TWO NOTES ON LOCALLY MACAULAY RINGS

BY

LOUIS J. RATLIFF, JR.

1. Introduction. In this paper all rings are assumed to be commutative rings with a unit. The undefined terminology used in this paper (height, altitude, etc.) will be the same as that in [1]. Throughout this paper a number of known properties of locally Macaulay rings are stated, and then are used in the remainder of the paper without explicit mention.

In §2 it is proven that if R is a locally Macaulay ring and if \((a_1, \ldots, a_n)\) is a prime sequence in \(R\), the kernel of the natural homomorphism from \(P = R[X_1, \ldots, X_{n-1}]\) onto \(R' = R[a_2/a_1, \ldots, a_n/a_1]\) is \((a_1X_1 - a_2, a_1X_2 - a_3, \ldots, a_1X_{n-1} - a_n)P\) (Lemma 2.3). As a consequence, \(R'\) is a locally Macaulay ring and \((a_1, a_2/a_1, \ldots, a_n/a_1)\) is a prime sequence in \(R'\) (Theorem 2.4). Further, if \(R[X_1]\) is a Macaulay ring, then \(R'\) is a Macaulay ring (Theorem 2.8). An example is given to show that the converses are not in general true.

In §3 it is proven that, with the same \(R\) and \(a_i\), the Rees ring \(R^* = R[t_a, \ldots, t_{a_n}, 1/t]\) (\(t\) an indeterminant) of \(R\) with respect to \(A = (a_1, \ldots, a_n)R\) is a locally Macaulay ring (a Macaulay ring if \(R[X_1]\) is) and \((1/t, t_{a_1}, \ldots, t_{a_n})\) is a prime sequence in \(R^*\) (Theorems 3.1 and 3.3). A form of the converses of Theorems 3.1 and 3.3 is true (Theorem 3.8). Also, for every \(e \geq 1, k \geq e, 2\) and \(i = 1, \ldots, n\), \((a_1^e, a_2^e, \ldots, a_i^e)\) \(A^k = (a_1^e, a_2^e, \ldots, a_i^e) \cap A^k\) (Corollary 3.6). Further, for all \(k \geq 1\), every prime divisor of \(A^k\) has height \(n\), and \(A^k : a_1R = A^{k-1}\) (Corollary 3.7). It is also proven that if the Rees ring \(R^*\) of a Noetherian ring \(R\) with respect to an ideal \(A = (a_1, \ldots, a_n)R\) is a locally Macaulay ring (a Macaulay ring), then \(R' = R[a_1/a, \ldots, a_n/a]\) is a locally Macaulay ring (a Macaulay ring) for every non-zero-divisor \(a \in A\) (Corollary 3.9).

2. Transformations of locally Macaulay rings by a prime sequence.

**Lemma 2.1.** Let \(R\) be a ring, let \(a, b\) be elements in \(R\) such that \(a\) is not a zero divisor, and let \(X\) be an indeterminant. If \(aR : bR = aR\), then the kernel \(K\) of the natural homomorphism from \(R[X]\) onto \(R[b/a]\) is generated by \(aX - b\).

**Proof.** Clearly \(aX - b \in K\). Let \(f(X) = r_nX^n + \cdots + r_0 \in K\). Then \(r_nb^n + r_{n-1}ab^{n-1} + \cdots + r_0a^n = 0\), so \(r_n \in aR : b^nR = aR\), say \(r_n = da\). Since \(a\) is not a zero divisor, \(g(X) = (db + r_{n-1})X^{n-1} + r_{n-2}X^{n-2} + \cdots + r_0 \in K\), and \(f(X) = (aX - b)dX^{n-1} + g(X)\). Hence, by induction on \(n\), \(f(X) \in (aX - b)R[X]\), so \(K\) is generated by \(aX - b\), q.e.d.

