

SOME RESULTS ON CRUMPLED CUBES⁽¹⁾

BY
LLOYD L. LININGER

Introduction. This paper deals with some problems on crumpled cubes. In [6] R. H. Bing raised the following questions: (1) Is the union of a crumpled cube and a 3-cell topologically S^3 ? (2) Under what conditions will the union of two crumpled cubes sewn together along their boundaries be S^3 ? (3) Under what conditions will the union of two copies of the same crumpled cube sewn together along their boundaries by the identity homeomorphism be S^3 ? In [2] Bing proves that the union of two copies of the Solid Alexander Horned Sphere sewn together along their boundaries by the identity homeomorphism is S^3 . In [1] B. J. Ball proves that there exists a crumpled cube C such that two copies of C sewn together along their boundaries (not by the identity homeomorphism) is not topologically S^3 . In [7] Casler proves that the union of two copies of the Solid Alexander Horned Sphere sewn together along their boundaries is S^3 .

In this paper it is proved in §2 that question (1) has an affirmative answer. This result has recently been obtained independently by Norman Hosay [9]. Using this result, it is shown in §3 that if C is a crumpled cube and B is a unit ball in E^3 , then there exists an upper semi-continuous decomposition G of B such that (1) B/G is homeomorphic to C , and (2) if g is a nondegenerate element of G , then g is point-like and the intersection of g with the boundary of B is a one-point set. Corollary 2 of §3 shows that if C and D are crumpled cubes, S is a tame 2-sphere in S^3 , and C and D are sewn together along their boundaries, then there exists an upper semi-continuous decomposition G of S^3 such that the intersection of each nondegenerate element of G with S is a one-point set and S^3/G is homeomorphic to the union of C and D .

In §4 it is shown that if two copies of a particular "bad" crumpled cube are sewn together by the identity homeomorphism along their boundaries the result is S^3 , and it is shown that if two copies of a particular "nice" crumpled cube are sewn together by the identity homeomorphism along their boundaries the result is not S^3 .

1. Notation and terminology. If A is a set in a topological space, then $C1 A$ and $Bd A$ denote the closure and boundary, respectively, of A . If S is a 2-sphere in

Presented to the Society, January 26, 1964; received by the editors February 3, 1964.

(¹) This paper consists of the author's doctoral dissertation at the State University of Iowa, prepared under the supervision of Professor S. Armentrout. The author wishes to express his appreciation to Professor Armentrout for his assistance and encouragement.

E^3 , then $\text{Int } S$ and $\text{Ext } S$ denote the interior and exterior, respectively, of S . The usual metric for E^3 is d .

A set C is a crumpled cube if and only if C is homeomorphic to the union of a 2-sphere and its interior in E^3 .

A space X is a disk-with-holes if and only if X is a disk or there exist disjoint disks D_1, \dots, D_n in the interior of a disk D such that X is homeomorphic with $D - \bigcup_{i=1}^n \text{Int } D_i$.

For terms not defined in this paper the reader may see [4] or [6].

2. The sum of a crumpled cube and a 3-cell. In this section it is proved that the sum of a crumpled cube and a 3-cell is topologically S^3 . Theorem 1 is used to obtain this result. The proof of Theorem 1 given here relies heavily on the Side Approximation Theorem proved by Bing. For completeness this theorem is stated below.

BING'S SIDE APPROXIMATION THEOREM. *For each 2-sphere S in E^3 , each positive number e , and each component U of $E^3 - S$, there exists a homeomorphism h from S onto a polyhedral 2-sphere such that (1) if x is in S , then $d(x, h(x)) < e$, (2) there exists a finite collection of disjoint disks D_1, \dots, D_n on $h[S]$, each of diameter less than e such that $h[S] - \bigcup_{i=1}^n D_i \subset U$, and (3) there exists a finite collection of disjoint disks E_1, \dots, E_m on S , each of diameter less than e such that $S - \bigcup_{i=1}^m E_i$ is contained in $\text{Int } h[S]$ if U is the exterior of S or is contained in $\text{Ext } h[S]$ if U is the interior of S [4].*

THEOREM 1. *If C is a crumpled cube in E^3 with boundary S , A is a compact subset of $\text{Int } C$, and e is a positive number, then there exist a homeomorphism h' from C into E^3 and a polyhedral 2-sphere S' such that S' is homeomorphically within e of S , $h'[C] \subset \text{Int } S'$, if $x \in C$ then $d(x, h'(x)) < e$, and if $x \in A$ then $h'(x) = x$.*

Proof. There exists a positive number e' such that if T is a 2-sphere within e' of S , then $A \subset \text{Int } T$. Let e'' denote $(1/11) \min \{e, e', (1/10) \text{diam } C\}$. Since S is a 2-sphere, then by Bing's Side Approximation Theorem, there exists a polyhedral 2-sphere S_1 such that (a) S_1 is homeomorphically within e'' of S , (b) there exists a finite number of disjoint disks E_1, \dots, E_m on S_1 , each of diameter less than e'' , such that $\text{Cl}(S_1 - \bigcup_{i=1}^m E_i)$ is contained in $\text{Ext } S$, and (c) there exists a finite number of disjoint disks D_1, \dots, D_n on S , each of diameter less than e'' , such that $\text{Cl}(S - \bigcup_{i=1}^n D_i)$ is contained in $\text{Int } S_1$. For each i , $i = 1, 2, \dots, m$, let K_i denote a polyhedral disk such that (a) $\text{Bd } K_i = \text{Bd } E_i$, (b) $K_i - \text{Bd } K_i$ is contained in $\text{Ext } S_1$, (c) $(S_1 - \bigcup_{i=1}^m E_i) \cup (\bigcup_{i=1}^m K_i)$ is a 2-sphere S' , and (d) S' is homeomorphically within e'' of S .

It will be shown that there exist a polyhedral 3-manifold-with-boundary M and a homeomorphism h with the following properties: (1) $(S \cup \text{Int } S) \subset \text{Int } M$, (2) for each i , $i = 1, 2, \dots, m$, no simple closed curve in M links $\text{Bd } E_i$, (3) if D'_i is a polyhedral disk such that $\text{Bd } E_i = \text{Bd } D'_i$ then there does not exist a 2-sphere in M that contains D'_i in its interior, (4) $M - S'$ has one and only one component

V with diameter more than one half the diameter of C , (5) S' , S_1 , and M are in relative general position, (6) if W is a component of $M \cap S'$ such that W does not separate M , and W_1, W_2, \dots , and W_k are components of $M \cap S'$ such that (i) $\bigcup_{i=1}^k W_i$ does not separate M , (ii) $\bigcup_{i=1}^k W_i \cup W$ does separate M , and (iii) $W \cup \bigcup_{i=1}^k W_i$ is contained in some component of $\text{Cl}(M - V)$, then there exists a simple closed curve J in M such that J intersects $\bigcup_{i=1}^k W_i$ in a one-point set, J intersects W in a one-point set, and the diameter of each component of $J \cap V$ is less than e'' , (7) if x is in $V \cap \text{Int } S_1$, then $h(x) = x$, (8) h is from M into $\text{Int } S'$, and (9) for each x in M , $d(x, h(x))$ is less than $11e''$. This will complete the proof of Theorem 1 since if h' denotes the restriction of h to C then h' and S' satisfy the conclusion of Theorem 1.

There exists a positive number e_1 such that if T is a triangulation of E^3 with mesh e_1 , N is the union of all 3-simplexes of T which intersect C , and M is the union of all 3-simplexes is the 2nd barycentric subdivision of T which intersect N , then M will be a polyhedral 3-manifold-with-boundary which satisfies properties (1)–(6).

