^* \) and

\[
f^* \in \mathcal{D} ((S^*)^{1/2} F^*).
\]

Then \( Ff = f \) and \( f \) vanishes off \([0, 1]\). Let \( \mathcal{H}_1 \subseteq \mathcal{H} \) be all functions \( f^* \in \mathcal{H} \) such that \( f \) vanishes off \([0, \theta]\) for some \( \theta, 0 < \theta < 1 \).

Let \( f^* \in \mathcal{H}_1 \) and set \( f^*(t) = g_\lambda(t)f^*(t) \). Then

\[
f^*(y) = g_\lambda \ast f \cdot (y) = \int_0^\lambda \int_0^\lambda g_\lambda(z)f(x) D(z, y, x) \, d\mu(z) \, d\mu(x),
\]

and since \( D(z, y, x) = 0 \) if \( z + x > y \), for \( \lambda \) sufficiently small \( f^*(y) \) vanishes off \([0, 1]\) and \( F^* f^*(t) = f^*(t) \). Using (9), (10), and Lebesgue’s theorem, we obtain

\[
\|f^* - f^*\| \to 0
\]
as \( \lambda \to 0 \).

Since \( g_\lambda(y) \) is infinitely differentiable, \( g_\lambda(t) = O(t^{-r}) \) as \( t \to \infty \) for all \( r \). This and (9) imply that \( f^*(t) \in \mathcal{D} ((S^*)^{1/2} F^*) \). Using (9), (10) and the fact that \( f^* \in \mathcal{D} ((S^*)^{1/2} F^*) \) in conjunction with Lebesgue’s theorem we get

\[
\| (S^*)^{1/2} F^* (f^* - f^*) \| \to 0
\]
as \( \lambda \to 0 \).

Since \( g_\lambda(t) = O(t^{-r}) \) as \( t \to \infty \) for all \( r \) and \( f^* \in \mathcal{D} ((S^*)^{1/2} F^*) \) the same
proof as in Theorem 8c shows that

\[ \| (S_A^{-})^{1/2} F^{-} f^\ast - (S^{-})^{1/2} F^{-} f^\ast \| \to 0. \]

Therefore if \( f^\ast \in \mathcal{H} \), there exists an \( f^\ast \in \mathcal{D} ((S^{-})^{1/2} F^{-}) \) such that \( \| f^\ast - f^\ast \| < \epsilon, \) \( \| (S^{-})^{1/2} F^{-} (f^\ast - f^\ast) \| < \epsilon \) and \( (S_A^{-})^{1/2} F^{-} f^\ast \to (S^{-})^{1/2} F^{-} f^\ast \).

Now let \( f^\ast \in \mathcal{H} \). Set \( g_\theta(y) = f(\theta y^{-1}), \) \( 0 < \theta < 1. \) Then \( g_\theta(y) \) vanishes off \([0, \theta]\) and we have

\[ g_\theta^\ast(t) = \int_0^\infty f(\theta y^{-1})J(yt) \, d\mu(y) = \theta^{\alpha + 1} f^\ast (\theta t). \]

Thus \( g_\theta^\ast \in \mathcal{D} ((S^{-})^{1/2} F^{-}) \). It is easy to see that \( \| f - g_\theta \| \to 0 \) as \( \theta \to 1 \) and hence \( \| f^\ast - g_\theta^\ast \| \to 0 \) as \( \theta \to 1. \) Similarly \( \| (S^{-})^{1/2} F^{-} (f^\ast - g_\theta^\ast) \| \to 0 \) as \( \theta \to 1. \)

Since \( g_\theta^\ast \in \mathcal{H} \), we see that if \( f^\ast \in \mathcal{H} \) there exists an \( h^\ast \in \mathcal{H} \) such that \( \| f^\ast - h^\ast \| < \epsilon \) and \( \| (S^{-})^{1/2} F^{-} (f^\ast - h^\ast) \| < \epsilon. \)

Thus we have the theorem for functions in \( \mathcal{H} \). If \( F^\ast f^\ast = 0 \) the proof is trivial.

9. The asymptotic formula—Case II. Let \( S_F^{-} \) be constructed from \( F^{-} \) and \( S \) as in Theorem 3a. We recall that \( S_F^{-} \) is a self-adjoint operator on \( \mathcal{L}^{-} \), the closure in \( \mathcal{L}^- \) of \( \mathcal{J}^- \), where \( \mathcal{J}^- = \{ f | f \in \mathcal{L}^-; F^{-} f \in \mathcal{D} ((S^{-})^{1/2}) \} \), and \( \mathcal{D} (S_F^{-}) \subseteq \mathcal{J}^- \).

Because \( \Omega = [0, 1] \), \( \mathcal{J}^- \) is dense in \( \mathcal{L}^- \) and \( S_F^{-} \) is a self-adjoint operator on \( \mathcal{L}^- \). Let

\[ S_F^{-} = \int_0^\infty \lambda d\psi^- (\lambda) \]

be the spectral resolution of \( S_F^{-} \) on \( \mathcal{L}^- \) and let

\[ S_A^{-} F^{-} = \int_0^\infty \lambda d\psi^- (\lambda) \]

be the spectral resolution of \( S_A^{-} F^{-} = F^{-} S_A^{-} F^{-} \). Then using Theorems 8c and 8d in conjunction with Theorem 3d we have

\[ \psi^- (\lambda) f \to \psi^- (\lambda) f \]

for all \( f \in \mathcal{L}^- \) and \( 0 \leq \lambda < \infty \), \( \lambda \) not in the point spectrum of \( S_F^- \). We define \( R^- \) as the restriction of \( S_F^- \) to \( F^- \mathcal{L}^- \) and \( R_A^- \) as the restriction of \( S_A^{-} F^{-} \) to \( F^{-} \mathcal{L}^- \).

It is easy to see that \( R^- > 0 \) and \( R_A^- > 0. \) Thus \( R^- \) has the spectral resolution on \( F^- \mathcal{L}^- \)

\[ R^- = \int_0^\infty \lambda dG^- (\lambda), \]

where \( G^- (\lambda) = \psi^- (\lambda) - \psi^- (0) \) for \( 0 < \lambda < \infty \), and \( G^- (0) = 0, \) and \( R_A^- \) has
the spectral resolution on \( F^\sim \)
\[
R_A^\sim = \int_0^\infty \lambda dG_A^\sim (\lambda),
\]
where \( G_A^\sim (\lambda) = \psi_A^\sim (\lambda) - \psi_A^\sim (0) \) for \( 0 < \lambda < \infty \), and \( G_A^\sim (0) = 0 \). Since \( \psi^\sim (0) = I - F^\sim \) and \( \psi_A^\sim (0) = I - F^\sim \) it follows from (1) that
\[
G_A^\sim (\lambda) \to G(\lambda)
\]
for \( 0 \leq \lambda < \infty \) and \( \lambda \) not in the point spectrum of \( R^\sim \). Here "\( \to \)" is in \( L^\sim \).