Presented to the Society, November 14, 1964; received by the editors September 29, 1964.
A local (Noetherian) ring $R$ is a Macaulay local ring in case there exists a system of parameters $(a_1, \ldots, a_n)$ in $R$ such that $a_i$ is not in any prime divisor of $(a_1, \ldots, a_{i-1})R$ $(i = 1, \ldots, n)$. In particular $a_1$ is not a zero divisor. A Noetherian ring $R$ is a locally Macaulay ring in case $R_m$ is a Macaulay local ring for every maximal ideal $M$ in $R$. $R$ is a Macaulay ring in case $R$ is a locally Macaulay ring such that height $M = \text{altitude } R$ for every maximal ideal $M$ in $R$. It is known that if $R$ is a Macaulay local ring of altitude $n$ and if $(a_1, \ldots, a_k)$ is a subset of a system of parameters in $R$, then $R/(a_1, \ldots, a_k)R$ is a Macaulay local ring of altitude $n - k$ [3, p. 397]. Also, $R$ is a locally Macaulay ring if and only if the following theorem (the unmixedness theorem) holds: If an ideal $A$ in $R$ is generated by $k$ elements and if height $A = k$ $(k \geq 0)$, then every prime divisor of $A$ has height $k$ [1, p. 85]. These two facts immediately imply that if $R$ is a locally Macaulay ring (a Macaulay ring) and if $A$ is an ideal in $R$ which is generated by $k$ elements and has height $k$, then $R/A$ is a locally Macaulay ring (a Macaulay ring). Finally, it is known that if $X_1, \ldots, X_n$ are algebraically independent over a Noetherian ring $R$, then $R[X_1, \ldots, X_n]$ is a locally Macaulay ring if and only if $R$ is [1, p. 86].

These facts are used in the proof of

**Corollary 2.2.** Let $R$ be a locally Macaulay ring, and let $a, b$ be elements in $R$ such that $a$ is not a zero divisor and $aR : bR = aR$. Then $R[b/a]$ is a locally Macaulay ring.

**Proof.** $R[X]$ is a locally Macaulay ring, and the kernel of the natural homomorphism from $R[X]$ onto $R[b/a]$ is generated by $aX - b$ (Lemma 2.1). Since $aX - b$ is not a zero divisor in $R[X]$, $R[b/a]$ is a locally Macaulay ring, q.e.d.

Theorem 2.4 below generalizes the above corollary. To obtain the generalization the following definitions and lemma will be used.

An integral domain $R$ satisfies the altitude formula in case the following condition holds: If $R'$ is an integral domain which is finitely generated over $R$, and if $p'$ is a prime ideal in $R'$, then height $p' + \text{trd}(R'/p')(R'/p' \cap R) = \text{height } p' \cap R + \text{trd } R'/R$. It is known that if an integral domain $R$ is a homomorphic image of a locally Macaulay ring, then $R$ satisfies the altitude formula [1, p. 130].

If $R$ is a locally Macaulay ring, and if $p \subset q$ are prime ideals in $R$, then $R_q$ is a Macaulay local ring [1, p. 86], so height $p + \text{height } q/p = \text{height } q$ [3, p. 399]. This fact will be used in the future without explicit mention.

A sequence $(a_1, \ldots, a_n)$ of nonunits in a Noetherian ring $R$ is a prime sequence in case $a_1$ is not a zero divisor, $(a_1, \ldots, a_i)R: a_{i+1}R = (a_1, \ldots, a_i)R$ $(i = 1, \ldots, n - 1)$, and $(a_1, \ldots, a_n)R \neq R$. It is known that if $R$ is a semi-local ring, and if $(a_1, \ldots, a_n)$ is a prime sequence of elements in the Jacobson radical of $R$, then $(a_{\pi_1}, \ldots, a_{\pi_n})$ is a prime sequence for every permutation $\pi$ of $\{1, \ldots, n\}$ [3, pp. 394–395].

**Lemma 2.3.** Let $R$ be a locally Macaulay ring, let $(a_1, \ldots, a_n)$ be a prime sequence in $R$, and let $X_1, \ldots, X_{n-1}$ be algebraically independent over $R$. Then
the kernel $K$ of the natural homomorphism $\phi$ from $P = R[X_1, \ldots, X_n]$ onto $R' = R[a_2/a_1, \ldots, a_n/a_1]$ is generated by $(a_1X_1 - a_2, a_1X_2 - a_3, \ldots, a_1X_{n-1} - a_n)$.

**Proof** (1). The proof is by induction on $n$. The case $n = 1$ is trivial, and Lemma 2.1 proves the case $n = 2$. Let $n > 2$ and assume the conclusion holds for the case $n - 1$. Now $\phi = fg$, where $f$ and $g$ are the natural homomorphisms from $S = R[a_2/a_1, X_2, \ldots, X_{n-1}]$ onto $R'$ and from $P$ onto $S$ respectively. Since the kernel of $g$ is $(a_1X_1 - a_2)^R$ (Lemma 2.1), and since $R^* = R[a_2/a_1]$ is a locally Macaulay ring (Corollary 2.2), it is sufficient (by induction) to prove that $(a_1, a_3, a_4, \ldots, a_n)$ is a prime sequence in $R^*$. Since $R$ and $R^*$ have the same total quotient ring, $a_1$ is not a zero divisor in $R^*$, hence height $a_1R^* = 1$. Let $A_*^i = (a_1X_1 - a_2, a_1X_3, \ldots, a_1X_{n-1})^P$ $(i \geq 3)$. Then $A_*^i = (a_1, a_2, a_3, \ldots, a_i)^P$, hence height $A_*^i = i$. Consequently, by the unmixedness theorem $(a_1X_1 - a_2, a_1, a_3, \ldots, a_n)$ is a prime sequence in $P$, hence $(a_1, a_3, \ldots, a_n)$ is a prime sequence in $R^*$, q.e.d.

