A CHARACTERIZATION OF $\Delta_2$-SETS

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A $\Delta_2$-set $m$ is defined as follows: it is constructible and $O\delta^dm$ is isomorphic to an analytic well-ordering $W$ of the natural numbers which is expressible in both two-quantifier forms. J. R. Shoenfield has proved in [1] that a set of natural numbers is a $\Delta_2$-set if and only if it is analytic and expressible in both two-quantifier forms. In this paper two characterizations of $\Delta_2$-sets are given.

Let formulas of set theory be formulas of predicate-calculus with identity and the single predicate-variable e. A finite set $S$ of closed formulas of set theory will be called a finite system of axioms. Then we have the result: $m_0$ is a $\Delta_2$-set if and only if there is a finite system of axioms $S$ with the properties (a) $S$ has exactly one transitive model $m$, (b) $m$ is denumerable and $m_0 \in m$. Another characterization of $\Delta_2$-sets is obtained by giving a family $K$ of functions defined by transfinite induction such that (a) for $f \in K$, every $x$ belonging to the range of $f$ is a $\Delta_2$-set, (b) corresponding to every $\Delta_2$-set $x$ there is an $f$ in $K$ whose range contains $x$.

We use the terminology and notation of [2], [3] and [4]. The power of a set $m$ is denoted by $|m|$. A. We start with some preliminaries. A lemma is needed, which because of its simplicity is stated without proof (see for instance [5]). In what follows, $D(f)$, where $f$ is a function, will denote the domain of $f$ and $W(f)$ the range of $f$. We also need

**Definition 1.** A set $m$ is called simple if for $a, b \in m$,

$$
(x)(x \in m \cdot \quad \cdot x \in a \equiv x \in b) \equiv a = b.
$$

Then we have

**Lemma 1.** Let $m$ be nonvoid and simple. Then there is a function $g_m$ with $D(g_m) = m$ such that, for any $x \in m$, $g_m(x) = \{g_m(s) \mid s \in x \cap m\}$. The set $W(m)$ (to be denoted by $m^*$) is transitive. If $m_0$ is transitive and $m_0 \subseteq m$, then $g_m(x) = x$ for every $x \in m_0$. The function $g_m$ is uniquely determined by this property.

A transitive $m_0$ with $m_0 \subset m$, $m_0 \in m$, we call a transitive part of $m$. Thus, transitive parts are invariant under the isomorphism $g_m$.

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B. Let $m$ be infinite and let $R_1, R_2, \ldots, R_s$ be a system of $(n_1, n_2, \ldots, n_s)$-place relations over $m$. If $m$ has a well-ordering and $m_0 \subseteq m$ then by the Loewenheim-Skolem theorem $m$ has an elementary subsystem $\langle m', R_1', \ldots, R_s' \rangle$ such that $m_0 \subseteq m'$ and $|m'| = |m_0| + \chi$, where $R_i'$ is the restriction of $R_i$ to $m'$. This fact will be used in what follows. It is also convenient to introduce the notation $p[m]$ with the meaning: if $p(x_1, \ldots, x_n)$ is a formula of set theory with the free variables $x_1, \ldots, x_n$, then $p[m]$ denotes the $n$-ary relation defined by $p$ in the model $m$. In general, if $R$ is an $n$-ary relation, we write $R(x_1, \ldots, x_n)$ to mean that $\langle x_1, \ldots, x_n \rangle \in R$.

**Lemma 2.** Let $m, m_0$ be transitive and $m_0 \subseteq m$. There exists a transitive set $m_x$ with $m_0 \subseteq m_x$ such that for elements $a_1, \ldots, a_s$ of $m_0$ the equivalence $p[m_x](a_1, \ldots, a_s) \equiv p[m](a_1, \ldots, a_s)$ holds. If $m_0$ is infinite, then $m_x$ has the same power as $m_0$.

**Proof.** By the Loewenheim-Skolem theorem there is a set $m'$ with $m_0 \subseteq m'$ such that for elements $a_1, \ldots, a_s$ in $m'$ we have $p[m'](a_1, \ldots, a_s) \equiv p[m'](a_1, \ldots, a_s)$. For $a, b \in m'$,

$$(x) (x \in m' \rightarrow x \in a \equiv x \in b).$$

as $m$ is transitive. So $m'$ is simple. For $m_x = (m')^t$ and $a_1, \ldots, a_s \in m_0$, it follows as a consequence of Lemma 1 that

$$p[m](a_1, \ldots, a_s) \equiv p[m'](a_1, \ldots, a_s) \equiv p[m_1](g_m(a_1), \ldots, g_m(a_s)) \equiv p[m_1](a_1, \ldots, a_s),$$

as $m_0$ is a transitive part of $m'$ and hence invariant with respect to $g_m$. If $m_0$ is infinite, the set $m_x$ has the same power as $m_0$, by the Loewenheim-Skolem theorem.