**Lemma 9a.** Let \( A(1) < A(2) < \cdots \), \( A(k) \to \infty \) as \( k \to \infty \). Let \( f_n \in F^\sim L^\sim \), \( \|f_n\| = 1 \), and \( (R_A f_n f_n) \leq m < \infty \) for \( n \in p \). If \( f_n \to f \) as \( n \to \infty \) in \( p \), a subsequence of \( p \), then \( f \neq 0 \).

**Proof.** By Lemma 6a \( f_n(t) \to f(t) \) uniformly on any compact set. By Lemma 8b we have that given \( m_1 > 0 \), there exists \( A_0 > 0 \) and \( t_0 > 0 \) such that \( s_A(t) \geq m_1 \) for \( t > t_0 \), \( A > A_0 \). The rest of the proof is the same as that of Lemma 6b.

**Theorem 9b.** Let \( F \) satisfy (i), (ii), (iii), of §7 and \( \Omega = [0,1] \). If \( \lambda(A,1) \geq \lambda(A,2) \geq \cdots \) are the positive eigenvalues of \( B_A \) and \( 0 < \mu(1) \leq \mu(2) \leq \cdots \), \( \mu(k) \to \infty \) as \( k \to \infty \) are the positive eigenvalues of \( S_F^\sim \), then
\[
\lambda(A,k) = M - A^{-1} L(A^{-1})(\mu(k) + o(1)).
\]

**Proof.** Lemma 9a and (2) are the hypotheses of Theorem 3e. Hence
\[
(A(n))^{[L((A(n))^{-1})^{-1}(M - \lambda(A(n),k))] = \mu(k) + o(1) \text{ for } k = 1,2, \ldots \text{ and this is equivalent to (3).}
\]

10. **Definitions and preliminaries—Case III.** We shall assume that \( F \) satisfies the following conditions:
(i) \( F \) is a bounded real-valued function in \( L_{2,\omega} \);
(ii) \( F \) has a unique maximum \( M \) at \( \xi_0 \neq 0 \) and \( \lim \sup F < M \) as \( \xi \to \infty \);
(iii) \( M - F(\xi) \sim 0 \) as \( \xi \to \xi_0^+ \)
\[
L(\xi - \xi_0^{-}) \sim \sigma_1 |\xi - \xi_0^{-}|^{-1} L(\xi - \xi_0^{+})\text{ as } \xi \to \xi_0^{-}, \quad M - F(\xi) \sim 0 \text{ as } \xi \to \xi_0^{+}, \quad L(\xi) \text{ is an even, positive, continuous function that is slowly oscillating as } \xi \to 0 \text{ and is bounded and bounded away from zero as } \xi \to \infty.
\]
We also assume that \( \Omega = [0,2\pi] \).

We define four Hilbert spaces. Let \( \mathcal{H} \) and \( \mathcal{G}^- \) both be \( L_{2,\omega} \) and denote the norm by \( \| \|_\omega \). Let \( \mathcal{C} \) and \( \mathcal{C}^- \) both be \( L^2((\infty,\infty)) \) with respect to Lebesgue measure and denote the norm by \( \| \| \).

We define the unitary maps \( \phi \) of \( \mathcal{H} \) onto \( \mathcal{H}^- \) and \( \psi \) of \( \mathcal{C} \) onto \( \mathcal{C}^- \) by
\[
\phi f \cdot (\xi) = \int_0^\infty J(x\xi)f(x) \, d\mu(x)
\]
and
\[ \psi f \cdot (t) = \int_{-\infty}^{\infty} e^{2\pi i t} f(y) \, dy, \]

where the partial integrals converge in the metric of \( \mathcal{L}^\diamond \) and \( \mathcal{L}^\ast \), respectively.

Maps \( \chi_A \) of \( \mathcal{L}^\ast \) to \( \mathcal{L}^\ast \) and \( \chi_A^* \) of \( \mathcal{L}^\ast \) to \( \mathcal{L}^\ast \) are defined by
\[
\chi_A f \cdot (t) = f(\xi_0 + tA^{-1}) (C, A^{-1})^{1/2} (\xi_0 + tA^{-1})^* \quad \text{for} \quad t > -\xi_0 A,
\]
\[= 0 \quad \text{for} \quad t \leq -\xi_0 A \]

and
\[
\chi_A^* g \cdot (\xi) = g(A(\xi - \xi_0)) (C, A^{-1})^{-1/2} \xi^{-*} \quad \text{for} \quad \xi > 0,
\]
\[= 0 \quad \text{for} \quad \xi < 0. \]

It is easy to see that \( \chi_A \) is an isometric map into \( \mathcal{L}^\ast \) and that \( \chi_A^* \) is a partially isometric map onto \( \mathcal{L}^\ast \) whose partial domain is the range of \( \chi_A \). Thus \( \chi_A^* \chi_A = I \) and \( \chi_A \chi_A^* = I \) on the range of \( \chi_A \).

The operators \( E_A, E_A^*, F_A, T, T^*, \) and \( T_A^* \) are defined as in §4, using the maps \( \phi, \psi, \chi_A, \) and \( \chi_A^* \) of this section. Let the operators \( S_A^* \) and \( S^* \) be defined on \( \mathcal{L}^\ast \) by
\[
S_A^* f \cdot (t) = s_A(0)f(t)
\]
and
\[
S^* f \cdot (t) = s(t)f(t),
\]
where
\[
s_A(t) = A^\ast \left[ L(A^{-1}) \right]^{-1} \left[ M - F(\xi_0 + tA^{-1}) \right] \quad \text{for} \quad t > -\xi_0 A,
\]
\[= 0 \quad \text{for} \quad t < -\xi_0 A \]
and
\[
s(t) = \sigma_1 |t|^* \quad \text{for} \quad t < 0,
\]
\[= \sigma_2 |t|^* \quad \text{for} \quad t < 0. \]

We note that
\[
S_A^* f \cdot (t) = A^\ast \left[ L(A^{-1}) \right]^{-1} T_A^* f \cdot (t).
\]
Finally we define the projections \( F \) in \( \mathcal{L}^\ast \) and \( F^\ast \) in \( \mathcal{L}^\ast \) by
\[
F f \cdot (y) = f(y) \quad \text{for} \quad -1 \leq y \leq 1,
\]
\[= 0 \quad \text{for} \quad y > 1 \]
and
\[
F^\ast = \psi F \psi^{-1}. \]
In this case, \( f \ast g \) will denote convolution in \( L^1(-\infty, \infty) \) with respect to Lebesgue measure.

