

POWER SERIES WHOSE SECTIONS HAVE ZEROS OF LARGE MODULUS

BY

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1. **Introduction.** Several theorems in the theory of polynomials deal with the problem of obtaining bounds for the modulus of one or more zeros of a polynomial, $a_0 + a_1z + \cdots + a_nz^n$, when certain of the coefficients a_0, a_1, \dots, a_k , are regarded as fixed, and the remaining are arbitrary (cf. [2, Chapter 8]). In the present paper we apply results of this nature to partial sums of a power series $\sum a_pz^p$. For each positive integer n , r_n will denote the radius of the largest circle with center at $z = 0$ whose interior contains no zero of the n th section,

$$s_n(z) = \sum_{p=0}^n a_pz^p.$$

We shall be concerned primarily with growth properties of the sequence $\{r_n\}$. The most interesting case is that in which $\sum a_pz^p$ is the power series for an entire function which omits the value zero. It is not hard to show that this is equivalent to having $\lim r_n = \infty$. One can, however, construct other power series for which $\limsup r_n = \infty$.

Since nothing is lost by doing so, we shall always suppose that $a_0 = 1$. This assumption will be used freely and without explicit mention. For notational convenience, \sum and \sum' will denote sums taken over the nonnegative and positive integers, respectively.

In §2, upper bounds for r_n are obtained from algebraic relations between the zeros of $s_n(z)$ and the first "few" of the numbers a_1, a_2, a_3, \dots . From algebraic considerations alone, we show that

$$(1.1) \quad 1 + r_n = n^{o(1)},$$

except possibly for certain "exceptional" power series, and, further, that these exceptions must be power series for entire functions of the form $\exp\{P(z)\}$, where $P(z)$ is a polynomial.

In §3 we use analytic methods to obtain lower bounds for r_n in case $\sum a_pz^p$ is an entire function which omits the value zero. We are able to show that the "apparent exceptions" to (1.1) are actual exceptions, and thus characterize entire functions of the form $\exp\{P(z)\}$ for $P(z)$ a polynomial.

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Taken together, the upper and lower bounds yield a number of asymptotic properties of the sequence $\{r_n\}$. For $\sum a_p z^p = \exp\{g(z)\}$, where $g(z)$ is an entire function of order ρ , we show that

$$\limsup \frac{\log \log n}{\log r_n} = \rho.$$

This and similar results are discussed in §4.

In §5 we prove that

$$(1.2) \quad \limsup r_n |a_n|^{1/n} = 1,$$

provided that $\sum a_p z^p$ is an entire function of infinite order which has no zeros. Making use of (1.2) and the observation that $|a_n|^{-1/n}$ is the geometric mean of the zeros of $s_n(z)$, we deduce the following: *If $\sum a_p z^p$ is an entire function of infinite order without zeros, $\varepsilon > 0$, and $\varepsilon' > 0$, then, for infinitely many integers n , fewer than $n\varepsilon'$ zeros of $s_n(z)$ have moduli greater than $r_n(1 + \varepsilon)$.* This result is of some interest in connection with theorems of F. Carlson [1] and P. Rosenbloom [6] on zeros of sections of entire series of infinite order.

2. Upper bounds. Let $\sum' b_p z^p$ be the power series obtained formally from the identity

$$\frac{\sum p a_p z^{p-1}}{\sum a_p z^p} = \sum' p b_p z^{p-1}.$$

THEOREM 2.1. *If k is a positive integer such that $b_k \neq 0$, and $n \geq k$, then $s_n(z)$ has a zero in the disc*

$$|z| \leq \left\{ \frac{n}{k|b_k|} \right\}^{1/k}.$$

Proof. If one lets

$$\frac{s'_n(z)}{s_n(z)} = \sum' p b_p^{(n)} z^{p-1},$$

and observes that $b_p^{(n)} = b_p$ for $p \leq n$, the result then follows from a theorem of G. Sz.-Nagy ([3], [4], and [2, Example 2, p. 43]).

As a consequence of the above, we have

$$(2.1) \quad r_n = O(n^{1/k})$$

for every value of k for which $b_k \neq 0$. If $\sum a_p z^p = \exp\{P(z)\}$ for some polynomial $P(z)$, then (2.1) holds with k equal to the degree of $P(z)$. If $\sum a_p z^p$ is not a power series of this form, then (2.1) holds for infinitely many integers k . This establishes (1.1).

