PRIMARY IDEALS AND VALUATION IDEALS

BY

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1. Introduction. Let $D$ be an integral domain, let $\mathcal{P}$ denote the set of primary ideals of $D$, and let $\mathcal{V}$ denote the set of valuation ideals of $D$. The object of this paper is to investigate the significance of the relationships $\mathcal{V} \subseteq \mathcal{P}$, $\mathcal{P} \subseteq \mathcal{V}$, and $\mathcal{P} = \mathcal{V}$. Our point of departure was the observation in [8, p. 341, Example 2], that if $D$ is a Dedekind domain, then $\mathcal{P} = \mathcal{V}$. We prove here that $\mathcal{P} = \mathcal{V}$ if and only if $D$ is a Prüfer domain of dimension $\leq 1$. Also, $\mathcal{V} \subseteq \mathcal{P}$ if and only if every proper prime ideal of $D$ is maximal (i.e., $\dim D \leq 1$). However, these results are fairly immediate, and our main concern is with the implications of the containment $\mathcal{P} \subseteq \mathcal{V}$. If $D$ is a Prüfer domain, it is clear that $\mathcal{P} \subseteq \mathcal{V}$; and under the hypothesis that $D$ satisfy the ascending chain condition for prime ideals, we are able to prove $\mathcal{P} \subseteq \mathcal{V}$ implies $D$ is Prüfer. Moreover, in §5 we construct an example of a domain which satisfies the condition $\mathcal{P} \subseteq \mathcal{V}$ but which is not Prüfer.

Our terminology adheres to the conventions of [7], [8] with two exceptions: First, the domain $D$ is not included in the "ideals" of $D$, and second $\subseteq$ will denote containment and $\subset$ indicates proper containment.

2. Preliminary results on valuation ideals. We shall begin by reviewing some definitions (found in [8, Appendices 3, 4]).

(1) An ideal $A$ of a domain $D$ is called a valuation ideal if there exists a valuation ring $D_v \supseteq D$ and an ideal $A_v$ of $D_v$ such that $A_v \cap D = A$. When we want to specify the particular valuation ring $D_v$, we shall say $A$ is a $v$-ideal. If $A$ is a $v$-ideal, then $A \cdot D_v \cap D = A$.

(2) If $A$ is an ideal of $D$ and if $S$ is the set of all nontrivial valuations of the quotient field $K$ of $D$ which are non-negative on $D$, then $A' = \bigcap_{v \in S} A \cdot D_v$ is called the completion of $A$. If $A = A'$, then $A$ is called complete. $D' = \bigcap_{v \in S} D_v$ is the integral closure of $D$ (by [8, p. 15, Theorem 6]).

We now list some fundamental properties of complete ideals and valuation ideals.

2.1. Proposition. Let $D$ be a domain. If $A$ is any ideal of $D$, denote by $A'$ the completion of $A$ and by $A^*$ the intersection of those valuation ideals of $D$ which contain $A$. Then

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(2) There exists at least one valuation ideal ($\neq D$) containing $A$, since every prime ideal is a valuation ideal.
(a) \( A' \cap D = A^* \).
(b) \( (x)' = xD' \) for any \( x \in D \).
(c) If \( (x) = (x)' \) for some \( x \in D, x \neq 0 \), then \( D = D' \). ((b) proves the converse.)
(d) \( (x) = (x)^* \) for all \( x \in D \) if and only if \( D = D' \).

**Proof.**

(a) \( A' \cap D = \bigcap_{e \in \mathcal{S}} (A \cdot D_e) \cap D = \bigcap_{e \in \mathcal{S}} (A \cdot D_e \cap D) \supseteq A^* \).

Conversely, if \( B \) is a \( v \)-ideal such that \( B \supseteq A \), then \( B \cdot D_e \cap D = B \supseteq A \) implies \( A \cdot D_e \cap D \subseteq B \). Therefore, \( A' \cap D \subseteq A^* \).

(b) \([8, \text{p. 348, Proposition 1, (f)}]\).

(c) If \( t \in D' \), then \( tx \in D'x = (x)' \) by (b). Thus \( tx = sx \in (x) \) for some \( s \in D \).

Since \( x \neq 0 \), \( t = s \in D \) and \( D = D' \).

(d) \( D' = D \) implies \( (x) = (x)' \) by (b). Therefore, \( (x) = (x)^* \) by (a).

Conversely, \( (x) = (x)^* \) implies \( (x)' \cap D = x \cdot D' \cap D \), by (a) and (b). Then \( y/x \in D' \) implies \( y \in x \cdot D' \cap D = (x) \), so \( y \in (x) \) and hence \( y/x \in D \).

q.e.d.

A **Prüfer domain** is by definition a domain which satisfies any of the equivalent assertions of the next theorem.

**2.2. Theorem.** Let \( D \) be a domain. Then the following are equivalent:

(a) Every nonzero finitely generated ideal of \( D \) is invertible.
(b) \( D_P \) is a valuation ring for every prime ideal \( P \).
(c) Whenever \( A \not\subseteq (0), B, C \) are ideals such that \( A \) is finitely generated and \( AB = AC \), then \( B = C \).
(d) Every ideal of \( D \) is complete.
(e) Every ideal of \( D \) is an intersection of valuation ideals.

**Proof.**

(a) \( \iff \) (b) by \([4, \text{p. 554, Theorem 7}]\).

(a) \( \iff \) (c) by \([5, \text{p. 127, Krit. 3}] \) and \([2, \text{Theorem 2*}] \).

(b) \( \Rightarrow \) (d): If \( A \) is an ideal of \( D \) and \( A' \) denotes the completion of \( A \), then \( A' \subseteq \bigcap (A \cdot D_M) \), where the intersection ranges over all maximal ideals \( M \) of \( D \). But \( \bigcap (A \cdot D_M) = A \) by \([8, \text{p. 94, Lemma}] \). Therefore, \( A' \subseteq A \), so \( A' = A \).

(d) \( \Rightarrow \) (c): \( AB \subseteq AC \) implies \( (AB)' \subseteq (AC)' \), where \( ' \) denotes completion; so \( B' \subseteq C' \). But \( B' = B \), \( C' = C \). Similarly, \( AC \subseteq AB \) implies \( C \subseteq B \). (The properties of \( ' \) used here are found in \([8, \text{p. 384, Proposition 1}] \).)

(d) \( \Rightarrow \) (e): Immediate from the definition of a complete ideal.

(e) \( \Rightarrow \) (d): Apply 2.1–(d) and the remark of \([8, \text{p. 353}] \), that every intersection of valuation ideals of an integrally closed domain is complete. q.e.d.

