ON "ESSENTIALLY METRIZABLE" SPACES AND ON MEASURABLE FUNCTIONS WITH VALUES IN SUCH SPACES(1)

BY
ELIAS ZAKON

Introduction. In this paper we consider uniform spaces in which also a measure μ is defined and which possess certain topological properties either to the full extent or only "essentially" ("ess."), i.e. to within sets of measure zero, in a sense to be defined below. We shall define such notions as "ess. metrizability", "ess. separability", etc. Conditions for the ess. metrizability of a uniform space T will be established for the case where T is "ess. separable" and μ is a σ-finite extension of a Borel measure.

These concepts, and the "ess. metrization" theorem, will then be used to obtain some extensions of the theorems of Egoroff [2] and Lusin [9] on measurable functions. A variant of what we call "ess. uniform convergence" was already used by Frumkin [5] who showed that, with uniform convergence so modified, Egoroff's theorem can be extended to families $\{f(t,x)\}$ of real functions on a line interval, with t a continuous real parameter(2). A simpler proof of Frumkin's result was given in [17]. As regards Lusin's theorem, notable is Schaerf's result [11] extending it to functions with values in any space satisfying the 2nd axiom of countability. A further generalization seems to be only possible if ordinary continuity is replaced by its "ess." counterpart (cf. §1). Indeed, we shall show that, then, the 2nd countability requirement can be relaxed to separability (and even "ess. separability"), for uniform spaces with a nested (or "ess. nested") base. We shall also extend Frumkin's theorem to functions with values in such spaces and to Moore-Smith convergence of *nets* of functions (cf. [7, p. 62 ff.]). The ordinary theorem of Egoroff will be extended to separable pseudometrizable range spaces, thus covering a theorem formulated by Kvačko in [8] (3).

1. Preliminaries. Terminology and notation.

I. Let T be a topological space in which every point x has a fixed local

Received by the editors July 18, 1963 and, in revised form, October 17, 1964.

(1) This research was sponsored by the Summer Research Institute of the Canadian Mathematical Congress.

(2) Earlier, Tolstov [14] (and later Weston [16]), showed by a counterexample that such an extension fails for the ordinary notion of uniform convergence.

(3) Unfortunately, Kvačko's own proof contains an error. The pointwise convergence claimed in Lemma 3 (p.89) actually fails to materialize.

neighborhood base consisting of open sets N_x^i [also written as $N^i(x)$], where i runs over some index set J, one and the same for all x.(4) Such a space is said to be *uniform*, and J is called its *grader*, if J is a directed set(5) and if, in addition to "$x \in N_x^i$", the following postulates hold:

1.1. $i \leqq j$ $(i,j \in J)$ always implies $N_x^i \supseteq N_x^j$.

1.2. $x \in N_y^i$ implies $y \in N_x^i$.

1.3. For each $j \in J$, there is $k \in J$ such that $z, y \in N_x^k$ $(x \in T)$ implies $z \in N_y^j$.

The family of *all* N_x^i $(x \in T, i \in J)$ is called the *neighborhood system* of T, denoted by (N, J) or briefly by N. Accordingly, we write (T, N, J) for T. If another letter Q (boldfaced) replaces N, the same letter serves to denote the neighborhoods Q_x^i. J is said to be *quasi-countable* if it has a cofinal subset J' of type ω.(6) In a (pseudo) metric space, J is the set of all positive integers, and N_x^i is the sphere of radius 2^{-i} about x, as a convention.

II. We shall denote by m a non-negative countably additive measure on a σ-algebra M of subsets ("measurable sets") of a set $S \in M$. The triple (S, M, m) is called a *measure space*; m is said to be *topological* (with respect to a topology in S) if all Borel sets in S are measurable. Every mapping (function) $f: S \to T$ induces a measure μ in T if we define a set $B \subseteq T$ to be μ-measurable whenever $f^{-1}(B) \in M$, and put $\mu B = m(f^{-1}(B))$. We call μ the *f-induced* measure in T and denote it by m_f; it is topological iff f is *measurable*; i.e., iff $f^{-1}(G) \in M$ for every open set $G \subseteq T$.

III. Let μ be a measure in (T, N, J). We say that (N, J) is *μ-ess. finer* than another neighborhood system (Q, I) for T, and that (Q, I) is *ess. coarser* than (N, J), if, for each $i \in I$, there is $j \in J$ and a μ-measurable set $Z_i \subseteq T$ (depending on i only) such that $\mu Z_i = 0$ and $N_x^j - Z_i \subseteq Q_x^i$ for all $x \in T$. If (N, J) is both ess. finer and coarser than (Q, I), we say that they are *ess. equivalent*. If all Z_i can be chosen empty, we replace the term "ess." by "*uniformly*" in these definitions. (T, N, J) is said to be *μ-ess. metrizable* if (N, J) is ess. equivalent to some (Q, I) under which T is uniformly metrizable. Similarly for pseudometrizability. T is called *ess. separable* if (N, J) is ess. equivalent to some (Q, I) under which T has a dense countable subset.