If \( \lambda(A, 1) \geq \lambda(A, 2) \geq \cdots \) are the positive eigenvalues of \( B_A \), then \[ [M - \lambda(A, 1)] A^*[L(A^{-1})]^{-1} \leq [M - \lambda(A, 2)] A^*[L(A^{-1})]^{-1} \leq \cdots \] are the eigenvalues of \( F_A S_A F_A \) restricted to \( F_A \). As in the two previous cases we show that \( F_A S_A F_A \) converges to \( S_F \) of Theorem 3a such that if \( 0 < \mu(1) \leq \mu(2) \leq \cdots \) are the eigenvalues of \( S_F \) restricted to \( F \) then

\[
[M - \lambda(A, 1)] A^*[L(A^{-1})]^{-1} = \mu(k) + o(1).
\]

11. Convergence of \((S_A^{-1})^{1/2}\) to \((S^{-1})^{1/2}\) and \(F_A^{-1}\) to \(F^{-1}\) — Case III.

**Lemma 11a.** With the definitions of §10 we have

1. \( \lim s_A(t) = s(t) \);
2. for any \( \epsilon > 0 \), there is a constant \( M(\epsilon) \) independent of \( t, -\infty < t < \infty \) and \( A > 0 \) such that
   \[
s_A(t) \leq M(\epsilon) |t|^\nu [t^+ + |t|^{-\nu}];
\]
3. given \( m_1 > 0 \) there are numbers \( A_0 > 0, t_0 > 0 \) such that for \( A > A_0 \) and \( |t| > t_0 \), \( s_A(t) \geq m_1 \).

**Proof.** The proof is virtually the same as that of Lemma 8b.

**Theorem 11b.** \((S^{-1})^{1/2}\) is the closure of the strong limit of \((S_A^{-1})^{1/2}\).

**Proof.** Let \( f \in \mathcal{D} ((S^{-1})^{1/2}) \) and \( \epsilon > 0 \) be given. We must find a

\[
g \in \mathcal{D} ((S^{-1})^{1/2})
\]

such that \( \|f - g\| < \epsilon \), \( \|(S^{-1})^{1/2}(f - g)\| < \epsilon \) and \( (S_A^{-1})^{1/2} g \to (S^{-1})^{1/2} g \). If \( f_t(t) = e^{-t} f(t) \) then just as in Theorem 8c, \( g = f_\delta \) works if \( \delta \) is sufficiently small.

**Theorem 11c.** If \( F_A^{-1} \) and \( F^{-1} \) are defined as in §10, then \( F^{-1} \) is the strong limit of \( F_A^{-1} \) as \( A \to \infty \).

**Proof.** Let

\[
P(u, w, t, A) = J_{-1/2}(uA\xi_0 + t)) J_{-1/2}(uA\xi_0 + w))(u(A\xi_0 + t))^{1/2}(u(A\xi_0 + w))^{1/2}.
\]

A straightforward computation shows that

\[
F_A g \cdot (t) = \int_0^{2\pi} \int_{-A\xi_0}^{A\xi_0} P(u, w, t, A) g(w) dw du \quad \text{for } t > -A\xi_0,
\]

\[
= 0 \quad \text{for } t < -A\xi_0.
\]

In what follows \( C \) is a generic constant.
Let $W$ be the set of functions $f \in L^\infty$ which are continuous and have support in $|t| \leq a$ for some $a$. We first prove that if $g \in W$, then $F^*_A g \cdot (t)$ $\to F^* g \cdot (t)$ uniformly on $[-b, b]$ for any $b < \infty$.

For $A$ sufficiently large and $|t| \leq b$

$$F^*_A g \cdot (t) = \int_0^{2\pi} \int_{-a}^{a} P(u, w, t, A) g(w) \, dw \, du$$

and we can write $F^*_A g \cdot (t) = I_1 + I_2$, where

$$I_1 = \int_0^{a} \int_{-a}^{a} P(u, w, t, A) g(w) \, dw \, du,$$

$$I_2 = \int_{b}^{2\pi} \int_{-a}^{a} P(u, w, t, A) g(w) \, dw \, du.$$  

Now using (8) §2 and the well-known fact that $x^r J_r(x) = O(1)$ as $x \to 0$ we see that

$$J_{-1/2}(u(\xi_0 + t)) [u(\xi_0 + t)]^{1/2} \leq C$$

and hence

$$\left| \int_{-a}^{a} P(u, w, t, A) g(w) \, dw \right| < C$$

for $0 \leq u \leq 2\pi$, $A > 0$, and $t > -\xi_0 A$. Thus $I_1 \leq C\delta$ for $|t| \leq b$.

We will show in a moment that

$$\lim \int_{-a}^{a} P(u, w, t, A) g(w) \, dw = \int_{-a}^{a} g(w) \cos(u(t - w)) \, dw$$

uniformly for $|t| < b$. Using this and Lebesgue’s limit theorem, see (6), we will have

$$\lim I_2 = \int_{b}^{2\pi} \int_{-a}^{a} g(w) \cos(u(t - w)) \, dw \, du.$$  

From (8) §2 and a standard trigonometric identity,

$$P(u, w, t, A) = \pi^{-1} \{ \cos(u(t - w)) + \cos(u(2A\xi_0 + t + w) - \nu\pi) \} + O(A^{-1}),$$

where the $O(A^{-1})$ is uniform for $\delta \leq u \leq 2\pi$, $|t| \leq b$, $|w| \leq a$. Thus

$$\lim \int_{-a}^{a} P(u, w, t, A) g(w) \, dw = \pi^{-1} \{ \int_{-a}^{a} g(w) \cos(u(t - w)) \, dw$$

$$+ \lim \int_{-a}^{a} \cos(u(2A\xi_0 + t + w) - \nu\pi) g(w) \, dw + \lim \int_{-a}^{a} O(A^{-1}) g(w) \, dw \}.$$  

The second limit on the right is zero by the Riemann-Lebesgue theorem; the third integral is obviously zero, and it is easy to check that convergence is
uniform for \( \delta \leq u \leq 2\pi, \ |t| \leq b. \) Therefore we have (7).

Let

\[ I_3 = \pi^{-1} \int_{0}^{2\pi} \int_{-a}^{a} g(w) \cos(u(t-w)) \, dw \, du. \]

Then clearly \( F_A^* g \cdot (t) \to I_3 \) uniformly as \( A \to \infty \) for \( |t| \leq b. \)

Set \( A = [ -2\pi, 2\pi ] \). Then using the fact that \( C_s(u) \) is even we get

\[ I_3 = \int_{-\infty}^{\infty} C_s(2\pi u) e^{i2\pi ut} \int_{-\infty}^{\infty} g(w) e^{-i2\pi wu} \, dw \, du. \]

But since \( C_s(2\pi u) \) is the characteristic function of \( [-1,1] \), (9) is just

\[ h = \int C_s(2\pi u) e^{i2\pi ut} g(w) e^{-i2\pi wu} \, dw \, du. \]

Since \( F_A^* f \to F^* f. \)

Indeed, if \( g \in L^\infty \), we get

\[ |([F_A^* f - F^* f]|g)|^2 \leq 4 \|g\|^2 \|f\|^2 \]

using Schwarz's inequality and the fact that since \( F_A^* \) and \( F^* \) are projections, \( \|F^*\| = 1 \) and \( \|F_A^*\| = 1 \) for all \( A \). Hence there is a \( C > 0 \) such that

\[ \int_{|t| \geq C} |F_A^* f \cdot (t) - F^* f \cdot (t) | (g(t))^* \, dt \leq c/2 \]

for all \( A \). But by what we have just proved

\[ \int_{|t| \leq C} |F_A^* f \cdot (t) - F^* f \cdot (t) | (g(t))^* \, dt \to 0. \]

Now (10) follows trivially from (11) and (12).

