AN EXTENSION OF THE DENJOY-CARLEMAN-AHLFORS THEOREM IN SUBHARMONIC FORM(1)

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1. **Introduction.** We shall be concerned in this paper with the Denjoy-Carleman-Ahlfors theorem *in subharmonic form*, a theorem that can be expressed as follows [9]: *If* $v_1, \cdots, v_m$, $m \geqq 2$, *are nonconstant non-negative subharmonic functions on the finite plane which satisfy the condition* $\min\{v_i, v_j\} = 0$ *for* $i \neq j$, *then*

$$\liminf_{r \to +\infty} q(r) r^{-m/2} > 0,$$

where

$$q(r) = \left\{ \sum_{j=1}^{m} \int_0^{2\pi} v_j(re^{i\theta})^2 \, d\theta \right\}^{1/2}.$$

The classical Denjoy-Carleman-Ahlfors theorem is an immediate consequence of this subharmonic variant and a theorem of Lindelöf [5, p. 229].

We propose to extend the subharmonic form of the Denjoy-Carleman-Ahlfors theorem to a larger class of surfaces. In particular, we consider parabolic Riemann surfaces with finite ideal boundary in the sense of Kerékjártó-Stoïlow. Kuramochi [13] has extended Evans' theorem for the plane [6] to apply in general to parabolic Riemann surfaces with the result that we can be assured of the existence of a harmonic exhaustion function (in a sense to be made precise) on any such surface. We shall be interested in the case where the exhaustion function is such that the number of components of its level loci is bounded above, say by k (the "k-condition" of §6). For this reason attention is restricted to surfaces with finite ideal boundary. If the exhaustion function satisfies the k-condition for some k, we shall see that a strict analogue of the Denjoy-Carleman-Ahlfors theorem in subharmonic form can be obtained where the factor $m/2$ above is replaced by $[m - (k-1)]^2/2m$ and the condition $m \geqq 2$ is replaced by $m \geqq k + 1$.

It might be hoped that this result will yield a similar extension of the classical Denjoy-Carleman-Ahlfors theorem to admitted surfaces although we have not

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succeeded in establishing this. The difficulties here are due to the topological complexities introduced in going from the finite plane to surfaces of infinite genus. The extension is useful in other connections however as is indicated in §§7–8.

Our method of solution will follow that which Carleman [2] used in proving the Denjoy conjecture. We shall see that the Carleman method lends itself quite naturally to the study of growth problems on surfaces of higher topological structure not necessarily plane. Carleman's approach was to measure the growth of entire functions in terms of a quadratic integeral norm taken over the level loci of the exhaustion function $\log|z|$. This norm satisfies a differential inequality which is useful in the study of growth problems. Heins [9] has replaced this differential inequality by a notion of convexity which enables us to consider less "regular" situations than those to which Carleman's method applies. For surfaces of the type indicated above we shall see that the Carleman norm of a non-negative subharmonic function taken over the level loci of an exhaustion function possesses a certain convexity of the sort introduced by Heins. This convexity property will lead ultimately to the indicated extension.

§§2–4 of the present paper deal with material preliminary to the proposed extension of the Denjoy-Carleman-Ahlfors theorem. Here the level loci of harmonic functions together with certain integrals over these level loci are studied, the analysis being restricted in most instances to "finite" Riemann surfaces, i.e., relatively compact proper subregions of a Riemann surface which are bounded by a finite number of mutually disjoint Jordan curves. These results are then combined to yield a study of growth properties on parabolic Riemann surfaces that are admissible in the above sense.

2. **Harmonic functions—level analysis.** Throughout this section H will denote a Riemann surface admitting a nonconstant harmonic function u such that the set $\{s' < u < s''\}$ is relatively compact in H for s' and s'' satisfying $\inf u < s' < s'' < \sup u$. We shall be interested subsequently in certain integrals taken over the level loci of such u. To this end an analysis of the relevant properties of these loci is desirable. We establish the following notation:

$$\Gamma(s) = \{u = s\},$$

$$A(s', s'') = \{s' < u < s''\}.$$

Certain properties of the above sets are immediate. Thus, if $\inf u < s < \sup u$, then $\Gamma(s)$ is compact, nonempty and has a finite number of components. And, if $\inf u < s' < s'' < \sup u$, then $A(s', s'')$ and its components $\alpha(s', s'')$ are open and $\alpha(s', s'')$ contains at least one component of $\Gamma(s)$ for each $s \in (s', s'')$.

Let S denote the conformal structure of H and let δu denote the analytic differential generated by u, i.e., $\delta u\colon \sigma \to (u \circ \sigma)_x - i(u \circ \sigma)_y$, where $\sigma \in S$ and $z = x + iy$ is a point in the domain of σ. A point $p \in H$ is said to be a *critical point* of u if

there exists $\sigma \in S$, p an element of the image of σ, such that $\delta u_\sigma(\sigma^{-1}(p)) = 0$, where δu_σ denotes the image of σ under δu. Note that the critical points of u are well defined independently of the uniformizer σ. We denote the set of critical points of u by E. Since u is nonconstant on H the points of E cluster at no point of H. Thus E is at most countable and has at most a finite number of points in $A(s', s'')$, $\inf u < s' < s'' < \sup u$. Further, if $\mathscr{E} = \{s \mid \Gamma(s) \cap E \neq \varnothing\}$, then $\mathscr{E}$ is also at most countable and the points of $\mathscr{E}$ cluster at no value between $\inf u$ and $\sup u$. Thus $\mathscr{E}$ is the image of a monotone sequence of reals from one of the standard sets.

We focus attention on the components $\alpha(s', s'')$ of $A(s', s'')$ when the interval (s', s'') contains no points of $\mathscr{E}$. The following theorem will be of particular importance regarding these components.

THEOREM 2.1. *Let s' and s'' be finite and such that $\inf u \leqq s' < s'' \leqq \sup u$. If $(s', s'') \cap \mathscr{E} = \varnothing$ and $\alpha(s', s'')$ is a component of $A(s', s'')$, then $\alpha(s', s'')$ is conformally equivalent to a proper annulus, i.e., there exist R_1 and R_2, $0 < R_1 < R_2 < +\infty$, and a conformal univalent map ψ of $\{R_1 < |z| < R_2\}$ onto $\alpha(s', s'')$.*

Observe that $\alpha(s', s'')$ has a conformal universal covering ϕ with domain $\{|z| < 1\}$. We shall want the following two lemmas.