**Corollary.** If every ideal of \( D \) is an intersection of primary ideals and if every primary ideal is a valuation ideal, then \( D \) is Prüfer.

**Proof.** Apply (e).
If $A$ is a valuation ideal of a domain $D$, the elements of $A$ must satisfy certain relations. For example, we have the following:

2.3. Lemma. Let $A$ be a valuation ideal of the domain $D$, and let $R$, $S$ be arbitrary subsets of $D$. Then

(a) $RS \subseteq A$ implies $\{r^2 \mid r \in R\} \subseteq A$ or $\{s^2 \mid s \in S\} \subseteq A$.

(b) $R^2 + S^2 \subseteq A$ implies $RS \subseteq A$.

Proof. 

(a) Suppose there exists $s \in S$ such that $s^2 \notin A$, and let $D_v$ be a valuation ring such that $A \cdot D_v \cap D = A$. Then for any $r \in R$, $v(r^2) \geq v(rs)$ or $v(s^2) \geq v(rs)$; and accordingly, $r^2 \in A \cdot D_v$ or $s^2 \in A \cdot D_v$. Since $s^2 \notin A$, $s^2 \notin A \cdot D_v$ and hence $r^2 \in A \cdot D_v$. Thus, $\{r^2 \mid r \in R\} \subseteq A \cdot D_v \cap D = A$.

(b) Suppose $r \in R$, $s \in S$ and $D_v$ is a valuation ring such that $A \cdot D_v \cap D = A$. We may assume $v(r) \geq v(s)$. Then $v(rs) \geq v(r^2) \geq v(r^2 + s^2)$, so $rs \in (r^2 + s^2)D_v \subseteq AD_v$. Thus, $rs \in AD_v \cap D = A$. q.e.d.

These are the only such relations which we use in this paper, so we shall not dwell on the subject. We would like to mention, however, the following general result, based essentially on the fact that the ideals of a valuation ring are linearly ordered:

Theorem. Let $F(x) = \sum_{\sigma} x_{\sigma(1)}^{m_1} \cdots x_{\sigma(s)}^{m_s}$ and $G(x) = \sum_{\sigma} x_{\sigma(1)}^{m_1} \cdots x_{\sigma(t)}^{m_t}$ be symmetric polynomials, $\sigma$ ranging over all permutations of $1, \ldots, n$ and $1 \leq t \leq s \leq n$; and suppose $F$, $G$ have the same degree and $m_1 + \cdots + m_{t-j} \leq n_1 + \cdots + n_{t-j}$ for $j = 1, \ldots, t-1$. If $A_1, \ldots, A_n$ are ideals of a domain $D$ such that $G(A_1, \ldots, A_n)$ is an intersection of valuation ideals, then

$$F(A_1, \ldots, A_n) \subseteq G(A_1, \ldots, A_n).$$

One might hope to characterize a valuation ideal by relations of the form $F(A_1, \ldots, A_n) \subseteq G(A_1, \ldots, A_n)$, but the above theorem tells us that such relations only characterize ideals which are intersections of valuation ideals. As a case in point, every ideal of a Prüfer domain is an intersection of valuation ideals but we shall presently see that such an ideal need not be a valuation ideal.

2.4. Corollary. If every principal ideal of a domain $D$ is a valuation ideal, then $D$ is a valuation ring.

Proof. By 2.3–(a), $x^2 \in (xy)$ or $y^2 \in (xy)$, for any nonzero elements $x, y \in D$. But then $x/y$ or $y/x \in D$, so $D$ is a valuation ring. q.e.d.

2.5. Proposition. Let $Q$ be a primary ideal of a domain $D$ and let $M$ be a multiplicative system in $D$ such that $Q \cap M = \emptyset$. Let $D_0$ be a domain containing $D$ such that $Q \cdot D_0 \cap D = Q$, and let $D_0^* = (D_0)_M$, $D^* = D_M$, $Q^* = D_M \cdot Q$. Then $D_0^* \cdot Q^* \cap D^* = Q^*$. 
Proof. \( Q^* \subseteq D^*_0 \cdot Q^* \cap D^* \) is clear. Suppose then \( x \in D^*_0 \cdot Q^* \cap D^* = D^*_0 \cdot Q \cap D^* \),
\[
x = \frac{t}{m} = \frac{r}{s}, \quad t \in D_0 Q, \ m, s \in M, \ r \in D.
\]
Therefore, \( st = rm \).
But \( st \in D_0 \cdot Q \) and \( rm \in D \), so \( rm \in D_0 \cdot Q \cap D = Q \). Since \( Q \cap M = \emptyset \), \( m \in M \) implies \( m \notin Q \). Thus \( r \in Q \) and \( x = \frac{r}{s} \in Q^* \). q.e.d.

2.6. Corollary. Let \( Q, D, \) and \( M \) be as above. If \( Q \) is a valuation ideal, then \( Q^* = Q \cdot D_M \) is also a valuation ideal.

Proof. If \( D_v \) is a valuation ring such that \( D_v \cdot Q \cap D = Q \), and if \( D^*_v = (D_v)_M \), then \( D^*_v \cdot Q^* \cap D^* = Q^* \) by 2.5. Thus, \( Q^* \) is a valuation ideal. q.e.d.

2.7. Corollary. Let \( D \) be a domain and let \( M \) be a multiplicative system in \( D \). If every primary ideal \( Q \) of \( D \) such that \( Q \cap M = \emptyset \) is a valuation ideal, then every primary ideal of \( D_M \) is a valuation ideal.

Proof. Let \( Q^* \) be any primary ideal of \( D_M \), and let \( Q = Q^* \cap D \). Then \( Q \cdot D_M = Q^* \) and \( Q \) is a primary ideal of \( D \) such that \( Q \cap M = \emptyset \). Therefore, by 2.6, \( Q^* \) is a valuation ideal. q.e.d.

2.8. Lemma. Let \( D \) be a domain, and let \( A_1, \ldots, A_n \) be \( v \)-ideals of \( D \) (for a fixed \( v \)). If \( d_i \) is an element of \( D \) such that \( d_i \notin A_i \), \( i = 1, \ldots, n \), then \( d = d_1 \cdots d_n \notin A_1 \cdots A_n \).

Proof. Since \( A_i D_v \cap D = A_i \), \( d_i \notin A_i \) implies \( d_i \notin A_i D_v \). Therefore, \( v(d_i) < v(a_i) \) for all \( a_i \in A_i D_v \). But then \( v(d) = v(d_1 \cdots d_n) < v(a_1 \cdots a_n) \) for any \( a_1, \ldots, a_n, a_i \in A_i D_v \). This means \( d \notin A_1 D_v \cdots A_n D_v \); and since \( A_1 \cdots A_n \subseteq A_1 D_v \cdots A_n D_v \), \( d \notin A_1 \cdots A_n \).