**Theorem 2.4.** With the same notation as Lemma 2.3, $R'_i = R[a_2/a_1, \ldots, a_i/a_1]$ $(2 \leq i \leq n)$ is a locally Macaulay ring, and $(a_1, a_{i+1}, \ldots, a_i, b_1, \ldots, b_k)$ is a prime sequence in $R'_i$, where $\{b_1, \ldots, b_k\}$ is a subset of $\{a_2, a_3, \ldots, a_i\}$, and $0 \leq j \leq n - i$. (For $j = 0$ the sequence is $(a_1, b_1, \ldots, b_k)$.)

**Proof.** That $R'_i$ is a locally Macaulay ring follows immediately from Lemma 2.3 and the remarks preceding the proof of Corollary 2.2. Let $A_*^i = (a_1, a_{i+1}, \ldots, a_i, b_1, \ldots, b_k)R'_i$. Since $(a_1X_1 - a_2, \ldots, a_1X_{i-1} - a_i, a_i)^R[X_1, \ldots, X_{i-1}]$ is generated by $(a_1, \ldots, a_i)$, $A_*^i$ is a proper ideal. Hence by the unmixedness theorem, since $j$ and $k$ are arbitrary, it is sufficient to prove height $A_*^i = j + k + 1$. Let $p'$ be a minimal prime divisor of $A_*^i$, let $q'$ be a (minimal) prime divisor of zero in $R'_i$ such that $q' \subset p'$ and let $p = p' \cap R$, $q = q' \cap R$. By the altitude formula (for $R'/q'$ over $R/q$), height $p'/q' + \text{trd} R'/p'R/p = \text{height} p/q$ (since $a_1 \notin q$). Also, height $p'/q' \leq j + k + 1$, $\text{trd} R'/p'R/p \leq i - 1 - k$, and height $p/q = \text{height} p \geq i + j$. Hence, height $p' = \text{height} p'/q' = j + k + 1$. Therefore height $A_*^i = j + k + 1$, q.e.d.

**Remark 2.5.** The last step in the proof of Theorem 2.4 shows the following results. For every (minimal) prime divisor $p'$ of $A_*^i$ and for every prime divisor $q'$ of zero contained in $p'$, $p'/q'$ is a minimal prime divisor of $(A_*^i + q')/q'$. Since height $p'/q' = j + k + 1$, none of the elements $a_1, \ldots, a_i, b_1, \ldots, b_k$ are in $q'$. Also the elements $a_2/a_1, \ldots, a_i/a_1$ which are not in $p'$ are such that their $p'$ residues are algebraically independent over $R/(p' \cap R)$.

**Remark 2.6.** In Theorem 2.4, if every permutation of $(a_1, \ldots, a_n)$ is a prime sequence in $R$ (for example, if $R$ is a semi-local locally Macaulay ring and $a_1, \ldots, a_n$ are in the Jacobson radical of $R$), then every permutation of $(a_1, a_{i+1}, \ldots, a_n, a_2/a_1, \ldots, a_i/a_1)$ is a prime sequence in $R'_i$.

---

(1) The author is indebted to the referee for the following proof which is considerably simpler than the author's original proof, and which leads to a more direct proof of Theorem 2.4.
Proof. Let \((c_1, \ldots, c_n)\) be a permutation of \((a_1, a_{i+1}, \ldots, a_n, a_2/a_1, \ldots, a_i/a_1)\). Since no \(a_i\) is a zero divisor in \(R\), \(c_1\) is not a zero divisor in \(R'\). Also \((c_1, \ldots, c_n)R'_i \neq R'_i\). Therefore, by the unmixedness theorem, it remains to prove height \((c_1, \ldots, c_n)R'_i = h\) \((h = 2, \ldots, n - 1)\). Let \(p'\) be a minimal prime divisor of \((c_1, \ldots, c_n)R'_i\), let \(q'\) be a prime divisor of zero in \(R\) which is contained in \(p'\), and let \(p = p' \cap R, q = q' \cap R\).