The proof that there exists a homeomorphism h from M into $\text{Int } S'$ that satisfies conditions (7), (8) and (9) is broken into three parts. Let F_1, \dots , and F_p be the components of $M - V$. In Part I it is proved that if for some j , there exists a positive integer i less than or equal to p such that $(\text{Cl } V) \cap F_i$ is contained in K_j , then there exists a homeomorphism h_i from M into E^3 such that $h_i[F_i] \subset \text{Int } S'$, $h_i[M]$ satisfies conditions (1)–(6), if V' is the large component of $h_i[M] - S'$, then $h_i[M] - V'$ has at most $p - 1$ components, and if x is in $V \cap \text{Int } S_1$ then $h_i(x) = x$. In Part II the following is proved: If for each j such that $j \leq p$, there does not exist an integer i such that $F_i \cap \text{Cl } V$ is contained in K_j , then there exists an integer k , $1 \leq k \leq m$, and a homeomorphism h_1 from M into E^3 such that $h_1[M]$ satisfies conditions (1)–(7), if V'' denotes the large component of $h_1[M] - S'$, then $h_1[M] - V''$ has at most p components, and $h_1[F_1] \cap \text{Cl } V''$ is contained in K_k . In Part III, by using induction and Parts I and II it is shown that there exists a homeomorphism h with the desired properties.

PROOF OF PART I. In this part, it is supposed that for some j , there exists an integer i such that $(\text{Cl } V) \cap F_i$ is contained in K_j . A homeomorphism h_i from M into E^3 will be constructed such that $h_i[F_i] \subset \text{Int } S'_2$, $h_i[M]$ satisfies conditions (1)–(7), and if V' denotes the large component of $h_i[M] - S'$ then $h_i[M] - V'$ has at most $p - 1$ components.

Let $X^1_1, \dots, X^1_{r(1)}$ be the distinct components of $F_i - K_j$ such that for each u , $(\text{Cl } X^1_u) \cap \text{Cl } V \neq \emptyset$, and if W is a component of $F_i - K_j$ such that $(\text{Cl } W) \cap \text{Cl } V \neq \emptyset$, then there exists a z such that $W = X^1_z$. Let X^2_1, \dots , and $X^2_{t(2)}$ be the distinct components of $F_i - \text{Cl}(K_j \cup [\bigcup_{k=1}^{t(1)} X^1_k])$ such that for each u less than or equal to $t(2)$, $(\text{Cl } X^2_u) \cap \text{Cl}(V \cup \bigcup_{k=1}^{t(1)} X^1_k) \neq \emptyset$. Since $F_i - K_j$ has only a finite number of components, there exists an integer r such that if the above process is continued, for each u and w , $X^w_u \neq \emptyset$ and $F_i = \text{Cl}(\bigcup_{w=1}^r \bigcup_{u=1}^{t(w)} X^w_u)$.

For each u and w such that w is greater than 1, $(Cl X_u^w) \cap Cl(\bigcup_{k=1}^{(w-1)} X_k^{w-1})$ is a collection of disk-with-holes. Let Y_{wu1}, \dots , and $Y_{wuq(w,u)}$ denote the components of $(Cl X_u^w) \cap (Cl[\bigcup_{j=1}^{(w-1)} X_j^{w-1}])$ and let Y_{wu} denote $\bigcup_{j=1}^{q(w,u)} Y_{wuj}$. For each w, u and z , let T_{wuz} be a disk in K_j^z such that (1) $Y_{wuz} \subset T_{wuz}$, and (2) $Bd T_{wuz} \subset Bd Y_{wuz}$. Let T'_{wuz} be a slight thickening of T_{wuz} with the following properties: (1) T'_{wuz} is polyhedral and topologically equivalent to $T_{wuz} \times [0,1]$, (2) T'_{wuz} is disjoint from the 0-skeleton of M , and (3) If $T_{wuv} \subset T_{wuz}$ then $T_{wuv} \subset Int T_{wuz}$. Let T'_{wu} denote $\bigcup_{z=1}^{q(w,u)} T'_{wuz}$, let T''_{wuz} denote the component of $T'_{wuz} - K_j$ that intersects $\bigcup_{u=1}^{(w-1)} X_u^{w-1}$, let T''_{wu} denote $\bigcup_{z=1}^{q(w,u)} T''_{wuz}$, and let T''_w denote $\bigcup_{u=1}^{(w)} T''_{wu}$. The remainder of the proof of Part I is now broken into two steps.

Step I. In Step I the following is proved: There exists a homeomorphism f_1 from M into E^3 with the following properties: (1) $f_1[M]$ has properties (1)-(7), (2) If V' is the large component of $f_1[M] - S'$, then $f_1[M] - V'$ has at most p components, (3) $f_1[F_1]$ is contained in $(Int S') \cup (F_1 - Cl X_1^w) \cup T''_{w1}$, and (4) If $k \neq 1$, then either $(f_1[Cl X_k^w]) \cap K_j = f_1[(Cl X_k^w) \cap K_j]$ or $(f_1[Cl X_k^w]) \cap K_j = \emptyset$.

PROOF OF STEP I. *Case 1.* $Y_{w1} = Y_{w11}$. Let Y_{w1}^* denote the disk on the boundary of T''_{w1} that is disjoint from K_j and corresponds to Y_{w1} . See Figure 1. A homeo-

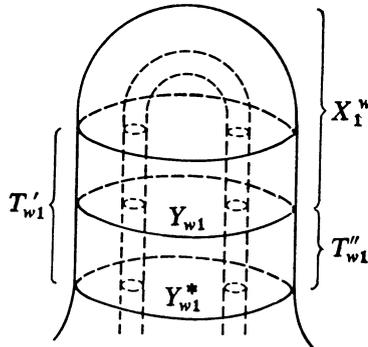


FIGURE 1

morphism f_1 is constructed as follows: Let W_1, \dots , and $W_{g(1)}$ be the components of $M - Bd T''_{w1}$, and assume $Int S_1 \cap V$ is contained in W_1 and $X_1^w = W_2$. If x is in W_1 then $f_1(x) = x$. If $(Cl W_1) \cup Int T''_{w1} = \emptyset$ and W_1 is contained in the unbounded component of $E^3 - (T''_{w1} \cup (Cl X_1^w))$, then $f_1(x) = x$. If $W = \{W_i : W_i \cap Int T''_{w1} = \emptyset \text{ and either } W_i = W_2 \text{ or } W_i \text{ is not contained in the unbounded component of } E^3 - (T''_{w1} \cup (Cl X_1^w))\}$ and $W' = \bigcup \{Cl W_i : \text{there exists an integer } s \text{ such that } W_s \in W \text{ and } Cl W_s \cup Cl W_i \neq \emptyset\}$, then $f_1[W']$ is contained in T''_{w1} and f_1 leaves W' pointwise fixed on $Bd T''_{w1}$. Each of the remaining W_k 's (if there are any) is contained in T''_{w1} and the homeomorphism f_1 can be extended to them as shown in Figure 2.

Case 2. $Y_{w1} = Y_{w11} \cup \dots \cup Y_{w1q(w,1)}$. There exists a simple closed curve J on $Bd M$ such that (1) $J \subset Cl(V \cup \bigcup_{k=1}^{w-1} \bigcup_{u=1}^{(k)} X_u^k)$, (2) The diameter of each compo-

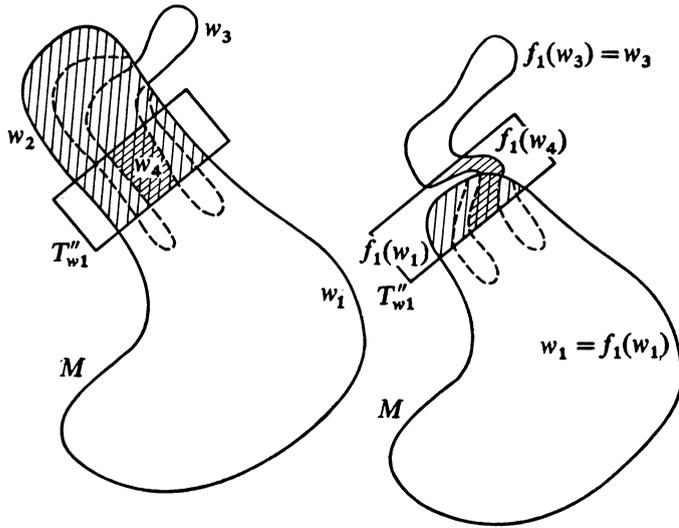


FIGURE 2

ment of $J \cup V$ is less than ϵ , and (3) $J \cap Y_{w_{11}}$ is a one-point set, and it can be assumed that $J \cap Y_{w_{12}}$ is a one-point set while J is disjoint from each of $Y_{w_{13}}, \dots$, and $Y_{w_{1q(w,1)}}$. Shrink F_1 slightly so that $J \cap F_1$ misses the modified F_1 and if J intersects V , pull J off V slightly. See Figure 3. Then there exists a homeomor-

