PROBABILITY AND THE (C, r) SUMMABILITY OF FOURIER SERIES(1)

BY
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1. **Introduction.** The Martin exit and entrance boundaries of a transient Markov chain on a denumerable state space was first constructed by Doob in [4], using a combination of probabilistic and potential theoretic methods; a direct probabilistic construction was presented later in an important paper by G. A. Hunt [12]. These two papers raised the interesting and important problem of illustrating the general theory of Markov chains and Martin boundaries by considering specific examples, i.e., the problem of constructing the Martin boundary and representation for specific Markov chains, and the more delicate problem of proving the corresponding fine limit theorems (see Doob, *op. cit.*, p. 452, and Naim's thesis [16] for the probabilistic and potential theoretic meaning of this term). Specific examples have been discussed by Lamperti and Snell [14], Watanabe [18], [19] (who did not discuss the fine limit theorems) and by Doob, Snell and Williamson [10]. This paper also contains a brief summary of the main results in [4].

Apart from these papers, not much else has been done in the way of constructing and analyzing specific examples. It is the purpose of this paper to make a contribution in this direction by considering a specific Markov process which is associated in a natural way with (C, r) summability of Fourier series (see [20] for an exposition of Cesaro summability of sequences and its applications to Fourier series). The construction of the potential theory of this process along the lines of [1], [4], and [16] yields limit theorems of an apparently new type in the (C, r) summability of Fourier series. These limit theorems will be called, following standard terminology, fine limit theorems. In §8 of this paper, we shall prove a theorem which links the notion of "fine limit" of a Fourier series to the notion of (C^*, r) summability of a Fourier series, an account of which is to be found in [21].

Before we outline the main results of this paper, it will be convenient and necessary to recall some definitions and set down some notational conventions, most of which have been taken from Zygmund [20].

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(1.1) DEFINITION. Let F be a function of bounded variation on $(-\pi,\pi)$. Then by the Fourier-Stieltjes series of dF, denoted by $S(dF)$, we mean the formal trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \tag{1.2}$$

where

$$a_n = \pi^{-1}\int_{-\pi}^{\pi} \cos nt\, dF(t), \qquad b_n = \pi^{-1}\int_{-\pi}^{\pi} \sin nt\, dF(t), \quad n = 0,1,2,\cdots.$$

Let F then be a function of bounded variariot on $(-\pi,\pi)$ with Fourier-Stieltjes series $S(dF)$. Then, as is well known, see [20], the nth partial sum of $S(dF)$ with respect to (C,r) summability, which we denote by $\sigma_n^r(x;dF)$, is given by the following equation:

$$\sigma_n^r(x,dF) = \frac{1}{\pi}\int_{-\pi}^{\pi} K_n^r(x-t)\, dF(t), \tag{1.3}$$

where

$$\begin{aligned} K_n^r(t) &= \frac{1}{2} + \sum_{j=0}^{n} \frac{A_{n-j}^r}{A_n^r}\cos jt, \qquad n \geqq 1, \\ &= \frac{1}{2}, \qquad n = 0, \end{aligned} \tag{1.4}$$

and where

$$A_k^r = \binom{k+r}{k} \approx \frac{(k)^r}{\Gamma(r+1)}. \tag{1.5}$$

We remark that k is an integer $\geqq 1$, $A_0^r = 1$, and that for k large enough A_k^r is approximately equal to $(k)^r/\Gamma(r+1)$, where $\Gamma(r)$ is Euler's Gamma function (cf. [20]).

We now look for functions $(P_{n,n+1}^r(x,y),\ n = 0,1,\cdots)$ Borel measurable in the pair (x,y), $|x| \leqq \pi$ and $|y| \leqq \pi$, with the following property:

$$\int_{-\pi}^{\pi} K_{n+1}^r(y-t) P_{n,n+1}^r(x,y)\, dy = K_n^r(x-t), \tag{1.6}$$

or equivalently, for all F of bounded variation on $(-\pi,\pi)$ we have:

$$\int_{-\pi}^{\pi} \sigma_{n+1}^r(y;dF) P_{n,n+1}^r(x,y)\, dy = \sigma_n^r(x,dF). \tag{1.7}$$

The reason why we look for such operators is that if they satisfied the following two conditions

$$P_{n,n+1}^r(x,y) \geqq 0, \qquad n = 0,1,2,\cdots, \tag{1.8}$$

$$\int_{-\pi}^{\pi} P^r_{n,n+1}(x,y)\,dy = 1, \qquad n = 0,12,\cdots, \tag{1.9}$$

then we could construct a Markov process $\{X_n(\omega),\ n = 0,1,2,\cdots\}$ with transition probability densities given by $\{P^r_{n,n+1}(x,y),\ n = 0,1,2,\cdots\}$ relative to which $\{\sigma^r_n(X_n(\omega); dF),\ F_n,\ n = 0,1,2,\cdots\}$ would be a martingale, where F_n is the Borel field generated by the random variables $(X_k(\omega),\ 0 \leqq k \leqq n)$. The notions of probability theory and the results of martingale theory that we shall use are to be found in the treatise of Doob [9].

Now we can always find $(P^r_{n,n+1}(x,y),\ n = 0,1,\cdots)$ satisfying (1.6), (1.7), and (1.9) for each $r \geqq 1$, but, as was shown by G.G. Lorentz (see pp. 5–7 of [17]), (1.8) necessarily fails for each such r. In [17] it was demonstrated that for each $r > 1$ we can nevertheless select a subsequence $\{n_k, k = 0,1,2,\cdots\}$ tending to infinity fast enough so that (1.6) through (1.9) are valid for a corresponding subsequence of operators $\{P^r_{n_k n_{k+1}}(x,y),\ k = 0,1,\cdots\}$. The subsequence is of course constructed independently of the function F. In order to be complete I shall now give a complete statement of this theorem; for the proof the reader is referred to Chapter I, Part 2 of [17].

(1.10) THEOREM. *For any order of (C,r) summability, $r > 1$, there exists a subsequence of integers $(n_k)_{k=0}^{\infty}$ and a corresponding sequence of operators $\{P^r_{n_k n_{k+1}}\}_{k=0}^{\infty}$ satisfying the following three conditions*:

(a) $P^r_{n_k n_{k+1}}(x,y) > 0$ *for* $x, y \in [-\pi,\pi]$.

(b) $\int_{-\pi}^{\pi} P^r_{n_k n_{k+1}}(x,y)\,dy \equiv 1$ *for all* $k \geqq 0$.

(c) $\int_{-\pi}^{\pi} P^r_{n_k n_{k+1}}(x,y) K^r_{n_{k+1}}(y-t)\,dy = K^r_{n_k}(x-t)$, $k \geqq 0$.