IV. A net of functions $\{f_k \mid k \in K\}$ from a measure space (S, M, m) to a uniform space (T, N, J) is said to converge *ess. uniformly* to f on a set $D \subseteq S$ if, for each $j \in J$, there is k_j and a set $Z_j \in M$ $(mZ_j = 0)$, depending on j only, such that $f_k(x) \in N^j(f(x))$ for all $x \in D - Z_j$ and $k \geqq k_j$. Notation: $f_k \to f$ (ess. unif.)(7), f is

(4) The assumption that the N_x^i are open, and Axiom 1.2 below, are redundant and have been introduced for convenience only. Cf. [15, pp. 7–9]; [1, p. 131, U_{II}].

(5) I.e. a partially ordered set in which any 2 elements have an upper bound.

(6) It suffices that J have a countable cofinal subset with no last element. This definition applies to all directed sets J (e.g. also in *nets*).

(7) Frumkin, [5], permits the sets Z_j to depend on k (i.e. on f_k) as well. In this case we shall speak of ess. uniform convergence in the *weaker sense* and write $f_k \to f$ (ess. unif. w.). A similar notation will be used for ordinary uniform (unif.), pointwise (ptw.) and (a.e.) convergence.

ess. uniformly continuous on $D \subseteq S$ (with respect to a neighborhood system (Q,I) in S) if, for each $j \in J$, there is $i = i(j) \in I$ and $Z_j \in M$ $(mZ_j = 0)$, depending on j only, such that $f(x) \in N^j(f(y))$ whenever $y \in D$ and $x \in (D \cap Q^i_y) - Z_j$(8).

2. **Essential metrization.** We say that (T, N, J) is *first-countable* if (N, J) is uniformly equivalent to some (Q,I) with a quasi-countable I(9). T is said to be *uniformly nested* (*ess. nested*) if (N,J) is uniformly (ess., respectively) equivalent to some (Q,I), with a totally ordered I(10).

2.1. LEMMA. *Let μ be a σ-finite topological measure in an ess. nested, but not first-countable, uniform space (T,N,J). Then, for each $p \in T$, there is $i \in J$ such that $\mu(N^i_p - N^j_p) = 0$, for all $j \in J$.*

Proof. As μ is σ-finite, $T = \bigcup_{k=i}^{\infty} G_k$, $\mu G_k < \infty$, $k = 1,2,\cdots$. Since μ is topological, all N^j_x are measurable. Let $p \in T$, and $A^j_k = G_k \cap N^j_p$. Then

2.1.1. $$N^j_p = \bigcup_{k=i}^{\infty} A^j_k, \quad \mu A^j_k < \infty, \quad \text{for each } j \in J.$$

For every k, let $m_k = \inf_{j \in J} \mu A^j_k$, and choose $j_{kr} \in J$, $r = 1,2,\cdots$, with

2.1.2. $$\lim_{r \to \infty} \mu A^{j_{kr}}_k = m_k, \quad k = 1,2,\cdots.$$

The j_{kr} $(k,r = 1,2,\cdots)$ form a countable set J', *not cofinal in* J (T is not first-countable!). Hence, if J is totally ordered, J' has an upper bound $i \in J$, so that, by 1.1, $N^i_p \subseteq N^{j_{kr}}_p$, $k,r = 1,2,\cdots$(11). If (N,J) is only ess. nested, a simple argument shows that $N^i_p \subseteq N^{j_{kr}}_p \cup Z$ $(k,r = 1,2,\cdots)$ for some $i \in J$ and some $Z \subseteq T$ $(\mu Z = 0)$. Hence $A^i_k = G_k \cap N^i_p \subseteq G_k \cap (N^{j_{kr}}_p \cup Z) = A^{j_{kr}}_k \cup (G_k \cap Z)$, $\mu(G_k \cap Z) = 0$; so $\mu A^i_k \leqq \mu A^{j_{kr}}_k$, $r = 1,2,\cdots$. By 2.1.2, $\mu A^i_k \leqq m_k = \inf_{j \in J} \mu A^j_k$ whence $\mu A^i_k = m_k$. It easily follows that $\mu(A^i_k - A^j_k) = 0$, for all k and j. Hence, by 2.1.1, $\mu(N^i_p - N^j_p) = 0$, $(j \in J)$, and the lemma is proved(12).

