

λ -CONTINUOUS MARKOV CHAINS⁽¹⁾

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0. Introduction and summary. We call a Markov operator P which has a representation $P(x, A) = \int_A p(x, y) \lambda(dy)$ with $p(x, y)$ bivariate measurable a λ -continuous Markov operator. It is a special kind of λ -measurable Markov operator of E. Hopf. If the state space is discrete, every Markov operator is λ -continuous where λ assigns measure 1 to every state. In §I various definitions and preliminaries are given. In §II the existence of invariant measures for a λ -continuous conservative P is proved. It is shown that the space is decomposed into at most countably many indecomposable closed sets C_1, C_2, \dots . For each C_i there is a σ -finite invariant measure μ_i which is equivalent to λ on C_i and vanishes outside C_i . Every invariant measure is shown to be of the form $\sum \alpha_i \mu_i$. In §III convergence properties of $\sum_{n=1}^N p^n(z, x) / \sum_{n=1}^N p^n(z, y)$ are studied. It is shown that for a conservative ergodic P the limit of the ratio is $f(x)/f(y)$ where f is the derivative of the invariant measure with respect to λ . All these theorems are well known for a discrete state space (cf. [2, 1.9]).

§IV treats laws of large numbers. The approach used here is similar to that of Harris and Robbins [7]. It contains generalizations of theorems of Chung for discrete state spaces (cf. [2, 1.15]). The theory of λ -measurable Markov operators is extensively used here.

§VI is devoted to some new results on λ -measurable Markov operators which are used in this paper. In §V the theory of Martin boundaries is investigated. The kernel $K(x, y)$ used here is

$$K(x, y) = \lim_{N \rightarrow +\infty} \frac{\sum_{n=1}^N p^n(x, y)}{\int \pi(dz) \sum_{n=1}^N p^n(z, y)},$$

where π is a finite measure equivalent to λ . By using this kernel the space X is embedded in a compact Hausdorff space \tilde{X} . Every π -integrable invariant function h is shown to have the representation $h = \int \tilde{y} \tilde{\eta}(d\tilde{y})$ for a Baire measure $\tilde{\eta}$ on \tilde{X} . The techniques used here are essentially extensions of that of G. A. Hunt [9]. The space X is assumed to be irreducible, i.e., X is the support of measure $\sum_{n=1}^{\infty} 2^{-n} \pi P^n$.

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I. Preliminaries. Let X be a nonempty set, \mathcal{X} , a σ -algebra of subsets of X and λ , a σ -finite measure on X . Let $p(x, y)$ be an $\mathcal{X} \times \mathcal{X}$ measurable function defined on $X \times X$ satisfying the following conditions:

1. $p(x, y) \geq 0$ for $(\lambda \times \lambda)$ almost all (x, y) ,
2. for (λ) almost all x , $\int p(x, y) \lambda(dy) \leq 1$.

Let $L_\infty(\lambda)$ be the collection of λ -essentially bounded functions and $\mathcal{A}(\lambda)$, the collection of all finite real valued, countably additive functions on \mathcal{X} which are absolutely continuous with respect to λ . Let $\mathcal{A}^+(\lambda)$ be the collection of all non-negative elements of $\mathcal{A}(\lambda)$. For any $f \in L_\infty(\lambda)$ define Pf by

$$(1.1) \quad Pf(x) = \int p(x, y)f(y) \lambda(dy).$$

For any $\nu \in \mathcal{A}(\lambda)$ define νP by

$$(1.2) \quad \nu P(A) = \int \nu(dx) \int_A p(x, y) \lambda(dy).$$

The operator P here is a special case of λ -measurable Markov operators of E. Hopf (cf. Appendix). We shall call it a λ -continuous Markov operator and $(X, \mathcal{X}, \lambda)$, the state space of P . The iterates of P are then given by

$$P^n f(x) = \int p^{(n)}(x, y)f(y) \lambda(dy),$$

$$\nu P^n(A) = \int \nu(dx) \int_A p^n(x, y) \lambda(dy),$$

where $p^{(n)}(x, y)$ are defined inductively by

$$p^{(n)}(x, y) = \int p^{(n-1)}(x, z)p(z, y) \lambda(dz).$$

The function p shall be called the density function of the operator P . (1.1), (1.2) remain meaningful for non-negative f not necessarily λ -essentially bounded and non-negative, σ -finite measure ν . All subsets of X discussed in this paper are elements of \mathcal{X} and all functions on X are \mathcal{X} -measurable functions. For two sets A, B , $A \subset B$, $A = B$ mean that $\lambda(A - B) = 0$, $\lambda(A \Delta B) = 0$ respectively. For two functions f, g on X , $f = g$, $f \leq g$ mean that the equality and the inequality, respectively, are satisfied except on a λ -null set. For any set A , 1_A represents the function which equals to 1 on A and 0 on the complement A' of A . For any function f and any additive set function ν define

$$I_A f(x) = 1_A(x)f(x),$$

$$\nu I_A(B) = \nu(A \cap B).$$

I_A is a λ -measurable Markov operator. Define

$$(1.3) \quad P_A^* = \sum_{n=0}^{\infty} (I_A \cdot P)^n,$$

$$(1.4) \quad P_A = \sum_{n=0}^{\infty} P(I_A \cdot P)^n.$$

P_A^* , P_A operating on either non-negative functions or measures have well-defined meanings. In our case of λ -continuous P ,

$$P_A f(x) = \int p_A(x, y) f(y) \lambda(dy),$$

$$\nu P_A(B) = \int \nu(dx) \int_B p_A(x, y) \lambda(dy),$$

where

$$(1.5) \quad p_A(x, y) = \sum_{n=1}^{\infty} p_{A,n}(x, y) \text{ with}$$

$$p_{A,n}(x, y) = \int_{A'} \cdots \int_{A'} p(x, z_1) p(z_1, z_2) \cdots p(z_{n-1}, y) \lambda(dz_1) \cdots \lambda(dz_{n-1}).$$

However $P_A^* I_A$ and $P_A I_A$ are λ -measurable Markov operators and $P_A I_A$ has density function $p_A(x, y) 1_A(y)$ [Appendix, Theorem 6.1]. Following E. Hopf and J. Feldman we call a set A a *conservative* set if, for every λ -non-null subset B of A , $P_B 1_B = 1$ on B . Let C be the largest conservative set which is then called the conservative part of X . $D = X - C$ is called the dissipative part of X . A set A is said to be *transient* if there is a non-negative number $q < 1$ such that $P_A 1_A \leq q$ on A . Then D is the union of at most countably many transient sets [Appendix, Theorem 6.3]. For any finite measure ν which is equivalent to λ , $\sum_{n=1}^{\infty} \nu P^n(A) = \infty$ if $A \subset C$ and A is λ -non-null and $\sum_{n=1}^{\infty} \nu P^n$ is σ -finite on D . A set A is *closed* if $P 1_A = 1$ on A . C is closed. It follows that $\sum_{n=1}^{\infty} p^n(x, y) < \infty$ for $(\lambda \times \lambda)$ almost all $(x, y) \in X \times D$, in particular, $\sum_{n=1}^{\infty} p^n(x, y) = 0$ a.e. $(\lambda \times \lambda)$ on $C \times D$. The collection of all closed subsets of C form a σ -algebra of subsets of C which we shall designate by \mathcal{C} . P is *conservative* if $C = X$. P is *dissipative* if $D = X$. An extended real valued non-negative function h is said to be *P-excessive* if $Ph \leq h$, *P-invariant* if equality holds. h is a *P-potential* if $P^n h \downarrow 0$ a.e. (λ) . A σ -finite measure μ is *P-excessive* if $\mu P \leq \mu$, *P-invariant* if equality holds. μ is a *P-potential* if $d\mu P^n / d\lambda \downarrow 0$ a.e. (λ) . Let

$$(1.6) \quad g(x, y) = \sum_{n=1}^{\infty} p^n(x, y).$$

Then $P^n g(\cdot, y) = \sum_{k=n+1}^{\infty} P^k(\cdot, y)$. Hence for (λ) almost all y , $g(\cdot, y)$ is P -excessive, in particular, if $y \in D$, $g(\cdot, y)$ is a P -potential.

For a finite valued P -excessive function h we define

$$(1.7) \quad p_h(x, y) = \frac{1}{h(x)} p(x, y).$$

p_h is well defined, a.e., $(\lambda_h \times \lambda_h)$ where λ_h is given by

$$\lambda_h(A) = \int_A h(x) \lambda(dx).$$

Since $\int p_h(x, y) \lambda_h(dy) = (1/h(x)) \int p(x, y) h(y) \lambda(dy) \leq 1$, a λ_h -continuous Markov operator P_h may be defined by:

$$(1.8) \quad P_h f(x) = \int p_h(x, y) f(y) \lambda_h(dy) = \frac{1}{h(x)} \int p(x, y) h(y) f(y) \lambda(dy),$$

$$(1.9) \quad \nu P_h(A) = \int \nu(dx) \int_A p_h(x, y) \lambda_h(dy) = \int \nu(dx) \frac{1}{h(x)} \int_A p(x, y) h(y) \lambda(dy).$$

The iterates P_h^n are then given by

$$P_h^n f(x) = \int p_h^{(n)}(x, y) f(y) \lambda_h(dy),$$

$$\nu P_h^n(A) = \int \nu(dx) \int_A p_h^{(n)}(x, y) \lambda_h(dy),$$

where

$$p_h^{(n)}(x, y) = \frac{1}{h(x)} p^{(n)}(x, y).$$

For the operator P_h it is easy to see that the space still has the same decomposition $C \cup D$ although sets of λ_h measure 0 have no significance.