In the following, $\mathcal{F}_i (i = 1, \ldots, 8)$, $J, K_1, K_2$, and $F$ denote the functions defined in [3, Definitions 9.1, 9.21, 9.24, 9.3]. By the methods developed, e.g. in [6], one can easily prove

**Lemma 3.** There is a formula $A_0(x_1, x_2, x_3, x_4, x_5, y)$ such that for any nonvoid transitive $m$ and any elements $O_0, O'_0, J_0, F_0, v, \xi$ of $m$, $A_0[m](O_0, O'_0, J_0, F_0, v, \xi)$ if and only if $O_0$ is an ordinal greater than zero, $O'_0 = 9 \times O'_0, J_0 = J^{-1} \upharpoonright O_0, F_0 = F \upharpoonright O_0$ and $F_0 \upharpoonright v = \xi$.

As a consequence of Lemma 3 we have:

**Theorem 1.** If $m_0$ is infinite, transitive and constructible and if $a \in m_0$ then $|Od^\alpha| \leq |m_0|$.

**Proof.** First, one easily shows that there exists a constructible transitive set $m$ with $m_0 \subseteq m$ such that $(Ex_1 x_2 x_3 x_4 x_5) A_0[m](x_1, x_2, x_3, x_4, x_5, a)$
holds with $A_0$ as in Lemma 3. By Lemma 2 there is a transitive set $m_1$ with $m_0 \subseteq m_1 \subseteq m$ and $|m_0| = |m_1|$ such that $(Ex_1 \cdots x_5)A_0[m_1](x_1, \ldots, x_5, a)$.

By Lemma 3, $Oda \in m_1$ and consequently $|Oda| \leq |m_1| = |m_0|$.

**Remark.** Lemmas 1–3 and Theorem 1 can be formalized without difficulty in the system of Gödel-Bernays with the aid of the methods of [5], [6]. If we add the axiom $V = L$, we obtain a new proof of the statement $P(F^n\omega_2) \subseteq F^n\omega_{a+1}$ which implies the continuum hypothesis. For this purpose it is sufficient to show: $x \subseteq F^n\omega_2 \Rightarrow |Oda| \leq \omega_2$. But this is an immediate consequence of Theorem 1, if we put there $m_0 = x \cup \{x\} \cup F^n\omega_2$. As was pointed out to the author by the referee, the above variant of Gödel's proof was observed by D. Scott three years ago and given him in a seminar a year ago.

C. As shown in [1] the following statement holds:

I. If $\gamma_0$ is a constructible function, then

$$(\alpha)(E\beta)(x)A(\bar{a}, \bar{\beta}, \bar{\gamma}_0) \equiv (\alpha)\mathcal{L}(E\beta)(x)A(\bar{a}, \bar{\beta}, \bar{\gamma}_0).$$

Here, $A$ is a recursive relation and the index $\mathcal{L}$ on the right side indicates that the quantifiers are restricted to constructive functions.

Functions which are recursive in $\Sigma^1 \cap \Pi^1_k$-predicates (or equivalently whose graphs are $\Sigma^1 \cap \Pi^1_k$-predicates) are called $\Sigma^1 \cap \Pi^1_k$-functions. By I, the notion of $\Sigma^1 \cap \Pi^1_k$-function is absolute and by [1] the set of $\Sigma^1 \cap \Pi^1_k$-functions coincide with the set of $\Delta_2$-functions (and similarly for predicates).

In [4] it was shown, that under the assumption $V = L$ we have:

II. The set of $\Sigma^1_{k+1} \cap \Pi^1_{k+1}$-functions is a basis for $\Pi^1_k$-predicates. From I and II we immediately obtain

**Lemma 4.** If there is a function $\gamma_0$ with $(Ex)(\beta)(Ex)A(\bar{a}(x), \bar{\beta}(x), \bar{\gamma}_0(x))$ then there exists a $\Delta_2$-function $\gamma_1$ with $(Ex)(\beta)(Ex)A(\bar{a}(x), \bar{\beta}(x), \bar{\gamma}_1(x))$.

