MEROMORPHIC MULTIVALENT CLOSE-TO-CONVEX FUNCTIONS

BY
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1. Introduction. Recently the author [5] has discussed some properties of the class of \( p \)-valent regular close-to-convex functions, called \( \mathcal{K}(p) \). It is the purpose of this paper to generalize some of these results to the meromorphic case.

Let \( f(z) \) be meromorphic for \( |z| < 1 \) with \( q \) (\( 1 \leq q \leq p \)) poles at the origin and \( f(z) \neq 0 \) for \( |z| < 1 \). We shall say that \( f(z) \) is in \( S^*_p(p) \) if there exists a \( \rho \) (\( 0 < \rho < 1 \)) such that for \( z = re^{\theta} \) (\( \rho < r < 1 \)),

\[
\text{Re} \left( \frac{zf''(z)}{f(z)} \right) < 0
\]

and

\[
\int_0^{2\pi} d \arg f(z) = \int_0^{2\pi} \text{Re} \left( \frac{zf''(z)}{f(z)} \right) d\theta = -2p\pi.
\]

We shall say that \( f(z) \) is in \( S^*_2(p) \) if it is regular on \( |z| = 1 \) and if (1.1) and (1.2) hold for \( |z| = 1 \). If \( f(z) \) is in \( S^*_p(p) \), there exists a \( \delta \) (\( 0 < \delta < 1 \)) such that \( f(rz) \) is in \( S^*_2(p) \) if \( \delta < r < 1 \).

We set \( S^*(p) = S^*_1(p) \cup S^*_2(p) \) and say that a function in \( S^*(p) \) is starlike of order \( p \).

Condition (1.2) along with the argument principle implies that a function in \( S^*(p) \) has exactly \( p \) poles in \( |z| < 1 \). It is easily seen that a function \( f(z) \), meromorphic in \( |z| < 1 \), is in \( S^*(p) \) if and only if the function \( [f(z)]^{-1} \) is regular and \( p \)-valently starlike in \( |z| < 1 \). Since the reciprocal of a \( p \)-valent function is \( p \)-valent, a function in \( S^*(p) \) is \( p \)-valent in \( |z| < 1 \). Also, using the fact that a regular \( p \)-valent starlike function can be written as the \( p \)th power of a regular univalent starlike function, it is easily seen that a function in \( S^*(p) \) with \( p \) poles at the origin can be written as the \( p \)th power of a meromorphic univalent starlike function.

Let \( F(z) \) be meromorphic in \( |z| < 1 \) with \( q \) (\( 1 \leq q \leq p \)) poles at the origin and with at most \( p \) poles in \( |z| < 1 \). We shall say that \( F(z) \) is in \( \mathcal{K}^*_q(p) \) if there exists a function in \( S^*(p) \) and a \( \rho \) (\( 0 < \rho < 1 \)) such that for \( z = re^{\theta} \) (\( \rho < r < 1 \))

\[
\text{Re} \left( \frac{zF'(z)}{F(z)} \right) > 0.
\]

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We shall say that $F(z)$ is in $\mathcal{K}^*(p)$ if $F(z)$ is regular on $|z| = 1$ and if there exists a function $f(z)$ in $S^*_2(p)$ such that (1.3) is satisfied for $|z| = 1$. If $F(z)$ is in $\mathcal{K}^*_r(p)$, there exists a $\delta$ ($0 < \delta < 1$) such that $F(rz)$ is in $\mathcal{K}^*_r(p)$ if $\delta < r < 1$.

We set $\mathcal{K}^*(p) = \mathcal{K}^*_1(p) \cup \mathcal{K}^*_2(p)$ and say that a function in $\mathcal{K}^*(p)$ is close-to-convex of order $p$.

The class $\mathcal{K}^*(1)$ was defined by Libera and Robertson [4] and Pommerenke [7]. It was shown in both papers that a function in $\mathcal{K}^*(1)$ need not be univalent. To show that a function in $\mathcal{K}^*(p)$ need not be $p$-valent, let $F(z)$ be such that

$$\frac{zF'(z)}{z^{-p}} = \frac{1 + z^{2p}}{1 - z^{2p}} \quad (|z| < 1).$$

Then

$$F(z) = -\frac{1}{pz^p} + \frac{2}{p} z^{2p} + \frac{2}{3p} z^{3p} + \cdots \quad (0 < |z| < 1).$$

If $F(z)$ was $p$-valent, then

$$F(z^{1/p}) = -\frac{1}{pz} + \frac{2}{p} z + \cdots \quad (0 < |z| < 1)$$

would be univalent, and so would

$$-pF(z^{1/p}) = \frac{1}{z} - 2z + \cdots \quad (0 < |z| < 1).$$

But this is impossible, since the coefficient of $z$ has modulus greater than 1. Thus $F(z)$ is at least $2p$-valent.