2.2. THEOREM. *A uniform space (T,N,J) which is ess. separable, with respect to a σ-finite topological measure μ, is μ-ess. pseudometrizable if, and only if, it is μ-ess. nested.*

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(8) If i and Z_j may depend on y as well, we simply say that f is *ess. continuous on D*. This notion is defined also if S and T are not uniform.

(9) This is tantamount to the uniform pseudometrizability of T. (Cf. [7, p. 186]).

(10) Such spaces are closely related to Fréchet's "espaces à écart"; cf. [3], [4].

(11) An $i \in J$ satisfying the last condition exists also if (N,J) is only uniformly equivalent to some (Q,I) with I totally ordered. Then I, too, has no countable cofinal subsets (we exclude the trivial case where I has a maximum); and we can replace the $N^{j_{kr}}_p$ by smaller $Q^{i_{kr}}_p \supseteq Q^{i_0}_p \supseteq N^i_p$.

(12) We can slightly modify the lemma to say that $N^i_p - N^i_p \subseteq Z_{pj}$ $(\mu Z_{pj} = 0)$ for all $j \in J$. Then it holds also if μ is only "ess. topological"; i.e. topological under some (Q,I) which is ess. equivalent to (N,J).

Proof. As every pseudometric space is uniformly nested, the condition is necessary. To prove sufficiency, assume that (N, J) is ess. nested, i.e., ess. equivalent to some uniform system (Q, I), with I totally ordered; and that (N, J) is also ess. equivalent to some uniform system (U, K) under which T has a dense countable subset $\{p_1, p_2, \cdots\}$. It suffices to show that (T, Q, I) is ess. pseudometrizable. Thus (recalling footnote 12) we lose no generality by assuming that J itself is totally ordered. We shall also assume that J and K have no last elements or countable cofinal subsets (otherwise, all is trivial).

By 2.1, we find, for each p_n, an $i_n \in J$, with $(N_{p_n}^{i_n} - N_{p_n}^{j}) \subseteq Z'_{nj}$ $(\mu Z'_{nj} = 0)$ for all $j \in J$. The i_n must have (as in 2.1) an upper bound $i \in J$, so that $N_{p_n}^{i} - N_{p_n}^{j} \subseteq Z'_{nj}$, for all j and n. The essential equivalence of (N, J) and (U, K) then easily yields an $\alpha \in K$ (henceforth fixed) such that, for all $k \in K$,

2.2.1. $$U^{\alpha}(p_n) - U^{k}(p_n) \subseteq Z_{nk}, \quad \mu Z_{nk} = 0, \qquad n = 1, 2, \cdots.$$

By Axiom 1.3, we can associate each $k \in K$ with another index in K (call it $k+1$) such that $k < k+1$, $\alpha + 1 \leqq k+1$, and such that, for all $x, y, z \in T$

2.2.2. $$y, z \in U_x^{k+1} \text{ implies } z \in U_y^{k} \text{ and } y \in U_z^{k}.$$

For each $k \in K$, we now put $X_k = \bigcup_{x \in T}(U_x^{\alpha+1} - U_x^{k})$, $Y_k = \bigcup_{n=1}^{\infty}(U_{p_n}^{\alpha} - U_{p_n}^{k+1})$. Then, by 2.2.1, $Y_k \subseteq \bigcup_{n=1}^{\infty} Z_{nk} = Z_k$, $\mu Z_k = 0$. We shall now show that

2.2.3. $$X_k \subseteq Y_k \subseteq Z_k \quad (k \in K, \mu Z_k = 0).$$

Indeed, if $y \in X_k$, then $y \in U_x^{\alpha+1}$ and $y \notin U_x^{k}$ for some $x \in T$. By density, there is $p_n \in U_x^{k+1} \subseteq U_x^{\alpha+1}$ (for $\alpha + 1 \leqq k+1$). Thus, by 2.2.2, $y \in U_{p_n}^{\alpha}$. Also, $y \notin U_{p_n}^{k+1}$; for $y, x \in U_{p_n}^{k+1}$ would imply $y \in U_x^{k}$, whereas $y \notin U_x^{k}$. Thus $y \in U_{p_n}^{\alpha} - U_{p_n}^{k+1}$, i.e., $y \in Y_k$, and 2.2.3 is proved.

Now let $K' = \{k_1, k_2, \cdots\}$ where $k_1 = \alpha + 1$ and $k_{n+1} = k_n + 1 > k_n$, $n = 1, 2, \cdots$. The sets U_x^{k}, with $k \in K'$, form a first-countable neighborhood system for T, graded by K'; we denote it by (V, K'). It is uniform since it inherits the properties 1.1, 1.2, and 2.2.2 (hence 1.3) from (U, K). Thus (T, V, K') is a first-countable uniform space, hence uniformly pseudometrizable. By definition, $V_x^{k} = U_x^{k}$ for $k \in K'$; thus (V, K') is *uniformly coarser* than (U, K). To show that it is also ess. *finer* (hence ess. equivalent to (U, K) and (N, J)), it suffices to prove that $U_x^{k} \supseteq V_x^{k_1} - Z_k = U_x^{\alpha+1} - Z_k$, for $k \in K$ and $x \in T$.