II. Invariant measures.

THEOREM 2.1. *Let P be λ -continuous and \mathcal{C} be the σ -algebra of all closed subsets of the conservative part C . Then λ is purely atomic on \mathcal{C} . Let \mathcal{C} be generated by distinct atoms C_1, C_2, \dots and $P_i = P I_{C_i}$; then $P_i P_j = 0$ if $i \neq j$ and $I_C P = I_C(P_1 + P_2 + \dots)$.*

Proof. Suppose that \mathcal{C} contained a nonatomic set A . We may assume $0 < \lambda(A) < \infty$. For every positive integer n there is a partition $E_1^{(n)}, \dots, E_{k_n}^{(n)}$ of A such that every $E_i^{(n)}$ belongs to \mathcal{C} and $\lambda(E_i^{(n)}) < n^{-1}$. We may assume that each $E_i^{(n+1)}$ is a subset of $E_j^{(n)}$ for some j . Let $A_n = \bigcup_i [E_i^{(n)} \times E_i^{(n)}]$. Since both $E_i^{(n)}$ and $A - E_i^{(n)}$ are closed sets, $p(x, y) = 0$ a.e. $(\lambda \times \lambda)$ on $A \times A - A_n$. Since A_n is monotonically decreasing and $\lambda \times \lambda(A_n) \leq n^{-1} \lambda(A) \rightarrow 0$ as $n \rightarrow \infty$, $p(x, y) = 0$

a.e. $(\lambda \times \lambda)$ on $A \times A$ which contradicts the fact that $\int_A \int_A p(x, y) \lambda(dx) \lambda(dy) = \lambda(A) > 0$.

For every closed set A , $I_A P = I_A P I_A$, hence $P_i P_j = P I_{C_i} P I_{C_j} = P I_{C_i} I_{C_j} P I_{C_j} = 0$ and $I_C P = I_C P I_C = I_C (P_1 + P_2 + \dots)$.

A conservative Markov operator P is said to be *ergodic* if the only λ -non-null set in \mathcal{C} is C .

THEOREM 2.2. *If a λ -continuous Markov operator P is conservative and ergodic then P possesses a σ -finite invariant measure μ which is unique up to a constant multiple. Furthermore μ is equivalent to λ .*

Proof. The proof of the existence of invariant measure shall be essentially that of T. E. Harris adapted to the present situation [6]. Since P is conservative $\int p(x, y) \lambda(dy) > 0$ implies that $\sum_{n=1}^{\infty} p^n(x, y) = \infty$ for (λ) almost all y on a non-null closed set. Since X is the only non-null closed set, $\sum_{n=1}^{\infty} p^n(x, y) = \infty$ for (λ) almost all y . Hence $\sum_{n=1}^{\infty} p^{(n)}(x, y) = \infty$ a.e. $(\lambda \times \lambda)$. Let M, a be two positive numbers with $2^{-1} < a < 1$. We shall show that there is a set A with $0 < \lambda(A) < \infty$ and a positive integer N such that for (λ) almost all $x \in A$,

$$(2.1) \quad \lambda \left[y: \sum_{n=1}^N p^{(n)}(x, y) > M, y \in A \right] > a\lambda(A).$$

Let E be an arbitrary set with $0 < \lambda(E) < \infty$. Let

$$f_n(x) = \lambda \left[y: \sum_{i=1}^n p^{(i)}(x, y) > M \right] \cap E.$$

Then $f_n \uparrow \lambda(E)$ a.e. (λ) . Determine N so that $\lambda[x: f_N(x) > (4/5)\lambda(E), x \in E] > (4/5)\lambda(E)$. Let $A = [x: f_N(x) > (4/5)\lambda(E), x \in E]$. Then A satisfies (2.1). Consider P_A and p_A given by (1.4) and (1.5) respectively. $P_A I_A$ is a λ -continuous Markov operator with density function $r(x, y) = p_A(x, y) 1_A(y)$. Let $r^n(x, y)$ be the n th iterate of $r(x, y)$,

$$q(x, y) = N^{-1} \sum_{n=1}^N r^{(n)}(x, y)$$

and Q be the corresponding λ -continuous Markov operator with density function $q(x, y)$. (2.1) implies that for (λ) almost all $x \in A$,

$$(2.2) \quad \lambda[y: q(x, y) > M/N, y \in A] > a\lambda(A).$$

(2.2) implies that there is a probability η such that

$$\text{ess sup}_x |Q^n 1_E(x) - \eta(E)| \rightarrow 0 \text{ uniformly in } E$$

(see Appendix of [6]). η is $P_A I_A$ -invariant. Let $\mu = \eta P_A$. Then

$$\mu P = \eta P_A P = \eta P_A I_A P + \eta P_A I_{A^c} P = \eta P + \eta \sum_{n=1}^{\infty} P(I_A \cdot P)^n = \eta P_A = \mu.$$

Hence μ is P -invariant. Since P is ergodic, μ is σ -finite by Theorem 6.4 of Appendix, μ has density with respect to $\lambda: d\mu/d\lambda = \int \eta(dx) p_A(x, \cdot)$. The support of an invariant measure is necessarily a closed set. Hence the support of μ is X and μ is equivalent to λ . Uniqueness of μ follows from Corollary 6.2 of Appendix.

THEOREM 2.3. *Let P be λ -continuous and conservative. Let C_1, C_2, \dots be distinct atoms which generate \mathcal{C} . For each C_i there is a σ -finite P -invariant measure μ_i which is equivalent to λ_{C_i} and every P -invariant measure is of the form $\sum \alpha_i \mu_i$.*

Proof. Let $P_i = P I_{C_i}$. It follows from Theorem 2.1 that there is a P_i -invariant measure which is equivalent to λ_{C_i} which is unique up to a constant multiple. Now $\mu_i P = \mu_i I_{C_i} P = \mu P_i = \mu_i$, hence μ_i is also P -invariant. Conversely, if μ is P -invariant, let $v_i = \mu I_{C_i}$; then $v_i P_i = \mu I_{C_i} P = \mu P I_{C_i} = \mu I_{C_i} = v_i$. v_i is P_i -invariant, therefore a constant multiple of μ_i . Hence μ is of the form $\sum \alpha_i \mu_i$.

Let

$$(2.3) \quad B = [P_C 1_C > 0] - C.$$

J. Feldmann showed that every excessive measure for a λ -measurable Markov operator P is necessarily absolutely continuous to λ_{X-B} . In our case of λ -continuous Markov operators we have the following corollary.

COROLLARY 2.1. *If P is a λ -continuous Markov operator then there exists an excessive measure μ which is equivalent to λ_{X-B} where B is given by (2.3).*

Proof. Let $\pi \in \mathcal{A}^+(\lambda)$ and be equivalent to $\lambda_{X-(B \cup C)}$. Let $\eta = \sum_{n=0}^{\infty} \pi P^n$. Then η is equivalent to $\lambda_{X-(B \cup C)}$ and σ -finite. Let $\mu = \eta + \sum \alpha_i \mu_i$, $\alpha_i > 0$ where μ_i are invariant measures described in Theorem 2.3. μ is the desired excessive measure.

III. Ratio ergodic theorems. For a λ -measurable Markov operator P and $\nu, \eta \in \mathcal{A}^+(\lambda)$ the ratio ergodic theorem states that

$$\frac{\sum_{n=1}^N d\nu P^n / d\lambda}{\sum_{n=1}^N d\eta P^n / d\lambda}$$

converge a.e. (λ) on the set where the denominator is positive. The theorem was conjectured by E. Hopf and proved by Chacon and Ornstein [1]. The limit function was identified by J. Neveu [10]. For our case of λ -continuous operator P , because of \mathcal{C} being atomic the ratio ergodic theorem may take the form of

Theorem 3.1. But, first, we shall introduce functions u_{C_i} . Let C_i be an atom of C . Let $u_{C_i} = P_{C_i} 1_{C_i}$. u_{C_i} is the smallest excessive function which equals to 1 on C_i (Appendix, Corollary 6.1).

THEOREM 3.1. Let P be λ -continuous and π , a finite measure equivalent to λ ; then for $(\lambda \times \lambda)$ almost all (x, z)

$$(3.1) \quad \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N p^n(x, z)}{\int \pi(dy) \sum_{n=1}^N p^n(y, z)} = \frac{u_{C_i}(x)}{\int \pi(dy) u_{C_i}(y)} \quad \text{if } z \in C_i$$

$$= \frac{g(x, z)}{\int \pi(dy) g(y, z)} \quad \text{if } z \in D \cap \left[\bigcup_{n=1}^{\infty} \text{supp } \pi P^n \right].$$

We shall define kernel $K(x, z)$ by

$$(3.2) \quad K(x, z) = \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N p^n(x, z)}{\int \pi(dy) \sum_{n=1}^N p^n(y, z)} .$$

This kernel shall be used to obtain the exit boundary of P .

Because of the existence of invariant measures for conservative λ -continuous Markov operators we are able to derive the following theorem.