If \(a_1 \notin p'\), then \(\text{trd } R'/p'/R/p = 0\). Hence by the altitude formula (for \(R'/q'\) over \(R/q\)), height \(p'/q' = \text{height } p/q\). Now height \(p' \leq h\) and height \(p \geq h\) (by the assumption on \((a_1, \ldots, a_n)\)), so height \(p' = \text{height } p = h\). If \(a_1 \in p'\), let \(k\) of the elements \(c_1, \ldots, c_n\) be in \(\{a_2/a_1, \ldots, a_i/a_1\}\). Then height \(p \geq i + (h - 1 - k)\) (by the assumption on \((a_1, \ldots, a_n)\)), and \(\text{trd } R'/p'/R/p \leq i - 1 - k\). By the altitude formula for \(R'/q'\) over \(R/q\), height \(p' = \text{height } p'/q' = h\), q.e.d.

Remark 2.6 is of some interest because of the following

Lemma 2.7. Let \(R\) be a locally Macaulay ring, and let \((a_1, \ldots, a_n)\) be a prime sequence in \(R\) such that every permutation of \((a_1, \ldots, a_n)\) is a prime sequence in \(R\). Let \(A = (a_1, \ldots, a_n)R\). Then, for all \(k \geq 1\), (1) every prime divisor of \(A^k\) has height \(n\), and (2) \(A^k: a_1R = A^{k-1}\) \((i = 1, \ldots, n)\).

Proof. This can be proved in the same way as Lemmas 5 and 6 in [3, pp. 401-402]. Without assuming that every permutation of \((a_1, \ldots, a_n)\) is a prime sequence in \(R\), Corollary 3.7 below proves (1) is still true and (2') \(A^k: a_1R = A^{k-1}\) (for all \(k \geq 1\)), q.e.d.

It is known that if \(R\) is a Macaulay ring and if \(X_1, \ldots, X_n\) are algebraically independent over \(R\), then \(R[X_1, \ldots, X_n]\) is a Macaulay ring if and only if there does not exist an ideal \(p\) in \(R\) such that \(R/p\) is a semi-local integral domain of altitude one [1, p. 87]. Hence if \(R[X_1]\) is a Macaulay ring, then \(R[X_1, \ldots, X_n]\) is a Macaulay ring. This fact is used in the proof of the next theorem.

Theorem 2.8. If \(R\) and \(R[X]\) are Macaulay rings (\(X\) transcendental over \(R\)), and if \((a_1, \ldots, a_n)\) is a prime sequence in \(R\), then \(R' = R[a_2/a_1, \ldots, a_n/a_1]\) is a Macaulay ring.

Proof. The kernel \(K\) of the natural homomorphisms from \(P = R[X_1, \ldots, X_{n-1}]\) onto \(R'\) has height \(n - 1\). Since \(P\) is a Macaulay ring, if \(M\) is a maximal ideal in \(P\) which contains \(K\), then altitude \(R + n - 1 = \text{altitude } P = \text{height } M = \text{height } M/K + \text{height } K\). Hence, if \(M'\) is a maximal ideal in \(R'\), then height \(M' = \text{altitude } P - \text{height } K = \text{altitude } R\). Since \(R'\) is a locally Macaulay ring by Theorem 2.4, \(R'\) is a Macaulay ring, q.e.d.

Remark 2.9. If \(R\) is a locally Macaulay ring (a Macaulay ring such that \(R[X]\) is a Macaulay ring), and if \((a_1, \ldots, a_n)\) is a prime sequence in \(R\), then \(R[a_1/a, \ldots, a_n/a]\) is a locally Macaulay ring (a Macaulay ring) for every non-zero-divisor \(a \in (a_1, \ldots, a_n)R\). This follows from Theorems 3.1 and 3.3 and Corollary 3.9 below.