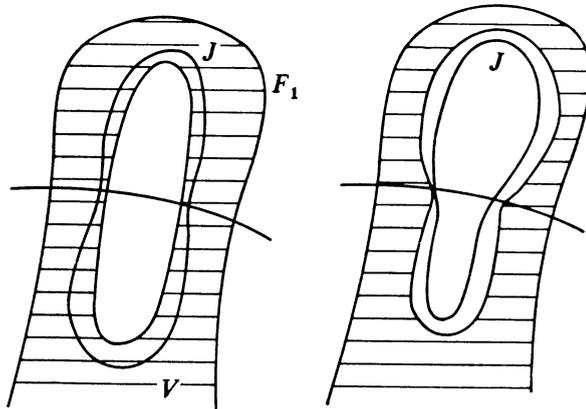


FIGURE 3

phism f^* from the modified M into E^3 with the following properties: (1) If x is in $M - T_{w_{12}}$ then $f^*(x) = x$, (2) f^* maps $T_{w_{12}} \cap [\text{modified } M]$ as shown in Figure 4. Note that $J \cap (E^3 - T_{w_{12}})$ is the center of $f^*[T_{w_{12}} \cap \text{modified } M] \cap (E^3 - T_{w_{12}})$, and (3) $f^*[\text{modified } M]$ satisfies conditions (1)-(7), and if V^* is the large component of $f^*[\text{modified } M] - S'$, then $f^*[\text{modified } M] - V^*$ has at most p components. After $q(w,1) - 1$ repetitions of this argument, the modified Y_{w_1} is contained in $Y_{w_{11}}$. See Figure 5. An argument like the one given

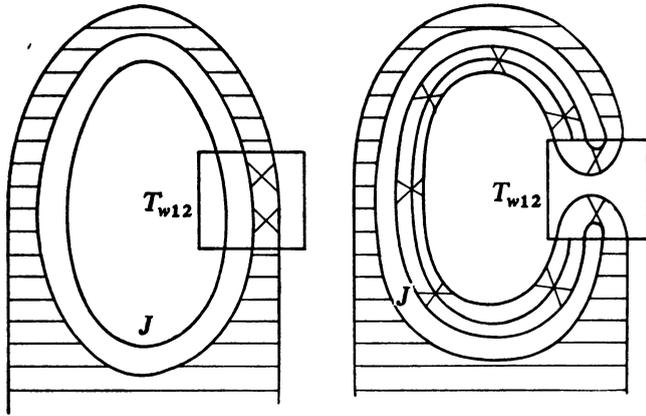


FIGURE 4

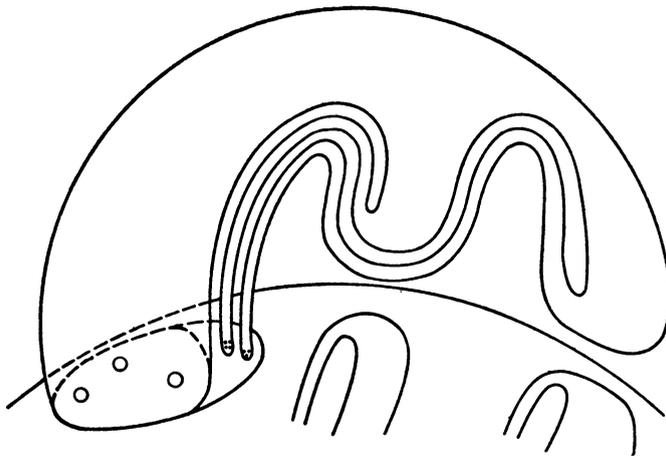
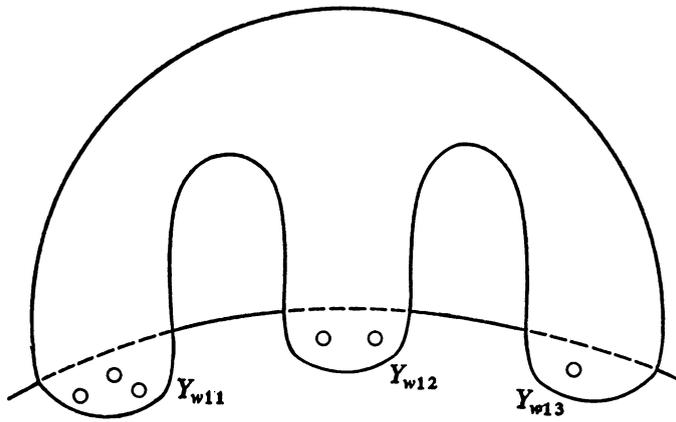


FIGURE 5

in Case 1 using T''_{w11} in place of T''_{w1} gives the existence of a homeomorphism f_1 with the desired properties. This concludes the proof of Step I.

It follows that there exist homeomorphisms f_2, \dots , and $f_{t(w)}$ satisfying the following: (1) $f_{t(w)} \cdots f_2 f_1$ is a homeomorphism from M into E^3 , (2) $f_{t(w)} \cdots f_1 [M]$ satisfies conditions (1)–(7) (3) If V'' is the large component of $f_{t(w)} \cdots f_1 [M] - S'$ then $f_{t(w)} \cdots f_1 [M] - V''$ has at most p components, and (4) $f_{t(w)} \cdots f_1 [M]$ is contained in

$$(M - \text{Cl}[\bigcup_{k=1}^{t(w)} X_k^w]) \cup \bigcup_{k=1}^{t(w)} T''_{wk}.$$

Step II. In Step II it is proved that there exists a homeomorphism h_i from M into E^3 such that $h_i[F_i] \subset \text{Int } S'$, $h_i[M]$ satisfies conditions (1)–(7), and if V' is the large component of $h_i[M] - S'$ then $K_i[M] - V'$ has at most $p - 1$ components. This will conclude the proof of Part I.

PROOF OF STEP II. Using an argument similar to the one given in Step I, it can be shown that there exists a homeomorphism $f_{t(w)+1}$ from $f_{t(w)} \cdots f_1 [M]$ into E^3 with the following properties: (1) $f_{t(w)+1} \cdots f_1 [M]$ satisfies conditions (1)–(7), (2) Each component of $f_{t(w)} \cdots f_1 [M] - K_j$ that intersects X_1^{w-1} is mapped into $(M - [\bigcup_{k=1}^{t(w)} X_k^w \cup X_1^{w-1}]) \cup [\bigcup_{k=1}^{t(w-1)} T''_{(w-1)k} \cup T''_w]$, and (3) If V''' is the large component of $f_{t(w)+1} \cdots f_1 [M] - S'$ then $f_{t(w)+1} \cdots f_1 [M] - V'''$ has at most p components. After $t(1) + t(2) + \dots + t(w - 1) - 1$ additional repetitions of this argument the homeomorphism h_i is obtained. Since $h_i[F_i]$ is contained in the union of T''_1 and a small neighborhood of V , it follows that $h_i[F_i]$ is contained in $\text{Int } S'$.

PART II. In this part, it is supposed that for each integer j less than or equal to p , there does not exist an integer i such that $F_i \cap \text{Cl } V$ is contained in K_j . It is shown that there exist an integer k , $1 \leq k \leq m$, and a homeomorphism h_1 from M into E^3 such that (1) $h_1[M]$ satisfies conditions (1)–(7), (2) If V'' denotes the large component of $h_1[M] - S'$ then $h_1[M] - V''$ has at most p components, and (3) $h_1[F_1] \cup \text{Cl } V''$ is contained in K_k .

PROOF OF PART II. An argument similar to the one given in the proof of Part I, Step I, Case 2 shows that there exists a homeomorphism h_1 from M into E^3 such that h_1 has the desired properties

PART III. In this part, it is shown that there is a homeomorphism h which satisfies the conclusion of Theorem 1.