COROLLARY 2.2.

$$(2.2) \quad \liminf \frac{\log r_n}{\log \log n} \leq \liminf \frac{\log(1/|b_k|)}{k \log k}.$$

Proof. From Theorem 2.1 we have

$$\frac{\log r_n}{\log k} \leq \frac{\log n}{k \log k} - \frac{1}{k} + \frac{\log(1/|b_k|)}{k \log k}.$$

Choose $n = n(k)$ so that $\log n \sim k$, and let $k \rightarrow \infty$.

The above result is of particular interest if $\sum' b_p z^p = g(z)$, where $g(z)$ is an entire function of order ρ . We then have $\sum a_p z^p = \exp\{g(z)\}$, and

$$(2.3) \quad \limsup \frac{\log \log n}{\log r_n} \geq \rho,$$

since the right-hand side of (2.2) is $1/\rho$. Later we shall see that equality holds in (2.3).

For certain entire functions $g(z)$, Theorem 2.1 can be used to obtain an extremely good upper bound for r_n . For this purpose we make use of the *maximum term* function, $\mu(r) = \mu(r, g)$, defined by

$$(2.4) \quad \mu(r) = \max_p \{|b_p| r^p\},$$

and the *central index*, $\nu(r)$, which is the largest integer m such that

$$\mu(r) = |b_m| r^m.$$

THEOREM 2.3. Let $\sum a_p z^p = \exp\{g(z)\}$, where $g(z) = \sum' b_p z^p$ is an entire function of finite order. For each n , let β_n be the positive number such that $\mu(\beta_n) = n$, where $\mu(r)$ is defined by (2.4). Then for all sufficiently large n , $s_n(z)$ has a zero in the disc $|z| \leq \beta_n$.

Proof. From Theorem 2.1,

$$r_n \leq \left\{ \frac{n}{k |b_k|} \right\}^{1/k} \leq \left\{ \frac{n}{|b_k|} \right\}^{1/k}.$$

Let $k = \nu(\beta_n)$. Then $|b_k| \beta_n^k = \mu(\beta_n) = n$. Therefore $r_n \leq \beta_n$.

It remains to show that $k \leq n$, or equivalently, that $\nu(\beta_n) \leq \mu(\beta_n)$. This is true provided the inequality

$$(2.5) \quad \nu(r) < \mu(r)$$

holds for all sufficiently large r . A proof of (2.5) follows easily from the relation [10, p. 34]

$$\limsup \frac{\log \nu(r)}{\log r} = \rho,$$

where ρ is the order of $g(z)$. The hypothesis that ρ is finite can, therefore, be replaced by (2.5).

3. Lower bounds. We obtain lower bounds for the numbers r_n under the assumption that $\sum a_p z^p = \exp\{g(z)\}$, where $g(z) = \sum' b_p z^p$ is an entire function.

$G(z)$ will denote the majorant of $g(z)$ defined by

$$(3.1) \quad G(z) = \sum' |b_p| z^p.$$

We note for future use that the order of $G(z)$ is the same as that of $g(z)$; we shall also need the inequality

$$(3.2) \quad \sum |a_p| r^p \leq \exp \{G(r)\} \quad \text{if } r \geq 0.$$

A proof of (3.2) follows from expanding $\exp \{G(r)\}$ as a power series in r and observing that the coefficient of r^p is at least as great as $|a_p|$.

THEOREM 3.1. *Let $\sum a_p z^p = \exp \{g(z)\}$, where $g(z)$ is an entire function with majorant $G(z)$ given by (3.1). If n is a positive integer, then*

$$(3.3) \quad r_n > r \exp \{-2G(r)/n\} \quad \text{for all } r \geq 0.$$

In particular, if α_n is the positive number such that $G(\alpha_n) = n$, then

$$(3.4) \quad r_n > \alpha_n / e^2.$$

Furthermore, if $g(z)$ is not a polynomial, then for $\varepsilon > 0$, one has

$$(3.5) \quad r_n > \alpha_n(1 - \varepsilon)$$

for all sufficiently large n .