Necessary and sufficient conditions for a function to be in $\mathcal{K}^*(1)$ have been given in [4] and [7]. In §2 we obtain necessary conditions for a function $F(z)$ to be in $\mathcal{K}^*(p)$ and show that these conditions with the added assumptions of regularity on $|z| = 1$ and $F'(z) \neq 0$ in $|z| \leq 1$ are sufficient.

Recently, Royster [8] has shown that if

$$f(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1)$$

is in $S^*(p)$ then $|a_n| = O(1/n)$. In §3 we will extend this result to functions in $\mathcal{K}^*(p)$ with $p$ poles at the origin. This result was obtained for $\mathcal{K}^*(1)$ by Libera and Robertson [4] and Pommerenke [7].

2. The class $\mathcal{K}^*(p)$.

**Theorem 1.** If $F(z)$ is in $\mathcal{K}^*(p)$, then there exists $\rho$ ($0 < \rho < 1$) such that for $z = re^{i\theta}$ ($r < 1$)
(2.1) \[ \int_0^{2\pi} d\arg d F(z) = \int_0^{2\pi} \frac{d}{d\theta} \arg \left[ re^{i\theta} F'(re^{i\theta}) \right] d\theta = -2\pi \]

and for any \( \theta_1 \) and \( \theta_2 \) with \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \)

(2.2) \[ \int_{\theta_1}^{\theta_2} d\arg d F(z) < \pi. \]

**Proof.** Suppose \( F(z) \) is in \( \mathcal{K}_1^*(p) \). Then there exists \( f(z) \) in \( S^*(p) \) and \( \rho \) \((0 < \rho < 1)\) such that (1.1), (1.2) and (1.3) hold for \( \rho < |z| < 1 \).

Since \( \text{Re} \left[ zF'(z)/f(z) \right] > 0 \) for \( |z| = r \) \((\rho < r < 1)\), we may define \( \arg \left[ zF'(z)/f(z) \right] \) to be single valued and continuous for \( |z| = r \) and such that

(2.3) \[ \left| \arg \frac{zF'(z)}{f(z)} \right| < \pi \quad (|z| = r). \]

Furthermore, since \( zF'(z) \neq 0 \) for \( |z| = r \), we may define \( \arg \left[ zF'(z) \right] \) to be single valued and continuous for \( |z| = r \). Since \( f(z) = \left[ f(z)/zF'(z) \right] \left[ zF'(z) \right] \), we may define \( \arg \left[ f(z) \right] = \arg \left[ zF'(z) \right] - \arg \left[ zF'(z)/f(z) \right] \) to be a single valued and continuous determination of \( \arg \left[ f(z) \right] \) for \( |z| = r \). Then

(2.4) \[ \left| \arg zF'(z) - \arg f(z) \right| = \left| \arg \frac{zF'(z)}{f(z)} \right| < \frac{\pi}{2} \quad (|z| = r). \]

It is easily seen that (2.4) implies

(2.5) \[ -\pi + \int_{\theta_1}^{\theta_2} d\arg f(z) < \int_{\theta_1}^{\theta_2} d\arg d F(z) < \pi + \int_{\theta_1}^{\theta_2} d\arg f(z) \]

for \( \theta_1 < \theta_2 \) and \( |z| = r \). Since \( f(z) \) is in \( S^*(p) \),

\[ \int_{\theta_1}^{\theta_2} d\arg f(z) < 0 \quad (|z| = r). \]

Thus we obtain (2.2) for \( |z| = r \) from the right side of (2.5). Letting \( \theta_1 = 0 \) and \( \theta_2 = 2\pi \) in (2.5) and noting that

\[ \int_0^{2\pi} d\arg f(z) = -2\pi \]

we obtain

(2.6) \[ -(2p + 1)\pi < \int_0^{2\pi} d\arg d F(z) < -(2p - 1)\pi. \]

However, the integral appearing in (2.6) is an integral multiple of \( 2\pi \). Thus (2.1) holds for \( |z| = r \). Since \( r \) was arbitrary \((\rho < r < 1)\) (2.1) and (2.2) hold for \( \rho < |z| < 1 \).
If $F(z)$ is in $\mathcal{X}_2^*(p)$, then the preceding argument with $r = 1$ shows that (2.1) and (2.2) hold for $|z| = 1$. But since $F(z)$ is regular near $|z| = 1$, we can show the existence of a $\rho$ ($0 < \rho < 1$) such that (2.1) and (2.2) hold for $\rho < |z| \leq 1$.