PROOF OF PART III. Assume the F_i 's are indexed so that there exists an integer u , $1 \leq u \leq p$, with the property that if $i < u$ then there exists an integer k , $1 \leq k \leq m$, such that $F_i \cap \text{Cl } V$ is contained in K_k and if $i \geq u$ then there does not exist an integer k , $1 \leq k \leq m$, such that $F_i \cap \text{Cl } V \subset K_k$. If u is greater than 1, apply Part I to F_1 and obtain a homeomorphism h_1 from M into E^3 that maps F_1 into $\text{Int } S'$. Then apply Part I to $h_1[F_2]$ obtaining h_2 . Continue this process to obtain homeomorphisms h_3, \dots , and h_{u-1} . Then by applying Part II to $h_{u-1} \cdots h_1 [F_u]$, a homeo-

morphism h_u is obtained, and the results of Part I are then applied to $h_u \cdots h_1[F_u]$ to obtain h_{u+1} . Then by using Part I or Part II, depending upon $h_{u+1} \cdots h_1[F_{u+1}]$, a homeomorphism h_{u+2} is obtained. Continue this process. Let h denote the composition of the h_i 's. Clearly h satisfies conditions (1)–(8). Condition (9) follows since for each i less than or equal to p , the diameter of F_i is less than $5e''$, the diameter of $h[F_i]$ is less than $5e''$, and h moves no point of V more than e'' . This concludes the proof of Theorem 1.

THEOREM 2. *If C is a crumpled cube in S^3 and e is a positive number, then there exists a homeomorphism h from C into S^3 such that the closure of $S^3 - h[C]$ is a 3-cell and if x belongs to C then $d(x, h(x))$ is less than e .*

Proof. Let f be a positive sequence such that the series whose sequence of terms is f converges to a number less than e , and let C_1, C_2, \dots be a sequence of closed sets such that for each $i, C_i \subset \text{Int } C_{i+1}$, and $\bigcup_{i=1}^{\infty} C_i = \text{Int } C$. Let S denote the boundary of C .

By Theorem 1 there exists a polyhedral 2-sphere S_1 and a homeomorphism h_1 from C into S^3 such that S_1 is homeomorphically within $f(1)$ of $S, h_1[C] \subset \text{Int } S_1$ if $x \in C$ then $d(x, h_1(x)) < f(1)$, and if $x \in C_1$ then $h_1(x) = x$.

Let e_2 denote $(1/3) \min \{f(2), d(S_1, h_1[S])\}$. Then by Theorem 1 there exists a polyhedral 2-sphere S_2 and a homeomorphism h'_2 from $h_1[C]$ into S^3 such that S_2 is homeomorphically within e_2 of $h_1[S], h'_2 h_1[C] \subset \text{Int } S_2$, if $x \in h_1[C_2]$ then $h'_2(x) = x$, and if $x \in h_1[C]$ then $d(x, h'_2(x)) < e_2$. Let h_2 be $h'_2 h_1$. It follows that h_2 maps C into $\text{Int } S_2$, if $x \in C$ then $d(x, h_2(x)) < f(1) + f(2)$, and $S_2 \subset \text{Int } S_1$.

Proceeding by induction, there exists a 2-sphere S_n and a homeomorphism h'_n from $h_{n-1}[C]$ into S^3 such that S_n is homeomorphically within $f(n)$ of $h_{n-1}[S], h'_n h_{n-1}[C] \subset \text{Int } S_n$, if $x \in h_{n-1}[C_n]$ then $h'_n(x) = x, S_n \subset \text{Int } S_{n-1}$, and if $x \in h_{n-1}[C]$ then $d(x, h'_n(x)) < f(n)$. Let h_n be $h'_n h_{n-1}$.

The sequence h_1, h_2, \dots is uniformly convergent. Let h be the limit of h_1, h_2, \dots . It follows that h is uniformly continuous and maps C into S^3 .

It is to be proved that h is one-to-one.

LEMMA 1. *The restriction of h to the interior of C is one-to-one.*

Proof. Suppose x and y are distinct points in $\text{Int } C$. There exists a positive integer j such that x and y are in C_j . Hence for each integer $k \geq j, h_k(x) = h_j(x), h_k(y) = h_j(y)$, and $h_j(x) \neq h_j(y)$. Then $h(x) \neq h(y)$.

LEMMA 2. *For each x in $h[\text{Int } C], \lim h_i^{-1}(x)$ exists and is equal to $h^{-1}(x)$.*

Proof. $\text{Sup } \{d(h_i^{-1}(x), h_{i+1}^{-1}(x)) : x \text{ in } h[C]\} \leq f(i)$. Hence the sequence $h_1^{-1}, h_2^{-1}, \dots$ is uniformly convergent on $h[C]$ and therefore, for each x in $h[C], \lim h_i^{-1}(x)$ exists. Suppose x belongs to $h[\text{Int } C]$. By Lemma 1 there is one and only one point y in $\text{Int } C$ such that $h(y) = x$ or $y = h^{-1}(x)$. There exists a positive integer j such that $y \in C_j$, hence $h(y) = h_j(y)$ and $\lim h_i^{-1}(x) = h_j^{-1}(x) = y$.

LEMMA 3. *The function h is a homeomorphism from C .*

Proof. It suffices to show h is one-to-one from C . Suppose h is not one-to-one. Then there exist distinct points x and y in C such that $h(x) = h(y)$. Let x_1, x_2, \dots , and y_1, y_2, \dots be sequences such that for each i , x_i and y_i belong to $\text{Int } C$, the sequence $\{x_i\}$ converges to x , and the sequence $\{y_i\}$ converges to y . Then the sequence $\{h(x_i)\}$ converges to $h(x)$ and the sequence $\{h(y_i)\}$ converges to $h(y)$ which is equal to $h(x)$. Let e' be a positive number less than $d(x, y)/2$. Since h is uniformly continuous on $h[\text{Int } C]$, there is a positive number r such that if $d(p, q) < r$, then $d(h^{-1}(p), h^{-1}(q)) < e'$. But there exists a positive integer i such that $d(h(x_i), h(y_i)) < r$ and $d(x_i, y_i) > e'$. This is a contradiction.

LEMMA 4. *If x belongs to C then $d(x, h(x))$ is less than e .*

Proof. $d(x, h(x)) \leq \lim_{n \rightarrow \infty} (d(x, h_1(x)) + d(h_1(x), h_2(x)) + \dots + d(h_{n-1}(x), h_n(x))) \leq \sum_{i=1}^{\infty} f(i) < e$.

LEMMA 5. *The closure of $S^3 - h[C]$ is a 3-cell.*

Proof. For each positive integer i , S_i is homeomorphically within $f(i)$ of $h_{i-1}[S]$ and $h_{i-1}[S]$ is homeomorphically within $\sum_{j=i}^{\infty} f(j)$ of $h[S]$. Therefore S_i is homeomorphically within $f(i) + \sum_{j=i}^{\infty} f(j)$ of $h[S]$. For each positive integer i , S_i is a polyhedral 2-sphere in $\text{Ext } h[C]$. It follows from Theorem 2.1 of [5] that the closure of $S^3 - h[C]$ is a 3-cell. This concludes the proof of Theorem 2.

3. Crumpled cubes as decomposition spaces. The following definitions and terminology will be used in §3. If r is a positive number, $B(r)$ will denote the ball in E^3 with center at the origin and radius r . If S and T are 2-spheres in E^3 and $S \subset \text{Int } T$, then $[S, T]$ will denote $\text{Cl}(\text{Int } T - \text{Int } S)$. Suppose r and s are positive numbers and $r < s$. A cellular subdivision $\{E_1, \dots, E_q\}$ of $[\text{Bd } B(r), \text{Bd } B(s)]$ is a canonical subdivision of $[\text{Bd } B(r), \text{Bd } B(s)]$ if and only if there exist disks D_1, \dots , and D_q on $\text{Bd } B(r)$ such that (1) for each i , there exists a j such that if p is in E_i , then there exists a point t in D_j and a real number c such that $p = ct$, (2) if $i \neq j$ and $D_i \cap D_j$ is nonempty, then $D_i \cap D_j \subset \text{Bd } D_i \cap \text{Bd } D_j$, (3) for each i , $E_i \cap B(r)$ is nonempty, and (4) for each i and j , the diameter of D_i is at most twice the diameter of D_j . Suppose S_1 and S_2 are tame 2-spheres in E^3 and S_2 is contained in $\text{Int } S_1$. A cellular subdivision $\{E_1, \dots, E_q\}$ of $[S_2, S_1]$ is an A -subdivision of $[S_2, S_1]$ if and only if there exists a homeomorphism h from $[S_2, S_1]$ onto $[\text{Bd } B(1), \text{Bd } B(2)]$ such that $\{h[E_1], \dots, h[E_q]\}$ is a canonical subdivision of $[\text{Bd } B(1), \text{Bd } B(2)]$. If X is a topological space and G is an upper semi-continuous decomposition of X , then X/G will denote the decomposition space associated with G .