Let then $y \in U_{x_0}^{\alpha+1} - Z_k$. By 2.2.3, $y \notin X_k$; so, by the definition of X_k, y is in none of the sets $U_x^{\alpha+1} - U_x^{k}$ $(x \in T)$. Hence, as $y \in U_{x_0}^{\alpha+1}$, we must have $y \in U_{x_0}^{k}$. Thus, indeed, $U_x^{k} \supseteq V_x^{k_1} - Z_k$, q.e.d. This completes the proof.

NOTE 1. Continuing the process, we can replace (V, K') by a first-countable system (Q, I) *such that* $I \subseteq J$ *and* $Q^{i}(x) = N^{i}(x)$ *for all* $x \in T$ *and* $i \in I$. Indeed, by the ess. equivalence of (V, K') and (N, J), there is, for each $k \in K'$, an $i_k \in J$ and $Z''_k \subseteq T$ $(\mu Z''_k = 0)$, with $N^{i_k}(x) - Z''_k \subseteq V^{k}(x)$ for all $x \in T$. This very formula

ensures that the $N^{i_k}(x)$ form a system which is ess. finer than (V,K') and (N,J), hence ess. equivalent to both (being coarser than (N,J)). As in the proof of 2.2, the i_k can be so chosen that they preserve Axioms 1.1–1.3. Setting $I = \{i_k \mid k \in K'\}$ and $Q^i(x) = N^i(x)$ for $i \in I$, we obtain the required first-countable uniform system (Q,I). Thus *T becomes uniformly pseudometrizable when its neighborhood system (N,J) is reduced to (Q,I), $Q \subseteq N$.*

NOTE 2. Theorem 2.2 holds also if the measure μ is only "ess. topological" (cf. footnote 12).

3. **Egoroff's Theorem.** A straightforward application of the definitions given in §1 yields the following lemma:

3.1. LEMMA. *Let f be a function from a measure space (S,M,m) to a uniform space (T,N,J). If f is the ess. uniform limit (possibly in the weaker sense) of a net of functions, or if f is ess. uniformly continuous, or ess. continuous (with respect to a neighborhood system (U,K) for S), then f remains so also when (U,K) is replaced by some m-ess. finer system, or (N,J) is replaced by an m_f-ess. coarser system (with m_f the f-induced measure in T).*

We can now obtain our intended extensions of Egoroff's theorem. Our objective is twofold: (1) to obtain these extensions, which seem to us of a certain interest themselves; (2) to illustrate the application of our "ess. concepts" in measure theory.

3.2. THEOREM. *Let $\{f_k \mid k \in K\}$ be a net of measurable functions from a measure space (S,M,m), with $mS < \infty$, into a uniform space (T,N,J). If T is separable and uniformly nested, and if $f_k \to f$ (a.e.) on S, then, for every $\varepsilon > 0$, there is a set $D \in M$ such that $m(S-D) < \varepsilon$ and, moreover:*

(i) *$f_k \to f$ (unif.) on D, provided K is countable;*

(ii) *$f_k \to f$ (ess. unif. w.) on D, provided K is quasi-countable.*

If, instead, T is only m_f-ess. separable and m_f-ess. nested, and if f is measurable then assertion (ii) *still holds, while* (i) *changes to*

"(i′) *$f_k \to f$ (ess. unif.) on D, provided K is countable.*"

Proof. If T is separable and uniformly nested, then it is uniformly pseudometrizable([13]). Let ρ be a pseudometric for T. No generality is lost by assuming that $f_k \to f$ (ptw.) on S. Then f is measurable([14]); so also is $g_k(x) = \rho(f_k(x), f(x))$

([13]) Indeed, we may assume that T is separated (otherwise replace it by its separated quotient space). If $\{p_n\}$ is a dense sequence in T, we find, for each $x \in T$, neighborhoods $N^{i_n}(x)$, with $p_n \notin N^{i_n}(x)$, $n = 1, 2, \cdots$. Then, unless T is metrizable, there is $N^i(x) \subseteq \bigcap_{n=1}^{\infty} N_x^{i_n}$ (cf. [7, p. 204, Ex. D]). As $N^i(x)$ contains no p^n, it consists of x alone. Thus T is discrete, hence metrizable after all. (We are indebted to the referee for this remark and several simplifications of the proof.).