Using (2.1) and the argument principle we immediately obtain the following corollary.

**Corollary 1.** If $F(z)$ is in $\mathcal{X}_2^*(p)$, then $F'(z)$ has at least $(p + 1)$ poles in $|z| < 1$ and if $F'(z) \neq 0$ for $|z| < 1$, then $F'(z)$ has exactly $(p + 1)$ poles in $|z| < 1$.

**Theorem 2.** Let $F(z)$ be meromorphic in $|z| < 1$ with $q$ ($1 \leq q \leq p$) poles at the origin. If $F'(z) \neq 0$ for $0 < |z| \leq 1$ and $F(z)$ is regular on $|z| = 1$ and if (2.1) and (2.2) hold for $|z| = 1$, then $F(z)$ is in $\mathcal{X}_2^*(p)$.

**Proof.** Consider the function $G(z)$, regular for $|z| = 1$, given by

$$G(z) = \int_0^z \frac{dz}{z^2 F'(z)} = b_0 + \cdots .$$

Since $zF'(z) \neq 0$ for $|z| = 1$ we may define $\arg [zF'(z)]$ to be single valued and continuous for $|z| = 1$. Since $zG'(z) = [zF'(z)]^{-1}$, we may define $\arg zG'(z) = - \arg zF'(z)$. Thus, for $|z| = 1$

$$\int_0^{2\pi} d \arg d G(z) = 2\pi$$

and

$$\int_{\theta_1}^{\theta_2} d \arg d G(z) = -\pi \quad (\theta_1 < \theta_2).$$

The author has shown (Theorem 3 [5]) that under these conditions $G(z)$ is in $\mathcal{X}(p)$. That is, there exists $g(z)$, regular for $|z| \leq 1$ such that

$$\Re \left[ \frac{zG'(z)}{g(z)} \right] > 0 \quad (|z| = 1)$$

and

$$\Re \left[ \frac{zG'(z)}{g(z)} \right] > 0 \quad (|z| = 1).$$

The function $f(z) = [g(z)]^{-1}$ is in $S^*(p)$ and

$$\frac{zG'(z)}{g(z)} = zG'(z)f(z) = \frac{f(z)}{zF'(z)}.$$
Therefore $F(z)$ is in $\mathcal{K}^*(p)$.

Using the same procedure as above and by appealing to Theorem 2 [5], we may remove the condition of regularity on $|z| = 1$, if $q = p$. We thus have the following theorem.

**Theorem 3.** Let

$$F(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1)$$

be meromorphic for $|z| < 1$ and $F'(z) \neq 0$. If there exists a $\rho$ $(0 < \rho < 1)$ such that (2.1) and (2.2) hold for $\rho < |z| < 1$, then $F(z)$ is in $\mathcal{K}^*(p)$.

We will have need of the next lemma in what follows.

**Lemma 1.** Let $F(z)$ be in $\mathcal{K}^*(p)$. Then, there exists a function

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_{-p}| = 1),$$

in $S^*_2(p)$, such that

$$\text{Re} \left[ \frac{zF'(z)}{f(z)} \right] > 0 \quad (|z| = 1).$$

**Proof.** There exists a function $g(z)$ in $S^*_2(p)$ with $s$ poles $(1 \leq s \leq p)$ at the origin such that,

$$\text{Re} \left[ \frac{zF'(z)}{g(z)} \right] > 0 \quad (|z| = 1).$$

The function $g(z)$ has $(p - s)$ nonzero poles in $|z| < 1$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{p-s}$ be these poles and let

$$h(z) = z^{s-p} \prod_{i=1}^{p-s} (z - \alpha_i) \left(1 - \bar{\alpha}_i z\right)$$