Moreover $n_0 = 0$ and $n_1 = 1$, and in general n_{k+1} satisfies the following "growth" condition:

$$n_{k+1} \geqq n_k + \frac{4n_k^2(r+n_k-1)^2}{r(r-1)}. \tag{1.11}$$

The functions $P^r_{n_k n_{k+1}}(x,y)$ are defined as follows:

$$P^r_{n_k n_{k+1}}(x,y) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{j=1}^{n_k} B^r_j[n_k, n_{k+1}]\cos j(x-y)\right], \tag{1.12}$$

where

$$B^r_j[n_k, n_{k+1}] = \frac{A^r_{n_k-j}/A^r_{n_k}}{A^r_{n_{k+1}-j}/A^r_{n_{k+1}}}, \qquad j = 1,2,\cdots,n_k. \tag{1.13}$$

It is to be observed that if m and n are arbitrary integers greater than j, $n \geqq m$, formula (1.13) is still well defined, i.e.

$$B_j^r[m,n] = \frac{A_{m-j}^r / A_m^r}{A_{n-j}^r / A_n^r}, \qquad j = 0,1,2,\cdots,m. \tag{1.13'}$$

We also have

$$P_{m,n}^r(x,y) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{j=1}^{m} B_j^r[m,n]\cos j(x-y)\right]. \tag{1.12'}$$

However, for arbitrary integers m,n it is not necessarily true that $P_{m,n}^r(x,y) \geqq 0$. Indeed, it is easily verified that the trigonometrical polymonials $P_{n,n+1}^r$, defined by (1.12′), satisfy properties (1.6), (1.7), (1.9), but do not satisfy (1.8) according to the theorem of Lorentz we mentioned before.

This leads naturally to the following:

(1.14) DEFINITION. A subsequence $(n_k)_{k=0}^{\infty}$ will be called an *admissible subsequence* if there exists a corresponding sequence of *positive* operators $(P_{n_k n_{k+1}}^r)_{k=0}^{\infty}$ of the form (1.12) satisfying (a), (b), (c) of (1.10).

For trigonometric polynomials of the form (1.12′), we shall state a useful identity. But first, we introduce some notation.

If $f \in L^1(-\pi,\pi)$, then we define the convolution of f and $P_{m,n}^r$ as follows:

$$(P_{m,n}^r * f)(t) = \int_{-\pi}^{\pi} f(x) P_{m,n}^r(t,x)\, dx, \tag{1.15}$$

where $P_{m,n}^r$ is of the form (1.12′).

The following identity is readily obtained after a routine calculation: If $m_0, m_1, \cdots, m_k$ is a sequence of nonnegative integers such that

$$0 \leqq m_0 < m_1 < \cdots < m_k,$$

then

$$P_{m_0,m_k}^r = P_{m_0,m_1}^r * P_{m_1,m_2}^r * \cdots * P_{m_{k-1},m_k}^r. \tag{1.16}$$

It should be noted that this identity is an algebraic one and that the sequence $\{m_j\}_{j=0}^k$ need not be elements of an admissible subsequence. However (1.16), when applied to an admissible subsequence yields the following useful fact.

(1.17) Every subsequence of an admissible subsequence is an admissible subsequence.

We are now ready to state the main results of this paper. Our first order of business is to construct a state space R and a Markov process, which we denote by $\{X_{n_k}(\omega), k \geqq 0\}$, such that the transition probability densities of this Markov process are given by (1.12). This is done in §2. Then we define the notion of regular (harmonic) function with respect to this random walk and obtain the following characterization of these functions:

(1.18) THEOREM. *The function $u(n_k,x)$ is a regular function if and only if there exists a trigonometrical series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

whose nth (C,r) *partial sum, denoted by* $\sigma_n^r(x)$, *has the property that* $\sigma_{n_k}^r(x) = u(n_k, x)$.

Pursuing this analysis further, we obtain the following additional results:

(1.19) THEOREM. *The Martin exit boundary of this random walk is* $C(0;1)$, *the circumference of the unit disc in the Euclidean plane, and the minimal positive regular functions are given by the* (C,r) *kernel* $\{K_n^r(x-t);\ t \in C(0;1)\}$

The Martin representation theorem turns out to be equivalent to the following classical result in the theory of Fourier series (cf. p. 82 of [20]):

(1.20) THEOREM. *The class of positive regular functions* $u(n_k, x)$ *is the same as the class of* (C,r) *sums of Fourier-Stieltjes series of positive measures, i.e, if* $u(n_k, x)$ *is a positive regular function, then there exists a monotone nondecreasing function* F *such that*

$$u(n_k, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} K_{n_k}^r(x-t)dF(t).$$

In §§7 and 8 I present some results on

(1) a Dirichlet problem that is naturally associated with the Markov process discussed in this paper,

(2) the *resolutivity* of the Martin boundary in the sense of Brelot (see [1], [4], [12]),

(3) the *fine topology* and corresponding *fine limit* theorems. Proofs are for the most part omitted, because the details have already been presented in a more general setting by Brelot, Doob, Hunt, and Naim (see [1], [4], [12], [16]).

In §8 I link the "probabilistic" notion of a fine limit with the classical notion of (C^*,r) limit, discussed by Zygmund and Marcinkiewicz in [21]. The main result here is based on the Martingale convergence theorem and the special nature of the (C,r) kernel. Moreover, we obtain a result, not known in the general case, on the rate of convergence of the minimal paths to their poles (see [4] for a discussion of minimal paths and their properties).

2. **The state space R and the random walk**(2). Let $[n_k]$ be an admissible subsequence with respect to a fixed order r $(r > 1)$ of Cesaro summability. Since $p_{01}^r(\xi,\eta) = 1/2\pi > 0$, we can, and do, choose $n_0 = 0$ and $n_1 = 1$, i.e., the first step of the random walk is distributed uniformly on the interval $(-\pi, \pi)$.

(2) It is worth pointing out that the state space R constructed in this paper is not discrete; i.e. I am presenting a specific example of Martin boundary theory in an area where the general theory has not yet been developed.

The state space R in which this random walk takes place is the following.

Denote by $C(0; r_n), n = 0, 1, \cdots$ the circumference of a circle centered at the origin in the Euclidean plane and of radius r_n. Let $r_n = 1 - 1/(n+2)$, $n = 0, 1, \cdots$, set $R_n = C(0; r_n)$, and denote by B_n the Borel subsets of R_n. We now make the following

(2.1) DEFINITION. If $\{n_k\}$ is an admissible subsequence with respect to (C, r) summability, then we define $R = \bigcup_{k=0}^{\infty} R_{n_k}$, with Borel field B' generated by $\{B_{n_k}, k \geqq 0\}$; we define on R a topology by giving in the relative topology of the Euclidean plane.

REMARK 1. It is to be observed that R so defined *depends* on the subsequence used so that, strictly speaking, we should have written $R(n_k)$. This is a case, however, where notational distinctions lead to unnecessary complications and confusions especially with regard to the symbol R_{n_k} already employed. Moreover, the discussion to be given in this paper is valid for any number $n > 1$ and any admissible subsequence.

REMARK 2. We will be engaged in much computation, and to avoid being lost in a forest of subscripts and superscripts we drop those already used in accordance with the following conventions:

(1) The points of R will be denoted by (k, ξ) instead of $r_{n_k} e^{i\xi}$, $\xi \in (-\pi, \pi)$.

(2) We shall write $p_{k,k+1}(\xi, \eta)$ for the more cumbersome $P_{n_k n_{k+1}}(\xi, \eta)$.

(3) R_k for R_{n_k} and B_k for B_{n_k}.

(4) The value of a real valued function u with domain R at the point (k, ξ) is denoted by $u(k, \xi)$.

Since the admissible subsequence and r remain fixed for the remainder of this paper, replacing the subscript n_k by k, and omitting the superscript r should cause no confusion.

(5) It will be helpful to the reader's intention to consider R as being a set of concentric circles in the plane, centered at the origin.

(2.2) The random walk on R with transition probability densities $p_{m,m+1}(\xi, \eta)$ is now defined in the following way: If we are at the point $(m, \xi) \in R$, then the probability distribution of going from (m, ξ) to a Borel set $A \in B_{m+1}$ is given by

$$\int_A p_{m,m+1}(\xi, \eta)\, d\eta. \tag{2.3}$$

Thus the random walk starting at the point (m, ξ) proceeds outward, step by step, along the successive rings R_m of R; the probability distribution of going from a point (m, ξ) to a Borel set $A \in B_{m+1}$ is given by (2.3).