Proof. Suppose $r > 0$ and let $f(z) = \sum a_p z^p$. We shall establish (3.3) by showing that

$$(3.6) \quad |z| \leq r \exp \{-2G(r)/n\}$$

implies

$$|1 - s_n(z)/f(z)| < 1,$$

and therefore that $s_n(z) \neq 0$. The latter is obviously true if $z = 0$; suppose z satisfies (3.6) and $z \neq 0$. Then $0 < |z| < r$, and $|1/f(z)| = |\exp\{-g(z)\}| < \exp\{G(r)\}$. Also,

$$\begin{aligned} |f(z) - s_n(z)| &\leq |z/r|^n \sum_{p=n+1}^{\infty} |a_p| r^p |z/r|^{p-n} \\ &< |z/r|^n \sum_{p=n+1}^{\infty} |a_p| r^p \\ &< |z/r|^n \exp \{G(r)\} \end{aligned}$$

by virtue of (3.2). Therefore

$$|1 - s_n(z)/f(z)| < \{|z/r| \exp\{2G(r)/n\}\}^n \leq 1,$$

by (3.6). This proves (3.3). If $r = \alpha_n$, we have (3.4).

The proof of (3.5) depends on the following property of $G(r)$ (cf. [5, Vol. 2, p. 4]): If $g(z)$ is not a polynomial, and $0 < c < 1$, then

$$(3.7) \quad \lim_{r \rightarrow \infty} \frac{G(cr)}{G(r)} = 0.$$

We now make use of (3.7) and the sequence $\{\alpha_n\}$ to construct a sequence $\{c_n\}$ such that

$$\lim c_n = 1 \quad \text{and} \quad \lim \frac{G(c_n \alpha_n)}{G(\alpha_n)} = 0.$$

In (3.3) let $r = c_n \alpha_n$ and replace n by $G(\alpha_n)$. Then

$$r_n > c_n \alpha_n \exp \{ - 2G(c_n \alpha_n) / G(\alpha_n) \}.$$

Observing that

$$\lim c_n \exp \{ - 2G(c_n \alpha_n) / G(\alpha_n) \} = 1$$

establishes (3.5).

4. Asymptotic properties. In a number of cases, fairly precise information about the sequence $\{r_n\}$ can be obtained by comparing the upper bounds of §2 with the lower bounds developed in §3.

THEOREM 4.1. *If $\sum a_p z^p = \exp \{P(z)\}$, where $P(z)$ is a polynomial of degree k , then there are positive numbers A and B such that*

$$A n^{1/k} < r_n < B n^{1/k}, \quad n = 1, 2, 3, \dots$$

The proof, which is omitted, follows easily from (2.1) and (3.4).

In view of (1.1), one sees that Theorem 4.1 characterizes power series for entire functions of the form $\exp \{P(z)\}$. Among all power series, the exponential series (more accurately, the series for ae^{bz}) is the only one for which r_n increases as rapidly as a linear function of n . Zeros of sections and remainders of this series have been investigated by G. Szegő [8].

Our next theorem is similar in some respects to a theorem of M. Tsuji [9] on the maximum modulus of zeros of sections of an entire series. If we let R_n denote the largest modulus of a zero of $s_n(z)$ (with the convention that $R_n = \infty$ if $a_n = 0$), Tsuji's theorem asserts that

$$\limsup \frac{\log n}{\log R_n}$$

is equal to the order of $\sum a_p z^p$. For an entire function $\sum a_p z^p$ which omits zero we obtain an analogous result involving r_n and the order of the logarithm of $\sum a_p z^p$.

THEOREM 4.2. *If $\sum a_p z^p = \exp \{g(z)\}$, where $g(z)$ is an entire function of order ρ ($0 \leq \rho \leq \infty$), then*

$$\limsup \frac{\log \log n}{\log r_n} = \rho.$$

Proof. Since the order of $G(z)$ is also ρ , we have

$$\begin{aligned} \rho &= \limsup_{r \rightarrow \infty} \frac{\log \log G(r)}{\log r} \geq \limsup_{n \rightarrow \infty} \frac{\log \log n}{\log \alpha_n} \\ &\geq \limsup \frac{\log \log n}{\log(r_n e^2)} \end{aligned}$$

by (3.4). Since $\log(r_n e^2) \sim \log r_n$, we can neglect the factor e^2 . The other half of the proof follows from (2.3).

If $g(z)$ is of finite positive order ρ and of type τ ($0 \leq \tau \leq \infty$), one can prove a sharper result, namely, that

$$\limsup \frac{\log n}{r_n^\rho} = \tau.$$

For this one needs (3.5) in place of (3.4); the “ \geq ” half of the result is obtained from Theorem 2.1 by a procedure similar to the proof of Corollary 2.2. In this case one chooses $n = n(k)$ so that $\log n \sim k/\rho$.