([14]) The measurability of the limit function, as is easily seen, holds also for measurable functions with values in a pseudometric space.

because ρ is continuous on $T \times T$ (a separable space)([15]). Let

$$E_k^j = \{x \in S \mid \rho(f_k(x), f(x)) \leqq 2^{-j}\} \in \boldsymbol{M},$$

and $D_n^j = \bigcap_{k \geqq n} E_n^j$, for $n, k \in K$, and $j = 1, 2, \cdots$. Since the directed set K is (at least) quasi-countable, it has a cofinal subset K' of order type ω (we discard the trivial case where K has a last element); for simplicity, we identify K' with the positive integers in standard order. Then the pointwise convergence $f_k \to f$ easily implies that, for each $j \in J$, $S = \bigcup_{n \in K} D_n^j = \bigcup_{n \in K'} D_n^j$ (the latter holds by the cofinality of K' in K since the sets D_n^j increase with n).

Now, if K is countable, the sets D_n^j are measurable, and the standard proof of Egoroff's theorem (cf. [10, p. 157], or [6, p. 88]) works, with purely notational changes, thus yielding our assertion (i). If K is only quasi-countable, then the procedure outlined in [17], with trivial changes, yields (ii). This establishes the first part of the theorem.

Next suppose that T is only m_f-ess. separable and m_f-ess. nested, with f a measurable function. Then m_f is a topological measure in T. By its definition, m_f is also finite since so is m. Hence, by Theorem 2.2 and Note 1 to it, the neighborhood system $(\boldsymbol{N}, \boldsymbol{J})$ is m_f-ess. equivalent to some $(\boldsymbol{Q}, \boldsymbol{I})$ under which T is uniformly pseudometrizable, and which satisfies the conditions $\boldsymbol{I} \subseteq \boldsymbol{J}$ and $\boldsymbol{Q} \subseteq \boldsymbol{N}$. Due to this, $(\boldsymbol{Q}, \boldsymbol{I})$ preserves the measurability of f_k and f, and the convergence $f_k \to f$. We shall now show that $(T, \boldsymbol{Q}, \boldsymbol{I})$ can be made separable by dropping from T a set of measure 0. In fact, recall from the proof of 2.2 that T has a dense sequence $\{p_n\}$ under the neighborhood system $(\boldsymbol{U}, K)$, hence also under the coarser system $(\boldsymbol{V}, K')$, constructed in that proof. From Note 1 to 2.2, we see that $(\boldsymbol{V}, K')$ is m_f-ess. equivalent to $(\boldsymbol{Q}, \boldsymbol{I})$. This easily implies that, for each $i \in \boldsymbol{I}$, there is $Z_i \subseteq T$ $(m_f Z_j = 0)$, with $T = \bigcup_{n=1}^{\infty} Q(p_n) \cup Z_i$. As $(T, \boldsymbol{Q}, \boldsymbol{I})$ is pseudometrizable, the grader $\boldsymbol{I}$ may be assumed countable; so, for $Z = \bigcup_{i \in T} Z_i$, we have $mZ = 0$ and $T - Z = \bigcup_{n=1}^{\infty} (Q_{p_n}^i - Z)$ for all $i \in \boldsymbol{I}$. Hence, unless $mS = m_f T = 0$, some of the $Q_{p_n}^i - Z$ are nonempty. Choosing $q_{in} \in Q_{p_n}^i - Z$, we construct a countable dense subset of $T - Z$, with $T - Z$ treated as a subspace of T. Thus, by dropping the null sets Z and $f^{-1}(Z)$ from T and S, respectively, all is reduced to the case where $(T, \boldsymbol{Q}, \boldsymbol{I})$ is separable and uniformly pseudometrizable, so that the first part of the theorem applies. The second part then follows by 3.1, and all is proved.

3.3. COROLLARY. *If, in Theorem* 3.2, *the functions f_k and f are continuous, and the measure m is topological, there is a closed set $D \in \boldsymbol{M}$, with $m(S - D) < \varepsilon$, and such that* (*with K quasi-countable in all cases*):

(i*) *$f_k \to f$ (unif.) on D, if T is uniformly nested and separable;*

(ii*) *$f_k \to f$ (ess. unif.) on D, if T is m_f-ess. nested and ess. separable.*

([15]) This already yields our assertion (i) for *sequences* of functions, by simply applying Egoroff's theorem to the *real* functions g_k (referee's remark). We need, however, a few additional details to prove 3.2 in full and to obtain our Corollary 3.3 below.

Formulae (i*) *and* (ii*), *with D not necessarily closed, result also if the convergence* $f_k \to f$ *(a.e.) is monotone, i.e., such that, whenever* $f_k(x) \in N^j_{f(x)}$ *holds for* $k = k_0$, *it also holds for* $k > k_0$.