It will now be shown that the converses of Theorems 2.4 and 2.8 are not in
general true. Let \( S = k[X, Y] \), where \( k \) is a field and \( X \) and \( Y \) are algebraically independent over \( k \). Let \( P = (X - 1, Y)S \), \( R_1 = S_P \), and \( N_1 = PR_1 \). Let \( Q = (X, Y)S \), \( R_2 = S_Q \), and \( N_2 = QR_2 \). Let \( R' = R_1 \cap R_2 \), \( M_1 = N_1 \cap R' \), and \( M_2 = N_2 \cap R' \). Further let \( R = k + (M_1 \cap M_2) \), and let \( M = (M_1 \cap M_2)R \). Then \( R' \) is the intersection of two regular local rings, hence \( R' \) is normal. The following statements are easily verified: (1) \( M_1 \) and \( M_2 \) are the maximal ideals in \( R' \), and \( R_{Mi} = R_i \) is Noetherian (\( i = 1,2 \)). Therefore \( R' \) is Noetherian \([1, \text{ p. 203}]\), so \( R' \) is a normal semi-local Macaulay domain. (2) Since \( R'/M_i = k \) (\( i = 1,2 \)), \( R \) is a local domain and \( R' \) is its derived normal ring \([1, \text{ p. 204}]\). (3) \( X, Y \in R \), \( X \notin R, \ R' = R[XY/Y] \), and \( (Y, X = XY/Y) \) is a prime sequence in \( R' \). (4) If \( p \) is a height one prime ideal in \( R \), then \( R_p \) is a regular local ring. Since \( R \neq R' \), \( M \) is an imbedded prime divisor of every nonzero element in \( M \) \([1, \text{ p. 41}]\), hence \( R \) is not a Macaulay domain.

3. The Rees ring of a locally Macaulay ring. Let \( R \) be a Noetherian ring, let \( A = (a_1, \ldots, a_n)^R \) be an ideal in \( R \), let \( t \) be an indeterminate, and set \( u = t^{-1} \). The graded Noetherian ring \( R^* = R[ta_1, \ldots, ta_n, u] \) is called the Rees ring of \( R \) with respect to \( A \).

**Theorem 3.1.** Let \( R \) be a locally Macaulay ring, and let \( a_1, \ldots, a_n \) be a prime sequence in \( R \). Then the Rees ring \( R^*_R \) of \( R \) with respect to \( (a_1, \ldots, a_i)^R \) (\( 1 \leq i \leq n \)) is a locally Macaulay ring, and \( (u, a_{i+1}, \ldots, a_j, b_1, \ldots, b_k) \) is a prime sequence in \( R^*_R \), where \( \{b_1, \ldots, b_k\} \) is a subset of \( \{ta_1, \ldots, ta_i\} \) and \( 0 \leq j \leq n - i \). (For \( j = 0 \) the sequence is \( (u, b_1, \ldots, b_k) \).

**Proof.** Since \( u \) is transcendental over \( R, R[u] \) is a locally Macaulay ring, hence \( (u, a_1, \ldots, a_n) \) is a prime sequence in \( R[u] \). Since \( ta_j = a_j/u, R^*_R \) is a locally Macaulay ring and \( (u, a_{i+1}, \ldots, a_j, b_1, \ldots, b_k) \) is a prime sequence in \( R^*_R \) by Theorem 2.4, q.e.d.

**Remark 3.2.** In Theorem 3.1, if every permutation of \( (a_1, \ldots, a_n) \) is a prime sequence in \( R \), then every permutation of \( (u, a_1, \ldots, a_n) \) is a prime sequence in \( R[u] \) (since \( R[u] \) is a locally Macaulay ring and \( u \) is transcendental over \( R \)), hence by Remark 2.6 every permutation of \( (u, a_{i+1}, \ldots, a_n, ta_1, \ldots, ta_i) \) is a prime sequence in \( R^*_R \).

**Theorem 3.3.** If \( R \) and \( R[X] \) are Macaulay rings (\( X \) transcendental over \( R \)), and if \( a_1, \ldots, a_n \) is a prime sequence in \( R \), then the Rees ring \( R^*_R \) of \( R \) with respect to \( (a_1, \ldots, a_n) \) is a Macaulay ring.

**Proof.** Considering the natural homomorphism from \( R[u, X_1, \ldots, X_n] \) onto \( R^*_R \) and the ideal \( (u, a_1, \ldots, a_n) \) of \( R^*_R \), the proof is the same as the proof of Theorem 2.8, q.e.d.

**Lemma 3.4.** Let \( R^*_R \) be the Rees ring of a locally Macaulay ring \( R \) with respect to a prime sequence \( (a_1, \ldots, a_n) \) in \( R \). Then \( (ta_1, \ldots, ta_i, u) \) is a prime sequence in \( R^*_R \) (\( i = 1, \ldots, n \)).
Proof. Since $R^*$ is a locally Macaulay ring and height $(u, ta_1, \ldots, ta_i)R^* = i + 1$ (Theorem 3.1), it is sufficient to prove height $(ta_1, \ldots, ta_i)R^* = i$. Let $p$ be a minimal prime divisor of $A^*_i = (ta_1, \ldots, ta_i)R^*$. Then height $p \leq i$, hence $u \notin p$. Let $T = R[u, i]$, so $T$ is a quotient ring of $R^*$. Since $pT$ is a minimal prime divisor of $A^*_i T = (a_1, \ldots, a_i)T$, and since height $(a_1, \ldots, a_i)R[u] = i$, height $A^*_i T = i$. Therefore height $p = i$, so height $A_i^* = i$, q.e.d.