LEMMA 2.2. *If f is analytic on $\{|z| < 1\}$ with real part $u \circ \phi$, then $g = \exp \circ f$ is a covering of the (proper) annulus $\{e^{s'} < |w| < e^{s''}\}$.*

Proof. Since $\alpha(s', s'')$ contains no critical points of u, it follows that f and therefore g are locally simple. On the other hand f has no asymptotic values in $\{s' < \operatorname{Re} z < s''\}$ and therefore g has no asymptotic values in $\{e^{s'} < |w| < e^{s''}\}$. It follows that g is a covering.

LEMMA 2.3. *Let $G(\phi)$ denote the group of conformal automorphisms on $\{|z| < 1\}$ which leave ϕ invariant. Then either $G(\phi) = \{i\}$, i the identity map, or $G(\phi)$ is cyclic hyperbolic (the terminology here is that in* [17]).

Proof. Assume $G(\phi) \neq \{i\}$ and let $\beta \in G(\phi)$, $\beta \neq i$. Then $f \circ \beta = f +$ an imaginary constant dependent on β and therefore (1) $g \circ \beta = e^{i\theta(\beta)} g$, $0 \leqq \theta(\beta) < 2\pi$. It is easily established that (2) $g \circ \beta^{-1} = e^{-i\theta(\beta)} g$. Now, if $G(g)$ denotes the group of conformal automorphisms on $\{|z| < 1\}$ which leave g invariant, then $G(g)$ is cyclic hyperbolic by Lemma 2.2 (cf. [17, pp. 253–254]). Thus all elements of $G(g)$, not the identity have the same fixed points, say a and b, $a \neq b$. Let $\gamma \in G(g)$, $\gamma \neq i$, and consider $\delta = \beta^{-1} \circ \gamma \circ \beta$. By (1) and (2) it follows that $\delta \in G(g)$. But then β must also be hyperbolic with fixed points a and b. Since β is an arbitrary element of $G(\phi)$, $\beta \neq i$, it follows that $G(\phi)$ is abelian and therefore cyclic (cf. [17, p. 259]). Lemma 2.3 follows.

Proof of Theorem 2.1. In view of Lemma 2.3 it suffices to show that $G(\phi)$ does not reduce to the identity (cf. [18, p. 233]). Assume the contrary. Then ϕ

is univalent and $\alpha(s',s'')$ is conformally equivalent to $\{|z|<1\}$. On the other hand, if $s\in(s',s'')$ then $\Gamma(s)\cap\alpha(s',s'')$ is compact. Therefore $\{u\circ\phi=s\}\subset\{|z|\leqq r\}$ for some $0<r<1$. But in $\{r<|z|<1\}$ there exist points whose $u\circ\phi$-values are arbitrarily close to s' and points whose $u\circ\phi$-values are arbitrarily close to s''. Since $\{r<|z|<1\}$ is connected and $u\circ\phi$ is continuous, this implies that $\{u\circ\phi=s\}\cap\{r<|z|<1\}\neq\varnothing$. Contradiction.

COROLLARY 2.4. *Let ψ be a conformal univalent map from $\{R_1<|z|<R_2\}$, $0<R_1<R_2<+\infty$, ontv $\alpha(s',s'')$, $\alpha(s',s'')$ as in Theorem* 2.1. *Then $u\circ\psi(z)=B\log|z|+C$, $B\neq 0, B, C$* constant. *It can be assumed that $B>0$ and, by the appropriate adjustment in the domain of ψ, C can be taken to be* 0.

Proof. Observe that $u\circ\psi$ can be extended continuously to $\{R_1\leqq|z|\leqq R_2\}$ with $u\circ\psi$ so extended taking the boundary values s' and s'' on the appropriate boundary components. We can assume $u\circ\psi$ has the boundary value s' on $\{|z|=R_1\}$ and the boundary value s'' on $\{|z|=R_2\}$ by considering $\psi\circ f$, if need be, where $f: z\to R_1R_2/z$. Then by the maximum principle for harmonic functions

$$u\circ\psi(z)=B\log|z|+C,$$

where

$$B=(s''-s')/(\log R_2-\log R_1)$$

and

$$C=(s'\log R_2-s''\log R_1)/(\log R_2-\log R_1).$$

Moreover, if $g: z\to Dz$ where $D=\exp(-C/B)$, then $\psi\circ g: \{R_1/D<|z|<R_2/D\}\to\alpha(s',s'')$ is univalent onto and $u\circ\psi\circ g(z)=B\log|z|$.

It follows from Corollary 2.4, whenever $s\in(s',s'')$ and $(s',s'')\cap\mathscr{E}=\varnothing$, that $\Gamma(s)$ has precisely one component in $\alpha(s',s'')$ and that component is the image of a simple closed regular analytic curve of the form $\theta\to\psi(re^{i\theta})$.

COROLLARY 2.5. *If ψ_1 and ψ_2 are conformal univalent maps from $\{r_1<|z|<r_2\}$ and $\{R_1<|z|<R_2\}$ respectively onto $\alpha(s',s'')$, with $\psi_1=B_1\log|z|+C_1$ and $\psi_2=B_2\log|z|+C_2$, $B_i>0$, $i=1,2$, then $B_1=B_2$ and $\psi_1=\psi_2\circ f$, where f is a homothetic transformation of the form $f(z)=Az$, $A\neq 0$.*

Proof. Since the domains of ψ_1 and ψ_2 are conformally equivalent it follows that $r_1/r_2=R_1/R_2$ and hence that $B_1=B_2$. Also, $f=\psi_2^{-1}\circ\psi_1$ is such that $\log|f(z)|=B\log|z|+C$, where $B=1$ since $r_1/r_2=R_1/R_2$. Thus $|f(z)/z|=e^C$ and it follows that $f(z)=Az$ for some $A\neq 0$.