Indeed, the continuity of f_k and f implies that the sets E^j_k, $D^j_n = \bigcup_{k \geq n} E^j_k$ and $D = \bigcup_{j=1}^{\infty} D^j_{n_j}$, defined in the proof of 3.2, are closed, hence measurable if m is topological. Hence the standard proof of Egoroff's theorem works also if K is only quasi-countable, and yields assertions (i*) and (ii*).

In case the convergence $f_k \to f$ (a.e.) is monotone, we extract from K a cofinal subset I of type ω, thus obtaining a subnet $\{f_k \mid k \in I\}$, to which then assertion (i) of 3.2 applies. The monotonicty of the convergence then easily implies that (i) holds not only for the subnet, but for the whole net. The rest follows exactly as in Theorem 3.2.

NOTE 1. Since the real axis is a quasi-countable directed set (under its standard order), Theorem 3.2 contains Frumkin's result [5] as a special case, with a considerable simplification of the proof.

NOTE 2. Part (i) of Theorem 3.2 holds also with the uniform nestedness of T replaced by the following condition: "For each noncofinal countable subset $I \subseteq J$, and each $x \in T$, there is $i_x \in J$, with $N^{i_x}_x \subseteq \bigcap_{j \in I} N^j_x$." The proof is the same as outlined in footnote 13, but the discrete case requires a separate treatment because the discrete metrization may not preserve the uniform structure of T in this case.

4. **Lustin's theorem.** In [11] and [12] Schaerf proved several variants of Lusin's theorem for functions with values in spaces satisfying the 2nd axiom of countability. His results apply, in particular, to separable pseudometric spaces and can be "translated", via 2.2 and 3.1, into propositions "essentially valid" in ess. pseudometrizable spaces, as it was done in 3.2. Since this "translation" is almost automatic, we shall limit ourselves to a few remarks and propositions in which Schaerf's own results can be improved, or his proofs can be replaced by simpler standard proofs.

A measure space $(S, \boldsymbol{M}, m)$, and m itself, are said to be *regular* [strongly regular], with respect to a topology in S, if, for any set $A \in \boldsymbol{M}$, and any $\varepsilon > 0$, there is an open set $G \supseteq A$ $(G \in \boldsymbol{M})$, with $mG \leqq mA + \varepsilon$ $[m(G - A) < \varepsilon$, respectively]. Every σ-finite regular measure is also strongly regular. So also is any Borel measure and its completion (least complete extension), in a space S with Urysohn's "F-property" ("every closed set is a G_δ-set"), provided that S is σ^0-*finite*, i.e. $S = \bigcup_{k=1}^{\infty} G_k$ for some *open* sets $G_k \in \boldsymbol{M}$, with $mG_k < \infty$, $k = 1, 2, \cdots$ ([16]). A

([16]) For a proof, see [18]. Schaerf in [13] obtains strong regularity only for metric spaces and therefore unnecessarily weakens his Theorems 2, 3, 4 and 8 in [11] and [12], as will be seen below.

function f on S is said to be *elementary* if it takes only countably many values, each on a measurable set. We need two very simple lemmas.

4.1. LEMMA. *Let f be an elementary function from a regular measure space (S,M,m) $(mS<\infty)$ to a topological space T. Then, for any $\varepsilon>0$, f is continuous on some closed set $F\in M$, with $m(S-F)<\varepsilon$. If, further, T is uniform, S is pseudometrizable and m is topological, then f can be made uniformly continuous on F.*

Proof. By assumption, f is constant on some disjoint sets $A_n\in M$, $n=1,2,\cdots$ with $S=\bigcup_1^\infty A_n$; so $\sum_1^\infty mA_n=mS<\infty$. Hence, given $\varepsilon>0$, there is an integer q with $m(S-\bigcup_{n=1}^q A_n)<\varepsilon/2$. By regularity, since $mS<\infty$, there are disjoint closed sets $F_n\subseteq A_n$, $n=1,2,\cdots,q$, with $m(A_n-F_n)<\varepsilon/4q, F_n\in M$. As f is constant on each F_n, it is continuous on $F=\bigcup_{n=1}^q F_n$, and $m(S-F)<\varepsilon$, q.e.d. The continuity of f will become uniform (with ρ a pseudometric for S, and T uniform) if the mutual distances between the sets F_n are made >0. It suffices to consider *two* of them, F_1,F_2. Put $X_r=\{x\in F_1\,|\,\rho(x,F_2)\geqq 1/r\}$, $r=1,2,3,\cdots$. The sets X_r are closed, hence measurable if m is topological. Also, $X_r\subseteq X_{r+1}$, and $F_1=\bigcup_{r=1}^\infty X_r$. Hence $mX_r\to mF_1<\infty$. Thus, for some r, $m(F_1-X_r)<\varepsilon/4q$, and $\rho(X_r,F_2)\geqq 1/r>0$. By reducing F_1 to X_r and leaving F_2 unchanged, we get the required result.