THEOREM 3.2. Let P be a λ -continuous Markov operator, C_i , an atom of \mathcal{C} , μ a P -invariant measure which is equivalent to $\lambda|_{C_i}$, and $f = d\mu/d\lambda$; then for $(\lambda \times \lambda \times \lambda)$ almost all $(x, y, z) \in C_i \times C_i \times C_i$,

$$(3.3) \quad \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N p^n(z, x)}{\sum_{n=1}^N p^n(z, y)} = \frac{f(x)}{f(y)} .$$

Proof. It is sufficient to prove the theorem for P being conservative and ergodic and $C_i = X$. We shall define another λ -continuous Markov Q which we shall call the μ -reverse of P (or just the reverse of P for in this case a P -excessive measure is essentially unique). For any $g \in L_{\infty}(\lambda)$ define Qg to be a function satisfying the following equality for every $h \in L_1(\mu)$:

$$\int h(Qg) d\mu = \int (Ph)g d\mu .$$

In other words, if $dv = g d\mu$, Qg is defined by $dvP = Qg d\mu$. The P -invariance

and λ -equivalence of μ imply that Q is a well-defined λ -measurable Markov operator (cf. Appendix). This Q has been called an "inverse" by S. Kakutani and " μ -adjoint" by J. Feldman. It follows from Theorem 3.1 [5] that Q is conservative. Q is also ergodic (Appendix, Lemma 6.2). Since P is λ -continuous, Q is also λ -continuous with density function $q(x, y)$:

$$(3.4) \quad q(x, y) = f(y) p(y, x) \frac{1}{f(x)}.$$

The iterates $q^{(n)}(x, y)$ are given by

$$(3.5) \quad q^{(n)}(x, y) = f(y) p^{(n)}(y, x) \frac{1}{f(x)}.$$

Applying (3.1) to $q(x, y)$ we have for $(\lambda \times \lambda \times \lambda)$ almost all (x, y, z) ,

$$\begin{aligned} 1 &= \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N q^{(n)}(x, z)}{\sum_{n=1}^N q^{(n)}(y, z)} \\ &= \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N p^{(n)}(z, x)}{\sum_{n=1}^N p^{(n)}(z, y)} \frac{f(y)}{f(x)} \end{aligned}$$

and (3.3) follows immediately.

COROLLARY 3.1. *Let P be λ -continuous, conservative and ergodic. Let μ be P -invariant and $f = d\mu/d\lambda$. If μ is finite and normalized to be a probability measure then for $(\lambda \times \lambda)$ almost all (x, y) ,*

$$(3.6) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N p^{(n)}(x, y) = f(y),$$

and for (λ) almost all x ,

$$(3.7) \quad \lim_{N \rightarrow \infty} \int \left| \frac{1}{N} \sum_{n=1}^N p^{(n)}(x, y) - f(y) \right| \lambda(dy) = 0.$$

If μ is not finite then for $(\lambda \times \lambda)$ almost all (x, y)

$$(3.8) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N p^{(n)}(x, y) = 0.$$

Proof. If μ is a probability measure, Theorem 3.1 implies that

$$\lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N p^{(n)}(x, y)}{Nf(y)} = 1,$$

hence (3.6) follows immediately. Since $p^n(x, y)$ are non-negative and

$$\int N^{-1} \left[\sum_{n=1}^N p^n(x, y) \right] \lambda(dy) = \int f(y) \lambda(dy),$$

(3.6) implies (3.7). If μ is not finite there is an increasing sequence $\{E_n\}$ of sets such that $\bigcup_n E_n = X$ and $\mu(E_n) < \infty$ for every n . The general ergodic theorem implies that for $(\lambda \times \lambda)$ almost all (x, y) ,

$$(3.9) \quad \lim_{N \rightarrow \infty} \sum_{n=1}^N p^n(x, y) \Big| \sum_{n=1}^N \frac{d\mu_{E_k} P^n}{d\lambda}(y) = \frac{1}{\mu(E_k)}.$$

Now for each n, k , $\mu_{E_k} P^n \leq \mu$. Hence every term in the summation appearing as the denominator of the left-hand side of (3.9) is $\leq f(y)$. Hence

$$\liminf_{N \rightarrow \infty} \frac{\sum_{n=1}^N p^n(x, y)}{Nf(y)} \leq \frac{1}{\mu(E_k)}.$$

Since $\mu(E_k) \rightarrow \infty$, it follows that

$$\lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N p^{(n)}(x, y)}{Nf(y)} = 0$$

and (3.8) is proved.

The following theorem follows immediately from Theorem 6.5 of Appendix.

THEOREM 3.3. *Let P be λ -continuous and conservative and C_1, C_2, \dots are the atoms of \mathcal{C} . Let μ be an invariant measure which is equivalent to λ and $\mu_i = \mu|_{C_i}$. Then if $f, g \in \bigcap_i L_1(\mu_i)$, $g > 0$ a.e. (λ) then there exists limit*

$$\lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N P^n f}{\sum_{n=1}^N P^n g} = \sum \frac{a_i}{b_i} 1_{C_i} \quad \text{a.e. } (\lambda),$$

where $a_i = \int f d\mu_i$, $b_i = \int h d\mu_i$. The limit is independent of particular μ chosen. Furthermore there exists

$$\lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{n=1}^N P^n f \right) = \sum a'_i 1_{C_i} \quad \text{a.e. } (\lambda),$$

where

$$\begin{aligned} a'_i &= a_i / \mu_i(C_i) & \text{if } \mu(C_i) < \infty, \\ a'_i &= 0 & \text{if } \mu(C_i) = \infty. \end{aligned}$$

Again the limit is independent of the particular μ chosen.

IV. Laws of large numbers. In this section we shall assume that $\int p(x, y)\lambda(dy) = 1$ for (λ) almost all x .

Let Ω be the infinite product space $\prod_{n=0}^{\infty} X$ and \mathcal{F} , the product σ -algebra $\prod_{n=0}^{\infty} \mathcal{X}$ of subsets of Ω . Let X_n be the function on Ω to X defined by $X_n(w) = x_n$ if $w = \{x_0, x_1, x_2, \dots\}$. Let T be the shift transformation on Ω to Ω defined by $Tw = \{x_1, x_2, \dots\}$ if $w = \{x_0, x_1, x_2, \dots\}$. For any function f defined on Ω we define function Tf by $Tf(w) = f(Tw)$. If f is a function on X then $Tf(X_n) = f(X_{n+1})$. For any measure ϕ on \mathcal{F} define measure ϕT by $\langle \phi T, f \rangle = \langle \phi, Tf \rangle$ for every non-negative \mathcal{F} -measurable function f ($\langle \phi, f \rangle = \int f d\phi$ if the integral is well defined).

A function $\mathcal{P}(x, E)$, $x \in X$, $E \in \mathcal{F}$ is defined in the following manner. If $E = [X_0 \in A_0, \dots, X_n \in A_n]$ where $A_i \in \mathcal{X}$,

$$(4.1) \quad \mathcal{P}(x, E) = \int_{A_0} \lambda(dx_0) \int_{A_1} \lambda(dx_1) \cdots \int_{A_n} \lambda(dx_n) p(x, x_0) p(x_0, x_1) \cdots p(x_{n-1}, x_n).$$

Then the definition of $\mathcal{P}(x, E)$ is extended to arbitrary $E \in \mathcal{F}$ by a well-known measure extension argument. We have that for (λ) almost all x $\mathcal{P}(x, \cdot)$ is a probability measure on \mathcal{F} and for every $E \in \mathcal{F}$ $\mathcal{P}(\cdot, E)$ is \mathcal{X} -measurable. For a real valued, bounded or non-negative \mathcal{F} -measurable function f we define

$$(4.2) \quad \mathcal{P}f = \int \mathcal{P}(\cdot, dw') f(\cdot, w').$$

In the above we write $w = \{x_0, x_1, \dots\}$ as a pair (x_0, w') where $w' = Tw$. $\mathcal{P}f$ is \mathcal{X} -measurable and

$$(4.3) \quad \mathcal{P}T^n f = P^n \mathcal{P}f.$$

(4.3) may be first proved for f being of the form 1_E where $E = [X_0 \in A_0, \dots, X_n \in A_n]$ and then extended to arbitrary f . If f is of the form $f(X_n)$ with f being \mathcal{X} -measurable then $\mathcal{P}f(X_n) = P f$. For any measure η absolutely continuous with respect to λ , a measure η is defined on \mathcal{F} by letting

$$(4.4) \quad \langle \eta, f \rangle = \langle \eta, \mathcal{P}f \rangle = \int \eta(dx) \int \mathcal{P}(x, dw') f(x, w')$$

for every non-negative \mathcal{F} -measurable function f . η is σ -finite. Since $\langle \eta, f \rangle = 0$ if and only if $\mathcal{P}f = 0$ a.e. (η) , $\langle \lambda, f \rangle = 0$ implies that $\langle \eta, f \rangle = 0$ so that η is absolutely continuous with respect to λ . Since

$$\langle \lambda T, f \rangle = \langle \lambda, Tf \rangle = \langle \lambda, \mathcal{P}Tf \rangle = \langle \lambda, P\mathcal{P}f \rangle = \langle \lambda P, \mathcal{P}f \rangle$$

and since λP is absolutely continuous with respect to λ , λT is absolutely continuous with respect to λ . Hence T is a λ -measurable Markov operator acting on $(\Omega, \mathcal{F}, \lambda)$ and the theory of λ -measurable Markov operator is applicable.

We shall begin with the ergodic decomposition of Ω for T . In the following all subsets of Ω are understood to be \mathcal{F} -measurable sets and two sets are equal if they are equal modulo λ -null sets.