2.9. Corollary. Let \( A, B \) be ideals of the domain \( D \), such that \( A \) is a valuation ideal and \( B^\infty \subseteq A^\infty \). Then \( B \subseteq A \).

Proof. If \( B \ni A \), then there exists \( b \in B \), \( b \notin A \). Therefore, by 2.8, \( b^\infty \notin A^\infty \); so \( B^\infty \nsubseteq A^\infty \). q.e.d.

2.10. Lemma. Let \( D \) be a domain, and let \( A \) be an ideal of \( D \) such that \( A^n \) is a valuation ideal for all \( n \). Then \( B = \bigcap_{n=1}^{\infty} A^n \) is prime.

Proof. \( xy \in B \) implies \( xy \in A^{2n} = (A^n)^2 \), for all \( n \). But then by 2.8, \( x \in A^n \) or \( y \in A^n \). Thus, \( x \in B \) or \( y \in B \). q.e.d.

2.11. Corollary. Let \( Q \) be a primary ideal of a domain \( D \), and suppose \( Q^{(i)} \) is a valuation ideal for all \( i \) (where \( Q^{(i)} \) denotes the \( i \)th symbolic power of \( Q \)). Then \( A = \bigcap_{i=1}^{\infty} Q^{(i)} \) is prime.

Proof. Let \( P = \sqrt{Q} \). By applying 2.6 and well-known properties of quotient
rings [7, p. 223], we may assume $D = D_P$ and hence that $P$ is maximal and $Q^{(0)} = Q$. Now apply 2.10.

2.12. Lemma. Let $P$ be a prime ideal of a valuation ring $D$, and let $A$ be the intersection of the primary ideals belonging to $P$. Then $A$ is prime, and there exists no prime ideal $P_1$ such that $A \subseteq P_1 \subset P$.

**Proof.** There is no loss of generality in assuming $D = D_P$ so that $D$ is quasi-local and $P$ is maximal in $D$. If $A = P$, the lemma holds. If $A \subsetneq P$, then there exists a $P$-primary ideal $Q \subsetneq P$, then given $x \in P$, $x \notin Q$, $Q \subsetneq (x) \subsetneq P$. Thus if $Q_i$ is any $P$-primary ideal of $D$, then $x_i \in Q_i$ for some $i$ so that $(x_i) \subseteq Q_i$. Further, $\sqrt{(x_i)} = \sqrt{(x)} = P$, and thus $(x_i)$ is $P$-primary. It follows that $A = \bigcap_{i=1}^{n}(x_i)$ is prime by 2.10. Further, if $B$ is an ideal of $D$ such that $A \subsetneq B \subsetneq P$, then $B \notin (x_i)$ for some $n$, so that $(x_i) \subsetneq B$. Therefore $B \subsetneq P = \sqrt{(x_i)} \subseteq \sqrt{B}$ and $B$ is not prime. q.e.d.

2.13. Lemma. Let $\{A_x\} = S$ be a set of valuation ideals of a domain $D$, and suppose for any $A_1, A_2 \in S$ there exists an $A_3 \in S$ such that $A_3 \subseteq A_1 \cap A_2$. If $A = \bigcap A_i$, then $\sqrt{A}$ is prime.

**Proof.** $xy \in \sqrt{A}$ implies $(xy)^n \in A$ for some $n$. Then $x^n \cdot y^n \in A_\lambda$ for all $\lambda$; so by 2.3-(a), $x^{2n} \in A_1$ or $y^{2n} \in A_\lambda$. If $x^{2n} \notin A_1$ and $y^{2n} \notin A_2$ for some $A_1, A_2 \in S$, then there exists $A_3 \in S$ such that $A_3 \subseteq A_1 \cap A_2$; and then $x^{2n} \notin A_3$, $y^{2n} \notin A_3$, a contradiction. Thus, we may assume $x^{2n} \in A_1$ for all $\lambda$. But then $x^{2n} \in A$ and hence $x \in \sqrt{A}$. q.e.d.

2.14. Proposition. Let $P$ be a prime ideal of a domain $D$, and let $\{Q_\lambda\}$ be the set of primary ideals belonging to $P$. If $A = \bigcap Q_\lambda$ and every $Q_\lambda$ is a valuation ideal, then $A$ is prime.

**Proof.** Let $Q$ be a primary ideal of $D$, and suppose $D_\nu$ is a valuation ring such that $Q \cap D = Q$, where $Q_\nu = Q \cdot D_\nu$. If $P_\nu = \sqrt{Q_\nu}$, $P_\nu$ is prime and $P_\nu \cap D = P$. (The radical of an ideal is the intersection of all prime ideals which contain it [5, p. 9]. The prime ideals of a valuation ring are linearly ordered so that every ideal of a valuation ring has prime radical.) Let $P_\nu^*$ be the intersection of the $P_\nu$-primary ideals of $D_\nu$, and let $P^* = P_\nu^* \cap D$. $P_\nu^*$ is prime by 2.12, so $P^*$ is also prime. Then $A \subseteq P^* \subseteq Q$, and thus $\sqrt{A} \subseteq P^* \subseteq Q$. Since this is true for any $P$-primary ideal $Q$, $\sqrt{A} \subseteq A$ and hence $\sqrt{A} = A$.

If $Q_1$ and $Q_2$ are $P$-primary ideals, then $Q_3 = Q_1 \cap Q_2$ is also $P$-primary. Therefore, we may apply 2.13 to conclude $A = \sqrt{A}$ is prime. q.e.d.

Thus, if $Q$ is a primary ideal of a domain $D$ having $\sqrt{Q} = P$ and if $\{Q_\lambda\}$ is the set of primary ideals belonging to $P$, then both $A_1 = \bigcap Q^{(i)}$ and $A_2 = \bigcap Q_\lambda$ are prime provided every $Q_\lambda$ is a valuation ideal. If $Q \subsetneq P, A_2 \subseteq A_1 \subsetneq P$. If $D$ is a valuation ring, it is easily seen that $A_2 = A_1$; but we do not know if this is true in general. More important, we know of no case where there exists
a prime ideal $P_1$ such that $A_2 \subset P_1 \subset P$, although it seems likely that this may happen.

3. Relationships between $\mathcal{P}$ and $\mathcal{V}$. Let $\mathcal{P}(D)$ be the set of primary ideals of the domain $D$ and let $\mathcal{V}(D)$ be the set of valuation ideals of $D$. $\mathcal{V}(D)$ contains, in particular, all prime ideals of $D$ [8, p. 12, Theorem 5]. When no confusion can result, we shall simply write $\mathcal{P}$ and $\mathcal{V}$.