3. **Line integrals—convexityp roperties.** H will now denote a finite Riemann surface whose frontier consists of a finite number $\geqq 2$ of mutually disjoint Jordan curves. To avoid unessential difficulties we shall assume the boundary curves are regular analytic. Let the components of $\operatorname{fr}H$ be divided into two disjoint non-

empty sets A and B whose union is the collection of all components of fr H. Let u denote the harmonic measure with respect to H of $\bigcup \beta$, $\beta \in B$, i.e., u has the boundary value 0 on $\alpha \in A$ and the boundary value 1 on $\beta \in B$. Observe that the pair (u, H) satisfies the conditions in §2. The notation from that section will continue to be in effect. The results that will be established in this section for the pair (u, H) extend trivially to any pair (U, H) where U is harmonic on H with boundary values b_0 and b_1, $b_0 \neq b_1$, replacing 0 and 1.

Since the components of fr H are regular analytic curves, it follows by the Schwarz reflection principle that u can be extended to a function harmonic in an open set Ω containing $\bar{H}$. Hence the sets E and $\mathscr{E}$ are finite and we can write $\mathscr{E} = \{s_j \mid s_j < s_{j+1},\ j = 1, \cdots, m\}$. For brevity, let $s_0 = 0$, $s_{m+1} = 1$ and let $I_j = (s_j, s_{j+1})$, $A_j = A(s_j, s_{j+1})$, $j = 0, \cdots, m$. For each j we denote the components of A_j by α_j^i, $i = 1, \cdots, n_j$, and for each $s \in I_j$, we denote the unique component of $\Gamma(s)$ in α_j^i by γ_s^i, $i = 1, \cdots, n_j$. Further, let ψ_j^i denote a conformal univalent map from an annulus onto α_j^i. By Lemma 2.4 we can assume that $u \circ \psi_j^i(z) = B_j^i \log |z|$, $B_j^i > 0$. Then the domain of ψ_j^i is $\{\exp(s_j/B_j^i) < |z| < \exp(s_{j+1}/B_j^i)\}$ and $\psi_j^i(\{|z| = \exp(s/B_j^i)\}) = \gamma_s^i$ if $s \in I_j$. Note that B_j^i is independent of admitted ψ_j^i.

If ω is a locally bounded upper or lower semicontinuous first order differential, we denote the line integral of ω over $\Gamma(s)$ by

$$\int_{\Gamma(\ )} \omega$$

(cf. [1] and [18]). All integrals are to be taken in the sense of Lebesgue. In particular, if $s \in I_j$, then

$$\int_{\Gamma(s)} \omega = \sum_{i=1}^{n_j} \int_{\gamma_s^i} \omega,$$

where we take the positive orientation on γ_s^i to be that induced by $\theta \to \psi_j^i \circ \exp([s/B_j^i] + i\theta)$. It follows from the Gauss-Green theorem that $\int_{\Gamma(s)} \omega = a$ constant whenever ω is a closed differential on H. Thus, if $* \, du$ denotes the conjugate differential of u, then $\int_{\Gamma(s)} * \, du = A$, A constant. In fact, if $s \in I_j$, then in the logarithmic plane

$$\int_{\gamma_s^i} * \, du = \int_0^{2\pi} B_j^i \, d\theta, \qquad i = 1, \cdots, n_j, \tag{3.1}$$

and therefore $A = 2\pi \sum_i B_j^i > 0$. Moreover, if h is harmonic on H, then $h * du - u * dh$ is a closed differential and therefore

$$\int_{\Gamma(s)} h * du$$

is linear in s. It follows from this result and the positive character of $* \, du$ on $(s)\Gamma$ ((3.1) above) that

$$\int_{\Gamma(s)} v * du$$

is convex in s for v subharmonic, $\not\equiv -\infty$, on H. We shall be concerned only with the case $v \geqq 0$. The following additional properties which will be wanted in §§4–5 are easily established.

(3.2) (HADAMARD THREE CIRCLE THEOREM). *If v is non-negative subharmonic on H and $\sigma(v;s) = \max v$ on $\Gamma(s)$, then $\sigma(v;s)$ is convex in s.*

(3.3) If $m(v;s) = \int_{\Gamma(s)} v * du$, then $\sigma(v;s) = 0$ if and only if $m(v;s) = 0$.

4. **λ-convexity of the Carleman norm.** We recall the following notion of convexity introduced by Heins [9], a notion which replaces a differential inequality of Carleman [2] and provides useful information in the study of growth problems.

(4.1) Let λ be a non-negative function on a connected subset B of the real line such that λ is integrable on each compact subset of B. A function L with domain B is said to be *λ-linear* if an only if L is non-negative, L' is absolutely continuous on each compact subinterval of B and $L'' = \lambda L$ almost everywhere.

For each non-negative integrable λ on $[a, b]$ there exists a function y on $[a, b]$ which is λ-linear and takes arbitrary preassigned non-negative values at a and b. The function y with these preassigned values is unique.

(4.2) Let λ be as in (4.1) and let f denote a finite valued non-negative function with domain B. Then f is said to be *λ-convex* provided that for each $[a, b] \subset B$ the unique λ-linear function on $[a, b]$ which agrees with f at a and b dominates f throughout $[a, b]$. (A slight gloss in terminology will be made in the future when no confusion is likely to arise, i.e., we shall speak of f being λ-convex on A, A a proper subset of B.)

Observe that

(4.3) If λ_1 and λ_2 are admissible on B and $\lambda_1 \leqq \lambda_2$, then each λ_2-convex function is also λ_1-convex.

The reader is referred to [9] for a further discussion of λ-convex functions and their properties. The following additional properties are readily verified.

(4.4) Let λ be non-negative and integrable on $[a, b]$. If f is λ-convex on (a, b) and continuous on $[a, b]$, then f is λ-convex on $[a, b]$.

(4.5) Let λ be admissible on B, B as above, and let f be defined on B and λ-convex on the subintervals of $B - N$, where N is a countable subset of B clustering at no point of B. If $f'(b^-) \leqq f'(b^+)$ for all $b \in N$, then f is λ-convex on B. (By $f'(b^-)$ and $f'(b^+)$ we mean respectively the left and right hand derivatives of f at b.)