**Proof.** From $(E\gamma)(Ex)(\beta)(Ex)A(\bar{a}(x), \bar{\beta}(x), \bar{\gamma}(x))$ and I, we obtain $(E\gamma)_L(Ex)_L(\beta)_L(Ex)_L\mathcal{L}(\bar{a}(x), \bar{\beta}(x), \bar{\gamma}(x))$. By II there are $\Delta_2$-functions $\gamma_1, \alpha_1$ with $(\beta)_L(Ex)_L\mathcal{L}(\bar{a}(x), \bar{\beta}(x), \bar{\gamma}(x))$. This implies $(Ex)_L(\beta)_L(Ex)_L\mathcal{L}(\bar{a}(x), \bar{\beta}(x), \bar{\gamma}(x))$ and consequently by I also $(Ex)(\beta)(Ex)A(\bar{a}(x), \bar{\beta}(x), \bar{\gamma}_1(x))$.

**Definition 2.** We call a binary number-theoretic function $\alpha$ a founded $\varepsilon$-function if (a) $(x, y)(\alpha(x, y) = 1 \lor \alpha(x, y) = 0)$, (b) there does not exist an infinite sequence $i_1, i_2, \ldots, i_n, \ldots$ of numbers such that $\alpha(i_2, i_1) = 1, \alpha(i_3, i_2) = 1, \ldots$, (c) $(x)(\alpha(x, i) = 1) \equiv \alpha(x, k) = 1 \cdot \cdots \cdot i = k$. Functions for which only (a) hold will be called characteristic functions.

Let $\omega$ denote the set of natural numbers. As shown in [5, Theorem 3], the following statement holds.

**Lemma 5.** Corresponding to a founded $\varepsilon$-function $\alpha$, there exists exactly one transitive set $m_{\alpha}$ and one biunique function $g_{\alpha}$ with $D(g_{\alpha}) = \omega$ and
$W(g_a) = m_a$ such that $w(i, k) = 1 \equiv g_a(i) \in g_a(k)$. If $w$ is constructible, then so are $g_a$ and $m_a$.

In the proof of the following theorem only the essential steps are given; the necessary details are easily supplied by using the methods of [4].

**Theorem 2.** If $w$ is a founded $e$-function belonging to $\Sigma_2^1 \cap \Pi_2^1$, then the set $m_a$ of Lemma 5 is a $\Delta_2^1$-set.

**Proof.** 1. By hypothesis there are two recursive predicates $P, Q$ with

$$(x, y)(\alpha(x, y) = 1 \equiv (E\psi)(\phi)(E\iota)P(\tilde{\psi}(t), \tilde{\psi}(t), x, y)),$$

$$(x, y)(\alpha(x, y) = 1 \equiv (\psi)(E\phi)(t)Q(\tilde{\psi}(t), \tilde{\phi}(t), x, y)).$$

The predicate $R$ which applies to $\zeta$ if and only if

$$(x, y)(\zeta(x, y) = 1 \lor \zeta(x, y) = 0)$$

and

$$(x, y)(\zeta(x, y) = 1 \lor \zeta(x, y) = 0)$$

has the form $(E\phi)(\psi)(E\psi)R(\phi(x), \tilde{\psi}(x), \zeta(x))$ with $R$ recursive. $R$ is satisfied by exactly one $\xi$, namely $x$.

2. Let $S$ be the relation defined as follows: $S(k, \lambda, \mu, \phi)$ if and only if $\lambda, \mu$ are founded $e$-functions such that

(a) $(x, y)(\lambda(x, k) = 1 \lor \lambda(y, x) = 1 \lor \lambda(y, k) = 1)$,

(b) $D(\phi) = \omega, W(\phi) = \{x | \lambda(x, k) = 1\}$ and

$$(x, y)(\phi(x) = \phi(y) \equiv x = y \lor \mu(x, y) = 1 \equiv \lambda(\phi(x), \phi(y)) = 1).$$

One shows without difficulty that $S$ is a $\Pi_1^1$-relation. Furthermore, it is easy to see that $S(k, \lambda, \mu, \phi)$ implies $g_a(k) = m_a$ with $g_a, m_a$ as in Lemma 5.

3. Let $A_0(x_1, x_2, \ldots, x_5, y)$ be the formula of Lemma 3. Define the relation $G$ as follows: $G(\lambda, i, k)$ if and only if $\lambda$ is a founded $e$-function such that

$$(Ex_1, \ldots, x_4)A_0[m_a][x_1, \ldots, x_4, g_a(i), g_a(k)].$$

Also $G$ can be shown to be a $\Pi_1^1$-relation.