The next theorem characterizes domains $D$ with the property that $\mathcal{V} \subseteq \mathcal{P}$.

3.1. Theorem. $\mathcal{V} \subseteq \mathcal{P}$ if and only if every proper prime ideal of $D$ is maximal.

Proof. Suppose every proper prime ideal of $D$ is maximal, and let $A$ be a valuation ideal. Then there exists a valuation ring $D_v \supseteq D$ and an ideal $A_v$ of $D_v$ such that $A_v \cap D = A$. If $P$ is the center of $D_v$ on $D$, then $D \subseteq D_P \subseteq D_v$, and $D_P$ is a one-dimensional quasi-local ring. Therefore, $A_v \cap D_P = A'$ is primary; and since $A' \cap D = A$, $A$ is also primary.

Conversely, assume $\mathcal{V} \subseteq \mathcal{P}$, and suppose there exist prime ideals $P, P'$ of $D$ such that $0 \subset P \subset P' \subset D$. By [6, p. 37], there exists a valuation ring $D_v$ having prime ideals $P_v, P'_v$ which lie over $P, P'$, respectively. Choose $x \in P', x \not\in P$ and $y \neq 0$ in $P$, and let $A = (xy)D_v \cap D$. Then $A$ is a valuation ideal and $A \subseteq P$. Claim: $A$ is not primary. For, if $A$ is primary, $xy \in A$ and $x \not\in P$ implies $y \in A$. But then $y = rxy$ for some $r \in D_v$, and hence $1 = rx \in P'_v$, a contradiction. q.e.d.

3.2. Lemma. Let $M$ be a prime ideal of a domain $D$, and suppose there exists a prime ideal $P \subset M$ such that there is no prime ideal $P_y$ with $P \subset P_y \subset M$. Then $P$ is the intersection of the $M$-primary ideals of $D$ which contain $P$.

Proof. By passage to $D_M/PD_M$, it suffices to prove the theorem under the assumption that $D$ is a one-dimensional quasi-local domain with maximal ideal $M$ and $P = (0)$. The proof follows easily in this case since every nonzero ideal is $M$-primary, and the intersection of all nonzero ideals of $D$, an integral domain, is $(0)$. q.e.d.

3.3. Theorem. Let $M$ be a prime ideal of a domain $D$, and suppose every $M$-primary ideal is a valuation ideal. If there exists a prime ideal $P \subset M$ such that there is no prime ideal $P_1$ with $P \subset P_1 \subset M$, then $P$ is unique (and is, in fact, the intersection of all $M$-primary ideals).

Proof. Let $P_0$ be the intersection of the $M$-primary ideals. By 2.14 and 3.2, $P_0$ is prime and $\subset M$. We shall show $P \subseteq P_0$; it then follows that $P = P_0$ and hence $P$ is unique. By 2.6 and the 1-1 correspondence between prime (primary) ideals of $D$ contained in $M$ and prime (primary) ideals of $D_M$, we may replace $D$ by $D_M$ and hence assume that $D$ is quasi-local with maximal ideal $M$. Let then $Q$ be any $M$-primary ideal of $D$, and we shall show $P \subseteq Q$. 
Choose $x \in Q$, $x \notin P$ and set $A = QP + (x^4)$. Then $A \subseteq Q$ and $\sqrt{A} \supseteq (P, x) \supseteq P$; so $\sqrt{A} = M$, and hence $A$ is $M$-primary. By hypothesis, $A$ is then a valuation ideal, so there exists a valuation ring $D_v$ and an ideal $A_v$ of $D_v$ such that $A_v \cap D = A$; and we may assume $A_v = AD_v$. Let also $P_v = P \cdot D_v$, $Q_v = Q \cdot D_v$.

Claim: $x^2 \notin P_v$.

For, $x^2 \in P_v$ implies $x \cdot x^2 \in Q_v \cdot P_v \cap D \subseteq A$. Then $x^3 = s + dx^4$, $s \in Q \cdot P$, $d \in D$. Therefore, $(1 - dx)x^3 = s \in P$.

Since $1 - dx$ is a unit of $D$, this implies $x^3 \in P$ and hence $x \in P$, a contradiction. Therefore, $x^2 \notin P_v$.

Because $D_v$ is a valuation ring, the ideals of $D_v$ are linearly ordered; so $x^2 \notin P_v$ implies $P_v \subseteq x^2 \cdot D_v$. Therefore, $P_v^2 \subseteq (x^2 \cdot D_v) \cdot P_v$. But $P^2 + (x^2)$ is a valuation ideal, so

$$x \cdot P \subseteq P^2 + (x^2) \text{ by 2.3-(b)}.$$

Therefore, $x \cdot P \subseteq P^2 + (x^2) \cdot P$ since $x \notin P$.

$$(x \cdot D_v) \cdot P_v \subseteq (P_v)^2 + (x^2 D_v) \cdot P_v = (x^2 D_v) \cdot P_v.$$

Thus, $(x \cdot D_v) \cdot P_v = (x^2 D_v) \cdot P_v$; and this implies

$$P_v = (x D_v) \cdot P_v = (x^2 D_v) \cdot P_v = (x^3 D_v) \cdot P_v = \cdots.$$

Therefore, $P_v \subseteq \bigcap_{i=1}^{\infty} (x^i D_v) = P_1$. $P_1$ is prime by 2.10, and $x \notin P_1$ implies $P_1 \cap D \subseteq M$. Therefore, $P \subseteq P_v \cap D \subseteq P_1 \cap D \subseteq M$, so by hypothesis, $P = P_1 \cap D$. This means both $A$ and $P$ are $v$-ideals for the same $v$. Since $A \notin P$, we must have $P = P_v \cap D \subseteq A_v \cap D = A$. Thus, $P \subseteq A \subseteq Q$. q.e.d.

A domain $D$ is said to satisfy the ascending chain condition for prime ideals provided any strictly ascending chain of prime ideals $P_1 \subset P_2 \subset P_3 \subset \cdots$ is finite. This is equivalent to saying that every nonempty family of prime ideals contains a maximal element. The remainder of this section is devoted to proving that if $D$ satisfies the a.c.c. for prime ideals and $2 \subseteq \mathcal{V}$, then $D$ is Prüfer.

3.4. Lemma. Let $D$ be a quasi-local domain, and suppose for any nonzero prime ideal $P$ of $D$ there exists a prime ideal $N(P)$ of $D$ such that $P_1 \subset N(P)$. Then $D$ satisfies the a.c.c. for prime ideals and the prime ideals of $D$ are linearly ordered (and conversely).