If $g(z)$ is of finite order, the asymptotic relation [5, Vol. 2, p. 8]

$$(4.1) \quad \log G(r) \sim \log \mu(r, G) = \log \mu(r, g)$$

yields information about the relative sizes of α_n and β_n . From (4.1) and the definitions of α_n and β_n , we have

$$(4.2) \quad \log G(\alpha_n) \sim \log G(\beta_n).$$

This supplies a connecting link between our upper and lower bounds for r_n .

THEOREM 4.3. *If $\sum a_p z^p = \exp\{g(z)\}$, where $g(z)$ is an entire function of finite order, then*

$$\log \alpha_n \sim \log r_n \sim \log \beta_n.$$

Proof. Since $\alpha_n/e^2 < r_n \leq \beta_n$ for large n , it suffices to prove that $\log \alpha_n \sim \log \beta_n$. If we note that $\alpha_n \leq \beta_n$, the result follows from (4.2) and the Hadamard three circle theorem (applied to circles with radii 1, α_n , and β_n).

THEOREM 4.4. *Let $\sum a_p z^p = \exp\{g(z)\}$, where $g(z)$ is an entire function of finite order. If, for some $\delta > 0$, there is a nondecreasing function $H(r)$ such that $\log G(r) \sim r^\delta H(r)$, then*

$$\alpha_n \sim r_n \sim \beta_n.$$

Proof. Suppose $\varepsilon > 0$. The condition on $G(r)$ implies that $g(z)$ is not a polynomial; therefore $\alpha_n(1 - \varepsilon) < r_n \leq \beta_n$ for large n , and we need only prove that $\alpha_n \sim \beta_n$.

From (4.2) and the condition on $G(r)$, we have

$$1 \geq \left[\frac{\alpha_n}{\beta_n} \right]^\delta \sim \frac{H(\beta_n)}{H(\alpha_n)} \geq 1,$$

since $\alpha_n \leq \beta_n$ and $H(r)$ is nondecreasing. Hence $\alpha_n \sim \beta_n$.

As a special case of the above, we note that the condition $\log G(r) \sim \tau r^\rho$ for positive numbers ρ and τ implies that

$$r_n \sim \left[\frac{\log n}{\tau} \right]^{1/\rho}.$$

For $g(z)$ of finite order, $\log G(r) \sim \log M_\rho(r)$, where $M_\rho(r)$ is the maximum modulus of $g(z)$ on $|z| = r$. Therefore the condition on $G(r)$ in the hypothesis of Theorem 4.4 is equivalent to the corresponding condition on $M_\rho(r)$. In addition, we note that Theorems 4.3 and 4.4 remain valid if α_n is replaced by α'_n , where α'_n is defined by $M_\rho(\alpha'_n) = n$.

5. Comparison with the coefficients. In this section we restrict our attention to the case $\sum a_p z^p = \exp \{g(z)\}$, where $g(z)$ is an entire function which is not a polynomial. This is equivalent to requiring that $\sum a_p z^p$ be an entire function of infinite order without zeros. We shall compare the lower bound (3.5) for r_n with the elementary upper bound

$$(5.1) \quad r_n \leq |a_n|^{-1/n} \quad \text{if } a_n \neq 0.$$

(The right-hand side of (5.1) is the geometric mean of the moduli of zeros of $s_n(z)$.) Our principal result is the following:

THEOREM 5.1. *If $\sum a_p z^p$ is an entire function of infinite order without zeros, then*

$$\limsup \alpha_n |a_n|^{1/n} = \limsup r_n |a_n|^{1/n} = 1.$$

Before proving Theorem 5.1 we shall consider two of its corollaries.

COROLLARY 5.2. *Let $\sum a_p z^p$ satisfy the hypothesis of Theorem 5.1. If $\varepsilon > 0$, then*

$$\alpha_n(1 - \varepsilon) < r_n \quad \text{and} \quad r_n |a_n|^{1/n} \leq 1$$

for all sufficiently large n , and

$$\alpha_n(1 + \varepsilon) > r_n > (1 - \varepsilon) |a_n|^{-1/n}$$

for infinitely many n .

Proof. Theorem 5.1 and inequalities (3.5) and (5.1).