4. The function $\zeta$ is constructible and so are the sets $m_a, m_a \cup \{m_a\}$. By Theorem 1 the cardinality of $\text{Ord}m_a$ is not greater than that of $m_a \cup \{m_a\}$; that is $\text{Ord}m_a$ is denumerable. Hence there is denumerable transitive set $m_1$ and an ordinal $v \in m_1$ such that $\text{(Ex}_1, \ldots, x_4)A_0[m_a][x_1, \ldots, x_4, v, m_a]$. Let $M$ be the predicate defined by the equivalence $M(\xi) \equiv (Ex, y, \mu)(G(x, y, \xi) \land S(y, \xi, \mu) \land R(\xi))$.

$M$ is a $\Sigma_2^1$-predicate by 1., 2., 3.., Since $m_1$ is denumerable, there is at least one founded $e$-function $\lambda$ with $m_1 = m_1$.

Obviously $M(\lambda)$ holds as a consequence of 1., 2., 3., and hence $(Ex)M(x)$. 
By Lemma 4 there is a $\Delta_2$-function $\beta$ satisfying Definition 2 with $M(\beta)$. From $M(\beta)$ it follows that there is a number $i$ with $(\exists x_1, \ldots, x_4) A_0[m_f](x_1, \ldots, x_4, g_\beta(i), m_a)$. Let $i_0$ be the smallest such number. Obviously $Od^x m_a$ is order-isomorphic to $\{\langle x, y \rangle \mid \beta(x, i_0) = 1 \land \beta(y, i_0) = 1 \land \beta(x, y) = 1\}$. By standard techniques it now easily follows that $Od^x m_a$ is order-isomorphic to a $\Delta_2$-well-ordering of $\omega$. q.e.d.

Here and later on we shall use a predicate Sat which is defined as follows:

$Sat(n, \lambda)$ if and only if (a) $\lambda$ is a founded $\epsilon$-function, (b) $n$ is the Gödel number of a closed formula $A$ of set theory which is true in the model $\langle \epsilon', \omega \rangle$ with $\epsilon' = \{\langle x, y \rangle \mid \lambda(x, y) = 1\}$. Obviously, $Sat(n, \lambda)$ if and only if $A$ is true in the model $\langle \epsilon^*=, m_{xy} \rangle$ with $\epsilon^* = \{\langle x, y \rangle \mid x \in m_1, y \in m_1, x \in y\}$. Again by the methods of [4] one shows without difficulty that Sat is in $\Sigma^1_2$. An easy consequence of Theorem 2 is:

**Corollary 1.** If $A$ is a finite system of axioms such that (a) $A$ has exactly one transitive model $m$ and (b) $m$ is denumerable, then $m$ is a $\Delta_2$-set.

**Proof.** Let $n$ be the Gödel number of $A$. From (a), (b) it follows that there exists at least one founded $\epsilon$-function $\lambda$ with $Sat(n, \lambda)$. Obviously by (a), for any founded $\epsilon$-function $\lambda'$ with $Sat(n, \lambda')$, $m_{\lambda'} = m$. By Lemma 4, there is a $\Delta_2$-function $\mu$ which satisfies Definition 2 such that $Sat(n, \mu)$. Again $m_\mu = m$, by (b). Hence, Theorem 2 applies, so $m$ is a $\Delta_2$-set.

With Corollary 1 the first half of the statement of the introduction is proved: if for a given set $m_0$ there exists a finite set of axioms $A$ and a transitive set $m$ satisfying conditions (a), (b) of Corollary 1 and such that $m_0 \in m$, then by Corollary 1 $m_0$ is obviously a $\Delta_2$-set.

**Remark.** It would suffice to replace (b) in Corollary 2 by (b') which states that $m$ can be well-ordered.

D. The statement given in the introduction is proved if we can show that to every $\Delta_2$-set $m_0$ there exists a finite system of axioms $A$ which satisfies (a), (b) of Corollary 1 and whose only transitive model $m$ contains $m_0$ as element. This will be proved in this section. Let $\alpha_1, \ldots, \alpha_n$ be $s_1, \ldots, s_n$-place number-theoretic functions, and let $W(\alpha_1, \ldots, \alpha_n)$ be the set of $(\alpha_1, \ldots, \alpha_n)$-well-orderings (that is, the set of Gödel numbers $e$ of $(\alpha_1, \ldots, \alpha_n)$-recursive functions such that the relation $\{\langle x, y \rangle \mid (e) (x, y) = 0\}$ is a well-ordering). Let, furthermore, $Rec(A)$ be the set of $A$-recursive functions and $P$ a binary $(\alpha_1, \ldots, \alpha_n)$-recursive relation. As proved in [2]:

III. If $(\phi)(\phi \in Rec(W(\alpha_1, \ldots, \alpha_n))) \rightarrow (Et) P(\phi(t), t)$ then $(\phi)(Et) P(\phi(t), t)$.