Proof. If $P_1 \subset P_2 \subset \cdots$ is an ascending chain of prime ideals of $D$, then $U = \bigcup P_i$ is also prime; so if $U \neq P_i$ for all $i$, then $P_i \subseteq N(U)$ for all $i$; and we would have $U = \bigcup P_i \subseteq N(U) \subset U$, a contradiction. Therefore, $D$ satisfies the a.c.c. for prime ideals.

Now suppose there exist prime ideals $P_1$, $P_2$ of $D$ such that $P_1 \notin P_2$ and $P_2 \notin P_1$. Since $D$ has the a.c.c. for prime ideals, there exists a prime ideal $M$, maximal with respect to the properties $P_1 \subseteq M$, $P_2 \notin M$. Since $P_2 \notin M$, $M$ is not the maximal ideal of $D$ and there exists a prime ideal $M_2 \supseteq M$. If $\{M_2\}$ is
the set of all such prime ideals, then \( M \neq \bigcap M_a \) since \( P_2 \subseteq \bigcap M_a \) and \( P_2 \nsubseteq M \).

Therefore, by Zorn's lemma, there is a prime ideal \( M_0 \) minimal with respect to the property that \( M_0 \supsetneq M \). Therefore, \( M \subseteq N(M_0) \subseteq M_0 \) implies \( M = N(M_0) \).

But then \( P_2 \subseteq M_0 \) means \( P_2 \subseteq N(M_0) = M \), a contradiction to the choice of \( M \).

q.e.d.

3.5. Corollary. Let \( D \) be a quasi-local domain such that \( D \) satisfies the a.c.c. for prime ideals. If \( \mathcal{Q} = \mathcal{V} \), then the prime ideals of \( D \) are linearly ordered.

Proof. If \( P \) is any nonzero prime ideal of \( D \), the set of all prime ideals \( P_1 \subset P \) contains a maximal element \( N(P) \), since \( D \) satisfies the a.c.c. for prime ideals.

By 3.3 \( N(P) \) is unique and hence contains every prime ideal \( P_1 \subset P \). Therefore, by 3.4 the prime ideals of \( D \) are linearly ordered. q.e.d.

3.6. Lemma. Let \( D \) be a quasi-local domain which satisfies the a.c.c. for prime ideals, and suppose \( \mathcal{Q} \subset \mathcal{V} \). Then \( D \) is integrally closed.

Proof. Let \( S = \{ x \in D \mid (x)' \cap D \ni (x) \} \) (where \( ' \) denotes completion). By 2.1-(d), \( S = \emptyset \) if and only if \( D \) is integrally closed; so assume \( S \neq \emptyset \). By 3.5, the prime ideals of \( D \) are linearly ordered; so there exists a least prime \( P \) containing \( S \) (i.e., \( P \) is the intersection of all prime ideals which contain \( S \)). Moreover, applying the a.c.c., there exists a prime ideal \( P_0 \subset P \) such that there is no prime ideal \( P_1 \) with \( P_0 \subset P_1 \subset P \). Since \( S \ni P_0 \), there exists \( x \in S \), \( x \notin P_0 \); and then \( \sqrt{(x)} = P \). \( x \cdot D_p \) is primary and hence a valuation ideal by 2.6. Therefore, \( (x \cdot D_p)' \cap D_p = x \cdot D_p \), by 2.1-(a). Thus, \( y \in (x)' \cap D \) implies \( y \in (x \cdot D_p)' \cap D_p = x \cdot D_p \). Since \( x \in S \), there exists \( y \in (x)' \cap D \) and \( y \notin (x) \); so

\[
y = (a/b) \cdot x, \quad b \notin P, \ a \notin (b).
\]

But \( y \in (x)' = x \cdot D' \) implies \( y/x \in D' \), where \( D' \) is the integral closure of \( D \) (see 2.1). Therefore, \( a = y/x \cdot b \in b \cdot D' = (b)' \), so \( a \notin (b) \) implies \( b \in S \). But \( S \subseteq P \) and \( b \notin P \), a contradiction. q.e.d.

3.7. Lemma. Let \( D \) be an integrally closed, quasi-local domain; and suppose \( x, y \) are nonzero elements of \( D \) such that \( xy \in (x^2, y^2) \). Then \( x/y \) or \( y/x \) is in \( D(\mathfrak{A}) \).

Proof. If \( x \) or \( y \) is a unit of \( D \), we are done; so assume \( x, y \) are nonunits. Then \( x y = d_1 x^2 + d_2 y^2 \), \( d_i \in D \). If \( d_1 \) is a unit in \( D \), the integral closure of \( D \) gives the proposition. Therefore, we may assume \( d_1, d_2 \) are nonunits also. Let \( K \) be the quotient field of \( D \). By [8, p. 12, Theorem 5], there exists a valuation ring \( D_v \subset K \) which dominates \( D \). Then \( x/y \) or \( y/x \in D_v \), say \( x/y \in D_v \).

\[
d_2 y/x = d_2 d_1 + (d_2 y/x)^2
\]

(\( \mathfrak{A} \)) This result can also be proved by putting together the proofs of [3, Theorem 2.5-(f)] and [4, p. 554, Satz 6].
Therefore, $d_2 = (x/y) \cdot d$ for some $d \in D$.

But $x/y(1 - d_1 x/y) = d_2$ implies

$$v(x/y) + v(1 - d_1 x/y) = v(d_2).$$

$1 - d_1 x/y$ is a unit of $D_v$ since $d_1$ is a nonunit of $D_v$. Therefore,

$$v(1 - d_1 x/y) = 0,$$

so

$$v(x/y) = v(d_2).$$

Combining (1) and (2), we get

$$v(d) = 0.$$}

Therefore, $d$ is a unit in $D$. Thus

$$x/y = d_2 \cdot 1/d \in D. \quad \text{q.e.d.}$$

3.8. Theorem. Let $D$ be a domain which satisfies the a.c.c. for prime ideals. If $\mathcal{Q} \subseteq \mathcal{V}$, then $D$ is a Prüfer domain (and conversely $^{(*)}$).

Proof. By 2.2-(b) it is sufficient to see $D_p$ is a valuation ring for any prime ideal $P$ of $D$. Therefore, by 2.7 we may assume $D$ is quasi-local, and by 3.6 and 2.1-(d) $D$ is integrally closed. Suppose then there exist nonzero $x, y \in D$ such that $x/y$ and $y/x \notin D$. $x, y$ are then nonunits of $D$, so the fact that the prime ideals of $D$ are linearly ordered (by 3.5) implies $\sqrt{(x,y)}$ is prime. Consider then the set $\mathcal{P}$ of all prime ideals of $D$ which are of the form $\sqrt{(x,y)}$ for such $x, y$. By the a.c.c., $\mathcal{P}$ contains a maximal element $P$ and suppose $x, y$ are the elements of the above type such that $P = \sqrt{(x,y)}$. $(x^2, y^2) \cdot D_P$ is then primary and hence by 2.7 a valuation ideal. Therefore, by 2.3-(b), $xy \in (x^2, y^2) \cdot D_P$. Applying 3.7, we may assume $x/y \in D_P$. Then $x/y = r/s$, $r, s \in D$, $s \notin P$. But this means $r/s, s/r \notin D$, and $s \notin P$ implies $\sqrt{(r,s)} \supset P$, a contradiction to the choice of $P$. q.e.d.