We return now to u and H as in §3. Let v be non-negative subharmonic on

H. We wish to construct a nontrivial function λ such that the Carleman norm of v given by

$$Q(s) = \left\{ \int_{\Gamma(s)} v^2 * du \right\}^{1/2} \tag{4.6}$$

is λ-convex on $(0,1)$. To this end it will be convenient to introduce the following notion of convexity—a notion designed to take advantage of (4.3), (4.5) and the fact that the zero function is λ-convex for arbitrary non-negative integrable λ. Given an arbitrary function λ from a connected interval B of the real line into $R^+ \cup \{+\infty\}$, R^+ the non-negative reals, a non-negative function f on B will be termed $[\lambda]$-*convex* if and only if f is λ^*-convex for any non-negative λ^* that is integrable on each compact subinterval of B and satisfies $\lambda^* \leqq \lambda$ on B. This concept will prove to be useful in §5. Observe that every λ-convex function is also $[\lambda]$-convex.

Returning to the function v above, we define θ_j^i on I_j by $\theta_j^i(s) =$ the angular measure of $\{|z| = \exp(s/B_j^i)\} \cap \{v \circ \psi_j^i = 0\}$. Define λ_j^i on I_j by

$$\lambda_j^i(s) = 0 \text{ if } \inf v > 0 \text{ on } \gamma_s^i,$$

$$\lambda_j^i(s) = \left\{ B_j^i \left[2 - \frac{\theta_j^i(s)}{\pi} \right] \right\}^{-2} \text{ if } \inf v = 0 \text{ on } \gamma_s^i \text{ and } \theta_j^i(s) < 2\pi,$$

$$\lambda_j^i(s) = +\infty \quad \text{if } \theta_j^i(s) = 2\pi,$$

i.e., if and only if $v \equiv 0$ on $\gamma_s{}^i$(cf. (3.3)).

Finally, define λ on $(0,1)$ by $\lambda(s) = \min\{\lambda_j^i(s) \mid i = 1, \cdots, n_j\}$ if $s \in I_j$ and $\lambda(s) = 0$ if $s \in \mathscr{E}$. (This latter condition is harmless and is motivated by the definition of the function μ to follow.) Observe that θ_j^i, λ_j^i and λ are independent of admitted ψ_j^i (Corollaries 2.4, 2.5).

The aim of the present section is to establish the following:

THEOREM 4.7. *In terms of the above notation and with the above assumptions on v, the Carleman norm of v given by* (4.6) *is* $[\lambda]$-*convex on* $(0,1)$.

We proceed by showing that Theorem 4.7 can be reduced to a theorem of more restricted type. Thus, under the same assumptions on v, define Φ_j^i on I_j by $\Phi_j^i(s) =$ the angular measure of $\{|z| = \exp(s/B_j^i)\} \cap$ interior $\{v \circ \psi_j^i(z) = 0\}$. Define μ_j^i on I_j by

$$\mu_j^i(s) = 0 \text{ if } \Phi_j^i(s) = 0,$$

$$\mu_j^i(s) = \left\{ B_j^i \left[2 - \frac{\Phi_j^i(s)}{\pi} \right] \right\}^{-2} \text{ if } 0 < \Phi_j^i(s) < 2\pi,$$

$$\mu_j^i(s) = +\infty \quad \text{if } \Phi_j^i(s) = 2\pi.$$

Define μ on $(0,1)$ by $\mu(s) = \min\{\mu_j^i(s) \mid i = 1, \cdots, n_j\}$ if $s \in I_j$ and $\mu(s) = 0$ if $s \in \mathscr{E}$, this latter convention being useful since μ is then lower semicontinuous.

We assert that Theorem 4.7 is a consequence of

THEOREM 4.8. *The Carleman norm of v is $[\mu]$-convex on* $(0,1)$.

For $\varepsilon > 0$, define $v_\varepsilon = (v - \varepsilon)^+$. Then v_ε satisfies the conditions of Theorem 4.8. Let Q_ε and μ_ε denote the corresponding terms when v is replaced by v_ε. By Theorem 4.8 Q_ε is $[\mu_\varepsilon]$-convex on $(0,1)$. Moreover $\lambda \leqq \mu_\varepsilon$. Thus, if $0 \leqq \lambda^* \leqq \lambda$ and λ^* is integrable on each compact subinterval of $(0,1)$, then $\lambda^* \leqq \mu_\varepsilon$ and hence Q_ε is λ^*-convex on $(0,1)$. Now $0 \leqq Q - Q_\varepsilon \leqq \varepsilon\sqrt{A}$, where $A = \int_{\Gamma(s)} * \, du$, and hence $\lim_{\varepsilon \to 0} Q_\varepsilon = Q$. It follows that Q is λ^*-convex on $(0,1)$ (cf. [9]). In other words, Q is $[\lambda]$-convex on $(0,1)$.

Note that, if $\sigma(v;s) = \max v$ on $\Gamma(s)$ is positive on $(0,1)$, then the functions λ and μ are themselves integrable on each compact subinterval of $(0,1)$. To see this we observe, first, that Φ_j^i and μ_j^i are lower semicontinuous on I_j. Hence μ is lower semicontinuous on I_j and, since $\mu \geqq 0$, it follows that μ is lower semicontinuous on $(0,1)$. Now μ_ε is nondecreasing in ε and $\lim_{\varepsilon \to 0} \mu_\varepsilon = \lambda$ (cf. [9]). Since μ_ε is also lower semicontinuous, it follows that λ is measurable. Finally, λ is bounded on each compact subinterval of $(0,1)$ and so also is μ, μ being dominated by λ. The condition $\sigma(v;s) >$ is essential here.

Proof of Theorem 4.8. We show, first, that Q is $[\mu]$-convex on certain subintervals of I_j, each j. If $s \in I_j$, write

$$Q(s) = \left\{ \sum_{i=1}^{n_j} \left[Q_j^i(s) \right]^2 \right\}^{1/2},$$

where

$$Q_j^i(s) = \left\{ \int_{\gamma_s^i} v^2 * \, du \right\}^{1/2}.$$

Observe that $[Q_j^i(s)]^2$ is non-negative and convex in s on I_j, each i. (In fact, Q_j^i itself is convex. This result will be wanted in the final stages of this proof. It will be convenient to establish the result at that time.) Therefore there exists a finite set N_j contained in I_j having the property that, if $(a,b) \subset I_j - N_j$, then either $Q_j^i \equiv 0$ on (a,b) or $Q_j^i > 0$ on (a,b), each i. We assert that Q is $[\mu]$-convex on each such subinterval (a,b). We put aside the trivial case where $Q_j^i \equiv 0$ on (a,b) for all i and turn to the case where $Q_j^i > 0$ on (a,b) for at least one i. In this latter case μ is itself integrable on each compact subinterval of (a,b). Here we proceed by showing that Q_j^i is μ-convex on (a,b), each i. This is trivially true if $Q_j^i \equiv 0$ on (a,b). If the index i is such that $Q_j^i > 0$ on (a,b), define

$$V_j^i(w) = v \circ \psi_j^i \; [\exp(w/B_j^i)], \qquad \{s_j < \operatorname{Re} w < s_{j+1}\},$$

and note that

$$\left[Q_j^i(s)\right]^2 = \int_0^{2\pi B_j^i} \left[V_j^i(s+it)\right]^2 dt, \qquad s \in I_j.$$