More formally we define a Markov process with state space R as follows:

Let $p_{m,m+2}(\xi, \eta) = p_{m,m+1}(\xi, \zeta) * p_{m+1,m+2}(\zeta, \eta)$ and define $p_{m,m+k}(\xi, \eta)$ inductively as follows:

$$p_{m,m+k}(\xi, \eta) = p_{m,m+k-1}(\xi, \zeta) * p_{m+k-1,m+k}(\zeta, \eta). \tag{2.4}$$

Let A be a Borel subset of $(-\pi,\pi)$. Set

$$P_{m,k}(\xi; A) = \int_A p_{m,m+k}(\xi,\eta)\, d\eta, \qquad k = 1,2,\cdots.$$

Define $P_{m,0}(\xi,A) = 1$ if $\xi \in A$, 0 otherwise. Then it is a standard result (see [9, p. 86]) that there exists a Markov process $\{\hat{X}_k(m,\xi)(\omega),\ k = 0,1,2,\cdots\}$ with state space $(-\pi,\pi)$, having the following two properties:

(2.5) (a) $$\Pr[\hat{X}_0(m,\xi)(\omega) = \xi] = 1,$$

and for $k \geqq 0$ we have

(b) $$\Pr[\hat{X}_{k+1}(m,\xi)(\omega) \in A \mid \hat{X}_k(m,\xi)(\omega)] = P_{m,k}(\hat{X}_k(m,\xi)(\omega); A)$$

with probability one.

We shall be interested in the space-time version of this process which is defined in the following way:

(2.6) $$X_k(m,\xi)(\omega) = (m+k, \hat{X}_k(m,\xi)(\omega)), \qquad k = 0,1,\cdots.$$

Then $\{X_k(m,\xi)(\omega);\ k \geqq 0\}$ is a Markov process on R, with stationary transition probabilities, whose densities are given by (2.4) (see [4, p. 454]).

For notational simplicity, we shall sometimes omit the ω and write $X_k(m,\xi)$ for $X_k(m,\xi)(\omega)$. This notational convenience is standard.

3. Regular and superregular functions.

(3.1) Definition. If u is a nonnegative measurable function on R or is merely bounded from below on each ring R_k of R, then we define the function $p * u$ at the point (k,ξ) as follows:

$$(p * u)(k,\xi) = \int_{-\pi}^{\pi} p_{k,k+1}(\xi,\eta)\, u(k+1,\eta)\, d\eta.$$

(3.2) Definition. Let A be a measurable subset of R. A measurable function u, bounded from below on each ring R_k $(-\infty < u \leqq +\infty)$ is said to be *superregular on A* if $p * u$ is finite on A and $u \geqq p * u$ on A.

Remark 1. It is to be observed that if

$$(p * u)(k,\xi) < +\infty, \quad \text{then } |u(k+1,\eta)| < +\infty$$

almost everywhere with respect to the measure whose density on R_{k+1} is $p_{k,k+1}(\xi,\eta)\, d\eta$. It follows at once that $|u(k+1,\eta)| < +\infty$ almost everywhere with respect to Lebesgue measure on R_{k+1}.

(3.3) Definition. We shall say that u is *subregular* on A if $-u$ is superregular on A. If u and $-u$ are both superregular on A, then u is said to be *regular*

on A. If $A = R$, then we shall drop the words "on R" and call u superregular, regular, or subregular.

(3.4) THEOREM. *If u is superregular then the stochastic process*

$$\{u[X_k(m,\xi)],\ F_k, k > 0\}$$

is a supermartingale, where F_k denotes the smallest Borel field with respect to which the random variables $\{X_0(m,\xi),\cdots,X_k(m,\xi)\}$ are measurable. (For the definition of supermartingale, see [9, *Chapter* VII].)

Proof. The following notation will be convenient. Define $p^{(2)} * u = p * (p * u)$ and in general we define $p^{(n)} * u = p * (p^{n-1} * u)$. It is a routine computation to verify that

$$(p^{(n)} * u)(m,\xi) = \int_{-\pi}^{\pi} p_{m,m+n}(\xi,\eta)\, u(m+n,\eta)\, d\eta. \tag{3.5}$$

Moreover, the fact that u is superregular implies that

$$u \geqq p * u \geqq p^{(2)} * u \geqq \cdots \geqq p^{(n)} * u,$$

and hence $(p^{(n)} * u)(m,\xi) < +\infty$ for all $n \geqq 1$. We now observe that the expected value of $u[X_n(m,\xi)]$, which we denote by $E\{u[X_n(m,\xi)]\}$, is finite for all $n \geqq 1$. This follows at once from the following equation, which is readily verified:

$$E\{u[X_n(m,\xi)]\} = (p^{(n)} * u)(m,\xi). \tag{3.6}$$

Since $X_0(m,\xi) = (m,\xi)$, it follows that $E\{u[X_0(m,\xi)]\}$ is finite if and only if $u(m,\xi)$ is finite. In any event it follows at once from the Markov property (see [9, p. 80]) that

$$\begin{aligned} E\{u[X_n(m,\xi)] \mid F_{n-1}\} &= E\{u[X_n(m,\xi)] \mid X_{n-1}(m,\xi)\} \\ &= (p * u)(X_{n-1}(m,\xi)) \\ &\leqq u[X_{n-1}(m,\xi)]. \end{aligned}$$

But this is precisely the condition that this stochastic process be a supermartingale.

An important class of regular functions are the nth (C,r) partial sums of an $S[dF]$. In particular, we have the following important corollary to Theorem (3.4).

(3.7) COROLLARY. *Let $\sigma_n^r(\xi; dF)$ be the nth (C,r) partial sum of $S(dF)$. Set $u(k,\xi) = \sigma_{n_k}^r(\xi; dF)$ where $\{n_k\}$ is the admissible subsequence. Then $u(k,\xi)$ is a regular function on R and hence $\{u[X_n(m,\xi)],\ F_n,\ n \geqq 0\}$ is a martingale.*

Proof. It suffices to show that $p * u = u$. Now

$$(p * u)(m,\xi) = \int_{-\pi}^{\pi} p_{m,m+1}(\xi,\eta)\, u(m-1,\eta)\, d\eta$$

$$= \int_{-\pi}^{\pi} p_{n_m,n_{m+1}}(\xi,\eta)\, \sigma^r_{n_{m+1}}(\eta; dF) = \sigma^r_{n_m}(\xi; dF).$$

This last equality follows at once from Definition (1.14), and (1.7). The proof is now completed by observing that $u(m,\xi) = \sigma^r_{n_m}(\xi;\ dF)$.

The converse to this theorem is a little more difficult and much more important.

(3.8) THEOREM. *The function u is a nonnegative regular function on R if and only if there exists a nondecreasing function F of bounded variation on $(-\pi,\pi)$ such that*

$$\sigma^r_{n_k}(\xi; dF) = u(k,\xi). \tag{3.9}$$

In other words the class of nonnegative regular functions is the same as the class of rth Cesaro means of Fourier-Stieltjes series of positive measures.