3.9. Corollary. A noetherian domain $D$ has the property $\mathcal{Q} \subseteq \mathcal{V}$ if and only if $D$ is a Dedekind domain.

Proof. $D$ is a Dedekind domain if and only if $D$ is a noetherian Prüfer domain (use (a) of 2.2). Now apply 3.8.

3.10. Corollary. Let $D$ be a noetherian domain and let $P$ be a prime ideal of $D$ such that every $P$-primary ideal is a valuation ideal. Then $P$ is a

\[ (*) \] The converse follows from the fact that $D_P$ is a valuation ring and $Q \cdot D_P \cap D = Q$ for any prime ideal $P$ of $D$ and any $P$-primary ideal $Q$.
minimal prime of \( D \) and \( D_P \) is a rank 1, discrete valuation ring (i.e., \( D_P \) is a noetherian valuation ring).

**Proof.** Let \( N \) be an ideal of \( D \) maximal with respect to the property that \( N \) is a prime ideal \( \subset P \). Then by 3.3, \( N \) is the intersection of all \( P \)-primary ideals. Since \( D \) is noetherian, this intersection is \( (0) \) (for example, the intersection of the symbolic powers of \( P \) is \( (0) \) [7, p. 216, Corollary 1]). Therefore, \( N = (0) \) and \( P \) is minimal. Also, \( D_P \) is a noetherian domain; and by 2.6, every primary ideal of \( D_P \) is a valuation ideal. Therefore, by 3.8, \( D_P \) is Prüfer and hence a valuation ring. q.e.d.

4. **Restricted \( \mathcal{V} \).** We shall now deal with some special cases which occur when the set \( \mathcal{V} \) is restricted.

4.1. **Proposition.** Let \( D \) be a domain with a.c.c. for prime ideals. Then the following assertions are equivalent:

(a) There exists a finite set \( D_{v_1}, \ldots, D_{v_n} \) of valuation rings such that every primary ideal of \( D \) is a \( v_i \)-ideal for some \( i \).

(b) \( D \) is a Prüfer domain with \( \leq n \) maximal ideals.

**Proof.** (a) \( \Rightarrow \) (b): \( D \) is a Prüfer domain by 3.8. If \( M \) is any maximal ideal of \( D \), \( M \) is a \( v_i \)-ideal for some \( v_i \) and hence \( M \) is the center of \( D_{v_i} \) on \( D \). There exist at most \( n \) such distinct centers.

(b) \( \Rightarrow \) (a): Let \( M_1, \ldots, M_t, t \leq n \), be the maximal ideals of \( D \). Since \( D \) is Prüfer, \( D_{M_i} \) is a valuation ring, and then \( D_{M_1}, \ldots, D_{M_t} \) are the required valuation rings. q.e.d.

4.2. **Corollary.** Let \( D \) be a domain with a.c.c. for prime ideals, and suppose every primary ideal of \( D \) is a \( v \)-ideal for a fixed \( v \). Then \( D \) is a valuation ring (and conversely).

**Proof.** By 4.1, \( D \) is a Prüfer domain with one maximal ideal \( M \), and hence \( D = D_M \) is a valuation ring. q.e.d.

In \( \S5 \) we shall construct an example which shows this corollary does not remain true when the a.c.c. hypothesis is dropped.

4.3. **Theorem.** Let \( D \) be a domain and \( M \supseteq N \) prime ideals of \( D \) such that \( M \) is a minimal prime of \( N + A \) for some finitely generated ideal \( A \) and such that every \( M \)-primary ideal is a valuation ideal. Let \( P \) be the intersection of the \( M \)-primary ideals. Then \( P \) is a prime ideal such that \( N \subseteq P \subset M \) and there exists no prime ideal \( P_1 \) with \( P \subset P_1 \subset M \).

**Proof.** Since \( M \) is a minimal prime of \( N + A \), \( M \) is not a union of prime ideals properly between \( N \) and \( M \). Therefore, we can apply Zorn's lemma to conclude there exists a prime ideal \( P \) such that \( N \subseteq P \subset M \) and there exists no
prime ideal $P_1$ with $P \subset P_1 \subset M$. By 3.3, $P$ is the intersection of all $M$-primary ideals. q.e.d.

4.4. Corollary. Let $D$ be a domain such that $\mathfrak{m} \subseteq \mathfrak{v}$, and suppose for every prime ideal $P$ of $D$ there exists a valuation ring $D_v$ of rank 1 such that $P$ is a $v$-ideal. Then $\dim D \leq 1$ and $D$ is a Prüfer domain.

Proof. Suppose there exist prime ideals $N \subset M$ in $D$. We shall show $N = (0)$. Choose $x \in M$, $x \notin N$; and let $M_0$ be a minimal prime of $N + (x)$. Then $N \subset M_0$; and by 4.3, $N \subseteq P$, where $P$ is the intersection of the $M_0$-primary ideals. There exists a rank 1 valuation ring $D_v$ such that $M_0 \cdot D_v \cap D = M_0$; so if $M_v$ is the maximal ideal of $D_v$, then $M_v \cap D = M_0$. Therefore, every $M_v$-primary ideal of $D_v$ contracts to an $M_0$-primary ideal of $D$. Since $D_v$ has rank 1, the intersection of the $M_v$-primary ideals of $D_v$ is $(0)$. Thus the intersection $P$ of the $M_0$-primary ideals of $D$ is also $(0)$. Therefore, $N \subseteq P = (0)$, so $\dim D = 1$. $D$ is then Prüfer by 3.8. q.e.d.

It is now natural to make the following conjecture:

Let $D$ be a domain such that $\mathfrak{m} \subseteq \mathfrak{v}$, and suppose for every prime ideal $P$ of $D$ there exists a rank $n$ valuation ring $D_v$ such that $P$ is a $v$-ideal. Then $\dim D \leq n$.

We have been unable to determine whether this is true or not.

4.5. Corollary. A domain $D$ with quotient field $K$ is almost Dedekind if and only if $\mathfrak{m} \subseteq \mathfrak{v}$ and for any prime ideal $P$ of $D$ there exists a rank 1, discrete valuation ring $D_v \subset K$ such that $P$ is a $v$-ideal.