Clearly V_j^i is subharmonic. Since $Q_j^i > 0$ on (a,b) it follows that Q_j^i is μ_j^i-convex on (a,b). The reasoning here is precisely that in [9, pp. 67–70] and we omit the details of the proof. (Since the upper limit $2\pi B_j^i$ replaces 2π, the corresponding change was required in defining μ_j^i.) Since $\mu \leqq \mu_j^i$, Q_j^i is μ-convex on (a,b) and this statement is now verified for all i. But then Q is μ-convex on (a,b) as follows: It suffices to show that

$$L = \left\{\sum_{i=1}^{n} f_i^2\right\}^{1/2}$$

is μ-convex on $[c,d] \subset (a,b)$ if the functions $f_i, i = 1, \cdots, n$, are μ-linear on $[c,d]$. Thus, note that each f_i satisfies either $f_i \equiv 0$ on $[c,d]$ or $f_i > 0$ on (c,d), i.e., $f_i > 0$ on $[c,d)$ or on $(c,d]$ (cf. [9, pp. 61–62]). Hence either $L \equiv 0$, in which case L is trivially μ-convex, or $L > 0$ on (c,d). In the latter case L is μ-convex on (c,d) as in [10, pp. 122–123]. Therefore L is μ-convex on $[c,d]$ by (4.4).

We have shown that Q is $[\mu]$-convex on $(0,1)$ less a certain finite set. We show now that Q is a convex on $(0,1)$. It will then follow by (4.5) that Q is $[\mu]$-convex on $(0,1)$. Theorem 4.7 and thereby Theorem 4.8 will then be established. Thus, let s' and s'' be such that $0 < s' < s'' < 1$ and let $\{g_n\}_1^\infty$ denote a sequence of positive continuous functions coming down to v on $\Gamma(s') \cup \Gamma(s'')$. Let h_n denote the solution to the Dirichlet problem on $\overline{A(s',s'')}$ with boundary values g_n. If Q_n denotes the Carleman norm of h_n, then $Q_n > 0$ and hence $Q_n'' \geqq 0$ on $(s',s'') \cap I_j$ for each j (cf. [10, pp. 122–123]). An application of (4.5) yields that Q_n is convex on $[s',s'']$. Note here that Q_n^2 is convex, $Q_n > 0$, and therefore $Q_n'(s^-) \leqq Q_n'(s^+)$ for all s. Moreover $Q_n \geqq Q$ for all n. Since $\lim_{n\to+\infty} Q_n(s') = Q(s')$ and $\lim_{n\to+\infty} Q_n(s'') = Q(s'')$, it follows that Q is convex on $(0,1)$.

5. **Evans-Kuramochi exhaustion functions.** Consider now a noncompact parabolic Riemann surface $\mathfrak{A}$. Let $D \subset \mathfrak{A}$ be compact and such that each frontier point of D is contained in a continuum which is also contained in D. Under these conditions Kuramochi [13] (see also [14] and [16]) has established the existence of a function u on $\mathfrak{A} - D$ such that

(a) $u \geqq 0$, harmonic on $\mathfrak{A} - D$,

(b) u vanishes continuously on $\operatorname{fr} D$,

(c) u approaches $+\infty$ at the ideal boundary of $\mathfrak{A}$ under the Alexandroff one-point compactification of $\mathfrak{A}$.

We term any such function u an *exhaustion function* for $\mathfrak{A} - D$. Clearly, if $0 < s' < s'' < +\infty$, then $A(s',s'') = \{s' < u < s''\}$ is relatively compact in $\mathfrak{A} - D$. Thus the results of §2 are valid and, if $s', s'' \notin \mathscr{E}$, then each component of $A(s',s'')$ is a finite Riemann surface and the situation is that of §3.

If ω is a locally bounded semicontinuous first order differential on $\mathfrak{A} - D$, let

$$\int_{\Gamma(s)} \omega$$

denote the common value of the expressions

$$\left[\int_{\Gamma(s)} \omega\right]_{(s',s'')}$$

determined by summing over the contributions to $\Gamma(s)$ from the components of $A(s',s'')$, any s', $s'' \notin \mathscr{E}$ such that $s \in (s',s'')$. We then have that $\int_{\Gamma(s)} * du = A$, $A > 0$, constant, $0 < s < +\infty$, and we can normalize u so that $A = 2\pi$. Note that if u is so normalized and I_j denotes a subinterval of $(0, +\infty)$ containing no points of $\mathscr{E}$, then $\sum_i B_j^i = 1$, B_j^i as defined in §3 relative to I_j. Note also that the functions $\sigma(v;s) = \max v$ on $\Gamma(s)$ and $m(v;s) = \int_{\Gamma(s)} v * du$ are convex in s for $v \geqq 0$ subharmonic on $\mathfrak{A}$. It follows that, if v is not identically constant, then $\lim_{s \to +\infty} \sigma(v;s) = \lim_{s \to +\infty} m(v;s) = +\infty$. ($\mathfrak{A}$ is parabolic and therefore v is unbounded above.) Moreover Theorem 4.7 remains valid for λ defined in terms of all the components of $\Gamma(s)$. In particular, if v is non-negative subharmonic on $A(s',s'')$, s', $s'' \notin \mathscr{E}$, and $\sigma(v;s) > 0$, $s \in (s',s'')$, then

$$Q(s) = \left\{\int_{\Gamma(s)} v^2 * du\right\}^{1/2}$$

is λ-convex on (s',s''). It suffices here to apply Theorem 4.7 to each component α of $A(s',s'')$ thereby obtaining a $[\lambda_\alpha]$-convexity statement for Q_α, λ_α and Q_α being the contributions to λ and Q from the component α. Then, since $\lambda \leqq \lambda_\alpha$ and λ is integrable, we have that Q_α is λ-convex. Finally note that

$$Q = \left\{\sum_\alpha Q_\alpha^2\right\}^{1/2}$$

and therefore Q is λ-convex. The final portion of the proof of Theorem 4.8 is apposite here.