Proof. If u is of the form (3.9) then, according to Corollary (3.7), u is regular. Thus we need prove only the converse. Suppose then that u is nonnegative and $p * u = u$ on R. I shall construct a formal trigonometric series of the form (1.2), with nth partial sums denoted by $S_n(\xi)$, such that applying (C,r) summability to $S_n(\xi)$ yields a new sequence $\sigma^r_n(\xi)$ with the property that $\sigma^r_{n_k}(\xi) = u(k,\xi)$. This construction will depend only on the fact that u is regular, and is not dependent on the fact that u is nonnegative. The proof of the theorem will then be completed if we can show that a formal trigonometric series whose nth (C,r) partial sums are nonnegative is an $S(dF)$, dF a nonnegative measure. But this is a classical theorem in Fourier series (see [20 p. 82]). We now proceed to give the details.

We first observe that $(p * u)(k,\xi)$ is a trigonometric polynomial of the n_kth degree. This is because the density $p_{k,k+1}(\xi,\eta)$ is a trigonometrical polynomial of the n_kth degree. It follows at once from the regularity of u that

$$u(k,\xi) = a_{n_k,0} + \sum_{j=1}^{n_k} a_{n_k,j} \cos j\xi + b_{n_k,j} \sin j\xi. \tag{3.10}$$

The regularity of u implies the following identity:

$$a_{n_k,0} + \sum_{j=1}^{n_k} a_{n_k,j} \cos j\xi + b_{n_k,j} \sin j\xi$$

$$= a_{n_{k+1},0} + \sum_{j=1}^{n_k} B^r_j[n_k,n_{k+1}](a_{n_{k+1},j} \cos j\xi + b_{n_{k+1},j} \sin j\xi).$$

We conclude that

$$
(3.11)\qquad \begin{aligned} a_{n_k,0} &= a_{n_{k+1},0} = \frac{a_0}{2}, && k=0,1,2,\cdots. \\ \frac{a_{n_k,j}}{a_{n_{k+1},j}} &= B_j^r[n_k,n_{k+1}] = \frac{b_{n_k,j}}{b_{n_{k+1},j}}, && k=0,1,2,\cdots, \\ & && j=0,1,\cdots,n_k. \end{aligned}
$$

Now, using (1.13) we get

$$
(3.12)\qquad \begin{aligned} \frac{a_{n_k,j}}{A_{n_k-j}^r/A_{n_k}^r} &= \frac{a_{n_{k+1},j}}{A_{n_{k+1}-j}^r/A_{n_{k+1}}^r} = a_j, \quad \text{and similarly} \\ \frac{b_{n_k,j}}{A_{n_k-j}^r/A_{n_k}^r} &= \frac{b_{n_{k+1},j}}{A_{n_{k+1}-j}^r/A_{n_{k+1}}^r} = b_j, \quad k=0,1,2,\cdots. \end{aligned}
$$

It is now readily verified, after a routine computation, that the formal trigonometric series

$$
(3.13)\qquad \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\xi + b_j \sin j\xi,
$$

where the $(a_j)_0^\infty$ and $(b_j)_1^\infty$ are obtained from u by formula (3.12), has the property that $\sigma_{n_k}^r(\xi) = u(k,\xi)$, where $\sigma_n^r(\xi)$ is the nth (C,r) partial sum of (3.13). Thus there is a one-one correspondence between the set of regular functions u and the (C,r) partial sums of formal trigonometric series.

Let us now recall the following classical result in theory of Fourier series [20, p. 82].

(3.14) THEOREM. *A necessary and sufficient condition that* (3.13) *be an* $S(dF)$, *with* F *nondecreasing, is that* $\sigma_n^r(\xi; dF) \geqq 0$ *for* $n = 0,1,2,\cdots$.

Actually the hypothesis of Theorem (3.14) can be weakened, as Zygmund points out, to the following: There exists a subsequence $(n_k)_0^\infty$, tending to infinity, such that $\sigma_{n_k}^r(\xi) \geqq 0$ for $k \geqq 0$. Now the hypothesis that u be nonnegative as well as regular implies that $\sigma_{n_k}^r(\xi) \geqq 0$, according to our representation theorem, and hence $u(k,\xi) = \sigma_{n_k}^r(\xi; dF)$.

REMARK. We shall obtain this theorem "probabilistically" in §5.

4. h-path processes. The following "minimum principle" is a useful one.

(4.1) THEOREM. *Let h be a nonnegative regular function on R and suppose there exists a point $(m,\xi) \in R$ such that $h(m,\xi) = 0$. Then h is identically zero.*

Proof. $0 = u(m,\xi) = \int_{-\pi}^{\pi} p_{m,m+1}(\xi,\eta)\, u(m+1,\eta)\, d\eta$. Hence $u(m+1,\eta) = 0$ for all $\eta \in (-\pi,\pi)$. In general $0 = u(m,\xi) = \int_{-\pi}^{\pi} p_{m,m+k}(\xi,\eta)\, u(m+k,\eta)\, d\eta$ and therefore u is identically zero on R_{m+k} for $k = 0,1,2,\cdots$. Now the above argument also proves that if $u \equiv 0$ on R_{m+1} then $u \equiv 0$ on R_m, and hence $u \equiv 0$ on R_k for all $k \geqq 0$. This completes the proof.

Suppose then that $h \not\equiv 0$ is a nonnegative regular function, and by Theorem (4.1) we may assume that h is never zero. Following Doob [4], we define the notion of an h-path process; a notion which plays such an important role in both classical and probabilistic discussions of potential theory (see, for example, [1], [4]–[6], [12], [16]).

(4.2) DEFINITION. If h is a nonnegative regular function on R ($h \not\equiv 0$), then we define a new transition probability density function $p^h_{k,k+1}$ as follows:

$$p^h_{k,k+1}(\xi,\eta) = p_{k,k+1}(\xi,\eta)\frac{h(k+1,\eta)}{h(k,\xi)}, \qquad k \geqq 0.$$

The random walk on R, starting at (m,ξ), with transition probability density function $p^h_{k,k+1}(\xi,\eta)$ will be denoted by

$$\{X^h_n(m,\xi),\ n \geqq 0\}. \tag{4.3}$$

REMARK. The Markov process (4.3) is called the "h-path process" and the corresponding probability paths are called "h-paths."

(4.4) DEFINITION. A measurable function u bounded from below on each ring R_m of R is said to be *h-superregular* on a subset A of R if $u \geqq p^h * u$ on A and $p^h * u$ is finite on A.

The corresponding definitions of h-subregular and h-regular functions are now obvious (see Definition (3.3)); we omit the details. The following analogue of Theorem (3.4) is useful; we omit the proof.

(4.5) Theorem. *If u is an h-(super) regular function then $\{u(X^h_n(m,\xi)), F_n, n \geqq 0\}$ is a (super) martingale.*

The next result, the verification of which we leave to the reader, follows at once from the definition.

(4.6) *The nonnegative function v is h-superregular if and only if there exists a superregular function u on R such that $v = u/h$. In particular $1/h$ is h-superregular.*

5. **The Martin exit boundary of the Markov chain.** Following Doob [4], we define the *Green's function* of the Markov process (2.6) as follows:

$$\begin{aligned} G_{k,m}(\xi,\eta) &= p_{k,m}(\xi,\eta), \qquad m \geqq k+1, \\ &= 0, \qquad 0 \leqq m \leqq k. \end{aligned} \tag{5.1}$$

In other words, $G_{k,m}(\xi,\eta)$ is the probability density of going from a point $(k,\xi) \in R_k$ to a point $(m,\eta) \in R_m$ in exactly $m-k$ steps.