LEMMA 4.1. *Let $X = C \cup D$ be the ergodic decomposition of X for P ; then $C = \prod_{n=0}^{\infty} C$ is the conservative part of Ω for T and $\Omega - C$ is the dissipative part of Ω .*

Proof. To show that C is T -conservative we shall show that, for every T -transient set E , $\lambda(E \cap C) = 0$. If E is T -transient then there is a number c for which $\sum_{n=0}^{\infty} T^n 1_E \leq c$ a.e. (λ). Let $f = \mathcal{P}1_E$. Then $P^n f = \mathcal{P}T^n 1_E$ so that $\sum_{n=0}^{\infty} P^n f = \mathcal{P}(\sum_{n=0}^{\infty} T^n 1_E) \leq c$ a.e. (λ). Hence $f = 0$ a.e. (λ) on C . Now

$$\lambda(E \cap C) = \int_C \mathcal{P}1_{E \cap C} d\lambda \leq \int_C \mathcal{P}1_E d\lambda = 0.$$

To show that $\Omega - C$ is T -dissipative it is sufficient to show that the set $[X_n \in D]$ is dissipative for $n = 0, 1, 2, \dots$. Let A be P -transient (therefore $A \subset D$) then $\sum_{n=0}^{\infty} P^n 1_A \leq d$ a.e. (λ) for some number d . Let η be a finite measure equivalent to λ . Then η is equivalent to λ . Now

$$\left\langle \eta, \sum_{k=0}^{\infty} T^k 1_{[X_n \in A]} \right\rangle = \left\langle \eta, \sum_{k=0}^{\infty} P^{k+n} 1_A \right\rangle < \infty.$$

Hence $\sum_{k=0}^{\infty} T^k 1_{[X_n \in A]} < \infty$ a.e. (λ). It follows from Theorem 2.1 [5] that $[X_n \in A]$ is T -dissipative. Since D is a countable union of P -transient sets, $[X_n \in D]$ is T -dissipative.

We shall designate by Γ the collection of T -closed subsets of C .

LEMMA 4.2. *Let P be conservative. Then T is also conservative. If a non-negative function f is P -invariant ($f = Pf$ a.e. (λ)) then $f(X_n)$ is T -invariant ($f(X_n) = f(X_{n+1})$ a.e. (λ)). Conversely if a non-negative function g on Ω is T -invariant then there is a P -invariant function f such that $g = f(X_n)$ a.e. (λ).*

Proof. If a P -invariant function f is of the form 1_A with $A \in \mathcal{C}$ then

$$\mathcal{P}1_A(X_n) \cdot 1_A(X_{n+1}) = P^n I_A P 1_A = P^n 1_A = 1_A = P^{n+1} 1_A.$$

Since $\mathcal{P}1_A(X_n) = P^n 1_A$ and $\mathcal{P}1_A(X_{n+1}) = P^{n+1} 1_A$ we have

$$\lambda[1_A(X_n) \neq 1_A(X_{n+1})] = \lambda[1_A(X_{n+1}) \neq 1_A(X_n) \cdot 1_A(X_{n+1})] = 0.$$

Hence $1_A(X_n) = 1_A(X_{n+1})$ a.e. (λ). In general f may be approximated from below by linear combinations of functions of the form 1_A . $f(X_n) = f(X_{n+1})$ a.e. (λ) is obtained by the usual limiting process.

Conversely if a bounded function g is T -invariant and $f = \mathcal{P}g$ then f is P -invariant. By the Martingale convergence theorem $\{f(X_n)\}$ converges a.e. (λ) to g

(cf. the proof of Theorem 1.1 on p. 460, [3]). Since $f(X_n) = f(X_0)$ a.e. (λ), $g = f(X_n)$ a.e. (λ). It follows that every set E in Γ is of the form $[X_n \in A]$ where $A \in \mathcal{C}$. Since every non-negative T -invariant function g is Γ -measurable, g is of the form $f(X_n)$ where f is \mathcal{C} -measurable.

COROLLARY 4.1. *A set E belongs to Γ if and only if E is of the form $[X_0 \in A]$ for some $A \in \mathcal{C}$.*

LEMMA 4.3. *If a measure μ on \mathcal{X} is P -invariant (P -excessive) then μ on \mathcal{F} is T -invariant (T -excessive).*

Proof. The lemma follows from the following equalities:

$$\begin{aligned} \mu(E) &= \langle \mu, \mathcal{P}1_E \rangle \cong \langle \mu P, \mathcal{P}1_E \rangle = \langle \mu, P\mathcal{P}1_E \rangle \\ &= \langle \mu, \mathcal{P}T1_E \rangle = \langle \mu, T1_E \rangle = \langle \mu T, 1_E \rangle \\ &= \mu T(E). \end{aligned}$$

THEOREM 4.1. *Let P be λ -continuous, C_1, C_2, \dots , atoms of the σ -algebra \mathcal{C} of P -closed subsets of C and μ_1, μ_2, \dots , the P -invariant measures equivalent to $\lambda_{C_1}, \lambda_{C_2}, \dots$ respectively. Then the σ -algebra Γ of T -closed subsets of C is also purely atomic with atoms C_1, C_2, \dots where $C_i = [X_0 \in C_i]$. Each μ_i is T -invariant and equivalent to λ_{C_i} .*

Proof. By Corollary 4.1, \mathcal{C} and Γ are isomorphic. Since \mathcal{C} is purely atomic, Γ is purely atomic with atoms $[X_0 \in C_i], i = 1, 2, \dots$. By Lemma 4.3, μ_i is T -invariant. μ_i is equivalent to λ_{C_i} since μ_i is equivalent to λ_{C_i} .

THEOREM 4.2. *Let P be λ -continuous and μ_1, μ_2, \dots be T -invariant measures of Theorem 4.1. Let $E_i = \bigcup_{n=0}^{\infty} [X_n \in C_i]$. If f, g are μ_i -integrable functions with $g \geq 0, \int g d\mu_i > 0$ then*

$$(4.4) \quad \lim_{N \rightarrow \infty} \frac{\sum_{n=0}^N T^n f}{\sum_{n=0}^N T^n g} = \frac{\int f d\mu_i}{\int g d\mu_i} \text{ a.e. } (\lambda) \text{ on } E_i.$$

The above limit is independent of the particular T -invariant, λ_{C_i} equivalent measure μ_i chosen. Furthermore

$$(4.5) \quad \lim_{N \rightarrow \infty} N^{-1} \left(\sum_{n=0}^N T^n f \right) = \int f d\mu_i / \mu_i(C_i) \text{ a.e. } (\lambda) \text{ on } E_i$$

(the right-hand side of (4.5) is interpreted to be 0 if $\mu_i(C_i) = \infty$).

Proof. Since C_i is an atom of Γ and μ_i is a T -invariant measure with support C_i , (4.4) is true a.e. (λ) on C_i by Theorem 6.5 of Appendix. Let

$$F = \left[\begin{array}{c} \sum_{n=0}^N T^n f(w) \\ w: \lim_{N \rightarrow \infty} \frac{\sum_{n=0}^N T^n f(w)}{\sum_{n=0}^N T^n g(w)} = \frac{\int f d\mu_i}{\int g d\mu_i} \end{array} \right].$$

Then $\mathcal{P}1_{F \cap C_i} = 1$ a.e. (λ) on C_i . Let $E^k = [X_0 \notin C_i, \dots, X_{k-1} \notin C_i, X_k \in C_i]$. To show that (4.4) is also true a.e. (λ) on $E_i - C_i$ it is sufficient to show $\lambda(E^k - E^k \cap F) = 0$ for $k = 1, 2, \dots$. We shall show this by showing $\mathcal{P}1_{E^k} = \mathcal{P}1_{E^k \cap F}$ a.e. (λ) . It is clear that $1_{E^k \cap F}(w) = 1_{E^k}(w) \cdot 1_{E \cap C_i}(T^{k+1}w)$. Hence for (λ) almost all x ,

$$\begin{aligned} \mathcal{P}1_{E^k \cap F}(x) &= 1_{X - C_i}(x) \int_{X - C_i} \dots \int_{C_i} p(x, y_1) \dots p(y_{k-1}, y_k) \lambda(dy_1) \dots \lambda(dy_k) \mathcal{P}1_{F \cap C_i}(y_k) \\ &= 1_{X - C_i} \int_{X - C_i} \dots \int_{C_i} p(x, y_1) \dots p(y_{k-1}, y_k) \lambda(dy_1) \dots \lambda(dy_k) \\ &= \mathcal{P}1_{E^k}(x). \end{aligned}$$

Any T -invariant λ_{C_i} equivalent measure is a constant multiple of μ_i . Hence the limit is independent of the particular μ_i chosen. If $\mu_i(C_i) < \infty$ and g is chosen to equal to 1 on C_i then (4.4) becomes (4.5). If $\mu(C_i) = \infty$ we may choose a monotone nondecreasing sequence $\{g_k\}$ such that $0 \leq g_k \leq 1$, $\int g_k d\mu_i < \infty$, $g_k \uparrow 1$ a.e. (λ) ; then

$$\limsup_{N \rightarrow \infty} \frac{\sum_{n=0}^N T^n f}{N} \leq \lim_{N \rightarrow \infty} \frac{\sum_{n=0}^N T^n f}{\sum_{n=0}^N T^n g_k} = \frac{\int f d\mu_i}{\int g_k d\mu_i} \text{ a.e. } (\lambda) \text{ on } E_i.$$

Since $\int g_k d\mu_i \uparrow \infty$ as $k \rightarrow \infty$, $\lim_{N \rightarrow \infty} N^{-1} \sum_{n=0}^N T^n f = 0$ a.e. (λ) on E_i .

THEOREM 4.3. Let $D = \bigcap_{n=0}^{\infty} [X_n \in D]$ and $\eta \in \mathcal{A}^+(\lambda)$. Let η be defined by (4.4) and $\mu = \sum_{n=0}^{\infty} \eta I_D T^n$. Then μ is T -excessive and $\sum_{n=0}^{\infty} T^n f$ converges a.e. (λ) on D for every μ -integrable function f .