and

$$f(z) = h(z)g(z) = \sum_{n=-p}^{\infty} c_n z^n \quad (0 < |z| < 1).$$

Since $[zh'(z)/h(z)]$ is purely imaginary on $|z| = 1$ and $\text{Re} \left[ zg'(z)/g(z) \right] < 0$ for $|z| = 1$, then $\text{Re} \left[ zf'(z)/f(z) \right] < 0$ for $|z| = 1$. Furthermore, since $f(z)$ has $p$ poles in $|z| \leq 1$, all of them at the origin, and since $f(z) \neq 0$ in $|z| \leq 1$,

$$\int_0^{2\pi} \text{Re} \left[ \frac{e^{i\theta}f'(e^{i\theta})}{f(e^{i\theta})} \right] d\theta = -2p\pi.$$
Thus, \( f(z) \) in \( S^*_2(p) \). Furthermore,

\[
\frac{zF'(z)}{f(z)} = \frac{z^{p-z}zF'(z)}{\prod_{i=1}^{p}(z - \alpha_i)(1 - \overline{\alpha}_i)zg(z)}.
\]

But \( z^{p-z}[\prod_{i=1}^{p}(z - \alpha_i)(1 - \overline{\alpha}_i)]^{-1} \) is real and positive on \( |z| = 1 \). Therefore

\[
\text{Re} \left[ \frac{zF'(z)}{f(z)} \right] > 0 \quad (|z| = 1).
\]

Replacing \( f(z) \) by \( 1/|f(z)|f(z) \), the proof of the lemma is completed.

**Theorem 4.** If \( f(z) \) is in \( \mathcal{H}^*(p) \) and has all its poles at the origin, then necessarily it has \( p \) poles there and \( F'(z) \neq 0 \) for \( |z| < 1 \).

**Proof.** Suppose

\[
F(z) = \sum_{n=-q}^{\infty} a_n z^n \quad (0 < |z| < 1) \quad (1 \leq q \leq p).
\]

There exists a \( \rho \) \( (0 < \rho < 1) \) such that \( F(rz) \) is in \( \mathcal{H}^*_2(p) \) if \( \rho < r < 1 \). By Lemma 1, there exists

\[
f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1)
\]

in \( S^*_2(p) \) such that

\[
\text{Re} \left[ \frac{rzF'(rz)}{f(z)} \right] > 0 \quad (|z| = 1).
\]

Since \( f(z) \neq 0 \) for \( |z| \leq 1 \),

\[
\frac{rzF'(rz)}{f(z)} = \sum_{n=-p}^{\infty} c_n z^n
\]

is regular for \( |z| \leq 1 \). Thus,

\[
\text{Re} \left[ \frac{rzF'(rz)}{f(z)} \right] > 0 \quad (|z| \leq 1).
\]

Therefore, we must necessarily have \( q = p \) and \( F'(rz) \neq 0 \) for \( |z| \leq 1 \). Thus, \( F'(z) \neq 0 \) for \( |z| \leq r \). Since \( r \) was arbitrary \( (\rho < r < 1) \), \( F'(z) \neq 0 \) for \( |z| < 1 \).

If \( F(z) \) has all its poles at the origin we may improve Lemma 1 by removing the condition of regularity on \( |z| = 1 \).

**Lemma 2.** Let

\[
F(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1)
\]
be in \( \mathcal{K}^*(p) \). Then there exists
\[
f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_p| = 1)
\]
in \( S^*(p) \) such that
\[
\text{Re} \frac{zF'(z)}{f(z)} > 0 \quad (|z| < 1).
\]

**Proof.** There exists a \( \rho (0 < \rho < 1) \) such that the function \( F_\rho(z) = F(\rho z) \) is in \( \mathcal{K}^*_\rho(p) \) if \( \rho < r < 1 \). Then by Lemma 1 there exists
\[
f_\rho(z) = \sum_{n=-p}^{\infty} c_n z^n \quad (0 < |z| < 1) \quad (|c_p| = 1)
\]
in \( S^*_\rho(p) \), such that
\[
\text{Re} \left[ \frac{zF'(z)}{f_\rho(z)} \right] > 0 \quad (|z| \leq 1).
\]