Thanks to these convexity and λ-convexity statements we arrive at the following lower estimate on the rate of growth relative to u of non-negative subharmonic functions on $\mathfrak{A}$.

THEOREM 5.1. *Let v be non-negative subharmonic and not identically constant on $\mathfrak{A}$. Let a be such that $\sigma(v;s) > 0$ if $s > a$ and let λ be as defined in* §4 *relative to v, λ with domain $\{s > a\}$. If*

$$Q(s) = \left\{\int_{\Gamma(s)} v^2 * du\right\}^{1/2},$$

then there exist c, c_1, A, *with* $a < c < c_1$, $A > 0$, *such that*

$$\int_c^s \exp\left\{ 2\int_c^t \sqrt{(\lambda(x))}\,dx \right\} dt \leqq A[Q(s)]^2 \qquad \textit{for } s \geqq c_1 .$$

Proof. By the above remarks Q is λ-convex on $\{s > a\}$ and $\lim_{s\to+\infty} Q(s) = +\infty$. Heins [9] (as an extension of a result due to Carleman) has established the conclusion to Theorem 5.1 under just these conditions.

It can be shown that $\lambda \equiv 0$ is possible even in nontrivial situations where, for example, $\inf v = 0$ on $\Gamma(s)$ with $s \to +\infty$. In such cases Theorem 5.1 yields no information not already known.

6. **The Denjoy-Carleman-Ahlfors theorem — subharmonic form.** An exhaustion function u on $\mathfrak{A} - D$ will be said to satisfy the *k-condition* if and only if there exists a positive integer k such that the number of components of $\Gamma(s)$ is bounded above by k independent of s, $0 < s < +\infty$. The following theorem extends the Denjoy-Carleman-Ahlfors theorem as stated in §1.

THEOREM 6.1. *If u is a normalized exhaustion function on $\mathfrak{A} - D$ which satisfies the k-condition for some positive integer k and if $h_1, \cdots, h_m$, $m \geqq k+1$, are non-negative subharmonic and not identically constant on $\mathfrak{A}$, $\min\{h_i, h_j\} = 0$, $i \neq j$, then*

$$\liminf_{r\to+\infty} q(r) r^{-\alpha^2/2m} > 0,$$

where

$$q(e^s) = \left\{ \sum_{=1}^{m} \int_{\Gamma(s)} h_i^2 * du \right\}^{1/2},$$

and $\alpha = m - (k-1)$.

We remark before proceeding that the classical Denjoy-Carleman-Ahlfors theorem is a consequence of Theorem 6.1 taking $\mathfrak{A} = K$ (the finite plane) and $D = \{|z| \leqq 1\}$. For then $u(z) = \log|z|$ is a normalized exhaustion function satisfying the k-condition for $k = 1$ and Theorem 6.1 states that

$$\liminf_{r\to+\infty} q(r) r^{-m/2} > 0,$$

a statement that is tantamount to the classical theorem (see, for example, [2]). We remark also that any n-sheeted algebroid Riemann surface over K satisfies the conditions imposed on $\mathfrak{A}$ and admits an exhaustion function satisfying the k-condition for some $k \leqq n$. The case of infinite genus is most interesting here. It seems plausible that there exist nonalgebroid surfaces that are admissible in the above sense although we have not succeeded in constructing an example. It is

known that there exist nonalgebroid surfaces that are parabolic, have finite ideal boundary and are "topologically similar" to algebroid surfaces [7].

Proof of Theorem 6.1. Let a be such that $\sigma(h_n\,;\,s) > 0$ for $s > a$, $n = 1, \cdots, m$, and let Q_n and λ_n be as defined in §4 with h_n replacing v, λ_n with domain $\{s > a\}$. Then Q_n is λ_n-convex, $n = 1, \cdots, m$, and by Theorem 5.1 there exists c, c_1, A, with $a < c < c_1$, $A > 0$, such that for $s \geqq c_1$, $n = 1, \cdots, m$,

$$\int_c^s \exp\left\{2\int_c^t \sqrt{(\lambda_n(x))}\,dx\right\} dt \leqq A[Q_n(s)]^2.$$

Now $[q(e^s)]^2 = \sum_{n=1}^m [Q_n(s)]^2$ and therefore, applying the inequality between arithmetic and geometric means, we have

$$A[q(e^s)]^2 \geqq \int_c^s m\exp\left\{\frac{2}{m}\int_c^t \sum_{n=1}^m \sqrt{(\lambda_n(x))}\,dx\right\}dt, \quad s \geqq c_1. \tag{6.2}$$

We obtain a lower bound for $\sum\sqrt{\lambda_n}$ as follows: Note that for each $s > a$, $s \notin \mathscr{E}$, at most $k-1$ of the functions h_n have positive lower bound on some component (depending on n) of $\Gamma(s)$. The k-condition is essential here as also is the condition $m \geqq k+1$. If for some $s > a$, $s \notin \mathscr{E}$, there exist k of the functions h_n with positive lower bound on some component (depending on n) of $\Gamma(s)$, then the remaining $m-k$ ($\geqq 1$) functions vanish identically on $\Gamma(s)$ by the nonoverlap condition together with the k-condition. This contradicts the fact that $\sigma(h_n\,;\,s) > 0$ for $s > a$, $n = 1, \cdots, m$. Hence for each $s > a$, $s \notin \mathscr{E}$, at least $\alpha = m - (k-1)$ of the functions h_n have zero infimum on *each* component of $\Gamma(s)$. It follows that for these α functions h_n we have $\lambda_n(s) > 0$. Now $\sum_{n=1}^m \sqrt{\lambda_n} = \sum' \sqrt{\lambda_n}$ where the summation on the right is taken over the nonzero λ_n's. But

$$(\textstyle\sum'\sqrt{\lambda_n})\left(\sum' \dfrac{1}{\sqrt{\lambda_n}}\right) \geqq \alpha^2 \qquad \text{on } \{s > a\} - \mathscr{E}. \tag{6.3}$$