THEOREM 3. *If C is a crumpled cube and $\text{Cl}(S^3 - C)$ is a 3-cell, then there exists an upper semi-continuous decomposition G of $B(1)$ such that the intersection*

of each nondegenerate element of G with the boundary of $B(1)$ is a one-point set, each nondegenerate element of G is point-like, and $B(1)/G$ is homeomorphic to C .

Proof. Let S denote the boundary of C , and let e_1 denote $\min\{1, 1/10$ diameter of $C\}$. There exists a polyhedral 2-sphere S_1 in $\text{Ext } S$ such that S_1 is homeomorphically within e_1 of S . Now let e_{11} be $(1/2)d(C, S_1)$. By the Side Approximation Theorem there exists a polyhedral 2-sphere S_{11} with the following properties: (1) S_{11} is homeomorphically within e_{11} of S , (2) there exist disjoint disks $D_{11}, \dots,$ and $D_{1n(1)}$ on S , each of diameter less than e_{11} , such that $\text{Cl}(S - \bigcup_{i=1}^{n(1)} D_{1i})$ is contained in $\text{Ext } S_{11}$, and (3) there exist disjoint disks $K_{11}, \dots,$ and $K_{1m(1)}$ on S_{11} , each of diameter less than e_{11} , such that $\text{Cl}(S_{11} - \bigcup_{i=1}^{m(1)} K_{1i})$ is contained in $\text{Int } S$. Let e_{12} denote

$$(1/3) \min \{ \min \{ d(D_{1i}, D_{1j}) : i \neq j, 1 \leq i \leq n(1), \text{ and } i \leq j \leq n(1) \}, \\ \min \{ d(K_{1i}, K_{1j}) : i \neq j, 1 \leq i \leq m(1), \text{ and } 1 \leq j \leq m(1) \}, \\ \min \{ d(D_{1i}, S_{11} - \bigcup_{j=1}^{m(1)} K_{1j}) : 1 \leq i \leq n(1) \} \}.$$

Let T_1 be a triangulation of S^3 with mesh less than e_{12} , let N be the union of all 3-simplexes of T which intersect $[S, S_1]$, and let M_{11} be the union of all 3-simplexes in the 2nd barycentric subdivision of T which intersect N . Then M_{11} is a polyhedral 3-manifold with boundary and it can be assumed that M_{11} and S_{11} are in relative general position.

Let M_1 denote the union of M_{11} and $[S_{11}, S_1]$. Now $M_1 - [S_{11}, S_1]$ has a finite number of components, and let $F_{11}, \dots,$ and $F_{1r(1)}$ be the components of $\text{Cl}(M_1 - [S_{11}, S_1])$. It follows from the Side Approximation Theorem that for each i , F_{1i} has diameter less than $4e_1 + 2e_{12}$ which is less than $5e_1$.

There exists an A -subdivision $\{E_{11}, \dots, E_{1q(1)}\}$ of $[S_{11}, S_1]$ such that (1) for each i , $i \leq n(1)$, there exists a positive integer j such that $\text{Bd } D_{1i} \subset \text{Int } E_{1j}$, and (2) for each i , $i \leq q(1)$, the diameter of E_{1i} is less than $5e_1$. Let A_{1i}^* and A_{1i} , $i \leq q(1)$, be defined as follows: $A_{1i}^* = \bigcup \{X : X \in \{E_{11}, \dots, E_{1q(1)}\} \text{ or } X \in \{F_{11}, \dots, F_{1r(1)}\} \text{ and } d(X, E_{1i}) < e_1\}$ and A_{1i} is the component of A_{1i}^* that contains E_{1i} . Note that for each p in S , there exists an integer j such that p is in A_{1j} . For each p in S , denote $\bigcup \{A_{1i} : p \in A_{1i}\}$ by A_{p1} . Now for each i , the diameter, of A_{1i} is at most $17e_1$, hence for each p , the diameter of A_{p1} is at most $34e_1$.

Then there exists a homeomorphism h_1 from $S_1 \cup \text{Int } S_1$ onto $B(1.1)$ with the following properties: (1) $h_1[[S_{11}, S_1]] = [\text{Bd } B(.9), \text{Bd } B(1.1)]$, (2) $h_1[C]$ is contained in $\text{Int } B(1.01)$, and (3) $\{h_1[E_{11}], \dots, h_1[E_{1q(1)}]\}$ is a canonical subdivision of $[\text{Bd } B(.9), \text{Bd } B(1.1)]$ into $q(1)$ cells.

Now M_1 is a 3-manifold-with-boundary and S is a compact subset in $\text{Int } M_1$. It follows that there exists a positive number e_{13} such that, if Q is a set with diameter less than $34e_{13}$ and the intersection of Q with S exists, then there exists a cell R such that $Q \subset \text{Int } R \subset R \subset \text{Int } M_1$.

Let e_2 denote $\min\{d(S, \text{Bd } M_1), e_{13}, 1/4\}$. There exists a polyhedral 2-sphere S_2 in $\text{Ext } S$ such that S_2 is homeomorphically within e_2 of S . Let e_{21} be $(1/3)\min\{d(S, S_2), \min\{d(\text{Bd } D_{1i}, \text{Bd } E_{1j}): \text{Bd } D_{1i} \subset \text{Int } E_{1j}\}\}$. By the Side Approximation Theorem there exists a polyhedral 2-sphere S_{22} such that (1) S_{22} is homeomorphically within e_{21} of S , (2) there exist disjoint disks $D_{21}, \dots,$ and $D_{2n(2)}$ on S , each of diameter less than e_{21} , such that $\text{Cl}(S - \bigcup_{i=1}^{n(2)} D_{2i}) \subset \text{Ext } S_{22}$, and (3) there exists disjoint disks $K_{21}, \dots,$ and $K_{2m(2)}$ on S_{22} , each of diameter less than e_{21} , such that $\text{Cl}(S_{22} - \bigcup_{i=1}^{m(2)} K_{2i}) \subset \text{Int } S$. Let e_{22} denote

$$(1/3) \min \{ \min \{ d(D_{2i}, D_{2j}) : i \neq j, 1 \leq i \leq n(2), \text{ and } 1 \leq j \leq n(2) \}, \\ \min \{ d(K_{2i}, K_{2j}) : i \neq j, 1 \leq i \leq m(2), \text{ and } 1 \leq j \leq m(2) \}, \\ \min \{ d(D_{2i}, S_{22} - \bigcup_{j=1}^{m(2)} K_{2j}) : 1 \leq i \leq n(2) \} \}.$$

If $M'_{22} = \{z : z \in S_2 \cup \text{Int } S_2 \text{ and } d(z, \text{Ext } S) \leq e_{22}\}$ then in the same way as M_{11} was obtained, a polyhedral 3-manifold-with-boundary M_{22} exists such that $[S, S_2] \subset \text{Int } M_{22}$ and $M_{22} \subset M'_{22}$. It can be assumed that M_{22} and S_{22} are in relative general position. Let M_2 denote the union of M_{22} and $[S_{22}, S_2]$. Now $M_2 - [S_{22}, S_2]$ has a finite number of components, and let $F_{21}, \dots,$ and $F_{2r(2)}$ be the components of $\text{Cl}(M_2 - [S_{22}, S_2])$. It follows that for each i , the diameter of F_{2i} is at most $5e_2$. It follows from the definition of M_2 that $S \subset \text{Int } M_2 \subset M_2 \subset \text{Int } M_1$ and that $\{x : x \in C \text{ and } d(x, S) > e_{22}\}$ is disjoint from $\text{Int } M_2$.