Let \( r_i (\rho < r_i < 1) \) be an increasing sequence tending to 1. The functions \( [f_\rho(z)]^{-1} \) are regular and \( p \)-valently starlike and have the moduli of their first \( p \) coefficients fixed. The class of regular and \( p \)-valently starlike functions with the moduli of their first \( p \) coefficients fixed forms a normal family of functions [1]. Thus, we can obtain a subsequence \( [f_{r_k}(z)]^{-1} \) tending uniformly in every closed subset of \( |z| < 1 \) to a function \( f(z) \) regular and \( p \)-valently starlike and such that
\[
f(z) = \sum_{n=p}^{\infty} d_n z^n \quad (|z| < 1) \quad (|d_p| = 1).
\]
Since \( F_{r_k}(z) \) tends to \( F(z) \) as \( r_k \) tends to 1 and since
\[
\text{Re}[zF_{r_k}(z)[f_{r_k}(z)]^{-1}] > 0 \quad \text{for } |z| < 1
\]
we have
\[
\text{Re} [zF'(z)f(z)] > 0 \quad \text{for } |z| < 1.
\]

But
\[
g(z) = [f(z)]^{-1} = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_p| = 1)
\]
is in \( S^*(p) \) and
\[
\text{Re} \left[ \frac{zF'(z)}{g(z)} \right] = \text{Re} [zF'(z)f(z)] > 0 \quad \text{for } |z| < 1.
\]

3. **The coefficients of a function in \( \mathcal{K}^*(p) \).** We will make use of the following lemma, proven by Royster [8] and the author [6].
Lemma 3. Let

\[ f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_p| = 1) \]

be in \( S^*(p) \), then for \( n \geq 1 \)

\[ |b_n| \leq \frac{2p}{(n+p)\sqrt{p}} \left( \sum_{k=-p}^{-1} |k| |b_k|^2 \right)^{1/2}. \]

The following lemma was proven for \( p = 1 \) by Pommerenke [7].

Lemma 4. Let

\[ F(z) = \frac{1}{z^p} + \sum_{n=-(p-1)}^{\infty} a_n z^n \quad \text{and} \quad f(z) = \frac{e^{i\theta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n, \quad (0 < |z| < 1) \]

and let \( U(z) = \text{Re} \left[ zF'(z)/f(z) \right] \), then for \( r < 1 \)

(3.1) \[ na_n = -pe^{-i\theta} b_n + \frac{1}{\pi} \int_0^{2\pi} \left[ U(re^{i\theta}) \frac{e^{-in\theta}}{r^n} \right] \]

\[ \times \left[ f(re^{i\theta}) - \sum_{k=n}^{\infty} b_k(re^{i\theta})^k \right] d\theta. \]

Proof. Let

\[ \frac{zF'(z)}{f(z)} = -pe^{-i\theta} + \sum_{k=1}^{\infty} C_k z^k \quad (|z| < 1). \]

Then

\[ \frac{-p}{z^p} + \sum_{n=-(p-1)}^{\infty} na_n z^n = \left[ -pe^{-i\theta} + \sum_{k=1}^{\infty} C_k z^k \right] \left[ \frac{e^{i\theta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n \right] \]

\[ = \frac{-p}{z^p} - pe^{-i\theta} \left[ \sum_{n=-(p-1)}^{\infty} b_n z^n \right] + e^{i\theta} \left[ \sum_{k=1}^{\infty} C_k z^{k-p} \right] \]

\[ + \sum_{n=-(p-2)}^{\infty} \left[ \sum_{k=1}^{n+1} C_k b_{n-k} \right] z^n. \]

Thus, for \( n \geq 1 \)

(3.2) \[ na_n = -pe^{-i\theta} b_n + e^{i\theta} C_{p+n} + \sum_{k=1}^{n+p-1} C_k b_{n-k}. \]

Now

\[ C_k = \frac{1}{rk\pi} \int_0^{2\pi} U(re^{i\theta}) e^{-ik\theta} d\theta. \]
Substituting into (3.2), we obtain

\[ na_n = -pe^{-i\theta}b_n + \frac{1}{\pi} \int_0^{2\pi} \left[ U(re^{i\theta}) \frac{e^{-i\theta}}{r^n} \right] \times \left[ e^{i\theta} + \sum_{k=0}^{n+p-1} r^{n-k}e^{i(n-k)\theta}b_{n-k} \right] d\theta \]

\[ = -pe^{-i\theta}b_n + \frac{1}{\pi} \int_0^{2\pi} \left[ U(re^{i\theta}) \frac{e^{-i\theta}}{r^n} \right] \times \left[ f(re^{i\theta}) - \sum_{k=n}^{\infty} b_k(re^{i\theta})^k \right] d\theta. \]

**Theorem 5.** Let

\[ F(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1) \quad (a_{-p} \neq 0) \]

be in \( \mathcal{X}^*(p) \), then \( |a_n| = O(n^{-1}) \).