On the other hand, by the nonoverlap condition and the normalization of u, we have

$$\sum' \frac{1}{\sqrt{\lambda_n}} \leqq 2 \qquad \text{on } \{s > a\} - \mathscr{E}. \tag{6.4}$$

(Note here that $\sum_i B_j^i = 1$.) Inequalities (6.3) and (6.4) imply

$$\sum\sqrt{\lambda_n} \geqq \alpha^2/2 \qquad \text{on } \{s > a\} - \mathscr{E}.$$

Returning to (6.2) we have

$$A[q(e^s)]^2 \geqq \int_c^s m\exp\left\{\frac{\alpha^2}{m}\int_c^t dx\right\} dt.$$

Elementary calculations now yield that

$$A[q(e^s)]^2 \geqq \frac{m^2}{\alpha^2}[\exp\{(\alpha^2/m)\ (s-c)\} - 1]$$

and it follows that

$$\liminf_{r\to+\infty} q(r)r^{-\alpha^2/2m} > 0.$$

Theorem 6.1 is established.

It would be of interest to determine necessary and sufficient conditions in order that a noncompact parabolic $\mathfrak{A}$ admit an exhaustion function satisfying the k-condition for some k. A necessary condition is that $\mathfrak{A}$ have finite ideal boundary in the sense of Kerékjártó-Stoïlow (cf. [1, pp. 81–87]). We show in §8 that there exist surfaces with finite ideal boundary admitting some exhaustion functions that do not satisfy the k-condition for any k. The possibility remains however that these surfaces also admit other exhaustion functions that do satisfy the k-condition for some k.

An important question is: How sharp is the lower estimate of growth given by the Carleman method? That is, given k and m $(> k)$ can an example be constructed where the growth is correctly estimated by Theorem 6.1? We have not succeeded in treating the general case here. However an interesting example is afforded in the following case where $k = 1$ and $m = 4$:

Let $\mathfrak{A}$ denote the algebroid Riemann surface associated with $w^2 = \sin z$. $\mathfrak{A}$ can be realized by taking two copies of the finite plane, say E_1 and E_2, slit along the intervals $[n\pi, (n+1)\pi]$, $n = \pm 1, \pm 3, \pm 5, \cdots$, of the real axis and joining E_1 and E_2 in the usual manner, i.e., by identifying the upper edges of the slits of E_1 with the corresponding lower edges of the slits of E_2 and vice versa. Each point $p \in \mathfrak{A}$ corresponds to a unique point $c(p)$ of the finite plane under the obvious correspondence. The function $u: p \to \frac{1}{2}\log|c(p)|$ is a normalized exhaustion function on $\mathfrak{A}$ less the points over $\{|z| \leqq 1\}$ which satisfies the k-condition for $k = 1$. The function $v: p \to \log^+|\sin c(p)|$ is non-negative subharmonic on $\mathfrak{A}$ and has four components of positivity. The Carleman estimate of Theorem 6.1 yields that

$$\liminf_{t\to+\infty} q(t)t^{-2} > 0,$$

where

$$q(e^s) = \left\{\int_0^{2\pi} [\log^+ |\sin re^{i\theta}|]^2 d\theta\right\}^{1/2} \qquad \text{(sic!)}$$

with $2s = \log r$. On the other hand, it is easily seen that

$$\limsup_{t\to+\infty} q(t)t^{-2} < +\infty.$$

Thus the growth estimate is sharp in this case.

7. **Applications.** In what follows it will be assumed that $\mathfrak{A}$ is a noncompact parabolic Riemann surface that admits an exhaustion function u on $\mathfrak{A} - D$, D as in §5, which satisfies the k-condition for some positive integer k. We assume also that u is normalized. Thanks to an argument of Kjellberg [11] we shall see that Theorem 6.1 yields positive information regarding the structure of harmonic functions on $\mathfrak{A}$ which possess a certain slowness of growth relative to u.

Consider, first, a proper subregion Ω of $\mathfrak{A}$. It is assumed that Ω is not relatively compact. The results that follow hold trivially for relatively compact Ω. Let P denote the collection of non-negative harmonic functions on Ω which vanish continuously at each finite boundary point of Ω. Clearly P is a semimodule with operator (multiplication by non-negative constants). For each $v \in P$ we define its standard subharmonic extension $\bar{v}$ by $\bar{v} = \bar{v}$ on Ω, $\bar{v} = 0$ on $\mathfrak{A} - \Omega$. A function $v \in P$ will be said to be of *order* ρ with respect to the exhaustion function u if

$$\rho = \rho(v) = \limsup_{s \to +\infty} \frac{\log^+ \left\{ \int_{\Gamma(s)} \bar{v}^2 * du \right\}^{1/2}}{s}.$$

For each finite $A \geqq 0$ we denote by P_A the collection of elements v of P such that $\rho(v) \leqq A$. Again P_A is a semimodule with operator (multiplication by non-negative constants).

Define the *dimension* of P_A, written $\dim P_A$, as the minimum number of elements of P_A which generate P_A as a semimodule provided such a finite set exists, otherwise as ∞ (cf. [7]). If P_A consists of the zero function alone we adopt the convention that $\dim P_A = 0$. The fundamental result regarding $\dim P_A$ is the following:

THEOREM 7.1. *For each finite* $A \geqq 0$, $\dim P_A \leqq \max\{k, A(k+1) + (k-1)\}$, *$k$ being the upper bound on the number of components of the level loci of the exhaustion function u.*

The proof rests upon the concept of a minimal harmonic function due to Martin [15] and the cited argument of Kjellberg. We remark before proceeding that, if $A \geqq 0$ and $\dim P_A$ is finite, $= m$ say, then any set of generators $\{v_1, \cdots, v_m\}$ of P_A has the following properties.

(i) v_i is minimal, $i = 1, \cdots, m$.

(ii) If $v \in P_A$ is minimal, then $v = cv_i$ for some i, $c > 0$.

(iii) If $\{w_1, \cdots, w_m\}$ is a second set of generators of P_A, then $w_i = c_i v_{n(i)}$, $c_i > 0$, $i = 1, \cdots, m$.

(iv) $v \in P_A$ admits a unique representation of the form

$$\sum_{i=1}^{m} c_i v_i, \quad c_i \text{ real.}$$

The properties above are stated in [7] for a slightly different situation. It is easily verified that they remain valid here. The following lemma is due to Kjellberg [11].