Proof. It is clear that D is T -closed and dissipative. μ is σ -finite with D as its support. The convergence of $\sum_{n=0}^{\infty} T^n f$ then follows from Theorem 6.5 of Appendix.

The following corollaries are special cases of Theorem 4.2 and Theorem 4.3.

COROLLARY 4.2. Let C_i be an atom of \mathcal{C} and μ_i a P -invariant measure equivalent to λ_{C_i} . If f, g are μ_i -integrable functions with $g \geq 0$, $\int g d\mu_i > 0$ then

$$\lim_{N \rightarrow \infty} \frac{f(X_1) + \dots + f(X_N)}{g(X_1) + \dots + g(X_N)} = \frac{\int f d\mu_i}{\int g d\mu_i} \quad \text{a.e. } (\lambda) \text{ on } E_i,$$

where $E_i = \bigcup_{n=0}^{\infty} [X_n \in C_i]$ and

$$\lim_{N \rightarrow \infty} \frac{f(X_1) + \dots + f(X_N)}{N} = \frac{\int f d\mu_i}{\mu_i(C_i)} \quad \text{a.e. } (\lambda) \text{ on } E_i,$$

where $1/\infty$ is 0.

COROLLARY 4.3. *Under the same assumption as in Theorem 4.3 if f is a function on X and $g = \mathcal{P}1_D$ and if fg is integrable with respect to $\sum_{n=0}^{\infty} \eta I_D (PI_D)^n$ then $\sum_{n=0}^{\infty} f(X_n)$ converges a.e. (λ) on D .*

Proof. We only need to point out that $\int f(X_0) d\mu = \int fgd(\sum_{n=0}^{\infty} \eta I_D (PI_D)^n)$. Then the corollary follows immediately from Theorem 4.3.

V. Martin boundaries. Let π be a finite measure equivalent to λ which is fixed all through this section and h , a P -excessive π -integrable function. Let measure π_h, λ_h be defined by $\pi_h(A) = \int_A h d\pi, \lambda_h(A) = \int_A h d\lambda$. π_h is finite and λ_h is σ -finite. Let P_h be given by (1.8), (1.9). P_h is a λ_h -continuous Markov operator of which p_h of (1.7) is the density with respect to λ_h . Let the kernel $K(x, y)$ be given by (3.2). We shall construct measures on the product σ -algebra \mathcal{F} of subsets of the product space Ω as in §IV. However in this section the component spaces of Ω will be $X \cup [\rho]$. ρ is usually called an "absorbing state." An element w of Ω is a sequence $\{x_0, x_1, x_2, \dots\}$ with x_n being elements of X or equal to ρ . X_n shall be the function on Ω to $X \cup [\rho]$ defined by $X_n(w) = x_n$ if $w = \{x_0, x_1, x_2, \dots\}$. For a π -integrable P -excessive function h a function $\mathcal{P}_h(x, E)$ ($x \in X, E \in \mathcal{F}$) is defined as follows. If $E = [X_0 \in A_0, \dots, X_n \in A_n]$ where $A_i \in \mathcal{X}$, let

$$\mathcal{P}_h(x, E) = \int_{A_0} \lambda_h(dx_0) \int_{A_1} \lambda_h(dx_1) \dots \int_{A_n} \lambda_h(dx_n) p_h(x, x_0) p_h(x_0, x_1) \dots p_h(x_{n-1}, x_n).$$

If $E \in \mathcal{F}$ and is of the form $\Omega - E$ we let $\mathcal{P}_h(x, F) = 1 - \mathcal{P}_h(x, E)$. Then the definition of $\mathcal{P}_h(x, E)$ is extended to arbitrary $E \in \mathcal{F}$ by the usual measure extension argument. Measure π_h on \mathcal{F} is defined by

$$(5.1) \quad \pi_h(E) = \int_{A \cap X} \pi_h(dx_0) \mathcal{P}(x_0, E_1)$$

if E is of the form $[X_0 \in A, \{X_1, X_2, \dots\} \in E_1]$. Then the definition is again extended to arbitrary E . π_h is a finite measure on \mathcal{F} .

THEOREM 5.1. *Let h be P -excessive and π -integrable. Let π_h be defined by (5.1) and*

$$\begin{aligned}
 K(x, X_n(w)) &= K(x, y) \text{ if } X_n(w) = y \in X, \\
 &= 0 \quad \text{if } X_n(w) = \rho.
 \end{aligned}$$

Then $\{K(x, X_n(w))\}$ converges for $(\lambda \times \pi_h)$ almost all (x, w) . (Note that $\pi_h\{X_n \in D \cap [\bigcup_{n=1}^\infty \text{supp } \pi P^n]\} = 0$ for $n \geq 1$. Hence $K(x, X_n(w))$ are well defined a.e. $(\lambda \times \pi_h)$ for $n \geq 1$.)

Proof. If $X_n(w)$ are in C for $n \geq 1$ the convergence of $K(x, X_n(w))$ is obvious since $K(x, \cdot)$ is constant on every C_i and C, C_i are also P_h -closed. Since $K(x, y)1_D(y)$ is the kernel for operator PI_D it is sufficient to prove the theorem for a dissipative P .

Let $\mu = \sum_{n=1}^\infty \pi P^n$. μ is a σ -finite, P -excessive measure. Let $f = d\mu/d\lambda$; then $f(y) = \int \pi(dx) g(x, y)$ where $g(x, y)$ is given by (1.6). Let

$$q^n(y, x) = f(x)p^n(x, y)/f(y).$$

Then we may define a μ -continuous Markov operator Q with $q(x, y)$ as its density function with respect to μ . Q is actually the μ -reverse of P (cf. Appendix). Since P is dissipative so is Q . Hence for any finite measure ν absolutely continuous with respect to μ , $\int \nu(dy) \sum_{n=1}^\infty q^n(y, x)$ is finite for (μ) almost all x . Since

$$\sum_{n=1}^\infty q^n(y, x) = f(x)g(x, y)/f(y) = f(x)K(x, y),$$

hence $\int \nu(dy) K(x, y) < \infty$ for (μ) almost all x .

Let a, b be two real numbers with $0 \leq a < b$. Let $\beta(w)$ be the number of down crossings of $[a, b]$ by the sequence $K(x, X_1(w)), K(x, X_2(w)), \dots$. Now we shall proceed to prove the theorem by showing that for (μ) almost all x , $\int \beta(w)\pi_h(dw) < \infty$.

Since P_h is also dissipative there is a monotone nondecreasing sequence of transient sets E of which the union is X . Let

$$\begin{aligned}
 \Omega_E &= [w: X_n(w) \in E \text{ for some } n > 0], \\
 (5.2) \quad \tau_E(w) &= \sup [n: X_n(w) \in E] \text{ if } w \in \Omega_E \\
 &= 0 \text{ otherwise.}
 \end{aligned}$$

τ_E is finite valued a.e. (π_h) . Let

$$\begin{aligned}
 L_E(x) &= \mathcal{P}_h \left(x, \bigcap_{n=1}^\infty X_n \notin E \right) \text{ if } x \in E \\
 &= 0 \text{ otherwise.}
 \end{aligned}$$

On Ω_E define functions Y_0, Y_1, Y_2, \dots with values in $X \cup [\rho]$ as follows:

$$\begin{aligned}
 Y_0(w) &= X_{\tau_E(w)}(w), \\
 Y_1(w) &= X_{\tau_E(w)-1}(w) \text{ if } \tau_E(w) \geq 2 \\
 &= \rho \text{ otherwise,} \\
 &\dots\dots\dots
 \end{aligned}$$

Let $A_0, \dots, A_n \in X$. We shall compute the following measure:

$$\begin{aligned}
 & \pi_h[Y_0 \in A_0, Y_1 \in A_1, \dots, Y_n \in A_n] \\
 &= \sum_{m=n+1}^{\infty} \pi_h[X_{m-n} \in A_n, X_{m-n-1} \in A_{n-1}, \dots, X_0 \in A_0] \cap [\tau_E = m] \\
 &= \sum_{m=n+1}^{\infty} \int_{A_n} \pi P^{m-n}(dy_n) \int_{A_{n-1}} \lambda(dy_{n-1}) \cdots \int_{A_0} \lambda(dy_0) p(y_n, y_{n-1}) \cdots \\
 & \quad \cdots p(y_1, y_0) h(y_0) L_E(y_0) \\
 (5.3) \quad &= \int_{A_n} \mu(dy_n) \int_{A_{n-1}} \lambda(dy_{n-1}) \cdots \int_{A_0} \lambda(dy_0) p(y_n, y_{n-1}) \cdots p(y_1, y_0) h(y_0) L_E(y_0) \\
 &= \int_{A_n} \mu(dy_n) \int_{A_{n-1}} \lambda(dy_{n-1}) \cdots \int_{A_0} \lambda(dy_0) q(y_{n-1}, y_n) \cdots q(y_0, y_1) f(y_0) h(y_0) L_E(y_0) \\
 &= \int_{A_0} f(y_0) h(y_0) L_E(y_0) \lambda(dy_0) \int_{A_1} \lambda(dy_1) \cdots \int_{A_n} \lambda(dy_n) q(y_0, y_1) \cdots q(y_{n-1}, y_n).
 \end{aligned}$$

Let $\eta_E(A) = \pi_h[Y_0 \in A]$. η_E is the distribution of Y_0 . η_E has density function $f(y_0)h(y_0)L_E(y_0)$ with respect to λ . The collection of x for which $\int \eta_E(dy)K(x, y) = \infty$ for some E in the sequence is a μ -null set. Let x belong to the complement of this μ -null set. Then $\int K(x, Y_0)d\pi_h < \infty$. Let

$$\begin{aligned}
 \beta_E(w) &= \text{the number of down crossings of } [a, b] \\
 & \quad \text{by } K(x, X_1(w)), \dots, K(x, X_{\tau_E}(w)) \text{ if } W \in \Omega_E, \\
 &= 0 \text{ otherwise.}
 \end{aligned}$$

Then $\beta_E(w)$ is also the number of up crossings of $[a, b]$ by $K(x, Y_0(w)), K(x, Y_1(w)), \dots$. Since $K(x, \cdot)$ is a Q -potential, (5.3) implies that $K(x, Y_n)$ is a supermartingale and $\int K(x, Y_n)d\pi_h \downarrow 0$. It follows from a well-known inequality of Doob [3, p. 316, Theorem 3.3] that $E[\beta_E] \leq b/(b-a)\pi_h(\Omega_E)$. $\int \beta_E d\pi_h \uparrow \int \beta d\pi_h$ as $E \uparrow X$. Hence $\int \beta d\pi_h \leq b/(b-a)$ and the theorem is proved.