Following Martin [15] and Doob [4], we define the Martin kernel $\hat{K}_{k,m}(\xi,\eta)$ as follows:

$$\begin{aligned} \hat{K}_{k,m}(\xi,\eta) &= \frac{G_{k,m}(\xi,\eta)}{G_{0,m}(0,\eta)}, \qquad m \geqq k+1, \\ &= 0 \quad \text{if } 0 \leqq m \leqq k. \end{aligned} \tag{5.2}$$

(5.3) THEOREM. *For* $m \geqq k+1$,

(a) $$G_{k,m}(\xi,\eta) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{j=1}^{n_k} B_j^r[n_k,n_m]\cos j(\xi-\eta)\right],$$

(b) $$G_{0,m}(\xi,\eta) = \frac{1}{2\pi}.$$

Proof. (a) follows at once from (1.16) and (1.12′), and (b) is just a special case of (a).

We thus have the following simple expression for $\hat{K}$:

$$\begin{aligned} \hat{K}_{k,m}(\xi,\eta) &= 2\pi p_{k,m}(\xi,\eta) \quad \text{for } m \geqq k+1, \\ &= 0 \quad \text{for } 0 \leqq m \leqq k. \end{aligned} \tag{5.4}$$

In order to construct the Martin exit boundary for the Markov process (2.6) we shall need some results on the asymptotic behavior of (5.4). For example, it is easy to show by a direct computation, using (1.5), that

$$\lim_{n\to\infty}(A_{n-j}^r / A_n^r) = 1. \tag{5.5}$$

It follows at once from this and (1.13) that

$$\lim_{n_m\to\infty} B_j^r[n_k,n_m] = A_{n_k-j}^r / A_{n_k}^r. \tag{5.6}$$

Applying these results to the expression (5.3a) for $G_{k,m}(\xi,\eta)$ and using (1.4) we get

$$\lim_{m\to\infty} G_{k,m}(\xi,\eta) = \frac{1}{\pi} K_{n_k}^r(\xi-\eta). \tag{5.7}$$

Now this last result is a very useful one in so far as the problem of determining the Martin exit boundary of the process is concerned. In particular if we consider (m,η) as a sequence of points in R then it is clear that

$$\lim_{m\to\infty}(m,\eta) = \lim_{n\to\infty} r_{n_m}e^{i\eta}, \tag{5.8}$$

where the limit is taken with respect to the topology induced on R by the Euclidean topology of the plane. This topology on R is generated by the following metric

$$d[(k,\xi);(m,\eta)] = \left| r_{n_k}e^{i\xi} - r_{n_m}e^{i\eta}\right|. \tag{5.9}$$

It is easy to verify that the completion of R with respect to d adjoins to R a boundary $R' = C(0;1)$, the circumference of the unit disc. In fact we have the following important result.

(5.10) THEOREM. *$\hat{K}_{k,m}(\xi,\eta)$ as a function of (m,η) possesses a continuous extension to $R \cup R'$. Denoting the point $e^{it} \in C(0;1)$ by t, $t \in (-\pi,\pi)$, we have*

$$\lim_{(m,\eta_m)\to t} \hat{K}_{k,m}(\xi,\eta_m) = 2K^r_{n_k}(\xi - t). \tag{5.11}$$

Proof. We observe that $\hat{K}_{k,m}(\xi,\eta)$ is a trigonometric polynomial of degree n_k; this follows at once from (5.3a).

Let $f_j(m,\eta) = (A^r_{n_m} / A^r_{n_m - j}) \cos j(\xi - \eta)$, ($k$ and ξ are held fixed). Now by (5.3a) we have that $\hat{K}_{k,m}(\xi,\eta)$ is a finite linear combination of the $f_j(m,\eta)$. Hence it suffices to prove that $\lim_{(m,\eta_m)\to t} f_j(m,\eta_m) = f_j(t)$ exists and moreover $f_j(t)$ is continuous on $C(0;1)$. The proof is simple. We first observe that $\lim(m,\eta_m) = t$ if and only if $m \to +\infty$ and $\lim_{m\to\infty} \eta_m = t$. It follows at once from (5.5) and the continuity of the cosine function that $\lim_{m\to\infty} f_j(m,\eta_m) = \cos j(\xi - t) = f_j(t)$. This proves that $f_j(m,\eta)$ possesses a continuous extension to $R \cup R'$ and hence so does $\hat{K}_{k,m}(\xi,\eta)$. Moreover since $R \cup R'$ is a compact metric space the extended function is uniformly continuous on $R \cup R'$.

The preceeding discussion may be summarized in the following

(5.12) THEOREM. *The Martin exit boundary of the Markov process* (2.6) *is* $C(0;1)$.

6. **The Martin representation theorem and the class of minimal regular functions.** Suppose v is a positive regular function on R. Then $F_n(x) = (1/2\pi)\int_{-\pi}^{x} v(n,\xi)\,d\xi$ is a monotone nondecreasing function which defines a measure on R the support of which is contained in R_n, and with density given by $v(n,\xi)/2\pi$. Since $F_n(\pi) = c$, a constant, it follows that $(dF_n,\ n \geqq 0)$ are uniformly bounded on R. In addition the regularity of v implies that for $k < n$ we have

$$\int_{R_n} \hat{K}_{k,n}(\xi,\eta)\,dF_n(\eta) = v(k,\xi).$$

Moreover by the uniform continuity of $\hat{K}$ on $R \cup R'$ and the Helly selection principle we obtain the following result: there exists a measure dF whose support is contained in R' such that

$$v(k,\xi) = \lim_{n\to\infty} \int_{R_n} \hat{K}_{k,n}(\xi,\eta)\,dF_n(\eta) = 2\int_{R'} K^r_{n_k}(\xi - t)\,dF(t).$$

Thus, except for a constant factor, we have derived theorem (3.14) by probabilistic methods. We restate this formally as follows:

(6.1) THEOREM. *If v is a positive regular function on R then there exists a measure dF on R' such that*

$$v(k,\xi) = \int_{-\pi}^{\pi} K^r_{n_k}(\xi - t)\,dF(t).$$

We now show that $K^r_{n_k}(\xi - t)$ for each $t \in (-\pi,\pi)$ is a minimal positive regular function in the sense of Martin (see [1], [4], and [5]).

(6.2) DEFINITION. A positive regular function u is said to be a *minimal positive*

regular function if for any positive regular function $v \leqq u$ we have $v = cu$ for some constant c.

Before proceeding to a characterization of the class of minimal positive regular functions we shall need the following uniqueness theorem for Fourier-Stieltjes series.

(6.3) THEOREM. *Suppose F is a monotone increasing function of bounded variation on $(-\pi,\pi)$, with Fourier-Stieltjes series $S(dF)$. Let*

$$F_n(t) = \frac{1}{\pi}\int_{-\pi}^{t} \sigma_n^r(x; dF)\,dx;$$

then F_n is also a monotone increasing function of bounded variation on $(-\pi,\pi)$. Moreover, if f is any continuous, periodic function on $(-\pi,\pi)$ we have

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\,dF_n(t) = \int_{-\pi}^{\pi} f(t)\,dF(t),$$

i.e., the measures dF_n converge weakly to the measure dF.