4.2. LEMMA. *If $f:S\to T$ is a measurable function from a measure space (S,M,m) to a separable uniform space (T,N,J), then there is a net of elementary functions $f_j:S\to T$ $(j\in J)$ such that $f_j\to f$(unif.) on S.*

Proof. Let $\{p_n\}$ be a dense sequence in T. Then, for each $j\in J$, $T=\bigcup_{n=1}^\infty N^j_{p_n}$ and $S=\bigcup_{n=1}^\infty f^{-1}(N^j_{p_n})$. Put $A^j_1=f^{-1}(N^j_{p_1})$ and

$$A^j_n=f^{-1}(N^j_{p_n})-\bigcup_{k=1}^{n-1}f^{-1}(N^j_{p_k}),\qquad n=2,3,\cdots,$$

and define $f_j:S\to T$ $(j\in J)$, setting $f_j(x)\equiv p_n$ for $x\in A^j_n$, $n=1,2,\cdots$ Then all f_j are elementary functions. Moreover, for each $j\in J$, every $x\in S$ is in some $A^j_n\subseteq f^{-1}(N^j_{p_n})$ so that $f(x)\in N^j_{p_n}=N^j(f_j(x))$; or, $f_j(x)\in N^j(f(x))$, for all $x\in S$ and $j\in J$. Hence it easily follows that $f_j\to f$ (unif.) on S, q.e.d.

4.3. THEOREM. *If $f:S\to T$ is a measurable function from a regular measure space (S,M,m) $(mS<\infty)$ into a uniform space (T,N,J), then, for any $\varepsilon>0$, there is a closed set $F\in M$ such that $m(E-F)<\varepsilon$ and, moreover:*

(i) *f is continuous on F, provided T is uniformly nested and separable;*

(ii) *f is ess.continuous on F, if T is m_f-ess. nested and m_f-ess. separable.*

This continuity can be made uniform (ess.uniform, resp.) if m is also a topological measure and if, furthermore, in case (i), *S is a pseudometrizable space,*

and, in case (ii), *S is either a pseudometrizable, or a separable m-ess. nested uniform space.*

Proof. We consider only the last part of the theorem, since the nonuniform case is immediate, by a similar argument. Let S be pseudometrizable, and let T be separable and uniformly nested, hence likewise pseudometrizable (cf. footnote 13); so we may identify the grader J with the positive integers. Then, given $\varepsilon > 0$, Lemmas 4.2 and 4.1 yield elementary functions f_j, $j = 1, 2, \cdots$ such that $f_j \to f$ (unif.) on S, and each f_j is uniformly continuous on a closed set $F_j \in \boldsymbol{M}$, with $m(S - F_j) < \varepsilon / 2^j$. Let $F = \bigcap_{j=1}^{\infty} F_j \in \boldsymbol{M}$. Then F is closed, $m(S - F) < \varepsilon$, and all f_j (hence f) are uniformly continuous on F, q.e.d.([17]).

If T is only m_f-ess. separable and m_f-ess. nested, assertion (ii) follows from (i) via 2.2 and 3.1, exactly as in 3.2. Finally, if S is not first-countable but separable and m-ess. nested, with neighborhood system $(\boldsymbol{U}, K)$, let $\{p_n\}$ be a dense sequence in S. Then, as in the proof of 2.2, $(\boldsymbol{U}, K)$ is m-ess. equivalent to some $(\boldsymbol{V}, K')$ such that $(S, \boldsymbol{V}, K')$ is uniformly pseudometrizable, $K' \subseteq K$ and $V_x^k = U_x^k$ for $k \in K'$ and all $x \in S$. Moreover, there is $\alpha \in K'$, with $m(U_{p_n}^{\alpha} - U_{p_n}^{k}) = 0$ for all $k \in K$ and $n = 1, 2, \cdots$. As $\boldsymbol{V} \subseteq \boldsymbol{U}$, the measure m is topological also under $(\boldsymbol{V}, K')$. Thus all will be reduced to previous cases, if we show that *m is regular under* $(\boldsymbol{V}, K')$. Let $A \in \boldsymbol{M}$ and $\varepsilon > 0$. As m is regular under $(\boldsymbol{U}, K)$, we have $A \subseteq G$ and $m(G - A) < \varepsilon$ for some $\boldsymbol{U}$-open set G. By separability, G is the union of all neighborhoods $U_{p_n}^{k} \subseteq G$ $(n = 1, 2, \cdots)$, with $k \geqq \alpha$, so that $U_{p_n}^{k} \subseteq U_{p_n}^{\alpha} = V_{p_n}^{\alpha}$ (since $\alpha \in K'$). Let G_n be the union of all such $U_{p_n}^{k}$ for a *fixed* n. Then $G_n \subseteq V_{p_n}^{\alpha}$ and $m(V_{p_n}^{\alpha} - G_n) \leqq m(U_{p_n}^{\alpha} - U_{p_n}^{k}) = 0$. Also, $A \subseteq G = \bigcup_{n=1}^{\infty} G_n \subseteq \bigcup_{n=1}^{\infty} V_n^{\alpha} = G'$, with G' open under (ϕ, K') and $m(G' - G) = 0$. Since $A \subseteq G'$ and