Proof. Suppose there exists a proper prime ideal $P$ of $D$. Then by 4.4 $\dim D = 1$ and $D$ is a Prüfer domain. Therefore, $D_P$ is a rank 1 valuation ring, and hence $D_P$ is a maximal subring of $K$. But if $P$ is a $v$-ideal, then $D_P \subseteq D_v \subset K$; so $D_P = D_v$. Therefore, $D_P$ is rank 1, discrete and thus $D$ is almost Dedekind. q.e.d.

5. Counterexamples. We saw in 3.8 that when $D$ has the a.c.c. for prime ideals, $\mathfrak{m} \subseteq \mathfrak{v}$ is equivalent to the assertion that $D$ is a Prüfer domain. We construct in this section an example to show $\mathfrak{m} \subseteq \mathfrak{v}$ does not necessarily imply $D$ is Prüfer without the additional a.c.c. hypothesis.

5.1. Proposition. Let $D$ be a domain with quotient field $K$; let $A \neq (0)$ be an ideal of $D$; let $D_0$ and $D_1$ be subrings of $D$ such that $D_0 \subseteq D_1 \subseteq D$. Let $S = D_0 + A$, $T = D_1 + A$. Then

(a) If $Q$ is a primary ideal of $S$ such that $\sqrt{Q} \subset A$, then $Q$ is an ideal in $D$.

(b) If $D$ is quasi-local and $D_1$ is a field, then $T$ is quasi-local with maximal ideal $A$.

(c) If $D_0 \subseteq D_1$ and $D_0, D_1$ are fields, then $S$ is not a valuation ring.

(5) A domain $D$ is said to be almost Dedekind if for every proper prime ideal $P$ of $D$, $D_P$ is a rank 1, discrete valuation ring [1].
Proof. (a) We shall show that for any \( x \in Q \) and \( d \in D \), \( dx \in Q \). \( dx \in A \) since \( x \in A \) so \( dx \in S \). Now choose \( a \in A \), \( a \notin Q \). Then
\[
a(dx) = (ad) \cdot x \in Q
\]
since \( ad \in A \leq S \) and \( x \in Q \). But then \( a \in S \), \( dx \in S \), and \( a \notin Q \), \( a(dx) \in Q \) implies \( dx \in Q \).

(b) \( A \) is clearly a maximal ideal in \( T \). Suppose \( x + a \in T \), \( x \neq 0 \) in \( D_1 \), \( a \in A \).
\[
(x + a)^{-1} = x^{-1}(1 - a(x + a)^{-1})
\]
\( a(x + a)^{-1} \in A \) since \( x + a \) is a unit of \( D \). Therefore, \( (x + a)^{-1} \in T \).

(c) Since \( D_0 \) is a field \( \leq D_1 \), there exists \( y \in D_1 \) such that \( y, 1/y \notin D_0 \). If \( y \in S \), then \( y = z + a \) for some \( z \in D_0 \), \( a \in A \). But \( y - z = a \in D_1 \) implies \( a = 0 \) and hence \( y = z \in D_0 \), a contradiction. Therefore \( y \notin S \), and similarly \( 1/y \notin S \); so \( S \) is not a valuation ring. q.e.d.

Let \( k_0 \) and \( k \) be fields with \( k_0 \subseteq k \), and let \( x_1, x_2, \ldots, x_n, \ldots \) be elements from an extension field of \( k \) such that \( x_1, x_2, \ldots, x_n, \ldots \) are algebraically independent over \( k \). There exists a valuation \( v \) of \( K = k(x_1, x_2, \ldots, x_n, \ldots) \) over \( k \) such that \( v(x_i) > v(x_i^{m+1}) \) for all \( i, m \) and such that \( m_v = (x_1, x_2, \ldots, x_n, \ldots) \) is the maximal ideal of the valuation ring \( D_v \). Let \( T = k + m_v \) and \( S = k_0 + m_v \). \( S \) is quasi-local with maximal ideal \( m_v \) by (b). \( S \) is not a valuation ring by (c), so \( S = S_{m_v} \) is not a valuation ring and hence \( S \) is not Prüfer.

Claim: Every primary ideal \( Q \) of \( S \) is an ideal of \( D \) (and hence is a \( v \)-ideal).

Proof. If \( \sqrt{Q} \subseteq m_v \), \( Q \) is an ideal of \( D \) by (a). On the other hand, if \( \sqrt{Q} \neq m_v \), \( x_i^{m+1} \in Q \) for every \( i \) and some \( m(i) \). But \( v(x_i/x_i^{m+1}) > 0 \) implies \( x_i/x_i^{m+1} \in m_v \subseteq S \). Therefore, \( x_i \in x_i^{m+1} \cdot S \subseteq Q \). Since this is true for all \( i \), \( m_v \subseteq Q \); so \( m_v = Q \) and \( Q \) is an ideal of \( D_v \). q.e.d.

Thus, \( S \) is a domain such that every primary ideal is a valuation ideal (in fact, a \( v \)-ideal for a fixed \( v \)), but \( S \) is not a Prüfer domain. By choosing \( k \) algebraic over \( k_0 \), we see that \( S \) is not even integrally closed, since \( k \cap S = k_0 \subset k \).

The following addition to Proposition 5.1 shows that whenever \( P \) is a prime ideal of \( S \) and \( P \subseteq m_v \), then \( S_P \supseteq D_v \) and hence every such \( S_P \) is a valuation ring:

(d) (5.1 continued). If \( P \) is a prime ideal of \( S \) such that \( P \subseteq A \), then \( D \subseteq S_P \).

Proof. Choose \( x \in A \), \( x \notin P \). Then \( 1/x \in S_P \), so for any \( y \in D \), \( yx \in A \subseteq S \) and \( y = (yx) \cdot 1/x \in S_P \). Therefore, \( D \subseteq S_P \). q.e.d.

Thus, the above example has the property that for any prime ideal \( P \) of \( S \),

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(6) Such a \( v \), having value group the weak direct sum of the integers ordered lexicographically, may be constructed as follows: Define \( v(x_1^1 \cdot x_2^2 \cdot \ldots \cdot x_n^n) = (r_1, \ldots, r_n, 0, \ldots) \); \( v(f(x)) = \) minimum value of the power products occurring in \( f(x) \), for any \( f(x) \in k[x_1, \ldots, x_n, \ldots] \); and \( v(\xi) = v(f) - v(g) \) for any \( \xi = fg \in k[x_1, \ldots, x_n, \ldots] \).
either $S_p$ is a valuation ring or $P$ is the only ideal having radical $P$ (which indeed happens for $P = m_0$). It seems reasonable then to make the following conjecture:

If $D$ is a domain such that $2 \subseteq \mathcal{V}$, then for any prime ideal $P$ of $D$ either $D_P$ is a valuation ring or $P$ is the only ideal having radical $P(?)$.