The kernel $K(x, y)$ has the property that for (λ) almost all $y \in \bigcup_{n=1}^{\infty} \text{supp } \pi P^n$, $K(\cdot, y)$ is an element of $L_1(\pi)$ with L_1 norm = 1. For an element ϕ of $L_{\infty}(\pi)$ define

$$(5.4) \quad l_{\phi}(y) = \int \phi(x) K(x, y) \pi d(x).$$

COROLLARY 5.1. *Let h be P -excessive and π -integrable. Let $\phi \in L_{\infty}(\pi)$ and $l_{\phi}(y)$ be defined by (5.4) and $l_{\phi}(\rho) = 0$. Then $\{l_{\phi}(X_n)^k\}$ converges a.e. (π_h) and also in $L_1(\pi_h)$ for every positive integer k . Furthermore, if P is dissipative, E is transient and $E \uparrow X$ and τ_E be defined by (5.2), then $\{l_{\phi}(X_{\tau_E})^k\}$ converges a.e. (π_h) and also in $L_1(\pi_h)$ as $E \uparrow X$ where the value of $l_{\phi}(X_{\tau_E})$ is taken to be 0 if*

$w \notin \Omega_E$. The limit is independent of the particular sequence of transient sets E chosen.

Proof. It is sufficient to prove the corollary for a non-negative ϕ . For the a.e. (π_n) -convergence of $\{l_\phi(X_n)\}$ the same proof of Theorem 5.1 goes through since l_ϕ is a Q -potential if P is dissipative. Let $\tau(w) = \lim \tau_E(w)$ as $E \uparrow X$. If $\tau(w)$ is finite the convergence of $l_\phi(X_{\tau_E(w)})^k$ is trivial and the limit is $l_\phi(X_{\tau(w)}(w))^k$. If $\tau(w) = \infty$, $l_\phi(X_{\tau_E(w)}(w))^k$ is a subsequence of $l_\phi(X_n(w))^k$, therefore, its convergence follows from the convergence of $l_\phi(X_n(w))^k$ and both sequences have the same limit. The $L_1(\pi_n)$ -convergence follows from the fact that $|l_\phi(y)| \leq \|\phi\|_\infty$.

For the rest of this section we shall assume that $X = \bigcup_{n=1}^\infty \text{supp } \pi P^n$. We shall also assume that for every $y \in X$, $K(\cdot, y)$ is a non-negative element of $L_1(\pi)$ with L_1 norm 1. This may be accomplished by discarding a λ -null set. Consider the map T :

$$(5.5) \quad T(y) = K(\cdot, y).$$

This is a map of X into $L_1(\pi)$. We shall consider $L_1(\pi)$ as a subset of its second dual. Let \tilde{X} be the weak closure of the image of X under T . \tilde{X} is a compact Hausdorff space under the weak topology.

LEMMA 5.1. *Let $\tilde{\mathcal{X}}$ be the σ -algebra of all Baire subsets of \tilde{X} . Then T given by (5.5) is a measurable transformation of (X, \mathcal{X}) to $(\tilde{X}, \tilde{\mathcal{X}})$.*

Proof. We shall use the symbol $\langle \tilde{y}, \phi \rangle$ for $\phi \in L_\infty(\pi)$, $\tilde{y} \in \tilde{X}$ to denote the value of \tilde{y} at ϕ . To prove Lemma 5.1 we shall show that for every real valued continuous function f on \tilde{X} , fT is \mathcal{X} -measurable. We shall show this by two steps. First, we shall show that every real continuous function on \tilde{X} may be uniformly approximated by polynomials of functions of the type $\langle \cdot, \phi \rangle$. Second, we shall show that $\langle T(\cdot), \phi \rangle$ is \mathcal{X} -measurable for every $\phi \in L_\infty(\pi)$.

Let Λ be the collection of all functions on the unit ball S of $L_\infty(\pi)$ into the closed interval $[-1, +1]$. The topology on Λ shall be the Tychonoff topology. Then \tilde{X} may be considered as a closed subset of Λ . For any $\eta \in \Lambda$, $\phi \in S$ we shall denote the value of η at ϕ by $\langle \eta, \phi \rangle$. The polynomials of functions of the type $\langle \cdot, \phi \rangle$ form a subalgebra of the algebra of all real valued continuous functions on Λ . This subalgebra separates points of Λ and contains the constant functions. By the Stone-Weierstrass theorem every continuous function on Λ may be uniformly approximated by elements of this subalgebra. Now every continuous real valued function on \tilde{X} may be extended to be real continuous function on Λ . Hence the first step is finished. The \mathcal{X} -measurability of $\langle T(\cdot), \phi \rangle$ follows from the equality:

$$\langle T(\cdot), \phi \rangle = \int \pi(dx) \phi(x) K(x, \cdot).$$

THEOREM 5.2. *For every non-negative P -invariant π -integrable function h there is a Baire measure $\tilde{\eta}$ on $\tilde{\mathcal{X}}$ for which the following formula holds,*

$$(5.6) \quad h = \int \tilde{y} \tilde{\eta}(d\tilde{y}).$$

(5.6) is interpreted to be that for every $\phi \in L_\infty(\pi)$

$$\langle h, \phi \rangle = \int \langle \tilde{y}, \phi \rangle \tilde{\eta}(d\tilde{y}).$$

Proof. Let

$$\tilde{y}_{C_i} = \frac{u_{C_i}}{\int \pi(dy)u_{C_i}(y)}.$$

\tilde{y}_{C_i} is then the image of all points y in C_i under the transformation T . If h is of the form $\sum a_i u_{C_i}$, then $\tilde{\eta}$ in (5.6) is atomic which assigns measure $a_i \int \pi(dy)u_{C_i}(y)$ to \tilde{y}_{C_i} and the formula is valid. In general if $h = a_i$ on C_i then $h - \sum a_i u_{C_i} = 0$ on C . $\sum a_i u_{C_i}$ is P -invariant and, by Theorem 6.2 of Appendix, is the smallest P -excessive function which is equal to h on C . Hence $h - \sum a_i u_{C_i} \geq 0$ and is P -invariant. Hence we only need to establish (4.4) for h which equals to 0 on C .

Since $h = 0$ on C , we may replace P by PI_D . In other words we may assume P to be dissipative. Then P_h is also dissipative. Let E be P_h -transient and $E \uparrow X$. Let τ_E be defined by (5.2). Let us consider X_{τ_E} to be defined on Ω_E and η_E to be its distribution. Let $\tilde{X}_E = TX_{\tau_E}$ and $\tilde{\eta}_E$ be its distribution. We shall show that $\tilde{\eta}_E$ converges weakly, i.e., for every continuous function f on \tilde{X} , $\int f d\tilde{\eta}_E$ converges as $E \uparrow X$. Since $\{\tilde{\eta}_E(\tilde{X})\}$ is a bounded sequence, it is sufficient to show the convergence for f being a polynomial of functions of the form $\langle \cdot, \phi \rangle$ where $\phi \in L_\infty(\pi)$. It is clear that

$$\int (\langle \tilde{y}, \phi \rangle)^k \tilde{\eta}_E(d\tilde{y}) = \int_{\Omega_E} l_\phi(X_{\tau_E})^k d\pi_h$$

where l_ϕ is given by (5.4). The convergence of $\{\int (\langle \tilde{y}, \phi \rangle)^k \tilde{\eta}_E(d\tilde{y})\}$ is then a consequence of Corollary 5.1. Let $\tilde{\eta}$ be the limit measure of $\{\tilde{\eta}_E\}$.

Now we shall compute $\mathcal{P}_h(x, [X_{\tau_E} \in A, \tau_E > 0])$. For (π_h) almost all x

$$\begin{aligned} \mathcal{P}_h(x, [X_{\tau_E} \in A, \tau_E > 0]) &= \frac{1}{h(x)} \sum_{m=1}^\infty \int_A p^{(m)}(x, y) h(y) L_E(y) \lambda(dy) \\ &= \frac{1}{h(x)} \int_A g(x, y) h(y) L_E(y) \lambda(dy) \\ &= \frac{1}{h(x)} \int_A K(x, y) \eta_E(dy). \end{aligned}$$

Hence

$$(5.7) \quad \begin{aligned} \int \phi(x) h(x) \mathcal{P}_h(x, [\tau_E > 0]) \pi(dx) &= \iint \phi(x) K(x, y) \pi(dx) \eta_E(dy) \\ &= \int \langle \tilde{y}, \phi \rangle \tilde{\eta}(d\tilde{y}). \end{aligned}$$

Now $\mathcal{P}_h(x, [\tau_E > 0]) \geq \mathcal{P}_h(x, [X_1 \in E]) \uparrow 1$ a.e. (π_n) since h is P -invariant. (5.6) is then obtained by letting $E \uparrow X$ in (5.7).