Since \( W \) is dense in \( L^\infty \), again using the fact that \( \|F^*\| = 1 \) and \( \|F_A^*\| = 1 \) we see that \( F_A^* f \to F^* f \) for all \( f \in L^\infty \). But weak convergence of projections implies strong convergence and our theorem is proved.

12. Convergence of \( (S_A^*)^{1/2} F_A^* \) to \( (S^*)^{1/2} F^* \)—Case III. Consider the rectangle \( R \) shown in Figure 1.

\[ \text{Figure 1} \]
Let $\gamma_k(i\theta)$ be the harmonic measure of the side $\sigma_k$ at the point $i\theta$, $k = 1, 2, 3, 4$. That is, $\gamma_k(z)$ is harmonic on $R$ and $\gamma_k(z) = 1$ on the interior of $\sigma_k$ and $\gamma_k(z) = 0$ on $\sigma_j$, $j \neq k$.

**Lemma 12a.** With the above notations we have the inequalities

(i) $\gamma_1(i\theta) \geq 1 - \theta - 2\theta\xi_0^{-1} \cosh \pi t\xi_0^{-1}$,

(ii) $\gamma_2(i\theta) < \theta$,

(iii) $\gamma_3(i\theta) < \theta\xi_0^{-1} \cosh \pi t\xi_0^{-1}$,

(iv) $\gamma_4(i\theta) < \theta\xi_0^{-1} \cosh \pi t\xi_0^{-1}$.

**Proof.** The demonstration of this result follows routine lines. See [3] where it is given in detail.

A straightforward computation shows that

$$F_{\theta}^* g \cdot (t) = \int_0^{2\pi} Q(u, t, A) a(u, A) \, du \quad \text{for } t > - A\xi_0,$$

$$= 0 \quad \text{for } t \leq - A\xi_0,$$

where

$$a(u, A) = \int_{-\xi_0 A}^{\xi_0 A} C J(u(A\xi_0 + w)) \left[ A\xi_0 + w \right] g(w) \, dw,$$

$$Q(u, t, A) = J_{-1/2} \left[ u(A\xi_0 + t) \right] (A\xi_0 + t)^{1/2} u^{r^{+1/2}}.$$

Let $\mathcal{D}$ be the set of functions $h(y)$ in $C$ which are infinitely differentiable and have compact support and let $\mathcal{D}^\circ = \psi \mathcal{D}$. Let $\mathcal{D}_1$ be the subset of $\mathcal{D}$ consisting of those functions that have support on $|y| \leq C$ for some $C < 1$ and let $\mathcal{D}_1^\circ = \psi \mathcal{D}_1$. Let $\mathcal{D}_2$ be the subset of $\mathcal{D}$ consisting of those functions that have support on $|y| \geq C$ for some $C > 1$, let $\mathcal{D}_2^\circ = \psi \mathcal{D}_2$.

**Theorem 12b.** If $g \in \mathcal{D}_1^\circ$ or $g \in \mathcal{D}_2^\circ$ and if $a(u, A)$ is defined by (2) then

$$\Delta^N a(2\pi, A) = O(\Delta)$$

and

$$\frac{d}{du} \Delta^N a(2\pi, A) = O(\Delta)$$

as $A \to \infty$ for all $r$ and $N = 0, 1, 2, \ldots$, where

$$\Delta = \left( \frac{d}{du} \right)^2 + \frac{2\nu}{u} \frac{d}{du}.$$

**Proof.** Let $g \in \mathcal{D}_1^\circ$. Using (5) §2 we see that

$$\Delta^N a(u, A) = \left( -1 \right)^N \int_{-\xi_0 A}^{\xi_0 A} C_j J(u(A\xi_0 + w)) (A\xi_0 + w)^{-2N} g(w) \, dw.$$

We write
where

\[ \Delta^N a(u, A) = (-1)^N(a_1 + a_2 + a_3), \]

\[ a_1 = \int_{-iA}^{iA} C, J(u(A \xi_0 + w)) \left( A \xi_0 + w \right)^{r+2N} g(w) \, dw, \]

\[ a_2 = \int_{-i_1A}^{i_2A} C, J(u(A \xi_0 + w)) (A \xi_0 + w)^{r+2N} g(w) \, dw, \]

\[ a_3 = \int_{i_2A}^{iA} C, J(u(A \xi_0 + w)) (A \xi_0 + w)^{r+2N} g(w) \, dw. \]

Here \( -\xi_0 A < -\delta_1 A < 0 < \delta_2 A < \infty \) and \( \delta_1, \delta_2 > 0 \). \( \delta_1 \) and \( \delta_2 \) will be chosen precisely later.

It is well known that if \( g \in D^- \) then \( g(t) = O(t^{-r}) \) as \( t \to \infty \) for all \( r \).

Using this and (2) \( \S 2 \) we easily obtain that \( |a_1| = O(A^{-r}) \) and \( |a_3| = O(A^{-r}) \) as \( A \to \infty \) for all \( r \).

From the standard relation

\[ J_{-1/2} = \frac{1}{2} \left[ H_{1/2}^{(1)}(z) + H_{-1/2}^{(2)}(z) \right] \]

it follows that \( a_2 = a_2^+ + a_2^- \), where

\[ a_2^+ = \frac{1}{2} u^{1/2} \int_{-i1A}^{i1A} H_{1/2}^{(1)}(u(A \xi_0 + w)) (A \xi_0 + w)^{2N+1/2} g(w) \, dw \]

and

\[ a_2^- = \frac{1}{2} u^{1/2} \int_{-i1A}^{i1A} H_{-1/2}^{(2)}(u(A \xi_0 + w)) (A \xi_0 + w)^{2N+1/2} g(w) \, dw. \]

Applying Cauchy's theorem to the integrand of \( a_2^+ \) with respect to the top curve of Figure 2 and to the integrand of \( a_2^- \) with respect to the bottom curve of Figure 2.
curve we see that  \( a_2^+ = I_1^+ + I_2^+ + I_3^+ \) and  \( a_2^- = I_1^- + I_2^- + I_3^- \), where

\[
I_1^+ = \int_{-\delta_1}^{\delta_1} \theta \frac{e^{-i\theta}}{2\pi} d\theta, \quad I_2^+ = \int_{-\delta_1}^{\delta_1} \theta \frac{e^{-i\theta}}{2\pi} d\theta, \quad I_3^+ = \int_{-\delta_1}^{\delta_1} \theta \frac{e^{-i\theta}}{2\pi} d\theta
\]

\[
I_1^- = \int_{-\delta_1}^{\delta_1} \theta \frac{e^{i\theta}}{2\pi} d\theta, \quad I_2^- = \int_{-\delta_1}^{\delta_1} \theta \frac{e^{i\theta}}{2\pi} d\theta, \quad I_3^- = \int_{-\delta_1}^{\delta_1} \theta \frac{e^{i\theta}}{2\pi} d\theta
\]

the integrand of  \( I_1^+ \) being that of  \( a_2^+ \) and the integrand of  \( I_1^- \) being that of  \( a_2^- \).