Remark 3.5. Let $(a_1, \ldots, a_n)$ be a prime sequence in a locally Macaulay ring $R$. Then the radical of $(a_1, \ldots, a_n)R$ is the radical of $(a_1^{e_1}, a_2^{e_2}, \ldots, a_n^{e_n})R$ $(e_i \geq 1, i = 1, \ldots, n)$. Hence, by the unmixedness theorem, $(a_1^{e_1}, a_2^{e_2}, \ldots, a_n^{e_n})$ is a prime sequence in $R$. Therefore $R[a_2^{e_2}/a_1^{e_1}, \ldots, a_n^{e_n}/a_1^{e_1}]$ and $R[t, a_1^{e_1}, \ldots, a_n^{e_n}, u]$ are locally Macaulay rings.

Let $R$ be a Noetherian ring and let $R^*$ be the Rees ring of $R$ with respect to an ideal $A$ in $R$. Let $T = R[t, u]$, so $T$ is a quotient ring of $R^*$. For any ideal $B$ in $R$ let $B' = BT \cap R^*$. For any homogeneous ideal $B^*$ in $R^*$ let $[B^*]_k$ be the set of elements $r \in R$ such that $r^k \in B^*$. It is immediately seen that $[B^*]_k$ is an ideal in $R$ and $A^k \supseteq [B^*]_k \supseteq [B^*]_{k+1}$ for all integers $k$ (with the convention that $A^k = R$ if $k \leq 0$). Also, since $R^*$ is Noetherian, if $k$ is greater than or equal to the maximum degree of the generators of $B^*$, then $[B^*]_{k+1} = A[B^*]_k$, and if $k$ is less than or equal to the degree of the generators of $B^*$, then $[B^*]_{k-1} = [B^*]_k$ [2].

Let $B = (b_1, \ldots, b_i)R \subseteq A^*$. Clearly $B' = BT \cap R^* \supseteq (b_1 t^e, b_2 t^{e_2}, \ldots, b_i t^e)R^* = B^*$, and for $k \leq e$, $[B^*]_e \subseteq B \cap A^e = B \supseteq [B^*]_e \supseteq B$. Hence for $k > e$, $[B^*]_k = B \cap A^k \supseteq [B^*]_k = BA^{k-e}$. Since $B'T = B'T = BT$, $B^* = B'$ if and only if $u$ is not in any prime divisor of $B^*$. Hence if $(b_1 t^{e_1}, b_2 t^{e_2}, \ldots, b_i t^e, u)$ is a prime sequence in $R^*$, then $B' = B^*$. In particular, by Lemma 3.4 and Remark 3.5 we have proved the following

Corollary 3.6. Let $R$ be a locally Macaulay ring, let $(a_1, \ldots, a_n)$ be a prime sequence, and let $A = (a_1, \ldots, a_n)R$. Then, for every $e \geq 1$, $k \geq e$, and $i = 1, \ldots, n$, $(a_1^{e_1}, a_2^{e_2}, \ldots, a_i^{e_i})A^{k-e} = (a_1^{e_1}, a_2^{e_2}, \ldots, a_i^{e_i})R \cap A^k$.

Corollary 3.7. Let $(a_1, \ldots, a_n)$ be a prime sequence in a locally Macaulay ring $R$. Set $A = (a_1, \ldots, a_n)R$. Then, for all $k \geq 1$, (1) every prime divisor of $A^k$ has height $n$, and (2) $A^k : a_i R = A^{k-1}$.

Proof. By Corollary 3.6, $a_1 A^{k-1} = a_1 R \cap A^k$. Since $a_1$ is not a zero divisor in $R$, $A^{k-1} = a_1 A^{k-1} = a_1 R \cap A^k = (a_1 R \cap A^k)$ : $a_1 R = A^k : a_1 R$, hence (2) holds. For (1), $u^k R^* \cap R = A^k$, where $R^* = R[ta_1, \ldots, ta_n, u]$, and $k \geq 1$. Since $R^*$ is a locally Macaulay ring, every prime divisor of $uR^*$ has height one, and the prime divisors of $u^k R^*$ are the prime divisors of $uR^*$ (Remark 3.5). Let $p'$ be a prime divisor of $uR^*$, let $q'$ be a minimal prime divisor of zero in $R^*$ which is contained in $p'$, and let $p = p' \cap R$, $q = q' \cap R$. Applying Remark 2.5 (with $A^* = uR^*$) and the altitude formula for $R^*/q'$ over $R/q$, height $p = n$ (since $trdR^*/q'/R/q = 1$), so $p$ is a prime divisor of $A^k$. Since $u^k R^* \cap R = A^k$, (1) holds, q.e.d.
If \((a_1, \cdots, a_n)\) is a prime sequence in a locally Macaulay ring \(R\), then \((ta_1, \cdots, ta_n, u)\) is a prime sequence in the locally Macaulay ring \(R[ta_1, \cdots, ta_n, u]\) (Theorem 3.1 and Lemma 3.4). Theorem 3.8 contains the converse of this.