COROLLARY 5.3. *If $\sum a_p z^p$ is an entire function of infinite order without zeros, $\varepsilon > 0$ and $\varepsilon' > 0$, then for infinitely many integers n , fewer than $n\varepsilon'$ zeros of $s_n(z)$ have moduli greater than $r_n(1 + \varepsilon)$.*

Proof. Choose δ so that

$$0 < \delta < 1 - (1 + \varepsilon)^{-\varepsilon'}.$$

If n is a positive integer for which $r_n > (1 - \delta) |a_n|^{-1/n}$, then an easy calculation shows that fewer than $n\varepsilon'$ zeros of $s_n(z)$ have moduli greater than $r_n(1 + \varepsilon)$.

Proof of Theorem 5.1. From (3.5) and (5.1) it follows that

$$\limsup \alpha_n |a_n|^{1/n} \leq \limsup r_n |a_n|^{1/n} \leq 1;$$

consequently, we have only to prove that

$$(5.2) \quad \limsup \alpha_n |a_n|^{1/n} \geq 1.$$

To facilitate the proof of (5.2) we first establish two lemmas. These will enable us to obtain a lower bound for α_n in terms of the maximum modulus of $\sum a_p z^p$.

LEMMA 5.1a. *For all $r > 0$,*

$$(5.3) \quad \alpha_n > r \exp\{-2G(r)/n\}.$$

Proof. The function $u(r) = r \exp\{-2G(r)\}$ assumes its maximum at the number $r = \gamma_n$ such that $2\gamma_n G'(\gamma_n) = n$. Since $2rG'(r) > G(r)$ for all $r > 0$, we have

$$\alpha_n > \gamma_n > \gamma_n \exp\{-2G(\gamma_n)/n\}.$$

LEMMA 5.1b. *If $\varepsilon > 0$, then*

$$\alpha_n > \frac{r}{1 + \varepsilon} [M(r)]^{-8/n\varepsilon} \quad \text{for all } r > 0,$$

where

$$(5.4) \quad M(r) = \max_{|z|=r} \left| \sum a_p z^p \right|.$$

Proof. The proof depends on the following variant of the Borel-Carathéodory inequality (proved, but not explicitly stated, in [10, pp. 17-20]): If $0 < r < R$, then

$$(5.5) \quad G(r) = \sum' |b_p| r^p \leq \frac{4r}{R-r} A(R),$$

where

$$A(R) = \max_{|z|=R} \{\operatorname{Re} \sum' b_p z^p\}.$$

If we rewrite (5.3) in the form

$$\alpha_n > \frac{r}{1 + \varepsilon} \exp \left\{ -\frac{2}{n} G \left(\frac{r}{1 + \varepsilon} \right) \right\}$$

and in (5.5) replace r and R by $r/(1 + \varepsilon)$ and r respectively, we have

$$\alpha_n > \frac{r}{1 + \varepsilon} \exp \left\{ \frac{-8A(r)}{n\varepsilon} \right\}.$$

Since $A(r) = \log M(r)$, the result follows.

It is worth noting that the inequality of Lemma 5.1b remains valid if α_n is replaced by r_n . To see this one uses (3.3) in place of (5.3).

We are now in a position to establish (5.2). Let $\mu(r)$ and $\nu(r)$ denote (contrary to previous usage) the maximum term and central index of the series $\sum a_p z^p$. If we choose $n = \nu(r)$, the inequality of Lemma 5.1b can be written as

$$\log \{ (1 + \varepsilon) \alpha_n |a_n|^{1/n} \} > \frac{\log \mu(r)}{\nu(r)} - \frac{8}{\varepsilon} \frac{\log M(r)}{\nu(r)}.$$

Since $\sum a_p z^p$ is of infinite order, it follows that

$$\liminf \frac{\log M(r)}{\nu(r)} = 0$$

from a theorem of S. M. Shah [7]. Therefore

$$\limsup (1 + \varepsilon) \alpha_n |a_n|^{1/n} \geq 1,$$

which establishes (5.2) and completes the proof.

Theorem 5.1 adds an interesting footnote to certain more general results on zeros of sections of power series. If $\sum a_p z^p$ is an entire function of infinite order and $\varepsilon > 0$, it is known [1], [6] that, for infinitely many integers n , all but $o(n)$ zeros of $s_n(z)$ lie in the annulus

$$(1 - \varepsilon) |a_n|^{-1/n} < |z| < (1 + \varepsilon) |a_n|^{-1/n}.$$

If $\sum a_p z^p$ omits the value zero, we have shown that, for infinitely many integers n , no zero of $s_n(z)$ lies in the interior of the inner circle of the annulus.

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