Proof. By the theorem of Fubini on the interchange of the order of integration and (1.3) we have

$$\int_{-\pi}^{\pi} f(t)\,dF_n(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\int_{-\pi}^{\pi} K_n^r(t-x)\,dF(x)\right]dt = \int_{-\pi}^{\pi} \sigma_n^r(x;f)\,dF(x).$$

Now, for $r \geqq 1$, it is a well known theorem of Fejér [20, p. 48] that

$$\lim_{n\to\infty} \sigma_n^r(x;f) = f(x)$$

uniformly in x, hence

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} \sigma_n^r(x;f)\,dF(x) = \int_{-\pi}^{\pi} f(x)\,dF(x),$$

and therefore

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\,dF_n(t) = \int_{-\pi}^{\pi} f(t)\,dF(t)$$

for any continuous function f. This proves the theorem.

(6.4) COROLLARY. *Suppose $\lim_{n\to\infty} \sigma_n^r(x; dF) = 0$ for $a \leqq x \leqq b$, and where a and b are continuity points of F. Then the dF measure of $[a,b]$ is zero.*

Proof. By the preceding theorem dF_n tends weakly to dF. Hence F_n converges to F at all points x, which are continuity points of F. In particular $\lim_{n\to\infty} F_n(a) = F(a)$ and $\lim_{n\to\infty} F_n(b) = F(b)$. Since $F_n(b) - F_n(a) = \int_a^b \sigma_n^r(x; dF)\,dx$, it follows at once from the hypothesis that $\lim_{n\to\infty} \sigma_n^r(x; dF) = 0$ on $[a,b]$ that $\lim_{n\to\infty} F_n(b) - F_n(a) = 0 = F(b) - F(a)$. This concludes the proof.

(6.5) THEOREM. *If u is a minimal positive regular function then $u(k,\xi) = cK_n^r(\xi - t)$ for some point $t \in R'$ and some constant c.*

REMARK. The point t is called the *pole* of the minimal positive regular function (cf. [1]).

Proof. By Theorem (6.1) if u is a positive regular function on R we have the representation

$$u(k,\xi) = \int_{-\pi}^{\pi} K_{n_k}^r(\xi - t)\, dF(t).$$

We will show that if u is a minimal positive regular function then dF measure is concentrated at a single point.

Suppose then that there are two points $t_1 \neq t_2 \in R'$ such that each point is contained in a closed interval neighborhood E_1 and E_2 of positive dF measure. We can also suppose that the end points of E_1 and E_2 have zero measure and that E_1 and E_2 are disjoint. Now for $i = 1,2$ we have

$$\int_{E_i} K_{n_k}^r(\xi - t)\, dF(t) < \int_{R'} K_{n_k}^r(\xi - t)\, dF(t),$$

hence, since u is assumed to be minimal, we have

$$c_i u(k,\xi) = \int_{E_i} K_{n_k}^r(\xi - t)\, dF(t)$$

where c_i are constants.

It follows from a classical theorem of Lebesgue-Privaloff-Young (pp. 55–59 of [20]) that if (m_j, ξ_j) is a sequence of points tending either to a point $t \in E_1$ or a point $t \in E_2$ then $\lim_{j\to\infty} u(m_j, \xi_j) = 0$. By (6.4) it follows that the dF measures of E_1 and E_2 are zero. This contradicts the hypothesis that E_1 and E_2 were of positive dF measure. It should be remarked that this argument is similar to one used by Professor M. Brelot in a course of lectures (unpublished) on *Potential theory* given at the University of Illinois in the second semester of 1960–1961.

Conversely suppose that u is positive regular and $u(k,\xi) \leqq K_{n_k}^r(\xi - t)$. Then it is a well-known property of the (C,r) kernels (see p. 48 of [20]) that u tends to zero for those sequences (m_j, ξ_j) which approach a point $t' \neq t$, $t' \in R'$. The same reasoning used above shows that since

$$u(k,\xi) = \int_{R'} K_{n_k}^r(\xi - t)\, dF(t),$$

we must have dF measure concentrated at the point t. This concludes the proof that the class of minimal positive regular functions is given by

$$\{K_{n_k}^r(\xi - t),\ t \in (-\pi, \pi)\}.$$

7. **Resolutivity of the boundary** R'. The Dirichlet problem in the present context can be stated as follows:

(7.1) DIRICHLET PROBLEM. Given a continuous function f, domain R', find a regular function u, domain R, with "boundary values" f.

It is also natural, as was pointed out by Brelot [1], to consider a more general type Dirichlet problem which may be formulated as follows:

(7.2) Given f, satisfying the same hypotheses as in (7.1) find an h-regular function u, domain R with "boundary values" f.

In order to state the results we have obtained on problem (7.2) we need to define the notion of h-regular measure.

(7.3) DEFINITION. Let h be a positive regular function on R with the representation

$$h(k,\xi) = \int_{-\pi}^{\pi} K_{nk}^{r}(\xi - t)\, dH(t).$$

For each point $(k,\xi) \in R$ we define a probability measure on R', called h-regular measure, as follows:

$$\mu^h[(k,\xi); A] = \int_A \frac{K_{nk}^{r}(\xi - t)}{h(k,\xi)}\, dH(t). \tag{7.4}$$

REMARK. When $h = 1$ we shall omit the superscript. Incidentally, it is interesting to observe that μ^h is the analogue of classical h-harmonic measure, which was introduced into potential theory by Brelot in his fundamental paper [1].

Let us now return to problem (7.2). The usual way of solving a Dirichlet problem of this type is via the PWB (Perron-Wiener-Brelot) method, an account of which is to be found in [1] and [2]. It was shown in [17] that a variant of this method can be applied in the present context to yield the following result:

(7.5) THEOREM. *If $f \in L^1[\mu^h]$ then f is h-resolutive i.e. the PWB method yields a unique solution to the Dirichlet problem* (7.2) *and moreover the solution u has the following representation*:

$$\begin{aligned} u(k,\xi) &= \int_{-\pi}^{\pi} f(t)\mu^h[(k,\xi); dt] \\ &= \int_{-\pi}^{\pi} f(t)\, \frac{K_{nk}^{r}(\xi - t)}{h(k,\xi)}\, dH(t). \end{aligned} \tag{7.6}$$

REMARK. If for any continuous function f, domain R', we can find, via the PWB method, a *unique* h-regular function u corresponding to f then R' is said to be *h-resolutive*. A function f for which the PWB method yields a unique solution u to (7.2) is called *h-resolutive*. Brelot [1], [2] has characterized in the classical case the set of h-resolutive boundary functions. An important consequence of h-resolutivity is the

(7.7) THEOREM [4], [5]. *The h-resolutivity of R', for all positive regular functions h, implies that h-paths converge with probability one to unique points on R' i.e.*

$$\lim_{n\to\infty} X_n^h(m,\xi) = X_\infty^h(m,\xi) \in R'$$

exists, and moreover

$$\Pr[X_\infty^h(m,\xi) \in A] = \mu^h[(m,\xi); A].$$

Applying (7.7) to the special case where h is a minimal positive regular function we obtain the following result:

(7.8) THEOREM. *Let $h(k,\xi) = K_{n_k}^r(\xi - t)$ be a minimal positive regular unction "with pole t" (i.e. dH-measure is concentrated at the point t). Then the h-path process converges with probability one to t. We sometimes call this "the process conditioned to converge to t."*

Proof. According to (7.4) and (7.7), $\mu^h[(m,\xi); A] = 0$ if t is not in the closure of A, otherwise it equals one for every closed set containing t. This concludes the proof.

REMARK. This theorem will be strengthened later, by obtaining an estimate on the rate of convergence of minimal paths to their poles.