**Proof.** We may assume without loss of generality that \( a_{-p} = 1 \). There exists, by Lemma 2,

\[ f(z) = \frac{e^{i\theta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n \quad (0 < |z| < 1) \]

in \( \mathcal{S}^*(p) \) such that

\[ \left[ \text{Re} \frac{zF'(z)}{f(z)} \right] > 0 \quad (|z| < 1). \]

Let \( U(z) = \text{Re} \left[ zF'(z)/f(z) \right] \), then by a well-known result on harmonic functions,

\[ \frac{1}{\pi} \int_0^{2\pi} U(re^{i\theta}) d\theta = 2U(0) = -2p \cos \beta \leq 2p. \]

By Lemma 4, we have for \( n \geq 1 \)

\[ n |a_n| \leq p |b_n| + \frac{1}{\pi r^n} \left| \int_0^{2\pi} U(re^{i\theta}) e^{-i\theta} \sum_{k=n}^{\infty} b_k(re^{i\theta})^k d\theta \right| \]

\[ + \frac{1}{\pi r^n} \left| \int_0^{2\pi} U(re^{i\theta}) f(re^{i\theta}) e^{-i\theta} d\theta \right| \]

\[ \leq p |b_n| + \frac{2p}{r^n} \sum_{k=n}^{\infty} |b_k| r^k + \frac{1}{\pi r^n} \int_0^{2\pi} U(re^{i\theta}) |f(re^{i\theta})| d\theta. \]
The Area Theorems of Golusin [2] and Kobori [3] give for \( n \geq 1 \)

\[
\sum_{k=-n}^{\infty} k |b_k|^2 \leq \sum_{k=1}^{\infty} k |b_k|^2 \leq \sum_{k=-p}^{-1} |k| |b_k|^2.
\]

We thus have,

\[
\frac{2p}{r^n} \sum_{k=n}^{\infty} |b_k| r^k \leq \frac{2p}{r^n} \left[ \sum_{k=n}^{\infty} k |b_k|^2 \right]^{1/2} \left[ \sum_{k=n}^{\infty} \frac{r^{2k}}{k} \right]^{1/2}
\]

\[
\leq \frac{2p}{r^n} \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} \left[ \frac{1}{n} \sum_{k=n}^{\infty} r^{2k} \right]^{1/2}
\]

\[
= 2p \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} [n(1 - r^2)]^{-1/2}.
\]

Also for \( n \geq p \), by Lemma 3

\[
|b_n| \leq \frac{2p}{(p + n)^{1/2}} \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2}
\]

\[
\leq \frac{1}{\sqrt{p}} \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2}.
\]

Since \([f(z)]^{-1}\) is \( p \)-valently star like we have

\[
|f(re^{i\theta})|^{-1} \geq \frac{r^p}{1 + r^{2p}}
\]

or

\[
|f(re^{i\theta})| \leq \frac{(1 + r)^{2p}}{r^p}.
\]

Therefore, for \( n \geq p \)

\[
\frac{1}{\pi r^n} \int_0^{2\pi} U(re^{i\theta}) |f(re^{i\theta})| \, d\theta
\]

\[
\leq \frac{(1 + r)^{2p}}{r^{p + n}} \frac{1}{\pi} \int_0^{2\pi} U(re^{i\theta}) \, d\theta
\]

\[
\leq \frac{2p(1 + r)^{2p}}{r^{p + n}} \leq \frac{2p 4^p}{r^{2n}}.
\]

From (3.3), (3.4), (3.5) and (3.6) we have for \( n \geq p \) and any \( r < 1 \)

\[
|a_n| \leq \left[ \sqrt{p} + 2p \left[ n(1 - r^2) \right]^{-1/2} \right] \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} + 2p 4^p r^{-2n}.
\]
Let \( r^2 = (1 - 1/n) \), then for \( n \geq p + 1 \)
\[
|a_n| \leq (\sqrt{p} + 2p) \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} + 2p 4^p (1 + 1/(n - 1))^n \\
\leq (\sqrt{p} + 2p) \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} + 2p 4^p \frac{(p + 1)}{p} e.
\]
Thus, \( |a_n| = O(n^{-1}) \).

**References**


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