To show this conjecture is false, we must make some modifications in the example. Let then $D_w$ be a valuation ring having quotient field $k_0$; let $Y, x_1, \ldots, x_n, \ldots$ be quantities algebraically independent over $k_0$; and let $k = k_0(Y)$ and $K = k(x_1, \ldots, x_n, \ldots)$. As before, there exists a valuation $v$ of $K$ over $k$ such that the valuation ring $D_v$ has maximal ideal $m_v \neq 0$ with the property that $m_v$ is the only $m_v$-primary ideal of $D_v$, and moreover $D_v = k + m_v$ (see the preceding remarks). Let $S = D_w + m_v$.

The set $M$ of polynomials $f(Y) \in D_w[Y]$ such that the coefficients of $f$ generate all of $D_w$ (i.e., at least one coefficient is a unit in $D_w$) is multiplicatively closed in $D_w[Y]$, and the quotient ring $(D_w[Y])_M$ will be denoted by $D_w(Y)$ (see [6, p. 17]). Let $T = D_w(Y) + m_v$.

5.2. Lemma. Let $D_w$ be a valuation ring and let $Y$ be a transcendental element over $D_w$. Then $D_w(Y)$ is also a valuation ring.

Proof. Any element of the quotient field of $D_w(Y)$ has the form $f(Y)/g(Y), f, g \in D_w[Y]$. Let $c$ be the element of least value in the set of (non-zero) coefficients of $f$ and $g$. Then $f/c$ or $g/c$ is in the multiplicative system $M$ so $f/g = (f/c)/(g/c)$ or $g/f$ is in $D_w(Y)$. q.e.d.

Before proceeding with the example, we shall make some further additions to the proposition of 5.1. (5.1. continued):

(e) If $D_1$ is the quotient field of $D_0$ and if $T, D_0$ are valuation rings, then $S$ is a valuation ring also.

(f) If $D$ is quasi-local with maximal ideal $A$ and $B$ is an ideal of $S$ such that $B \not\subseteq A$, then $B = (B \cap D_0) + A$.

(g) If $A_0$ is an ideal of $D_0$ such that $A_0 \cdot D_1 \cap D_0 = A_0$ and if $D_1 \cap A = 0$, then $(A_0 + A) \cdot T \cap S = A_0 + A$.

Proof.

(e) Since $T$ is a valuation ring, we need only see that for any $y \neq 0$ in $T$, either $y \in S$ or $1/y \in S$. $A \subseteq S$, so we may assume $y \notin A$. Therefore, since $A$ is the maximal ideal of $T$, $y$ is a unit in $T$. If $y = x + a, x \neq 0$ in $D_1, a \in A$, then $1/y - 1/x = 1/x(-a/(x + a)) \in A$. But $D_0$ is a valuation ring with quotient field $D_1$, so $x \in D_0$ or $1/x \in D_0$. Thus, $y \in S$ or $1/y \in S$.

(f) If $x \in S, x \notin A$, then $1/x \in D$. Therefore, for any $a \in A, a/x \in A \subseteq S$; and then $a \in x \cdot S$. In particular, $B \supseteq A$ implies $A \subseteq B$. For any $b \in B, d_0 + a = b$ with $d_0 \in D_0, a \in A$. Therefore, $d_0 = b - a \in B$.

(?) Note that the converse is obviously true.
(g) \((A_0 + A) \cdot (D_1 + A) \cap (D_0 + A) \subseteq (A_0 \cdot D_1 + A) \cap (D_0 + A) \subseteq (A_0 \cdot D_1 \cap D_0) + A\)

since \(D_0 \subseteq D_1\) and \(D_1 \cap A = 0\).

But by hypothesis, \(A_0 \cdot D_1 \cap D_0 = A_0\); so we have \((A_0 + A) \cdot (D_1 + A) \cap (D_0 + A) \subseteq A_0 + A\). The opposite inclusion is obvious. q.e.d.

Continuing with the example, if \(Q\) is any primary ideal of \(S\), we next show that \(Q\) is a valuation ideal.

Case 1. \(Q \subseteq m_v\). If \(\sqrt{Q} = m_v\), then \(Q\) is an ideal in \(D_v\), by 5.1-(a), and hence \(Q\) is a \(v\)-ideal. If \(\sqrt{Q} = m_v\), then \(Q = m_v\). For, \(\sqrt{Q} = m_v\) implies \((Q \cdot D_v) = m_v\) and hence \(Q \cdot D_v = m_v\) since \(m_v\) is the only \(m_v\)-primary ideal of \(D_v\). But then for any \(x, y \in m_v\), \(xy = x(dq)\) for some \(d \in D_v\), \(q \in Q\). Therefore, \(xy = (xd)q \in m_v \cdot Q \subseteq Q\), so \(m_v^2 \subseteq Q\). But \(m_v^2 = m_v\) implies \(m_v = m_v^2\), so \(m_v \subseteq Q\) and thus \(m_v = Q\).

Case 2. \(Q \not\subseteq m_v\). By 5.1-(b), \(Q = Q_w + m_v\), where \(Q_w\) is a primary ideal of \(D_w\). \(Q_w \cdot D_v(Y) \cap D_w = Q_w\), by [6, p. 18, (6.17)].

Therefore, \(Q \cdot T \cap S = Q\), by 5.1-(f). Using the fact that \(D_v = k + m_v\) is a valuation ring and \(D_w(Y)\) is a valuation ring (by 5.2), we can apply 5.1-(e) to conclude that \(T\) is also a valuation ring. Thus, \(Q\) is a valuation ideal.

We have therefore proved that \(\mathfrak{B}(S) \subseteq \mathfrak{V}(S)\). Consider then the maximal ideal \(P = m_w + m_v\) of \(S\). If \(D_w\) is chosen to be, for example, rank 1, discrete, then \(m_w^2 \subseteq m_w\). Therefore, \(Q = m_w^2 + m_v\) is \(P\)-primary and \(Q \subseteq P\). However, \(S_p \subseteq S_{m_v} = k_0 + m_v\); and as before, \(k_0 + m_v\) is not a valuation ring by 5.1-(c). Therefore, \(S_p\) is not a valuation ring either. Thus \(S\) is a domain such that \(Q(S) \subseteq \mathfrak{V}(S)\), yet there exists a prime ideal \(P\) of \(S\) such that \(S\) has a \(P\)-primary ideal other than \(P\) and \(S_p\) is not a valuation ring.

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