There exists an A -subdivision $\{E_{21}, \dots, E_{2q(2)}\}$ of $[S_{22}, S_2]$ such that (1) for each $i, 1 \leq i \leq n(2)$, there exists a positive integer j such that $\text{Bd } D_{2i} \subset \text{Int } E_{2j}$, (2) for each $i, 1 \leq i \leq q(2)$, the diameter of E_{2i} is less than $5e_2$, and (3) $q(2) \geq 4q(1)$. Let A_{2i}^* and A_{2i} be defined as follows: $A_{2i}^* = \bigcup\{X : X \in \{E_{21}, \dots, E_{2q(2)}\} \text{ or } X \in \{F_{21}, \dots, F_{2r(2)}\} \text{ and } d(X, E_{2i}) < e_2\}$ and A_{2i} is the component of A_{2i}^* that contains E_{2i} . For each p in S , there exists an integer j such that p is in A_{2j} . For each p in S , denote $\bigcup\{A_{2i} : p \in A_{2i}\}$ by A_{p2} . Note that for each p in S , the diameter of A_{p2} is at most $34e_2$. It follows that for each p in $S, A_{p2} \subset \text{Int } A_{p1}$. Furthermore, if $p \in [S_{11}, S_1]$ and Z_{p1} is the component of $A_{p1} \cap [S_{11}, S_1]$ that contains p , then $[S_{22}, S_2] \cap A_{p2}$ is contained in Z_{p1} and for each p in S , there exists a 3-cell C_{p2} such that $A_{p2} \subset \text{Int } C_{p2} \subset C_{p2} \subset \text{Int } A_{p1}$.

Now $S_2 \cap \{\bigcup_{i=1}^{q(1)} \text{Bd } E_{1i}\}$ exists and has one and only one component with diameter greater than $1/2$ the diameter of C . Denote this component by W_1 . It follows that for each $i, 1 \leq i \leq q(1)$, the intersection of W_1 and E_{1i} exists.

There exists a homeomorphism h_2 from $S_2 \cup \text{Int } S_2$ onto $B(1.01)$ with the following properties: (1) $h_2[[S_{22}, S_2]] = [\text{Bd } B(.99), \text{Bd } B(1.01)]$, (2) $h_2[C]$ is contained in $\text{Int } B(1.001)$, (3) if $x \in W_1$ then $h_2(x)$ is equal to the radial projection of $h_1(x)$ onto $\text{Bd } B(1.01)$, (4) if x is in $C - M_2$, then $h_2(x) = h_1(x)$, (5) h_2 differs from h_1 only at points close to $[S_{22}, S_2]$, and (6) $\{h_2[E_{21}], \dots, h_2[E_{2q(2)}]\}$ is a canonical subdivision of $[\text{Bd } B(.99), \text{Bd } B(1.01)]$.

The process is continued by induction. Hence for each positive i , $e_i, e_{i1}, e_{i2}, e_{(i-1)3}, S_i, S_{ii}, D_{i1}, \dots, D_{in(i)}, K_{i1}, \dots, K_{im(i)}, M_{ii}, M_i, F_{i1}, \dots, F_{ir(i)}, E_{i1}, \dots, E_{iq(i)}, A_{i1}^*, \dots, A_{iq(i)}^*, A_{i1}, \dots, A_{iq(i)}, A_{pi}, Z_{p(i-1)}, C_{pi}, W_i$, and h_i are defined inductively as follows: (1) Since S is contained in $\text{Int } M_{i-1}$, there exists a positive number $e_{(i-1)3}$ such that if Q is a set of diameter less than $34e_{(i-1)3}$ and the intersection of Q with S exists, then there exists a 3-cell R such that $Q \subset \text{Int } R \subset R \subset \text{Int } M_{i-1}$. (2) Let e_i be $(1/3) \min \{e_{(i-1)3}, 2^{-i}, d(C, \text{Bd } M_{i-1})\}$. (3) S_i is a polyhedral 2-sphere in $\text{Ext } S$ such that S_i is homeomorphically within e_i of S . (4) Let e_{i1} be $(1/3) \min \{d(C, S_i), \min \{d(\text{Bd } D_{(i-1)j}, E_{(i-1)k}) : \text{Bd } D_{(i-1)j} \subset \text{Int } E_{(i-1)k}\}\}$. (5) S_{ii} is a polyhedral 2-sphere that is homeomorphically within e_{i1} of S , such that there exist disjoint disks D_{i1}, \dots , and $D_{im(i)}$ on S , each of diameter less than e_{i1} , with the property that $\text{Cl}(S - \bigcup_{j=1}^{n(i)} D_{ij})$ is contained in $\text{Ext } S_{ii}$, and there exist disjoint disks K_{i1}, \dots , and $K_{im(i)}$ on S_{ii} , each of diameter less than e_{i1} , with the property that $\text{Cl}(S_{ii} - \bigcup_{j=1}^{m(i)} K_{ij})$ is contained in $\text{Int } S$. Such a 2-sphere exists by Bing's Side Approximation Theorem. (6) Let e_{i2} be

$$\begin{aligned} & (1/3) \min \{ \min \{ d(D_{iu}, D_{iv}) : u \neq v, 1 \leq u \leq n(i), \text{ and } 1 \leq v \leq n(i) \}, \\ & \min \{ d(K_{iu}, K_{iv}) : u \neq v, 1 \leq u \leq m(i), \text{ and } 1 \leq v \leq m(i) \}, \\ & \min \{ d(D_{iu}, S_{ii} - \bigcup_{j=1}^{m(i)} K_{ij}) : 1 \leq u \leq n(i) \} \}. \end{aligned}$$

(7) Let M_{ii} be $\{x : x \in S_i \cup \text{Int } S_i \text{ and } d(x, \text{Ext } S) \leq e_{i2}\}$ and let M_{ii} be a polyhedral 3-manifold-with-boundary such that $[S, S_i] \subset \text{Int } M_{ii}$ and $M_{ii} \subset M'_{ii}$. It is assumed that M_{ii} and S_{ii} are in relative general position. (8) Let M_i be the union of M_{ii} and $[S_{ii}, S_i]$. (9) The closure of $M_i - [S_{ii}, S_i]$ has a finite number of components, and let F_{i1}, \dots , and $F_{ir(i)}$ be the components of $\text{Cl}(M_i - [S_{ii}, S_i])$. (10) $\{E_{i1}, \dots, E_{iq(i)}\}$ is an A -subdivision of $[S_{ii}, S_i]$ such that (a) for each $j, 1 \leq j \leq n(i)$, there exists a k such that $\text{Bd } D_{ij} \subset \text{Int } E_{ik}$, (b) for each $j, 1 \leq j \leq q(i)$, the diameter of E_{ij} is less than $5e_i$, and (c) $q(i) \geq q(i-1)$. (11) For each $j, 1 \leq j \leq q(i)$, A_{ij}^* and A_{ij} are defined as follows: $A_{ij}^* = \bigcup \{x : x \in \{E_{i1}, \dots, E_{iq(i)}\} \text{ or } x \in \{F_{i1}, \dots, F_{ir(i)}\} \text{ and } d(x, E_{ij}) < e_i\}$, and A_{ij} is the component of A_{ij}^* that contains E_{ij} . (12) For each p in S , let A_{pi} be $\bigcup \{A_{ij} : p \in A_{ij}\}$. (13) For each p in S , if p is in $[S_{ii}, S_i]$ then $Z_{p(i-1)}$ is the component of $A_{pi} \cap [S_{ii}, S_i]$ that contains p . (14) For each p in S , C_{pi} is a 3-cell that contains A_{pi} in its interior and is contained in the interior of M_{i-1} . (15) W_{i-1} is the component of $S_i \cap \{\bigcup_{j=1}^{q(i-1)} \text{Bd } E_{(i-1)j}\}$ with diameter greater than $(1/2) \text{diam } C$. (16) h_i is a homeomorphism from $S_i \cup \text{Int } S_i$ onto $B(1 + 10^{-i})$ such that (a) $h_i[[S_{ii}, S_i]] = [\text{Bd } B(1 - 10^{-i}), \text{Bd } B(1 + 10^{-i})]$, (b) $h_i[C] \subset \text{Int } B(1 - 10^{-i-1})$, (c) if $x \in W_{i-1}$ then $h_i(x)$ is equal to the radial projection of $h_{i-1}(x)$ onto $\text{Bd } B(1 + 10^{-i})$, (d) if x is in $C - M_i$ then $h_i(x) = h_{i-1}(x)$, (e) h_i differs from h_{i-1} only at points close to $[S_{ii}, S_i]$, and (f) $\{h_i[E_{i1}], \dots, h_i[E_{iq(i)}]\}$ is a canonical subdivision of $[\text{Bd } B(1 - 10^{-i}), \text{Bd } B(1 + 10^{-i})]$.