A consequence of III is:

**Lemma 5.** If $R$ is a recursive relation, then for any function $\alpha$

$$(\phi)(\phi \in Rec(W(\alpha))) \rightarrow (Et) R(\alpha(t), \phi(t)).$$

**Proof.** The implication from right to left is obvious. The implication from
left to right is implied by III, and the fact that the relation \( \{ \langle x, y \rangle \mid R(\tilde{a}(x,y)) \} \) is \( \alpha \)-recursive.

In the proof of Theorem 3 will again perform only the principal steps. The effective construction of certain formulas and the verification of their properties will be surpassed and can easily be supplied by standard methods (e.g., [4], [6]).

\( L(\alpha, \beta) \) will denote the set of Godel numbers of \( \alpha, \beta \) recursive linear orderings and for \( f \in L(\alpha, \beta) \), \( d(f) \) denotes the set \( \{ x \mid (E\psi)((f)^{x,\beta}(x,y) = 0. \vee \{ f \}^{x,\beta}(y,x) = 0) \} \).

If \( f \in L(\alpha, \beta) \) or if \( \xi \) is the characteristic function of a linear ordering, \( \leq_f \) and \( \leq_\xi \) will denote the corresponding order-relation.

**Theorem 3.** If \( m_0 \) is a \( \Delta_2 \)-set, then there exists a finite system of axioms \( A \) such that: (a) \( A \) has exactly one transitive model \( m_0 \), (b) \( m \) is denumerable and \( m_0 \in m \).

**Proof.** We proceed by steps.

1. Since \( m_0 \) is a \( \Delta_2 \)-set, we conclude as in the proof of Theorem 2 (no.1), that there is a characteristic function \( \alpha_0 \) and a recursive relation \( P \) such that (a) \( \alpha_0 \) is the only element in the set \( \{ \xi \mid (E\psi)((\phi)(E\psi)(\Xi(t), \Xi(t), \Phi(t))) \} \), (b) \( \alpha_0 m_0 \) is order-isomorphic to \( \{ \langle x, y \rangle \mid \alpha_0(x, y) = 1 \} \).

2. Let \( m \) be transitive and \( O_0, O_0', J_0, F_0 \) be elements of \( m \) such that (Ex, x2) \( A_0[m] (O_0, O_0', J_0, F_0, x_1, x_2) \) with \( A_0 \) as in Lemma 3. For any set \( m' \), let \( h(m') \) be the smallest transitive set containing \( m' \) as subset (the transitive hull of \( m' \)). Consider the formula \( x = y \vee x \in y \vee x \in \bigcup y \vee x \in \bigcup y \) (to be denoted by \( H_4(x, y) \)). Since \( O_0 \) is an ordinal and \( F \uparrow O_0 = F_0 \) (Lemma 3), \( \{ x \mid H_4[m](x, F_0) = h(O_0) \} \). Similarly there are formulas \( H_i(x, y) \) (i = 1, 2, 3) with \( \{ x \mid H_1[m](x, O_0) = h(O_0) \} \), \( \{ x \mid H_2[m](x, O_0) = h(O_0) \} \), \( \{ x \mid H_3[m](x, J_0) = h(J_0) \} \).

Consequently the formula

\[
(Ex_5, x_6) A_0(x_1, \cdots, x_4, x_5, x_6) \land (s)(H_1(s, x_1) \land \cdots \land H_4(s, x_4))
\]

(to be denoted by \( A_1(x_1, \cdots, x_4) \)) has the property:

\[
A_1[m] (O_0, O_0', J_0, F_0) \rightarrow m = h(O_0) \cup O_0' \cup J_0 \cup F_0).
\]

Obviously, for any ordinal \( v > 0 \), there exists exactly one transitive set \( m \) with \( v \in m \) and (Ex, x2, x3) \( A_1[m] (v, x_1, x_2, x_3) \), namely,

\[
m = h(v) \cup \{ x \times x \} \cup \{ J^{-1} v \} \cup \{ F \uparrow v \}.
\]

(to be denoted in the following by \( h_v \)). Furthermore, for transitive \( m, m = h_v \) for some \( v > 0 \) if and only if (Ex, x4) \( A_1[m] (x_1, \cdots, x_4) \).