VI. Appendix on λ -measurable Markov operators. Let λ be a σ -finite measure and $\mathcal{A}(\lambda)$ be the Banach space of the collection of all finite countably additive set functions which are absolutely continuous with respect to λ . Let $L_\infty(\lambda)$ be the dual of $\mathcal{A}(\lambda)$. Let $P: \nu \rightarrow \nu P$ be a positive linear operator and a contraction on $\mathcal{A}(\lambda)$ to $\mathcal{A}(\lambda)$. Let $P: f \rightarrow Pf$ acting on $L_\infty(\lambda)$ be its dual. Operator P on $L_\infty(\lambda)$ is characterized by (1) P is positive, (2) $P1 \leq 1$, (3) $f_k \downarrow 0$ implies that $Pf_k \downarrow 0$. For an extended real valued non-negative function f which is not λ -essentially bounded Pf shall be the limit of Pf_n where $f_n \in L_\infty(\lambda)$, $f_n \uparrow f$. Similarly for any measure ν absolutely continuous with respect to λ , νP shall be the limit of $\nu_n P$ where $\nu_n \in \mathcal{A}(\lambda)$, $\nu_n \uparrow \nu$. $\langle \nu, f \rangle$ is to designate the integral $\int f d\nu$ whenever the latter is well defined. We have $\langle \nu P, f \rangle = \langle \nu, Pf \rangle$. We shall call P a λ -measurable Markov operator. An extended real valued non-negative function h is excessive if $Pf \leq f$, a potential if $P^n f \downarrow 0$. A σ -finite measure μ is excessive if $\mu P \leq \mu$, a potential if $d\mu P^n / d\lambda \downarrow 0$. It should be emphasized that for a λ -measurable Markov operator λ -null sets are irrelevant and $=, \leq$ mean $=$ a.e. (λ) , \leq a.e. (λ) .

For any set A , A' designates its complement. Let

$$P_A^\# = \sum_{n=0}^{\infty} (I_{A'} \cdot P)^n, \quad P_A = \sum_{n=0}^{\infty} P(I_{A'} \cdot P)^n,$$

then $P_A = PP_A^\#$ and

$$(6.1) \quad P_A^\# I_A = I_A + I_{A'} \cdot P_A I_A.$$

LEMMA 6.1. *If h is an excessive function and l is an excessive function which is $\geq h$ on A then $P_A^\# I_A h \leq l$. It follows that $P_A^\# I_A h \leq h$.*

Proof. We need to show

$$(6.2) \quad \sum_{n=0}^N (I_{A'} \cdot P)^n I_A h \leq l$$

for $N = 0, 1, 2, \dots$. For $N = 0$ (6.2) becomes $I_A h \leq l$. Assume (6.2) to be true for N ; then

$$P \sum_{n=0}^N (I_{A'} \cdot P)^n I_A h \leq P l \leq l.$$

Hence

$$I_A h + I_{A'} \cdot P \sum_{n=0}^N (I_{A'} \cdot P)^n I_A h \leq l, \text{ i.e., } \sum_{n=0}^{N+1} (I_{A'} \cdot P)^n I_A h \leq l.$$

THEOREM 6.1. $P_A^\# I_A$ and $P_A I_A$ are λ -measurable Markov operators.

Proof. We need to prove the theorem for $P_A^\# I_A$ only. By Lemma 6.1, $P_A^\# I_A 1 \leq 1$. It follows that $f \in L_\infty(\lambda)$ implies that $P_A^\# I_A f \in L_\infty(\lambda)$. Positivity of $P_A I_A$ is obvious.

Let $f_k \in L_\infty(\lambda), f_k \geq 0, f_k \downarrow 0$. The theorem is proved if we can show that $P_A^* I_A f_k \downarrow 0$. We may assume $f_k \leq 1$ for all k . Then

$$(6.3) \quad P_A^* I_A f_k(x) \leq \sum_{n=0}^N (I_A \cdot P)^n I_A f_k(x) + \sum_{n=N+1}^{\infty} (I_A \cdot P)^n 1_A(x).$$

Let x be fixed. For any $\varepsilon > 0$ choose N so large that the second term at the right-hand side of (6.3) is $\leq \varepsilon/2$, then choose K so large that the first term at the right-hand side of (6.3) is $\leq \varepsilon/2$ whenever $k \geq K$. Then $P_A^* I_A f_k(x) \leq \varepsilon$ whenever $k \geq K$ and the assertion that $P_A^* I_A f_k \downarrow 0$ is proved.

THEOREM 6.2. *If h is excessive then $P_A^* I_A h$ is the smallest excessive function which is $\geq h$ on A .*

Proof. We need only to prove that $P_A^* I_A h$ is excessive. Then the rest of the theorem follows from Lemma 6.1. By Lemma 6.1, $P_A^* I_A h \leq h$. Hence $P_A I_A h \leq h$. By (6.1)

$$P_A^* I_A h = I_A h + I_A \cdot P_A I_A h \geq I_A P_A I_A h + I_A \cdot P_A I_A h = P_A I_A h = P P_A^* I_A h.$$

COROLLARY 6.1. *If A is conservative then $P_A 1_A$ is an invariant function and is the smallest excessive function which is $= 1$ on A .*

Proof. $P_A 1_A = P P_A I_A 1$. Hence $P_A 1_A$ is excessive and is $\leq P_A I_A 1$. However A being conservative implies $P_A 1_A = 1$ on A . Hence, by Theorem 6.2, $P_A 1_A 1 = P_A 1_A$. Hence $P_A 1_A$ is invariant and is the smallest excessive function $= 1$ on A .

A set A is said to be recurrent if $P_A 1_A = 1$ on A . A set B is conservative if every subset of B is recurrent. A set E is dissipative if every λ -non-null subset of E is not conservative. A set T is transient if there is a non-negative number $q < 1$ such that $P_T 1_T \leq q$ on T . A subset of a transient set is transient.

THEOREM 6.3. *Any dissipative set E is a union of at most countably many transient sets.*

Proof. First we shall show that every λ -non-null nonrecurrent set A contains a λ -non-null transient set. The set $(P_A 1_A < 1) \cap A$ is λ -non-null, hence, there is a non-negative number $q < 1$ such that $T = (P_A 1_A \leq q) \cap A$ is non-null. By Theorem 6.2, $P_T 1_T \leq P_A 1_A$. Hence $P_T 1_T \leq P_A 1_A \leq q$ on T and T is transient.

If E is transient then there is nothing to prove. We assume that E is not transient. For every countable ordinal number $\alpha > 0$ we shall define a transient subset T_α of E by transfinite induction. Since E is λ -non-null (because E is not transient) and dissipative, E contains a λ -non-null subset which is not recurrent. Hence E contains a λ -non-null transient set T_1 . Suppose T_β are defined for all $\beta < \alpha$. If $E - U_{\beta < \alpha} T_\beta$ is λ -null define T_α to be the null set. Otherwise $E - U_{\beta < \alpha} T_\beta$ contains a λ -non-null transient set T_α . There must be an α for which T_α is null. For, if not, E would contain uncountably many disjoint λ -non-null sets which is im-

possible. Let α_0 be the first ordinal number such that T_{α_0} is null. Then $E = \bigcup_{\alpha < \alpha_0} T_\alpha$ and the theorem is proved.

For any $\nu \in \mathcal{A}^+(\lambda)$ the *support* of ν , $\text{supp } \nu$, is the set for which $A \subset \text{supp } \nu$ and A being λ -non-null imply that $\nu(A) > 0$, and $B \subset X - \text{supp } \nu$ implies that $\nu(B) = 0$. For any set A , we define the *consequent* of A , $F(A)$ by

$$F(A) = \bigcup_{n=0}^{\infty} \text{supp } \nu P^n$$

where $\nu \in \mathcal{A}^+(\lambda)$ has A as its support. The particular ν chosen clearly does not matter. If A is recurrent then $F(A)$ is the smallest closed set containing A .

THEOREM 6.4. *If A is recurrent then for every $\nu \in \mathcal{A}^+(\lambda)$, νP_A is σ -finite on $F(A)$. In particular, if P is conservative and ergodic, then νP_A is σ -finite for every non-null set A , and every $\nu \in \mathcal{A}^+(\lambda)$.*

Proof. It is sufficient to prove that $\nu P_A^\#$ is σ -finite on $F(A)$. Since $I_A \cdot P$ is a λ -measurable Markov operator and $P_A^\# = \sum_{n=0}^{\infty} (I_A \cdot P)^n$ it is sufficient to show that $F(A)$ is dissipative under the operator $I_A \cdot P$.