It follows that $B(1) = \bigcap_{i=1}^{\infty} h_i[S_i \cup \text{Int } S_i]$. Let G be the following decomposition of $B(1)$: $G = \{g : \text{either (1) for some point } p \text{ of } B(1) \text{ there exists a positive}$

integer n such that $p \notin h_n[M_n]$ and $g = \{p\}$, or (2) there exists a q in S such that $g = \bigcap_{i=1}^{\infty} h_i[A_{qi}]$. Then G is an upper semi-continuous decomposition of $B(1)$. It follows from the way the A_{pi} 's and the h_i 's are defined that the intersection of each nondegenerate element of G with the boundary of $B(1)$ is a one-point set.

If g is a nondegenerate element of G , then there exists a q in S such that $g = \bigcap_{i=1}^{\infty} h_i[A_{qi}]$. But $\bigcap_{i=1}^{\infty} h_i[A_{qi}]$ is equal to $\bigcap_{i=1}^{\infty} h_i[C_{qi}]$. Hence g is point-like.

Define a function h from C into $B(1)/G$ as follows: If $x \in \text{Int } C$, $h(x) = \{\lim h_i(x)\}$ and if $x \in S$, $h(x) = \bigcap_{i=1}^{\infty} h_i[A_{xi}]$. Then h is one-to-one. If not, there exist points p and q in C such that $p \neq q$ and $h(p) = h(q)$. Clearly, neither p nor q can be in $\text{Int } C$. If p and q are in S then there exists an integer n such that A_{pn} and A_{qn} are disjoint, and hence $h_n[A_{pn}]$ and $h_n[A_{qn}]$ are disjoint. This is a contradiction. Clearly h is onto $B(1)/G$ and h is a homeomorphism on $\text{Int } C$. To show h is a homeomorphism it suffices to show h is continuous at each point of S . Suppose p is in S , $\{a_i\}$ is a sequence in C , and $\{a_i\}$ converges to p . If U is an open set in $B(1)/G$ that contains $h(p)$, then there exists a positive integer k such that $h_k[A_{pk}]$ is contained in U . Since $\{a_i\}$ converges to p , all but finitely many of the a_i 's belong to A_{pi} . Therefore all but finitely many of the $h_i(a_i)$'s belong to $h_i[A_{pi}]$. By the way the h_i 's are constructed, if $j \geq i$ and $h_i(a_i) \in h_i[A_{pi}]$, then $h_j(a_i) \in h_i[A_{pi}]$, and hence all but finitely many of the $h(a_i)$'s belong to U . Therefore h is continuous at p and h is homeomorphism.

COROLLARY 1. *If C is a crumpled cube then there exists an upper semi-continuous decomposition G of $B(1)$ such that the intersection of each nondegenerate element of G with the boundary of $B(1)$ is a one-point set, each nondegenerate element of G is point-like, and $B(1)/G$ is homeomorphic to C .*

Proof. This follows directly from Theorem 2 and Theorem 3.

COROLLARY 2. *If C and D are crumpled cubes, S is a tame 2-sphere in S^3 , and C and D are sewn together along their boundaries, then there exists an upper semi-continuous decomposition G of S^3 such that the intersection of each nondegenerate element of G with S is a one-point set and S^3/G is homeomorphic to the sum of C and D .*

4. Two examples.

EXAMPLE 1. In [3] Bing constructed a 2-sphere S with the property that S is wild at each point and one of the complementary domains of $S^3 - S$ is an open 3-cell. In this example, it is shown that if C is the union of S and the "bad" complementary domain of $S^3 - S$ then the union of two copies of C sewn together along their boundaries by the identity homeomorphism is topologically S^3 .

The notation and terminology used will be the same as that used by Bing in [3]. Let X denote the sum of two copies of C sewn together along their boundaries by the identity homeomorphism. An upper semi-continuous decomposition G of

S^3 will be described. It will then be shown that X is homeomorphic to S^3/G and S^3/G is homeomorphic to S^3 .

A decomposition of S^3 . Let S' denote $\{(x, y, z) : x = 0\} \cup \{\infty\}$, and let f be the homeomorphism defined by $f(x, y, z) = (-x, y, z)$ if $(x, y, z) \in E^3$ and $f(\infty) = \infty$. Let f_1 be a homeomorphism from S^3 onto S^3 such that $f_1[S_0] = S'$ and for each $i, 1 \leq i \leq n$, let A_i denote $f_1[T_i] \cup ff_1[T_i]$. Let f_2 be a homeomorphism from S^3 onto S^3 such that $f_2f_1[S_1] = S'$ and if $x \in S^3 - \bigcup_{i=1}^n A_i$ then $f_2(x) = x$, and for each i and $j, 1 \leq i \leq n, 1 \leq j \leq 15$, let A_{ij} denote $f_2[T_{ij}] \cup ff_2[T_{ij}]$. The process is continued by induction. Let A_0 be $(\bigcup A_i) \cap (\bigcup A_{ij}) \cap (\bigcup A_{ijk}) \cap \dots$. It is assumed that the f_i 's are defined so that each component of A_0 is a tame arc that intersects S' in a one-point set.

Let G denote $\{g : g \text{ is a component of } A_0 \text{ or there exists a } p \text{ in } S^3 - A_0 \text{ such that } \{p\} = g\}$. Then G is an upper semi-continuous decomposition of S^3 and S^3/G is homeomorphic to X .

To prove that S^3/G is homeomorphic to S^3 it will be assumed that S^3 is $\{(x, y, z, w) : |x| + |y| + |z| + |w| = 1\}$ and S' is $\{(x, y, z, w) : (x, y, z, w) \in S^3 \text{ and } w = 0\}$.

If B is the homeomorphic image of A_i then B_{ij} will denote the homeomorphic image of A_{ij}, B_{ijk} denotes the homeomorphic image of A_{ijk} , etc.

LEMMA 6. *If B is the homeomorphic image of A_i in S^3, F_1, \dots, F_m are polyhedral disks such that (1) for each $i, 1 \leq i \leq m$, if C is a component of $F_i \cap B$ then C is a polyhedral disk, $\text{Int } C \subset \text{Int } B$, and $B - C$ has exactly two components, the closure of each being a solid torus, and (2) for each pair of integers i and $j, 1 \leq i \leq m, 1 \leq j \leq m$, and $i \neq j$, some component of $B - F_i$ contains $B \cap F_j$ and some component of $B - F_j$ contains $B \cap F_i$. Then there is a homeomorphism h and a positive integer j such that h is from S^3 onto S^3 , if $x \in S^3 - B$ then $h(x) = x$, and for each $B_{i_1 \dots i_j}, h[B_{i_1 \dots i_j}]$ intersects at most one of the elements of $\{F_1, \dots, F_m\}$.*

Proof. Let $Q(m)$ be the statement that Lemma 6 is true if there are m elements in $\{F_1, \dots, F_m\}$.

If h is the identity homeomorphism and $j = 1$ then $Q(1)$ is true.