3. Consider the class \( C \) of ordinals such that \( v \in C \) if and only if (a) \( F^v \) contains \( \omega \) (the set of natural numbers), \( \omega^2 \) and the sets

\[
\{ \langle x_1, x_2 \rangle \mid x_1 \in \omega \land x_1 + x_2 = x_3 \} \}
\]

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as members, (b) if \( \alpha, \beta \) are one- or two-place number-theoretic functions or subsets of \( \omega \) and \( \alpha, \beta \in F^v \), then \( \text{Rec}(\alpha, \beta) \subseteq F^v \). The class \( C \) is nonvoid.

4. Let \( D \) be the subclass of \( C \) containing exactly those ordinals of \( C \) for which the following holds: \( F^v \) contains number-theoretic functions \( \alpha, \beta \) and functions \( G, H \) having the properties
   
   (a) \( D(H) \subseteq L(\alpha, \beta) \) and \( D(H) \in F^v \),
   
   (b) for \( f \in D(H) \), \( H'f \) is a function with \( D(H'f) = d(f) \), \( W(H'f) \in v \) and \( (x, y)(x \leq_{f} y \land y \leq_{f} x \equiv (H'f)^{y} \land (H'f)^{x}) \),
   
   (c) for \( f \in L(\alpha, \beta) - D(H) \), there is in \( F^v \) a number-theoretic function \( \xi \) with \( (x)(\xi(x + 1) \leq_{f} \xi(x) \land \xi(x) \leq_{f} \xi(x + 1)) \),
   
   (d) for all number-theoretic functions \( \xi \) in \( F^v \), \( (Et) R(\bar{\xi}(t), \bar{\beta}(t), \bar{\xi}(t)) \),
   
   (e) \( D(G) = \omega \), \( W(G) \in v \) and \( \alpha \) is the characteristic function of a linear ordering such that \( (x, y)(x \leq_{\alpha} y \land y \leq_{\alpha} x \equiv G^{x} \in G^{y}) \).

By taking ordinals of sufficiently high cardinality, it is easy to see that the class \( D \) is nonvoid. Let \( m = h_{v} \) for \( v \in D \), and let \( \alpha, \beta, G, H \) be the elements of \( m \) having the properties (a)–(e). From (a), (b), (c) it follows that \( D(H) \) is the set of \( (\alpha, \beta) \)-recursive well-orderings. Since \( F^v \) is closed under recursive operations, \( \text{Rec}(D(H)) \) is a subset of \( F^v \), and hence by (d)

\[ (\xi)(\xi \in \text{Rec}(D(H))) \rightarrow (Et) R(\bar{\xi}(t), \bar{\beta}(t), \bar{\xi}(t)). \]

This, together with Lemma 5, implies \( (\phi)(Et) R(\bar{\alpha}(t), \bar{\beta}(t), \bar{\phi}(t)) \) and hence \( (E\phi)(\phi)(Et) R(\bar{\alpha}(t), \bar{\beta}(t), \bar{\phi}(t)) \). Consequently, \( \alpha \) is the characteristic function \( \alpha_0 \) considered in no. 1. By (e), \( W(G) \) is an ordinal order-isomorphic with \( \{ (x, y) \mid \alpha_0(x, y) = 1 \} \),

that is, \( W(G) = Od'm_0 \). Hence, again by Lemma 3, for every \( v \in D \), \( m_0 \) is an element of \( h_v \).

5. By standard methods one constructs without difficulty a closed formula \( A_2 \) and a formula \( A_3(x) \) having \( x \) as its only free variable such that (a) \( A_2[h_v] \) if and only if \( v \in D \), (b) for \( \mu \in h_v \), \( A_3[h_v][x] \) if and only if \( x \in D \). It is now easy to see that the formula \( (Ex_1, \ldots, x_4) A_1(x_1, \ldots, x_4) \land \ldots A_2 \land \neg(Ex) A_3(x) \) has exactly one transitive model, namely \( h_\delta \), where \( \delta \) is the smallest ordinal in \( D \). Furthermore, the theorem of Löwenheim-Skolem implies that \( h_\delta \) is denumerable \( (h_\delta \text{ can be well-ordered}) \). Since \( m_0 \in h_\delta \), the theorem is proved.