**Theorem 3.8.** Let \(R\) be a Noetherian ring and let \(A\) be an ideal in \(R\). If the Rees ring \(R^*\) of \(R\) with respect to \(A\) is a locally Macaulay ring (a Macaulay ring), then \(R\) is a locally Macaulay ring (\(R\) and \(R[X]\) are Macaulay rings). If also there are elements \(b_1, \cdots, b_n\) in \(A\) such that \((b_1t^{e_1}, \cdots, b_nt^{e_n}, u)\) is a prime sequence in \(R^*\), then \((b_1, \cdots, b_n)\) is a prime sequence in \(R\).

**Proof.** Let \(R^*\) be a locally Macaulay ring. Then, since \(T = R^*[t]\) is a quotient ring of \(R^*\), \(T\) is a locally Macaulay ring. Let \(M\) be a maximal ideal in \(R\). Since \(T\) is a quotient ring of \(R^*\) and of \(R[u]\), \(T_M = R[u]_{MR[u]}\) is a Macaulay local ring. Since \(u\) is transcendental over \(R\), a system of parameters in \(R^*\) is a system of parameters in \(R[u]_{MR[u]}\). It is known that if a local ring has one system of parameters which form a prime sequence, then each system of parameters forms a prime sequence [3, p. 399]. Hence \(R\) is a locally Macaulay ring. Therefore, if \((b_1t^{e_1}, \cdots, b_nt^{e_n}, u)\) is a prime sequence in \(R^*\), then, for \(i = 1, \cdots, n\), every prime divisor of \((b_1t^{e_1}, \cdots, b_nt^{e_n})T = (b_1, \cdots, b_i)T\) has height \(i\). Hence height \((b_1, \cdots, b_i)R = i\), and so \((b_1, \cdots, b_n)\) is a prime sequence in \(R\). Let \(R^*\) be a Macaulay ring. By what has already been proved, \(R\) and \(R[X]\) are locally Macaulay rings. To prove that \(R\) is a Macaulay ring, let \(M\) be a maximal ideal in \(R\). Then \(N^* = (M, u - 1)T \cap R^*\) is a maximal ideal in \(R^*\). Therefore, altitude \(R + 1 = \text{altitude } R^* = \text{height } N^* = \text{height } N^*T = \text{height } M + 1\), hence \(R\) is a Macaulay ring. Finally, let \(N\) be a maximal ideal in \(R[u]\). If there is a maximal ideal \(N^*\) in \(R^*\) such that \(N^* \cap R[u] = N\), then altitude \(R[u] = \text{height } N^* = (\text{since } R^*/N^*\text{ is a field}) \text{ height } N^* + \text{trd } R^*/N^*/R[u]/N = (\text{altitude formula}) \text{ height } N \leq \text{altitude } R[u].\) If there does not exist such \(N^*\), then \(NT = T\), hence \(u \in N\). Therefore \(R/N \cap R = R[u]/N\) is a field, so altitude \(R[u] = \text{altitude } R + 1 = \text{height } N \cap R + 1 = \text{height } N^*\). Hence \(R[X] \cong R[u]\) is a Macaulay ring, q.e.d.

**Corollary 3.9.** Let \(R\) be a Noetherian ring. If there exists an ideal \(A = (a_1, \cdots, a_n)R\) in \(R\) such that the Rees ring \(R^*\) of \(R\) with respect to \(A\) is a locally Macaulay ring (a Macaulay ring), then for every non-zero-divisor \(a \in A\), \(R' = R[a_1/a, \cdots, a_n/a]\) is a locally Macaulay ring (a Macaulay ring).