We consider  \( I_2^+ \) first. It is easily seen that  \( g(\theta + i\tau A/2) = O(e^{C\tau A}) \) as  \( A \to \infty \). Using the estimate (6) §2 for  \( H_{r,1/2} \), we see that for  \( u = 2\pi \) the integrand of  \( I_2^+ \) is  \( O(A^{2N} e^{C\tau(1)} \) as  \( A \to \infty \) and as  \( C < 1 \) we see that  \( I_2^+ = O(A^{-\tau}) \) as  \( A \to \infty \) for all  \( r \). \( I_2^- \) is handled in exactly the same way using estimate (7) §2 for  \( H_{r,1/2} \).

We next examine  \( I_1^- \). Consider the rectangle  \( R \) of Figure 3.

Choose  \( \delta_i \) to satisfy  \( \xi_0/2 < \delta_i < \xi_0, \quad i = 1, 2 \). Then  \( |g(w)| \leq C_1 A^{-\tau} \) on the bottom side of  \( R \). On the other three sides we have  \( |g(w)| \leq C_1 e^{2\pi r A} \). Throughout this argument  \( C_1 \) will be a generic constant.

Now  \( R \) is conformally equivalent to the rectangle of Lemma 12a. Using the notation there and one of the standard principles of harmonic measure we get

\[
\log |g(-\delta_1 A + iA\tau\theta)| \leq \gamma_1(i\tau\theta) \log(C_1 A^{-\tau}) + \gamma_2(i\tau\theta) + \gamma_3(i\tau\theta) + \gamma_4(i\tau\theta) \log(C_1 e^{2\pi r A}).
\]

Using the estimates of Lemma 12a we can choose  \( \tau \) so small that  \( \gamma_1(i\tau\theta) \geq 1/3 \) and  \( \gamma_2(i\tau\theta) + \gamma_3(i\tau\theta) + \gamma_4(i\tau\theta) < 0 \) for  \( 0 \leq \theta \leq 1/2 \). Hence

\[
|g(-\delta_1 A + iA\eta)| \leq C_1 A^{-\eta/3} e^{2\pi r A},
\]

for  \( 0 \leq \eta \leq \tau/2 \). Using the estimate (6) §2

\[
(A\xi_0 + w)^{2N+1/2} H_{r,1/2} (2\pi(A\xi_0 + w)) \leq C_1 e^{-2\pi r A} A^{2N}
\]

for  \( 0 \leq \eta \leq \tau/2 \), where  \( w = -\delta_1 A + i\eta A \). Therefore the integrand of  \( I_1^- \) is
$O(A^{2N-r/3}e^{-2(A-1-\epsilon r)A})$ as $A \to \infty$ for $0 \leq r \leq \tau/2$. Since $1 - C > 0$ and $r$ is arbitrary, $I_r^+ = O(A^{-r})$ for all $r$ as $A \to \infty$.

$I_r^+, I_{r_1},$ and $I_r^-$ are handled in the same way. Thus we have proved (4) if $g \in \mathcal{D}_1^+$. Now let $g \in \mathcal{D}_2^+$. Then $\psi^{-1}g \cdot (y)$ has its support on $1 < C \leq |y| \leq C_0 < \infty$ and $g(w) = g_1(w) + g_2(w)$, where

$$g_1(w) = \int_0^C \psi^{-1}g \cdot (y) e^{it\psi y}dy, \quad g_2(w) = \int_C^\infty \psi^{-1}g \cdot (y) e^{it\psi y}dy.$$ 

Then $a_2 = a_{21} + a_{22}$, where

$$a_{21} = \int_{-t_1 A}^{t_1 A} J_{-1/2}(u(A \xi_0 + w))[A \xi_0 + w]^{2N+1/2} u^{1/2-t}g_1(w) dw,$$

$$a_{22} = \int_{-t_1 A}^{t_1 A} J_{-1/2}(u(A \xi_0 + w))[A \xi_0 + w]^{2N+1/2} u^{1/2-t}g_2(w) dw.$$ 

Applying Cauchy's theorem as in the previous case we obtain $a_{21} = I_{11} + I_{21} + I_{31}$ and $a_{22} = I_{12} + I_{22} + I_{32}$, where

$$I_{11} = \int_{-t_1 A}^{t_1 A} J_{-1/2}(u(A \xi_0 + w))[A \xi_0 + w]^{2N+1/2} u^{1/2-t}g_1(w) dw,$$

$$I_{12} = \int_{-t_1 A}^{t_1 A} J_{-1/2}(u(A \xi_0 + w))[A \xi_0 + w]^{2N+1/2} u^{1/2-t}g_2(w) dw.$$ 

The proof from this point is so similar to the previous case that we omit it. Finally we consider the proof of (5). By (4) $\S 2$ we have that

$$\frac{d}{du} A^N a(u, A)$$

$$= (-1)^{N+1} \int_{-t_0 A}^{t_0 A} u^{1/2-t} J_{-1/2}(u(A \xi_0 + w))[A \xi_0 + w]^{2N+3/2} g(w) dw.$$ 

But this is the same problem as (4) and of course has exactly the same solution.

**Lemma 12c.** Let $g \in \mathcal{D}_1^+$ or $g \in \mathcal{D}_2^+$. Then for every non-negative integer $N$ and $c > 0$

$$\limsup \int_{-A_0}^{cA} |t|^{2N} F_A g \cdot (t) t^{2N} dt \leq 2^{2N} \int_{-\infty}^{\infty} t^{2N}(2 + t)^N |g(t)|^2 dt.$$ 

**Proof.** Let $g \in \mathcal{D}_1^+$. Set $h(t, A) = 2A \xi_0 t(1 + t/2A \xi_0)$ and

$$g_N, A(t) = (h(t, A))^{N} g(t).$$ 

Let $\Delta$ be as in Theorem 12b and set $\Delta_A = \Delta + (A \xi_0)^2$. Finally let $a(u, A)$ be as in Theorem 12b. We first show that
(7) \[ F_{A\hat{g}}(t) = \left[ h(t, A) \right]^N F_{A\hat{g}}(t) + O(A^{-\gamma}) \]
for all \( r \) and \(-A_{\xi_0} \leq t \leq cA\).