Let \( M \) be the set of Gödel numbers of those closed formulas which admit at least one denumerable transitive model. The methods applied in the proof of Theorem 3 can be used to prove

**Theorem 4.** \( x \in M \) is a complete \( \Sigma^1_2 \)-predicate.

**Proof.** 1. Consider the predicate \( \text{Sat} \) introduced in \( C \) (following Theorem 2). Obviously, \( n \in M \equiv (E\lambda) \text{ Sat}(n, \lambda) \). Since \( \text{Sat} \) is a \( \Pi^1_2 \)-predicate, \( x \in M \) is a \( \Sigma^1_2 \)-predicate.
Let \( P \) be a predicate with \( P(n) \equiv (E\phi)(\psi)(Et)P_1(n, \phi(t), \psi(t)) \) with \( P_1 \) recursive, and let \( C \) be the same class of ordinals as defined in no. 3 in the proof of Theorem 3. Now for a given natural number \( n \), let \( D_n \) be the subclass of \( C \) containing exactly those ordinals \( \nu \) of \( C \) for which the following holds: \( F^*\nu \) contains a number-theoretic function \( \alpha \) and functions \( G, H \) having the properties

(a) \( D(H) \subseteq L(\alpha) \) and \( D(H) \in F^*\nu \),
(b) for \( f \in D(H) \), \( H^f \) is a function with \( D(H^f) = d(f) \), \( W(H^f) \in \nu \) and \((x, y)(x \leq_f y \land y \leq_f x \iff (H^f)^{xy})\),
(c) for \( f \in L(\alpha) - D(H) \), there is in \( F^*\nu \) a number-theoretic function \( \xi \) with \( (x)(\xi(x + 1) \leq_f \xi(x) \land \xi(x) \leq_f \xi(x + 1)) \),
(d) for all number-theoretic functions \( \xi \) in \( F^*\nu \), \( (Et) P_1(\alpha(t), \xi(t)) \). Using the same arguments as in no. 4, proof of Theorem 4, we conclude that \( D_n \) is non void if and only if \( P(n) \). Furthermore, as in no. 5 of the proof, a closed formula \( G_n \) can be given effectively which has the property: for a transitive set \( m \), \( G_n \) if and only if \( D_n \) is nonvoid and \( m = \alpha_v \) where \( v \) is the smallest ordinal in \( D_n \). Clearly, the function \( g \) which attributes to every \( n \) the Gödel number \( g(n) \) of \( G_n \) is recursive. Since obviously \( g(n) \in L(\alpha) \equiv P(n) \), the theorem follows.

E. In this section, another characterization of the \( \Delta_2 \)-sets is given. In order to avoid a mere repetition of the arguments given in previous sections, we will state the definitions and theorems and give only some indications. We start with some notations and definitions. Let \( m \) be a transitive set in what follows. A list \( m_1, m_2, \ldots, m_k \) of elements of \( m \) and of formulas \( G_i(x_0, x_1, \ldots, x_k, y) (i = 1, \ldots, s) \) whose only free variables are \( x_0, x_1, \ldots, x_k, y \) will be called an \( m \)-list.

Definition 3. For a given \( m \)-list \( m_1, \ldots, m_k, G_1, \ldots, G_s \) one can define by transfinite induction functions \( F_1, \ldots, F_s \) as follows:

\[
F_0^i = i \quad (i = 1, \ldots, s),
\]
and

\[
F_v^i = \{x \mid G_i[m_v](m, m_1, \ldots, m_k, x) \land x \in m_v^i\}
\]

for \( \nu > 0 \). Here, \( m_v \) and \( m_v^i \) are given by

(a) \( m_v = m \cup \{m\} \cup_k (F_k \uparrow \nu) \cup (F_k \uparrow \nu) \cup (\bigcup F_k \uparrow \nu), \)
(b) \( m_v^i = m \cup \{m\} \cup_k (F_k \uparrow \nu) \cup (\bigcup F_k \uparrow \nu) \cup (\bigcup (\bigcup F_k \uparrow \nu)). \)

The functions so defined are called the functions associated with the given \( m \)-list.

It is an immediate consequence of Definition 3 that the sets \( m_v, m_v^i \) considered there are transitive.

Definition 4. An \( m \)-list is said to be terminating if there is an ordinal \( \mu > 0 \) such that the associated functions \( F_1, \ldots, F_s \) have the following properties:

(a) for \( \lambda < \mu, \bigcup_i F_i^\lambda \neq 0, (b) \bigcup_i F_i^\mu = 0. \)

We say that the given \( m \)-list terminates at \( \mu \), and call the set \( m_\mu \) given by Definition 3, (a) the characteristic model of the list.
Definition 5. A set $m'$ is called $m$-definable, if there is an ordinal $v > 0$ and an $m$-list terminating at $v$ such that the associated functions $F_i$ ($i = 1, \ldots, s$) satisfy the condition $m' \in \bigcup F^{\prime}_i v$. We also say that $m'$ is $m$-defined by the given list. If $m = 0$, $m'$ is simply called definable.