LEMMA 7.2. *Let $v, v_1, \cdots, v_n$ be positive harmonic functions with domain a region Ω. If $v \leqq \sum_{i=1}^n v_i$, then there exist n non-negative harmonic functions w_i, $i = 1, \cdots, n$, such that $v = \sum_{i=1}^n w_i$, $w_i \leqq v_i$, $i = 1, \cdots, n$.*

Proof of Theorem 7.1. Let $M = \max\{k, \quad A(k+1) + (k-1)\}$ and let $[M]$ denote the largest integer that does not exceed M. First, we show that every element of P_A admits a representation as a sum of at most $[M]$ minimal positive harmonic functions of Ω in P_A. Second, we show there exist at most $[M]$ mutually nonproportional minimal positive harmonic functions of Ω in P_A. Thus, assume there exists an element v of P_A which does not admit a representation as a sum of fewer than $m = [M] + 1$ minimal positive harmonic functions of Ω. Then by a lemma of Heins [8] there exist m positive harmonic functions v_i, $i = 1, \cdots, m$, such that $\sum_{i=1}^m v_i \leqq v$ and G.H.M. $\min\{v_i, v_j\} = 0$, $i \neq j$. (G.H.M. here stands for greatest harmonic minorant.) Clearly $v_i \in P_A$, $i = 1, \cdots, m$. Define $h_i = v_i - \sum_{j \neq i} v_j$, $i = 1, \cdots, m$. By Kjellberg's lemma, $\{h_i > 0\} \neq \varnothing$, $i = 1, \cdots, m$. Moreover $\{h_i > 0\} \cap \{h_j > 0\} = \varnothing$, $i \neq j$. The functions h_i^+ defined on $\mathfrak{A}$ by $h_i^+ = \max\{h_i, 0\}$ on Ω and $h_i^+ = 0$ on $\mathfrak{A} - \Omega$, $i = 1, \cdots, m$, are therefore non-negative subharmonic, not identically constant, and have nonoverlapping domains of positivity. Hence the situation is that of Theorem 6.1. Since $v_i \in P_A$, $i = 1, \cdots, m$, it follows from Theorem 6.1 that $A \geqq [m - (k-1)]^2/2m$. Now $m \geqq k + 1$ and therefore $m \leqq A(k+1) + (k-1)$. Contradiction. Thus each element of P_A admits a representation as a sum of $[M]$ or less minimal positive harmonic functions of Ω. Clearly these minimal functions are elements of P_A.

Consider now the collection of minimal positive harmonic functions in P_A and assume there exist $m > [M]$ which are mutually nonproportional, say $v_1, \cdots, v_m$. Let $S = \sum_{i=1}^m v_i$. Then $S \in P_A$ and by the above conclusion S is representable as a sum of at most $[M]$ minimal harmonic functions. Applying Kjellberg's lemma to the inequalities $v_i \leqq S$, $i = 1, \cdots, m$, we find that $m \leqq [M]$. Contradiction. We conclude that $\dim P_A \leqq [M]$.

In particular, Theorem 7.1 implies that $\dim P_0 \leqq k$. It follows that the exhaustion function u, being of zero order on $\mathfrak{A} - D$, is representable as a finite sum of minimal harmonic functions at most k in number. This result will be of importance in §8. Note also that any collection of mutually nonproportional minimal harmonic functions of Ω having finite order is at most countable.

The argument that led to Theorem 7.1 yields more completely.

THEOREM 7.2. *Let Ω_i, $i = 1, 2, 3, \cdots$, be proper subregions of $\mathfrak{A}$ such that $\Omega_i \cap \Omega_j = \varnothing$, $i \neq j$, and let $P_A(i)$ denote the semi-module of non-negative harmonic functions on Ω_i which vanish continuously on $\operatorname{fr}\Omega_i$ and are of finite order $\leqq A$. Then for each finite $A \geqq 0$ at most $M = \max\{k, \ A(k+1) + (k-1)\}$ of the semimodules $P_A(i)$ are nontrivial. Moreover $\sum_i \dim P_A(i) \leqq M$.*

A particular consequence is

COROLLARY 7.4. *Let v be harmonic on $\mathfrak{A}$. If*

$$\limsup_{s\to+\infty} \frac{\log^+\left\{\int_{\Gamma(s)} v^2 * du\right\}^{1/2}}{s} = A,$$

A finite, then for each constant c the combined number of components of $\{v > c\}$ and $\{v < c\}$ does not exceed M. On each such component $v - c$ or $c - v$ is given as the sum of at most M mutually nonproportional minimal harmonic functions and the combined number of these minimal harmonic functions also does not exceed M.

8. **Exhaustion functions and the k-condition.** We are now in a position to establish the existence of parabolic Riemann surfaces with a single Kerékjártó-Stoïlow ideal boundary element which admit exhaustion functions not satisfying the k-condition for any positive integer k (see §6). The central concept here is that of harmonic dimension due to Heins [7].

Examples of parabolic surfaces with a single ideal boundary element and infinite harmonic dimension in the sense of Heins have been given by Kuramochi [12], Cornea [4], and Constantinescu and Cornea [3]. Let $\mathfrak{A}$ denote such a surface and consider an exhaustion function u on $\mathfrak{A} - D$, D a parametric disk say. The existence of u is assured by the result of Kuramochi. Since $\mathfrak{A}$ has infinite harmonic dimension there exists an infinite sequence $\{v_n\}_1^\infty$ of mutually nonproportional minimal positive harmonic functions of $\mathfrak{A} - D$. We can assume each v_n is normalized such that $v_n(z_0) = 1$ at a fixed $z_0 \in \mathfrak{A} - D$. The function $h = u + \sum_{n=1}^\infty a_n v_n$, where $a_n = (\frac{1}{2})^n$, is positive harmonic on $\mathfrak{A} - D$ by the Harnack convergence principle and the fact that $h(z_0) = u(z_0) + 1$. It is immediate that h has limit $+\infty$ at the ideal boundary. Moreover h vanishes continuously on the boundary of D. Thus h is an exhaustion function for $\mathfrak{A} - D$. Clearly h cannot be represented as a finite sum of minimals. Hence, by the remarks in §7, h does not satisfy the k-condition for any positive integer k.

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