We shall first show that $P_A 1_A = 1$ on $F(A)$. It follows that $P_A^\# 1_A = 1$ on $F(A)$ since $P_A^\# 1_A$ is P -excessive (Theorem 6.2) and $P_A 1_A = P P_A^\# 1_A$. Let $F_n(A) = \text{supp } \eta P^n$ where η has A as its support. We shall show that $P_A 1_A = 1$ on $F_n(A)$ for $n = 0, 1, 2, \dots$. By the definition of recurrency of A , $P_A 1_A = 1$ on $F_0(A) = A$. Assuming that $P_A 1_A = 1$ on $F_n(A)$ we proceed to show that $P_A 1_A = 1$ on $F_{n+1}(A)$.

$$\begin{aligned} \eta P^n(x) &= \eta P^n(F_n(A)) = \langle \eta P^n, P_A 1_A \rangle \\ &= \langle \eta P^n, P P_A^\# 1_A \rangle = \langle \eta P^{n+1}, P_A^\# 1_A \rangle \\ (6.5) \quad &= \langle \eta P^{n+1}, 1_A \rangle + \langle P^{n+1}, I_A \cdot P_A 1_A \rangle \\ &\leq \langle \eta P^{n+1}, 1_A \rangle + \langle \eta P^{n+1}, 1_{A'} \rangle = \langle \eta P^{n+1}, 1 \rangle \\ &= \eta P^{n+1}(X). \end{aligned}$$

The fifth equality in (6.5) is owing to (6.1) and the inequality following it is owing to the fact $P_A 1_A \leq 1$ (Lemma 6.1). Since $\eta P^n(X) \geq \eta P^{n+1}(X)$, equality holds all through (6.5). Hence

$$\langle \eta P^{n+1}, I_A \cdot P_A 1_A \rangle = \langle \eta P^{n+1}, 1_{A'} \rangle$$

and $P_A 1_A = 1$ on $\text{supp } \eta P^{n+1} - A = F_{n+1}(A) - A$. Hence $P_A 1_A = 1$ on $F_{n+1}(A)$.

Now we have the following equality:

$$P^n = \sum_{k=0}^{n-1} (I_A \cdot P)^k I_A P^{n-k} + (I_A \cdot P)^n.$$

Since $F(A)$ is closed, for $\nu \in \mathcal{A}^+(\lambda)$ with $\text{supp } \nu \subset F(A)$, we have

$$\begin{aligned} v(F(A)) &= vP^n(F(A)) = \langle vP^n, 1 \rangle \\ &= \left\langle v \sum_{k=0}^{n-1} (I_A \cdot P)^k I_A P^{n-k}, 1 \right\rangle + \langle v(I_A \cdot P)^n, 1 \rangle \\ &= \left\langle v \sum_{k=0}^{n-1} (I_A \cdot P)^k I_A, 1 \right\rangle + v(I_A \cdot P)^n(X) \\ &= \left\langle v, \sum_{k=0}^{n-1} (I_A \cdot P)^k 1_A \right\rangle + v(I_A \cdot P)^n(X). \end{aligned}$$

Now

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} (I_A \cdot P)^k 1_A = P_A^\# 1_A = 1 \text{ on } F(S).$$

Hence

$$(6.6) \quad \lim_{n \rightarrow \infty} v(I_A \cdot P)^n(X) = 0.$$

If $F(A)$ were not dissipative under the operator $I_A \cdot P$ then $F(A)$ would contain an $I_A \cdot P$ -recurrent set B . Let v have support B . Then $v(X) = v(I_A \cdot P)^n(X)$ for all positive integers n , which contradicts (5.6).

Let μ be a σ -finite excessive measure. Then, if v is absolutely continuous to μ , so is vP . P acting on $\mathcal{A}(\mu)$ is a μ -measurable Markov operator which we shall designate by P_μ . Let $g \in L_\infty(\mu)$ with $L_\infty(\mu)$ norm $\|g\|$ and μ_g be defined by $\mu_g(A) = \int_A g d\mu$. $\mu_g P$ is σ -finite and absolutely continuous with respect to μ . Define

$$Qg = d\mu_g P / d\mu.$$

Since $\pm \mu_g \leq \|g\| \mu$, $\pm \mu_g P \leq \|g\| \mu P \leq \|g\| \mu$, hence $Qg \in L_\infty(\mu)$ with $\|Qg\| \leq \|g\|$. Clearly $g_k \in L_\infty(\mu)$, $g_k \downarrow 0$ imply that $Qg_k \downarrow 0$, so that Q is a μ -measurable Markov operator. Q is characterized by the following equality:

$$(6.7) \quad \int P f \cdot g d\mu = \int f \cdot Qg d\mu$$

whenever one side of (6.7) is well defined. J. Feldman showed that the space X has the same decomposition $X = C \cup D$ for Q as that of P although μ -null sets are irrelevant to Q [5, Theorem 3.1]. We shall call Q the μ -reverse of P . Since for $g \geq 0$

$$\langle \mu, Qg \rangle = \int 1 \cdot Qg d\mu = \int P 1 \cdot g d\mu \leq \int g d\mu = \langle \mu, g \rangle,$$

μ is also Q -excessive. The symmetric roles played by Q, P in (6.7) imply that P_μ is the μ -reverse of Q .

LEMMA 6.2. *Let P be a conservative λ -measurable Markov operator which possesses an invariant measure μ . Let Q be its μ -reverse. If set A is P -closed then it is Q -closed. Conversely if A is a subset of the support of μ and A is Q -closed then it is P -closed.*

Proof. We shall prove the lemma for the case that μ is equivalent to λ . Then, applying it to P_μ and noting that a subset of the support of μ is P -closed if and only if it is P_μ -closed, we obtain the assertion for the general case.

When μ is equivalent to λ the roles played by P, Q are symmetric. Hence we only need to prove the sufficiency part. Let S be Q -closed. Let f be non-negative, bounded, μ -integrable and $[f > 0] = S$. Let μ_f be defined by $\mu_f(A) = \int_A f d\mu$. Then $S = \bigcup_{n=0}^\infty \text{supp } \mu_f Q^n$. Now (6.7) implies that $d\mu_f Q^n / d\mu = P^n f$. Hence $S = \bigcup_{n=0}^\infty [P^n f > 0]$. However $[P^n f > 0] = [P^n 1_S > 0]$. Hence

$$S = \bigcup_{n=0}^\infty [P^n 1_S > 0] = [P_S 1_S > 0].$$

Now by Corollary 6.1, $P_S 1_S$ is P -invariant, therefore \mathcal{C} -measurable [7, Theorem 9.1]. Hence the set $[P_S 1_S > 0] = S$ is P -closed.

LEMMA 6.3. *Let P possess an excessive measure μ and Q be its μ -reverse.*

1. *If h is a finite valued P -excessive function then the measure μ_h defined by $\mu_h(A) = \int_A h d\mu$ is Q -excessive. If, in addition, h is a P -potential then μ_h is a Q -potential.*

2. *If η is a P -excessive measure absolutely continuous with respect to μ then $d\eta/d\mu$ is Q -excessive. If, in addition, η is a P -potential then $d\eta/d\mu$ is a Q -potential.*

Proof. For any non-negative function f , μ_f shall always denote the measure defined by $\mu_f(A) = \int_A f d\mu$. Now if $0 \leq Ph \leq h$, $g \geq 0$ then by (6.7)

$$\langle \mu_h Q^n, g \rangle = \langle \mu_h, Q^n g \rangle = \langle \mu_g, P^n h \rangle \leq \langle \mu_h, g \rangle.$$

Hence μ_h is Q -excessive. It is clear that $P^n h = d\mu_h Q^n / d\mu$ so that h being a P -potential implies that μ_h is a Q -potential. If η is a P -excessive measure absolutely continuous to μ , then, by (6.7), for $g \geq 0$

$$\langle \eta, P^n g \rangle = \langle \mu_g, Q^n (d\eta/d\mu) \rangle.$$

However

$$\langle \mu_g, d\eta/d\mu \rangle = \langle \eta, g \rangle \geq \langle \eta P^n, g \rangle = \langle \mu_g, d\eta P^n / d\mu \rangle.$$

Hence

$$\langle \mu_g, d\eta/d\mu \rangle \geq \langle \mu_g, Q^n (d\eta/d\mu) \rangle,$$

$$Q^n (d\eta/d\mu) = d\eta P^n / d\mu$$

and the conclusion 2 follows.

COROLLARY 6.2. *If P is conservative and ergodic and if P possesses a nontrivial invariant measure μ then it is unique up to a constant multiple.*

Proof. By Lemma 6.2, Q , the μ -reverse of P , is also ergodic. Hence the only Q -excessive functions are constants. If η is another P -invariant measure then $d\eta/d\mu$ is Q -excessive and is a constant by Lemma 6.3. Hence η is a constant multiple of μ .

THEOREM 6.5. *If μ is a P -excessive measure and if $f, g \in L_1(\mu)$, $g > 0$ a.e. (μ) then $\sum_{n=1}^{\infty} P^n f$ converges a.e. (μ) on D and, on C , there exists*

$$\lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N P^n f}{\sum_{n=1}^N P^n g} = \frac{\bar{f}}{\bar{g}} \quad \text{a.e. } (\mu)$$

where \bar{f}, \bar{g} are \mathcal{C} -measurable and satisfy

$$\int_A \bar{f} d\mu_1 = \int_A f d\mu, \quad \int_A \bar{g} d\mu_1 = \int_A g d\mu$$

for every $A \in \mathcal{C}$ where μ_1 is a finite measure equivalent to μ .

Proof. Let Q be the μ -reverse of P . It follows from (6.7) that $d\mu_f Q^n/d\mu = P^n f$, $d\mu_g Q^n/d\mu = P^n g$. For Q , \mathcal{C} is still the σ -algebra of conservative Q -closed sets by Lemma 6.2. The theorem is then obtained immediately by applying the general ratio ergodic theorem.

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