We note the following evident facts:

(8) \[ \Delta^N a(u, A) = O(1), \quad \frac{d}{du} \Delta^N a(u, A) = O(u) \]
as \( u \to 0 \). Also an easy computation using (4) §2 and (5) §2 shows that

(9) \[ \frac{d}{du} u^{2r} \frac{d}{du} J(u(A_{\xi_0} + t)) = u^{2r} \Delta J(u(A_{\xi_0} + t)). \]

Now, since

\[ ( - \Delta_A)^N J(u(A_{\xi_0} + w)) = \left[ h(w, A) \right]^N J(u(A_{\xi_0} + w)) \]
we see that

\[ F_{A\hat{g}}(t) = C \int_0^{2\pi} J(u(A_{\xi_0} + t))(A_{\xi_0} + t)^r \Delta(u(A_{\xi_0} + t)) \ du. \]

Performing \( 2N \) integrations by parts and making repeated use of (8) and (9) we obtain

(10)
\[
F_{A\hat{g}}(t) = \sum_{j=0}^{N-1} K_j(A) k_j(t, A) (h(t, A))(A_{\xi_0} + t)^r u^{2r} \Delta (A_{\xi_0} + A_{\xi_0} t)^r A^r (A_{\xi_0} + A_{\xi_0} t)^r A^r \]

The terms \( K_j(A) \) and \( P_j(A) \) arise from the terms \( (- \Delta_A)^r a(2\pi, A) \) and

\[ \frac{d}{du} (- \Delta_A)^r a(2\pi, A), \]

respectively, and by Theorem 12b are \( O(A^{-\gamma}) \) as \( A \to \infty \) for all \( r \). The terms \( k_j(t, A) \) and \( p_j(t, A) \) arise from the terms \( (A_{\xi_0} + t)^r 2\pi(2\pi (A_{\xi_0} + t)) \) and \( (A_{\xi_0} + t)^r J \left[ 2\pi (A_{\xi_0} + t) \right] \), respectively, and are thus uniformly bounded for all \( t \) and \( A \). Thus we have proved (7). Now

(11) \[ h(t, A) \geq A_{\xi_0} t \]
for \( t > -A_{\xi_0} \) and

(12) \[ (h(t, A))^2 \leq (2A_{\xi_0} t)^2 (2 + t^2) \]
for \(-\infty < t < \infty \) and \( A > 2^{-1/2} \xi_0 \). Using (7), (12) and the fact that \( \| F_A \| = 1 \) we get

\[
\int_{-A_{\xi_0}}^{A_{\xi_0}} |h(t, A)|^{2N} F_A \cdot (t) \ dt \leq (2A_{\xi_0})^{2N} \int_{-\infty}^{\infty} t^{2N}(2 + t^2)^N |g(t)|^2 \ dt + O(A^{-\gamma})
\]
for \( A > 2^{-1/2} \xi_0 \). Now using (11) and dividing through by \((A \xi_0)^{2N}\) we have for
\( A \) sufficiently large and all \( r \)
\[ \int_{-A_0}^{cA} t^{2N} |F \hat{\Lambda} g \cdot (t)|^2 \, dt \leq 2^{2N} \int_{-\infty}^{\infty} t^{2N} (2 + t^2)^N |g(t)|^2 \, dt + O(A^{-\gamma}). \]
The lemma clearly follows.

**Theorem 12d.** If \( g \in D_1 \) then
\[ \lim \|(S^-)^{1/2} F^{-} \hat{g} - (S_\Lambda)^{1/2} F \hat{\Lambda} g\| = 0. \]

**Proof.** First we set
\[ \|(S^-)^{1/2} F^{-} \hat{g} - (S_\Lambda)^{1/2} F \hat{\Lambda} g\|^2 = I_1 + I_2, \]
where
\[ I_1 = \int_{-\infty}^{\infty} |(\sigma_2 t)^{1/2} F^{-} \hat{g} - (s_\Lambda(t))^{1/2} F \hat{\Lambda} g|^2 \, dt, \]
\[ I_2 = \int_{0}^{\infty} |(\sigma_1 t)^{1/2} F^{-} \hat{g} - (s_\Lambda(t))^{1/2} F \hat{\Lambda} g|^2 \, dt. \]

Now \( I_1 = I_{11} + I_{12} + I_{13} \), where
\[ I_{11} = \int_{-\infty}^{-A_0} \sigma_2 |t|^{1/2} |F^{-} \hat{g}|^2 \, dt, \]
\[ I_{12} = \int_{-A_0}^{-T} \sigma_2 |t|^{1/2} |F^{-} \hat{g} - (s_\Lambda(t))^{1/2} F \hat{\Lambda} g|^2 \, dt, \]
\[ I_{13} = \int_{-T}^{0} |(\sigma_1 t)^{1/2} F^{-} \hat{g} - (s_\Lambda(t))^{1/2} F \hat{\Lambda} g|^2 \, dt, \]
and \( I_2 = I_{21} + I_{22} + I_{23} \), where
\[ I_{21} = \int_{A_0}^{\infty} |(\sigma_1 t)^{1/2} F^{-} \hat{g} - (s_\Lambda(t))^{1/2} F \hat{\Lambda} g|^2 \, dt, \]
\[ I_{22} = \int_{T}^{T_0} |(\sigma_1 t)^{1/2} F^{-} \hat{g} - (s_\Lambda(t))^{1/2} F \hat{\Lambda} g|^2 \, dt, \]
\[ I_{23} = \int_{0}^{T} |(\sigma_1 t)^{1/2} F^{-} \hat{g} - (s_\Lambda(t))^{1/2} F \hat{\Lambda} g|^2 \, dt. \]

Clearly \( \lim I_{11} = 0 \). Next
\[ I_{12} \leq 2(K_1 + K_2), \]
where
\[ K_1 = \int_{-A_0}^{-T} \sigma_2 |t| |g(t)|^2 \, dt. \]
and
\[ K_2 = \int_{-N_0}^{-T} s_A(t) |F_A g \cdot (t)|^2 dt. \]

It is evident that for \( T \) sufficiently large and all \( A \) we have \( K_1 \leq \epsilon/8 \). Now choose \( N \) so that \( \omega + 1 = 2N - a \) where \( a > 0 \). Then using (2) \( \S 11 \) and Lemma 12c we have
\[ \limsup K_2 \leq C 2^{2N} T^{-a} \int_{-\infty}^{\infty} t^{2N} (2 + t^2)^N |g(t)|^2 dt. \]
Thus for \( T \) sufficiently large \( \limsup K_2 < \epsilon/8 \) and
\[ \text{(13)} \quad \limsup I_{12} \leq \epsilon/2. \]
In the same way, for \( T \) sufficiently large
\[ \text{(14)} \quad \limsup I_{22} \leq \epsilon/2. \]

Fix \( T \) so that (13) and (14) are satisfied. Since \( s_A(t) \to s(t) \) boundedly for \( -T \leq t \leq T \) and \( F_A g \to F g \) in \( L^\infty \) we have
\[ \text{(15)} \quad \lim(I_{13} + I_{23}) = 0. \]

Finally \( I_{21} \leq 2(K_1 + K_2) \), where here
\[ K_1 = \int_{-N_0}^{\infty} s_A(t) |s(t)|^2 |g(t)|^2 dt \]
and
\[ K_2 = \int_{-N_0}^{\infty} s_A(t) |F_A g \cdot (t)|^2 dt. \]