Assume $Q(k)$ is true for $k \leq m - 1$. Suppose F_1, \dots, F_m are polyhedral disks that satisfy the hypotheses of Lemma 6. Let $\{C_1, \dots, C_k\}$ be the set of components of $(\bigcup F_i) \cap B$ and assume the numbering is such that for each $j, 2 \leq j \leq k - 1, \bigcup_{i < j} C_i$ and $\bigcup_{i > j} C_i$ are in different components of $B - C_j$. Then there exists integers r, s, t and v such that $\bigcup_{i=1}^r C_i = F_s \cap B$ and $\bigcup_{i=t}^m C_i = F_v \cap B$. Then there exist a homeomorphism h' such that (1) h' is from S^3 onto S^3 , (2) if $x \in S^3 - B$ then $h'(x) = x$, (3) for each i and $j, 1 \leq j \leq 15$, and $1 \leq i \leq m$, if C is a component of $h'[B_j] \cap F_i$, then C is a polyhedral disk, $\text{Int } C \subset \text{Int } B$, and $h'[B_j] - C$ has exactly two components, the closure of each being a solid torus, (4) for each triple

of integers $i, j, k, 1 \leq i \leq m, 1 \leq j \leq m, 1 \leq k \leq 15$, and $i \neq j$, such that F_i and F_j intersect $h'[B_k]$, some component of $h'[B_k] - F_j$ contains $F_i \cap h'[B_k]$ and some component of $h'[B_k] - F_i$ contains $F_j \cap h'[B_k]$, and (5) for each $k, 1 \leq k \leq 15, h'[B_k]$ intersects at most $m - 1$ of the elements of $\{F_1, \dots, F_m\}$. Hence for each $k, 1 \leq k \leq 15$, there exists an integer $z \leq m - 1$ such that $h'[B_k]$ satisfies the hypotheses of Lemma 6 and $h'[B_k]$ intersects at most z of the elements in $\{F_1, \dots, F_m\}$. Therefore there exists a homeomorphism h_k and an integer $j(k)$ such that h_k is from S^3 onto S^3 , if $x \in S^3 - h'[B_k]$ then $h_k(x) = x$, and for each $h'[B_{i_1 \dots i_{j(k)}}], h_k[h'[B_{i_1 \dots i_{j(k)}}]$ intersects at most one of the elements in $\{F_1, \dots, F_m\}$. Let $h = h_1 h_2 \dots h_{15} h'$ and let $j = \max \{j(k) : 1 \leq k \leq 15\}$. Then h and j satisfy the conclusion of Lemma 6. Hence $Q(m)$ is true, and the Lemma is proved.

LEMMA 7. *If B is the homeomorphic image of A_i in S^3 and P_1, \dots, P_m are disjoint polyhedral 2-spheres in S^3 , then there is a homeomorphism h and a positive integer q such that h is from S^3 onto S^3 , if $x \in S^3 - B$ then $h(x) = x$, and for each $B_{i_1 \dots i_q}, h[B_{i_1 \dots i_q}]$ intersects at most one of the elements in $\{P_1, \dots, P_m\}$.*

Proof. There exists a homeomorphism k from S^3 onto S^3 such that if $x \in S^3 - B$ then $k(x) = x$, for each i and $j, 1 \leq j \leq 15$, and $1 \leq i \leq n, k[B_{ij}]$ is polyhedral and $k[B_{ij}], P_1, \dots, P_m$ are in relative general position, and for each i and $j, 1 \leq j \leq 15$, and $1 \leq i \leq n$, if C is a component of $k[B_{ij}] \cap (\bigcup_{i=1}^m P_i)$, then C is a polyhedral disk, $\text{Int } C \subset \text{Int } k[B_{ij}]$, and $k[B_{ij}] - C$ has exactly two components, the closure of each being a solid torus. Let $\{F_1, \dots, F_r\}$ be the collection of components of $(\bigcup_{j=1}^{15} k[B_{ij}]) \cap (\bigcup_{i=1}^m P_i)$.

Lemma 7 then follows from Lemma 6.

Suppose the 2-spheres P_1, \dots, P_m in the hypothesis of Lemma 7 satisfy the following conditions: (1) For each $r, 1 \leq r \leq m$, there exists a real number $t(r)$ such that $P_r = \{(x, y, z, w) : (x, y, z, w) \in S^3 \text{ and } w = t(r)\}$ and (2) For each $r, 2 \leq r \leq m, t(r) > t(r - 1)$. If v is a real number such that for each $r, 2 \leq r \leq m, t(r) - t(r - 1) < v$, then the homeomorphism h in Lemma 7 can be constructed such that if M is a subset of $h[B_{i_1 \dots i_q}]$, then the diameter of M is less than $3v$. Therefore Lemma 7 can be strengthened to obtain the following result.

LEMMA 8. *If B is the homeomorphic image of A_i in S^3 and e is a positive number, then there exists a homeomorphism h and a positive integer q such that h is from S^3 onto S^3 , if $x \in S^3 - B$ then $h(x) = x$, and for each $B_{i_1 \dots i_q}$, the diameter of $h[B_{i_1 \dots i_q}]$ is less than e .*

THEOREM 4. *If G is the upper semi-continuous decomposition of S^3 whose non-degenerate elements are the components of A_0 , then S^3 / G is homeomorphic to S^3 .*

Proof. It follows from Lemma 8 that for each positive integer j there exists a homeomorphism t_j and an integer $q(j)$ such that t_j leaves $S^3 - \bigcup A_{i_1 \dots i_j}$ pointwise

fixed, and for each nondegenerate element g of G the diameter of $t_j[g]$ is at most $1/j$. It follows that S^3 is homeomorphic to S^3/G .

EXAMPLE 2. A "nice" crumpled cube D with the property that the union of two copies of D sewn together along their boundaries by the identity homeomorphism is not homeomorphic to S^3 .

Description. Let P be the 2-sphere $\{(x, y, z) \in E^3 : x = 0\} \cup \{\infty\}$ and let K be the 3-cell $\{(x, y, z) \in E^3 : x \geq 0\} \cup \{\infty\}$. Let J be the Fox-Artin arc of Figure 6, contained in $\text{Int } K$ except for its left endpoint which lies on P . Let G'

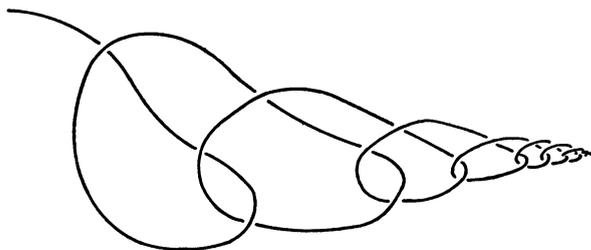


FIGURE 6

be the upper semi-continuous decomposition of S^3 whose only nondegenerate element is J . Since J is cellular, S^3/G' is homeomorphic to S^3 and the projection map ϕ from S^3 onto S^3/G' is a homeomorphism on P . Hence $\phi[K]$ is a crumpled cube.

Let G be the upper semi-continuous decomposition of S^3 whose only nondegenerate element is $\bigcup\{(x, y, z) : (|x|, y, z) \in J\}$. Then S^3/G is homeomorphic to the union of two copies of $\phi[K]$ sewn together along their boundaries by the identity homeomorphism. It follows from Example 1.3 of [8] that S^3/G is not homeomorphic to S^3 .

BIBLIOGRAPHY

1. B. J. Ball, *The sum of two solid horned spheres*, Ann. of Math. (2) **69** (1959), 253-257.
2. R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. **56** (1952), 354-362.
3. ———, *A wild surface each of whose arcs is tame*, Duke, Math. J. **28** (1961), 1-15.
4. ———, *Approximating surfaces from the side*, Ann. of Math. (2) **77** (1963), 145-192.
5. ———, *Conditions under which a surface in E^3 is tame*, Fund. Math. **47** (1959), 105-139.
6. ———, Mimeographed Notes, Topology Institute, University of Georgia, Summer, 1961.
7. B. G. Casler, *On the sum of two solid Alexander horned spheres*, Notices Amer. Math. Soc. **9** (1962), 108.
8. R. H. Fox and E. Artin, *Some wild cells and spheres in three dimensional space*, Ann. of Math. (2) **49** (1948), 979-990.
9. Norman Hosay, *The sum of a real cube and a crumpled cube is S^3* , Notices Amer. Math. Soc. **10** (1963), 666.

UNIVERSITY OF IOWA,
IOWA CITY, IOWA