In the following lemma, some properties of the concepts defined above are described.

Lemma 7. 1. If an $m$-list is terminating, then the ordinal at which it terminates is uniquely determined.

2. To any given $m$-list $m_1, \ldots, m_k, G_1, \ldots, G_s$ one can find effectively a formula $G(x_0, x_1, \ldots, x_k)$ with the property: for a transitive set $m'$ with $m \cup \{m\} \in m'$, $G[m'] (m, m_1, \ldots, m_k)$ if and only if the $m$-list terminates at an ordinal $p$ and $m'$ is the characteristic model $m^p$. If especially $m = 0$, then $G$ can be chosen so as to be closed.

Proof. Part 1 is an immediate consequence of Definitions 3, 4. The proof of Part 2 is based on the same standard techniques used in A–D and developed e.g. in [6].

Definition 7. The formula $G$ in Lemma 7 associated with the given $m$-list $m_1, \ldots, m_k, G_1, \ldots, G_s$ will be called the characteristic formula of the given list.

The connection between the concepts just defined and the considerations in A–D is given by

Theorem 5. A set $m$ is definable if and only if it is a $\Delta^2_2$-set.

Proof. Let $m$ be definable by a given 0-list whose characteristic formula and characteristic model are $G$ and $m_\mu$, respectively. Obviously, $m_\mu$ is denumerable. Since $m_\mu$ is the unique transitive model of $G$ by Lemma 7 and $m \in m_\mu$, $m$ is a $\Delta^2_2$-set by Theorem 2. The proof of the converse statement is tedious but does not involve anything new; for this reason we content ourselves with some indications. One can show that there is a special 0-list whose first associated function $F_1$ has the properties: (a) to every ordinal $v$ there is an ordinal $h(v)$ not greater than $v$ such that $F_1^v = F_1 h(v)$, (b) $W(F_1) = F$, (c) if $v_1 \leq v_2$ then $h(v_1) \leq h(v_2)$. It is not difficult now to enlarge this special 0-list to a second one which contains among its associated functions the function $F_1$ and which terminates at a certain ordinal $\mu$. The ordinal $\mu$ will be the smallest for which a certain set of conditions are satisfied, which are quite similar to (a)–(e) in no. 4, proof of Theorem 4; $m$ will then be a member of $W(F_1^\mu)$

Since an 0-list is completely determined by its characteristic formula (Lemma 7), we can equally characterize the given 0-list by the Gödel number of its characteristic formula. Let $M'$ be the set of Gödel numbers whose corresponding 0-lists are terminating. By reasoning which parallels the proof of Theorem 4 we obtain:
Theorem 6. The predicate $x \in M'$ is a complete $\Sigma^1_2$-predicate.

Remarks and problems. The notion of $m$-definability given here is in a loose sense somewhat analogous to the notion of computation in the theory of relative recursiveness: while in the latter case the process of computation terminates after a finite number of steps or continues a denumerable number of times, the process of transfinite induction connected with an $m$-list terminates at a certain ordinal $\mu$ or runs through the class of all ordinals. In this connection, it is worth noting that the following statement can be proved: if $m$, $m_1$ are transitive, and if $m_1$ is $m$-definable, then any $m_1$-definable set is also $m$-definable. It is also interesting that several schemes of transfinite induction which one could choose as alternatives to that one given by Definition 3 turn out to be equivalent with the latter.

We close by considering two problems. Assume for simplicity the axiom of choice. If the transitive hull $h(m)$ of $m$ is a $\Delta_2$-set, we cannot necessarily conclude that $m$ is also a $\Delta_2$-set, as simple examples show. However, $m$ is a $\Delta_2$-set if and only if $h(m \cup \{m\})$ is $\Delta_2$, as is easily seen. One can ask if the result obtained in A–D can be sharpened in the following way: $m$ is $\Delta_2$ if and only if there is a formula $G$ whose only transitive model is $h(m \cup \{m\})$. Another question arises in connection with the considerations of E. As a consequence of Lemma 7, if $m'$ is $m$-definable, there are elements $m_1, \ldots, m_k$ in $m$ and a formula $G(x_0, x_1, \ldots, x_k)$ with the property: there is exactly one transitive set $m''$ having $m$, $m_1$, $\ldots$, $m_k$ as members such that $G[m''](m, m_1, \ldots, m_k)$. The obvious question is: does the converse hold? If $m = 0$, then the answer is affirmative as a consequence of Corollary 1 and Theorem 5.

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References


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