Clearly
\[ \text{(16)} \quad \lim K_1 = 0. \]

Recalling the definition of \( s_A(t) \), (1) \( \S 10 \), we see that
\[ s_A(t) \leq CA^{u+1} \]
for \( A \) sufficiently large and all \( t \). Now if \( t \geq \xi_0 A \), then \( h(A, t) \leq 3t^2 \) and in this case (10) of Lemma 12c reduces to
\[ \text{(18)} \quad F_A g_{N,A} = o(A^{-1}) O(t^{2N-1}) + [h(t, A)]^N F_A g \cdot (t) \]
for all \( r \). We also note that for \( t \geq 0 \)
\[ \text{(19)} \quad h(A, t) \geq t^2. \]
From (18) we get that
Now applying (12), (17), (19) and the fact that \( \|F_A\| = 1 \), we obtain for sufficiently large \( A \) and all \( r \)

\[
K_2 \leq CA^{\omega+1-2N} \int_{-\infty}^{\infty} t^{2N}(2 + t^2)^N g(t) \, dt + O(A^{-\gamma}).
\]

Choosing \( N \) so that \( 2N > \omega + 1 \), we see that

\[
(20) \quad \lim K_2 = 0.
\]

Combining (13), (14), (15), (16) and (20) we finally get

\[
\lim sup \| (S^-)^{1/2} F^- g - (S_A^-)^{1/2} F_A^- g \| < \epsilon
\]

and as \( \epsilon \) is arbitrary the theorem is proved.

We remark that in the analogous theorems of Hirschman and Widom, the integral \( I_{2n} \) does not appear.

**Theorem 12e.** Let \( g \in \mathcal{D}_2^- \). Then

\[
\lim \| (S^-)^{1/2} F^- g - (S_A^-)^{1/2} F_A^- g \| = 0.
\]

**Proof.** The proof is the same as that of Theorem 12d. Note that in this case \( F^- g = 0 \).

**Theorem 12f.** \( (S^-)^{1/2} F^- \) is the closure of the strong limit of \( (S_A^-)^{1/2} F_A^- \).

**Proof.** Given \( f \in \mathcal{D} \), \( (S^-)^{1/2} F^- \) we must show there exists

\[
h \in \mathcal{D} \quad (S^-)^{1/2} F^- h
\]

such that

(i) \( \|f - h\| < \epsilon, \quad \| (S^-)^{1/2} F^- (f - h) \| < \epsilon, \)

(ii) \( (S_A^-)^{1/2} F_A^- h \to (S^-)^{1/2} F^- h \).

It is sufficient to consider two cases. \( F^- f = f \) and \( F^- f = 0 \).

Suppose that \( F^- f = f \). For \( 0 < \theta < 1 \), set \( g_\theta(t) = f(\theta t) \). Then \( \psi^{-1} g_\theta \cdot (y) = \theta^{-1} \psi^{-1} f \cdot (\theta y) \). Now \( F^- f = f \) implies \( \psi^{-1} f \cdot (y) = 0 \) for \( |y| > 1 \) and hence \( \psi^{-1} g_\theta \cdot (y) = 0 \) for \( |y| > \theta \). Clearly \( \psi^{-1} g_\theta \in \mathcal{D} \). It is also evident that for \( \theta \) sufficiently near 1,

\[
(21) \quad \| f - g_\theta \| < \epsilon/2, \quad \| (S^-)^{1/2} F^- (f - g_\theta) \| < \epsilon/2.
\]

For each \( \lambda > 0 \), let \( h_\lambda(y) \) be a non-negative, even, infinitely-differentiable function that vanishes for \( |y| \geq \lambda \) and \( \int \, h_\lambda(y) \, dy = 1 \).

Then \( |\psi h_\lambda \cdot (t)| \leq 1, \psi h_\lambda \cdot (t) \to 1 \) as \( \lambda \to 0 \) for all \( t \), and \( \psi h_\lambda \cdot (t) = O(t^{-\gamma}) \) as \( t \to \infty \) for all \( r \) and fixed \( \lambda \). Using this information and Lebesgue's theorem we get
(22) \[ \| (\psi h) \rho_s - g_s \| < \epsilon/2 \]

for \( \lambda \) sufficiently small. Next we note that \( \psi^{-1} g_s * h \cdot (y) = 0 \) for \( |y| > \lambda + \theta \).

If \( \lambda \) is such that

(23) \[ \lambda + \theta < 1 \]

then \( (\psi h) \rho_s \in \mathcal{D}_1^- \). Using \( |\psi h \cdot (t)| \leq 1 \), \( g_s(t) \psi h \cdot (t) \rightarrow g_s(t) \) as \( \lambda \rightarrow 0 \), and \( g_s \in \mathcal{D} ((S^-)^{1/2} F^-) \) we see that for \( \lambda \) sufficiently small

(24) \[ \| (S^-)^{1/2} F^- (g_s(\psi h) - g_s) \| < \epsilon/2. \]

Choose \( \theta \) so that (21) is satisfied and then choose \( \lambda \) so that (22), (23), and (24) are satisfied. Set \( h = (\psi h) \rho_s \). Then (i) is fulfilled. It is evident that \( h \in \mathcal{D} ((S^-)^{1/2} F^-) \) and since \( h \in \mathcal{D}_1^- \) by Theorem 12d we have (ii).

Suppose now that \( F \cdot f = 0 \). Then \( \psi^{-1} f^- (y) = 0 \) for \( |y| < 1 \). Choose \( 1 < C_1 < C_2 < \infty \) such that if \( g(y) = \psi^{-1} f^- (y) \) for \( C_1 < |y| < C_2 \) and \( g(y) = 0 \) otherwise, then \( \| g - \psi^{-1} f^- \| < \epsilon/2 \). Clearly \( F^- \psi g = 0 \) and \( \| \psi g - f \| < \epsilon/2 \). Now set \( h = (\psi h) \rho_s \), where \( \lambda \) is chosen so small that \( C_1 - \lambda > 1 \) and \( \| h - \psi g \| < \epsilon/2 \). Then \( h \in \mathcal{D}_2^- \), \( \| (S^-)^{1/2} F^- (f - h) \| = 0 \), and \( \| f - h \| < \epsilon \), and the result follows from Theorem 12e.

13. The asymptotic formula—Case III. Let \( S^- F^- \) be constructed from \( F^- \) and \( S^- \) as in Theorem 3a. Then as in Case II, \( S^- F^- \) is a self-adjoint operator on \( \mathcal{L}^- \).