**Proof.** Since \((a - u)R[t, u] = (at - 1)R[t, u]\) is the kernel of the mapping from \(R[t, u]\) onto \(R[1/a, a]\) (Lemma 2.1), and since \(R[t, u]\) is a quotient ring of \(R^*\), to prove the two statements about \(R'\) it is sufficient to prove that \(u\) is not in any prime divisor of \((ta - 1)R^*\). If \(u\) is in some (minimal) prime divisor \(p\) of \((ta - 1)R^*\), then \(p\) is a prime divisor of \(uR^*\). But \(uR^*\) is a graded ideal, hence \(p\) is a graded deal. This implies the contradiction \(1 \in p\). Therefore \(u\) is not in any prime divisor of \((ta - 1)R^*\), q.e.d.
Theorem 3.8 is of some interest, since the Rees ring $R^*$ of a locally Macaulay ring $R$ with respect to an ideal $A$ which cannot be generated by a prime sequence may be a locally Macaulay ring. For example, let $R$ be a semi-local Macaulay ring of altitude $n \geq 2$, and let $(a_1, \ldots, a_n)$ be a prime sequence in the Jacobson radical of $R$. Let $A = (a_1, \ldots, a_n)R$ and fix an integer $e \geq 2$. Then $A^e$ cannot be generated by $n$ elements, but the Rees ring of $R$ with respect to $A^e$ is a locally Macaulay ring. For convenience of notation this will be proved for the case $n = 2$ (the general case being exactly the same). Let $a = a_1$ and $b = a_2$, and let $N$ be a maximal ideal in $R^* = R[t^e, \ldots, t^eb^{-f}, \ldots, t^b, u]$. If $u \notin N$, then $R^*_N$ contains $T = R[t, u]$. Since $T$ is a locally Macaulay ring, $R^*_N$ is a Macaulay local ring. If $(t^e, \ldots, t^eb^{-f}, \ldots, t^b)R^*$ is not contained in $N$, say $t^fb^{-f} \notin N$. Then $t^{f+1}b^{-f-1}/t^fb^{-f} = a/b \in R^*_N$ (if $f < e$), and/or $b/a \in R^*_N$ (if $f > 0$). Since $(a, b)$ and $(b, a)$ are prime sequences in $R$, $R_e = R[a/b], R_0 = R[b/a]$, and $R_f = R[a/b, b/a]$ are locally Macaulay rings, and at least one of these rings (call it $R'$) is contained in $R^*_N$. Hence $S = R'[t^f b^{-f}]$ is a locally Macaulay ring contained in $R^*_N$, and $S$ contains $R[t^e, \ldots, t^eb^{-e}, \ldots, t^b]$. Since $t^fb^{-f} \notin N = NR^*_N \cap S$, $u = a^e b^{-f} t^fb^{-f} \in S_N$. Hence $R^*_N = S_N$ is a locally Macaulay ring.

Clearly the only maximal ideals in $R^*$ which contain $(t^e, \ldots, t^eb^{-f}, \ldots, t^b, u)R^*$ are the ideals $N_i = (M_i, t^e, \ldots, t^eb^{-f}, \ldots, t^b, u)R^*$, where $M_i$ is a maximal ideal in $R$. Therefore it remains to prove that the semi-local ring $R^*_R \cup N_i$ is a Macaulay ring. For this, it will be shown that $(a^e, b^e, u)$ is a prime sequence in $R^*$ (since the $N_i$ contain this sequence). Since $(a^e, b^e)$ is a prime sequence in the locally Macaulay ring $R^*[t]$, to prove $(a^e, b^e, u)$ is a prime sequence, it is sufficient to prove that $u$ is not in any prime divisor of either of the ideals $t^e R^*$ or $(a^e, t^b)R^*$. This is equivalent to proving $t^e R^* = a^e T \cap R^*$ and $(a^e, t^b)R^* = (a^e, b^e) T \cap R^*$, where $T = R[t, u]$. With the notation used in the proof of Corollary 3.6, these latter equalities are equivalent to $[a^e R^*]_k = [a^e T \cap R^*]_k$ and $[(a^e, t^b)R^*]_k = [(a^e, b^e) T \cap R^*]_k$ for all $k$. Since the degrees of the generators of the four ideals are all non-negative, and since $[a^e R^*]_0 = [a^e T \cap R^*]_0 = a^e R$ and $[(a^e, t^b)R^*]_0 = [(a^e, b^e) T \cap R^*]_0 = (a^e, b^e) R$, it must be shown that $a^e (A^e)^{k-1} = a^e R \cap (A^e)^k$ and $(a^e, b^e) (A^e)^{k-1} = (a^e, b^e) R \cap (A^e)^k$ for all $k \geq 1$. These equalities hold by Corollary 3.6.

References


University of California, Riverside, California