8. **Boundary behavior of nonnegative h-superregular functions.** As it stands Theorem (7.5) is incomplete; in particular we have not discussed in what way, if any, the function f is the "boundary value" of u. The following results, which we present without proof, have been discussed in detail by Doob and others in [3]–[6], [12]. The proofs rely heavily, of course, on the martingale convergence theorems.

(8.1) THEOREM. *If $f \in L^1[\mu^h]$ with PWB^h solution u (given by (7.6)) then $\lim_{n\to\infty} u\,[X_n^h(m,\xi)(\omega)] = f[X_\infty^h(m,\xi)(\omega)]$ with probability one.*

REMARK. The limit in (8.1), as has been pointed out by Doob [4] and Naim [16], can be put in a nonprobabilistic context, by introducing the notion of a *fine topology* at the Martin exit boundary of the process (2.6). This we now proceed to do. Proofs are, for the most part, omitted.

(8.2) THEOREM (0−1 LAW). *Let h be a minimal positive regular function with pole t and E a measurable subset of R. Then the h-paths lie in E infinitely often with probability zero or one. If the former is true then we say that E is "thin" (effilé) at t.*

(8.3) DEFINITION. A set E is said to be a *fine neighborhood* of the Martin boundary point t if $R - E$ is thin at t. For an interior point $(m,\xi) \in R$, the neighborhood system is that defined by the metric d (see (5.9)). This neighborhood

system defines a topology on $R \cup R'$ [13] which is called, following standard terminology, the *fine topology*.

(8.4) DEFINITION. If a function v has a fine limit (in the sense of the fine topology) $f(t)$ for almost all $t \in R'$, where almost all refers to h-regular measure μ^h, then f is called the *h-fine boundary function* of v.

(8.5) THEOREM. *If v is a nonnegative h-superregular function then v has an h-fine boundary function f, which is h-resolutive and such that*

$$v(m, \xi) \geqq E\{f(X^h_\infty(m, \xi))\}.$$

The conditions under which equality holds are discussed in [6]. It is also worth pointing out that, from the probabilistic point of view, if f is the h-fine boundary function of v then

$$\lim_{n \to \infty} v(X^t_n(m, \xi)) = f(t) \tag{8.6}$$

with probability one, for almost all $t \in R'$, h-regular measure. $\{X^t_n(m, \xi), n \geqq 0\}$ is of course the minimal path process "conditioned to converge to t" and starting at $(m, \xi) \in R$. Note also that the limit in (8.6) is independent of the starting position (m, ξ).

The h-fine boundary function f of Theorem (8.5) has certain properties which are worth pointing out. For example, we have the following

(8.7) THEOREM. *Suppose that in Theorem* (8.5) *v is an h-regular function of the form u/h, where $u(k, \xi) = \int_{-\pi}^{\pi} K^r_{n_k}(\xi - t)\, dU(t)$, and h is as in* (7.3). *Let $dU = dU_{ac} + dU_s$ be the Lebesgue decomposition of dU with respect to dH, where the subscripts ac and s refer to absolutely continuous part and singular part respectively. Then the h-fine boundary function f of v is the Radon-Nikodym derivative of dU_{ac} with respect to dH, i.e.*

$$f = \frac{dU_{ac}}{dH} \quad a.e.\ [\mu^h].$$

A ratio limit theorem similar to Theorem (8.7) was obtained by Doob in [8, Theorem 5.4]. He proved essentially the following result:

$$\lim_{n \to \infty} \frac{\sigma^r_n(\xi_n; dU)}{\sigma^r_n(\xi_n; dH)} = \frac{dU_{ac}}{dH}(\xi), \qquad |\xi_n - \xi| = O\left(\frac{1}{n}\right). \tag{8.8}$$

REMARK. It has been pointed about by Zygmund [20, Example 4, p. 64] that limit theorems of the above type are analogues of nontangential convergence for harmonic functions in the unit disc.

I shall now prove a theorem which links limit theorems of type (8.8) to the fine topology constructed in this paper. A connection with the notion of (C^*, r) summability of Zygmund-Marcinkiewicz [21] will also be discussed.

Our first step is to prove that a certain subset of R is not thin at the Martin boundary point $t \in R'$, $|t| \leqq \pi$.

(8.9) THEOREM. *Let* $I_n(t) = \{s: |s-t| < (1/n)^a\}$. *Consider* $I_{n_k}(t)$, *where* n_k *is an admissible subsequence, as a subset of* R_k, *"centered" at the point* $(k,t) \in R_k$. *Then* $\bigcup_{k=0}^{\infty} I_{n_k}(t)$ *is a fine neighborhood of* t, *for* $0 < a < r/(1+r)$.

Proof. We shall show that $R - (\bigcup_{k=0}^{\infty} I_{n_k}(t))$ is thin at t. To this end we shall use Theorems (4.5) and (4.6) and the following inequality which is to be found in Zygmund [20, p. 48].

$$K_n^r(t) \leqq Cn^{-r}t^{-r-1} \quad \text{for } \frac{1}{n} \leqq t \leqq \pi, \tag{8.10}$$

where C is a constant independent of n. It follows from (8.10) that

$$\frac{1}{K_n^r(s-t)} \geqq \frac{1}{C} n^r(|s-t|)^{1+r} \quad \text{for } \frac{1}{n} \leqq |s-t| \leqq \pi. \tag{8.11}$$

Now for a, $0 < a < 1$, we have $n^a < n$, or equivalently $1/n^a > 1/n$. Hence if $s \notin I_n$ we have $|s-t| \geqq 1/n^a > 1/n$. This implies that

$$\begin{aligned} \frac{1}{K_n^r(s-t)} &\geqq \frac{1}{C} n^r \left(\frac{1}{n^a}\right)^{1+r} \qquad s \notin I_n, \text{ i.e.} \\ \frac{1}{K_n^r(s-t)} &\geqq \frac{1}{C} n^{r-a(1+r)}, \qquad s \notin I_n. \end{aligned} \tag{8.12}$$

If we now choose a, so that $r - a(1+r) > 0$, i.e. $0 < a < r/(1+r)$, it follows that

$$\lim_{n\to\infty} \frac{1}{K_n^r(s-t)} = +\infty \quad \text{for } s \notin I_n. \tag{8.13}$$

Denote by $X_n^t(m,\xi) = (m+n, X_n^t(m,\xi))$ the minimal path process conditioned to converge to t. (recall that X_n^t is the space-time version of X_n^t, cf. (2.6)). By Theorems (4.5) and (4.6) and the martingale convergence theorem it follows at once that

$$\lim_{k\to\infty} \frac{1}{K_{n_{k+m}}^r(X_k^t(m,\xi)-t)} \tag{8.14}$$

exists and is finite with probability one. Now if $X_k^t(m,\xi)(\omega) \notin I_{n_{k+m}}$ infinitely often, then by (8.13) the limit (8.14) along this path would not exist, and this set of ω is of probability zero. It follows that with probability one

$$|X_k^t(m,\xi) - t| = O(n_{k+m}^{-a}), \quad \text{where } 0 < a < \frac{r}{1+r}. \tag{8.15}$$

Since in this paper $r > 1$, $r/(1+r)$ is increasing on $[0,\infty]$, we can choose $a = 1/2$. Of course the rate of convergence becomes better as r increases.

I shall now strengthen this result by showing that

$$\bigcup_{k=0}^{\infty} I_{n_k}(t), \quad \left(I_n(t) = \left\{s: |s-t| < \frac{1}{n}\right\}\right)$$

is not thin at t; whether or not it is a fine neighborhood of t is not yet known. The following lemma is needed; a probability interpretation will be given later.