Let

\[ S^- F^- = \int_0^\infty \lambda d\psi^\sim (\lambda) \]

be the spectral resolution of \( S^- F^- \) on \( \mathcal{L}^- \) and let

\[ S^- A^- F^- = \int_0^\infty \lambda d\psi^\sim (\lambda) \]

be the spectral resolution of \( S^- A^- F^- = S^- F^- S^- F^- \). Using Theorems 12e, 11c, and 11b in conjunction with Theorem 3d we have

(1) \[ \psi^\sim (\lambda) \rightarrow \psi^\sim (\lambda) \]

for every \( \lambda \) not in the point spectrum of \( S^- F^- \), \( 0 \leq \lambda < \infty \). Now define \( R^- \) as \( S^- F^- \) restricted to \( F^- \mathcal{L}^- = \mathcal{N}^- \) and let \( R^\sim A^- \) be \( S^- A^- F^- \) restricted to \( F^- \mathcal{L}^- = \mathcal{N}^- \). As in the previous cases \( R^- > 0 \) and \( R^\sim A^- > 0 \) and we have the spectral resolutions

\[ R^- = \int_0^\infty \lambda dG^\sim (\lambda) \]

on \( \mathcal{N}^- \), where \( G^\sim (\lambda) = \psi^\sim (\lambda) - \psi^\sim (0) \) and \( G^\sim (0) = 0 \) and

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on \( \mathcal{A}^* \), where \( G^*_A(\lambda) = \psi^*_A(\lambda) - \psi^*_A(0) \) and \( G^*_A(0) = 0 \). Since \( \psi^*_A(0) = I - F^* \) and \( \psi^*_A(0) = I - F^*_A \) it follows from (1) and Theorem 11c that

\[
G^*_A(\lambda) \to G^*(\lambda)
\]

for \( 0 \leq \lambda < \infty \) and \( \lambda \) not in the point spectrum of \( R^* \).

**Lemma 13a.** Let \( A(1) < A(2) < \cdots \), \( A(n) \to \infty \) as \( n \to \infty \). With the above definitions let \( f_n \in \mathcal{A}_{A(n)}, \| f_n \| = 1 \) and \( (R_A f_n) f_n \leq m < \infty \) for \( n \in p \), where \( p \) is a subsequence of \( 1, 2, 3, \ldots \). If \( f_n \to f \) as \( n \to \infty \) in \( p_v \), a subsequence of \( p \), then \( f \neq 0 \).

**Proof.** Let

\[
f^*(y) = \int_0^\infty f(x) J_{-1/2}(yx)(yx)^{1/2} dx.
\]

It is well known, see Titchmarsh [9, p. 473], that

\[
\int_0^\infty |f^*(y)|^2 dy = \int_0^\infty |f(x)|^2 dx.
\]

Noting that \( f_n \in \mathcal{A}_{A(n)} \) implies \( f_n(t) = F_A(n) f_n \cdot (t) \) a substitution gives

\[
f_n(t) = C^2 \int_0^{2\pi} ((A(n)\xi_0 + t)u)^* J(u(A(n)\xi_0 + t))
\]

\[
\times \int_{-A(n)} A(n)\xi_0 + w)^* J(u(A(n)\xi_0 + w)) f_n(w) dw du
\]

for \( t > -A(n)\xi_0 \) and zero otherwise. Setting \( f_{A,n}(w) = f_n(w - A(n)\xi_0) \) we write (3) as

\[
f_n(t) = (C f_{A,n})^* \cdot (t + A(n)\xi_0) \quad \text{for} \ t > -A(n)\xi_0,
\]

\[
= 0 \quad \text{for} \ t \leq -A(n)\xi_0.
\]

Then

\[
\|f_n\|^2 = \int_0^\infty |(C f_{A,n})^* \cdot (t)|^2 dt = \int_0^{2\pi} |f_{A,n}(w)|^2 dw = 1.
\]

Now using the Schwarz inequality on (3) in conjunction with (4) and the fact that \( (A(n)\xi_0 + t)u J(u(A(n)\xi_0 + t)) \leq C \) for \( 0 \leq u \leq 1, \ t > -A(n)\xi_0 \), and \( n \) in \( p_v \) we obtain

\[
|f_n(t)|^2 \leq C^2.
\]

Now, by (4) §2 we get
\[
\frac{d}{dt} \left( \left( (A\xi_0 + t) J(u(A\xi_0 + t)) \right) \right) = (A\xi_0 + t)^{-1} \left[ u J_{-1/2} (u(A\xi_0 + t)) \left[ u(A\xi_0 + t) \right]^{1/2} \right.
\]
\[
- u^2 J_{-1/2} (u(A\xi_0 + t)) \left[ u(A\xi_0 + t) \right]^{(r+1/2)} \right] \]
\]
and thus differentiating \( f_n(t) \) we see that for \( t \geq -A(n)\xi_0 \)
\[
|f_n'(t)| \leq C(A(n)\xi_0 + t)^{-1}.
\]

Hence, given \( t_0 \), there is an \( n_0 \) such that for \( n > n_0 \) and \( |t| < t_0 \)
\[
|f_n'(t)| \leq C.
\]

Now (5) and (6) imply that \( f_n, n \) in \( p_1 \), are uniformly bounded and equicontinuous on any interval \( |t| \leq t_0 \). Therefore since \( f_n \rightharpoonup f \) as \( n \to \infty \) in \( p_1 \), we have (if \( f \) is suitably redefined on a set of measure zero),
\[
(7) \quad f_n(t) \rightharpoonup f(t)
\]
uniformly for \( |t| \leq t_0 \).

By Lemma 11a, given \( m > 0 \) there exist \( A_0 > 0 \) and \( t_0 > 0 \) such that for \( A > A_0, |t| > t_0 \)
\[
s_A(t) \geq m_1.
\]

Taking \( m_1 > m \) and \( n > n_0 \), where \( A(n_0) > A_0 \), we have
\[
m \geq m_1 \int_{|t| > t_0} |f_n(t)|^2 dt.
\]
Hence
\[
\int_{|t| \leq t_0} |f_n(t)|^2 dt \geq 1 - m m_1^{-1} > 0
\]
and by (7)
\[
\int_{|t| \leq t_0} |f(t)|^2 dt > 0.
\]
That is, \( f \neq 0 \).

**Theorem 13b.** Let \( F \) satisfy (i), (ii), and (iii) of \( \S 10 \) and \( \Omega = [0, 2\pi] \). If \( \lambda(A, 1) \geq \lambda(A, 2) \geq \ldots \) are the positive eigenvalues of \( B_A \) and if \( 0 < \mu(1) \leq \mu(2) \leq \ldots, \mu(k) \to \infty \) as \( k \to \infty \), are the positive eigenvalues of \( S_F^* \) then
\[
\lambda(A, k) = M - L(A^{-1})A^* (\mu(k) + o(1))
\]
as \( A \to \infty \) for each fixed \( k = 1, 2, \ldots \).

**Proof.** Lemma 13a and (2) in conjunction with Theorem 3e yield
\[
(A(n))^{\mu} \left[ L((A(n))^{-1}) \right]^{-1} (M - \lambda(A(n), k)) = \mu(k) + o(1)
\]
as \( n \to \infty \) for \( k = 1, 2, \ldots \). The theorem follows immediately.

**Bibliography**


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