$$m(G' - A) = m(G' - G) + m(G - A) = 0 + \varepsilon,$$

regularity is proved, and the theorem is established.

NOTE 1. The last part of 4.3 (on unif. and ess. unif. continuity) holds, in particular, if m is a Borel measure or its completion. Indeed, if S is pseudometrizable, it has Urysohn's F-property, and thus m is regular, as explained above. The rest follows via 2.2 and 3.1. This strengthens Schaerf's Theorem 4. (Cf. also footnote 16.)

4.4. THEOREM. *The first part of* 4.3 (*assertions* (i) *and* (ii)) *holds also if the measure m is only regular and σ-finite.*

Proof. We consider only case (i), since (ii) follows from it via 2.2. As m is regular and σ-finite, it is also strongly regular and σ^0-finite. Thus

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([17]) This simple standard proof replaces Schaerf's proof of his Theorem 4 in [11], based on a more involved procedure.

$$S = \bigcup_{k=1}^{\infty} G_k,\ mG_k < \infty,$$

for some open sets $G_k \in M$, $k = 1, 2, \cdots$. Let $E_1 = G_1$, $E_k = G_k - \bigcup_{i=1}^{k-1} G_i$, $k = 2, 3, \cdots$; so $E_k \in M$ and $S = \bigcup_{k=1}^{\infty} E_k$, with $mE_k < \infty$, all disjoint. Hence, given $\varepsilon > 0$, we get, by 4.3, disjoint closed sets $F_k \subseteq E_k$ ($F_k \in M$) such that $m(E_k - F_k) < \varepsilon / 2^{k+1}$, and f is continuous on each F_k. Let $A = \bigcup_{k=1}^{\infty} F_k \in M$. Then $m(E - A) < \frac{1}{2}\varepsilon$, and it suffices to show that f is continuous on A. Fix $p \in A$. Then p is some G_k, call it G_k. By definition, all E_k ($k > k$) are disjoint from G_k. So are, a fortiori, all F_k ($k > k$). Hence $G_k \cap A \subseteq \bigcup_{k \leq k} F_k$. Now, as f is continuous on the disjoint closed sets F_k, it is continuous on the *finite* union $\bigcup_{k \leq k} F_k$, hence on $G_k \cap A$. Thus, for any neighborhood N of $f(p)$, there is a neighborhood G_p^* of p, with $A \cap G_k \cap G_p^* \subseteq f^{-1}(N)$. Since $G_k \cap G_p^*$ is a neighborhood of p, and $p \in A$ is arbitrary, f is continuous on A. By the strong regularity of m, there is also a closed set $F \subseteq A$ ($F \in M$), with $m(A - F) < \frac{1}{2}\varepsilon$. Then f is continuous on F and $m(S - F) < \varepsilon$, as required.

NOTE 2. The theorem applies. in particular, if m is a σ^0-finite Borel measure, or its completion, in a space S with Urysohn's F-property, or if m is a σ-finite Radon measure in a locally compact topological space (cf.[13]). Indeed, in both cases, m is strongly regular. Moreover, by Schaerf's Theorem 1, proved in [11], assertion (i) holds also if T is any topological space satisfying the 2nd axiom of countability. Thus we can strengthen Schaerf's Theorem 3 (plus supplement) and 8, by asserting what Schaerf calls the "strong Lusin property" instead of the "plain Lusin property" claimed in Theorem 3; and by extending Schaerf's Theorem 8 to spaces with Urysohn's F-property (instead of metric spaces).

The theorems proved so far constitute only a few examples of applications of our "ess." concepts. The definitions of §1 could be supplemented by such notions as "ess. compactness", "ess. uniform spaces", etc.; and ess. metrization theorems can be proved also for spaces which are not ess. separable (e.g. for Lindelöf spaces, etc.). This makes it possible to extend various theorems, valid in metric spaces, to nonmetrizable, but 'ess. metrizable", spaces, with ordinary topological concepts replaced by their "ess." counterparts. Since sets of measure 0 may be disregarded in many measure-theoretical problems, this seems to open an interesting field for investigations.

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University of Windsor,
Windsor, Ontario, Canada