(8.16) LEMMA. *For* $r \geqq 1$, $\liminf_{n\to\infty} \int_0^{1/n} K_n^r(t)\, dt \geqq 1/\pi > 0$.

Proof. Let $D_k(t)$, $|t| \leqq \pi$, denote the Dirichlet kernel,

$$D_k(t) = \begin{cases} \dfrac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}, & k \geqq 1 \\ \dfrac{1}{2}, & k = 0 \end{cases} \tag{8.17}$$

The following formula for the (C,r) kernel is well known [20, p. 48]:

$$K_n^r(t) = \sum_{k=0}^{n} \frac{A_{n-k}^{r-1}}{A_n^r} D_k(t). \tag{8.18}$$

Hence

$$\int_0^{1/n} K_n^r(t)\, dt = \sum_{k=0}^{n} \frac{A_{n-k}^{r-1}}{A_n^r} \int_0^{1/n} D_k(t)\, dt. \tag{8.19}$$

Moreover

$$\int_0^{1/n} D_k(t)\, dt = \int_0^{1/n} \frac{\sin\left(k+\frac{1}{2}\right)}{2\sin\frac{t}{2}}\, dt \geqq \int_0^{1/n} \frac{2k+1}{\pi} dt = \frac{2k+1}{n\pi}. \tag{8.20}$$

The last inequality follows at once from the fact that

$$D_k(t) \geqq \frac{2k+1}{\pi}, \qquad 0 \leqq t \leqq \frac{1}{n}, \qquad 0 \leqq k \leqq n. \tag{8.21}$$

To see this observe that $\sin(k+1/2)t$ attains its first maximum at the point $\pi/(2k+1)$, and is a concave function on the interval $[0,\pi/(2k+1)]$. Note also that $\pi/(2k+1) > 1/k \geqq 1/n$ for $1 \leqq k \leqq n$. It is easily verified that the chord $f(t) = ((2k+1)/\pi)t$ lies below $\sin(k+1/2)t$ on $[0,1/n]$, i.e.

$$\sin\left(k+\frac{1}{2}\right)t \geqq \left(\frac{2k+1}{\pi}\right)t \quad \text{for } 0 \leqq t \leqq \frac{1}{n} \leqq \frac{\pi}{2k+1}.$$

Similarly, $2\sin(t/2) \leqq 2 \cdot t/2 = t$ on the interval $[0, 1/n]$. Hence

$$D_k(t) \geqq \frac{\left(\frac{2k+1}{\pi}\right)t}{t} = \frac{2k+1}{\pi}$$

for $0 \leqq t \leqq 1/n$, and $0 \leqq k \leqq n$. It follows at once that

$$\int_0^{1/n} K_n^r(t)\,dt \geqq \sum_{k=0}^{n} \frac{A_{n-k}^{r-1}}{A_n^r}\,\frac{2k+1}{n\pi}. \tag{8.22}$$

Now an elementary computation shows that the right-hand side of (8.22) equals

$$\frac{2}{n\pi}\sum_{k=0}^{n} \frac{A_{n-k}^{r-1}}{A_n^r}(k+1) - \frac{1}{n\pi}. \tag{8.23}$$

Moreover, the first term in (8.23) is (C, r) summability applied to the sequence $s_k = k+1$, $k = 0, 1, \cdots$.

Let us denote the transformed sequence by σ_k^r. Thus (8.23) now becomes

$$\frac{2\sigma_n^r}{n\pi} - \frac{1}{n\pi}. \tag{8.24}$$

Hence

$$\liminf_{n\to\infty} \int_0^{1/n} K_n^r(t)\,dt \geqq \frac{2}{\pi}\liminf_{n\to\infty}\frac{\sigma_n^r}{n}. \tag{8.25}$$

Now for $r \geqq 1$, it is a well-known result [20, p. 43] that

$$\liminf_{n\to\infty}\frac{\sigma_n^r}{n} \geqq \lim_{n\to\infty}\frac{\sigma_n^1}{n} = \frac{1}{2}.$$

The last equality is readily checked by simple computation. This fact, together with (8.25), evidently proves the lemma(3).

(8.26) THEOREM. *The set* $\bigcup_{k=0}^{\infty} I_{n_k}(t)$, *where* $I_{n_k}(t) = \{(k,s): |s-t| < 1/n_k\} \subset R_k$, *is not thin at* t.

Proof. From the circular symmetry of the state space R it is clear that we need only prove this theorem for $t = 0$. We now begin the proof by observing that

(3) Lemma (8.16) may be used to construct a simple example of a function $f \in L^1(-\pi, \pi)$ which is (C, r) summable at a point but not (C^*, r) summable.

$$\int_{-1/n_k}^{+1/n_k} p_{m,k}(\xi,\eta)\frac{K_{nk}^r(\eta)}{K_{nm}^r(\xi)}\,d\eta \tag{8.27}$$

is the probability that the minimal path process, starting at (m,ξ) and conditioned to converge to the Martin boundary point $t=0$, lies in $I_{n_k}(0)$. Moreover, it is easy to verify, using the continuity of the integrand, (5.1) and (5.7), that for large k (8.27) is asymptotically equal to

$$\frac{1}{\pi}\int_{-1/n_k}^{+1/n_k} K_{n_k}^r(\eta)\,d\eta = \frac{2}{\pi}\int_0^{1/n_k} K_{n_k}^r(\eta)\,d\eta. \tag{8.28}$$

Now by lemma (8.16) we obtain that

$$\liminf_{k\to\infty}\int_{-1/n_k}^{+1/n_k} p_{m,k}(\xi,\eta)\frac{K_{nk}^r(\eta)}{K_{nm}^r(\xi)}\,d\eta \geqq \frac{2}{\pi^2} > 0, \tag{8.29}$$

and hence by the $0-1$ law, the minimal paths conditioned to converge to 0 lie in $\bigcup_{k=0}^{\infty} I_{n_k}(0)$ infinitely often. This concludes the proof.

In view of the above results it is natural to introduce the following:

(8.30) Definition. Let $E\subseteq R$ be not thin at the Martin boundary point $t\in R'$. If v is a function with domain R such that $\lim_{z\to t, z\in E} v(z)=a$ exists, then we say that a is a "fine cluster value" of v at t.

(8.31) Definition. Suppose $\sigma_n^r(t)$ is the nth (C,r) partial sum of a trigonometric series, which is not necessarily an $S[dF]$. If

$$\lim_{n\to\infty}\sigma_n^r(t+\alpha_n)=a \quad \text{for any sequence } \alpha_n = O\left(\frac{1}{n}\right), \tag{8.32}$$

we say that the trigonometric series is (C^*,r) summable at the point t to the value a.

As an example of the usefulness of the notion of (C^*,r) summability, Lebesgue's well-known theorem on the almost everywhere (C,r) convergence of $S(f)$, $f\in L^1[-\pi,\pi]$, can be strengthened to the following [20, p. 61, Example 4]:

(8.33) $S(f)$ is (C^*,r) summable almost everywhere to f, $r>0$, $f\in L^1[-\pi,\pi]$.

In terms of the fine topology (8.33) can be interpreted as follows:

(8.34) Theorem. *If a trigonometric series is (C^*,r) summable to the value a at the point t, then a is a fine cluster value of the function $u(k,\xi)=\sigma_{n_k}^r(\xi)$.*

Proof. The proof is an immediate consequence of the definitions and (8.16).

Finally, we note that Doob's result on Fourier series (Theorem 8.8) can also be interpreted as a fine limit theorem.

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