λ-CONTINUOUS MARKOV CHAINS. II(1)

BY

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Summary. Continuing the investigation in [8] we study a λ-continuous Markov operator P. It is shown that, if P is conservative and ergodic, P is indeed "periodic" as is the case when the state space is discrete; there is a positive integer δ, called the period of P, such that the state space may be decomposed into δ cyclically moving sets C₀, ..., Cₐ₁ and, for every positive integer n, Pⁿδ acting on each Cᵢ alone is ergodic. It is also shown that P maps Lq(μ) into Lq(μ) where μ is the non-trivial invariant measure of P and 1 ≤ q ≤ ∞. If μ is finite and normalized then it is shown that (1) if f ∈ L₁(μ), then {Pⁿδ+kf} converges a.e. (μ) to gₖ = ∑ₙ₌₀⁻¹ cₙ₊ₖ 1₁ₖ where cₙ = δ₁ₓ, f dμ if 0 ≤ j ≤ δ₁ and cₙ = cₙ if j = mδ + i, 0 ≤ i ≤ δ₁ - 1, (2) {Pⁿδ+kf} converges in L₂(μ) to gₖ if f ∈ L₂(μ), and (3) lim infₙ→∞ Pⁿδ+k f = gₖ a.e. (μ) if f ∈ L₁(μ) and f ≥ 0. If μ is infinite, then it is shown that (1) if f ≥ 0, f ∈ L₁(μ) for some 1 ≤ q < ∞, then lim infₙ→∞ Pⁿf = 0 a.e. (μ), (2) there exists a sequence {Eₖ} of sets such that X = ∪ₖ₌₁⁻¹ Eₖ and limₙ→∞ Pⁿ⁺₁ P⁻₁ Eₖ = 0 a.e. (μ) for i = 0, 1, ..., δ₁ and k = 1, 2, ...

I. Introduction. Let X be a nonempty set, X, a σ-algebra of subsets of X and λ, a σ-finite measure on X. Let p(x, y) be an X × X measurable function defined on X × X satisfying the following conditions:
1. p(x, y) ≥ 0 for (X × X) almost all (x, y),
2. ∫ p(x, y)λ(dy) ≤ 1 for (X) almost all x.

Let L₁(λ) be the collection of all λ-essentially bounded functions and A(λ), the collection of all finite, real-valued, countably additive functions on X which are absolutely continuous with respect to λ. Let A⁺(λ) be the collection of all non-negative elements of A(λ). For any f ∈ L₁(λ), P f is defined by

(1.1) P f(x) = ∫ p(x, y) f(y)λ(dy),

and for any v ∈ A(λ), vP is defined by

(1.2) vP(A) = ∫ v(dx) ∫ A p(x, y)λ(dy).

The operator P here is a special kind of λ-measurable Markov operator of E,
Hopf [7]. We call it a \( \lambda \)-continuous Markov operator. (1.1), (1.2) remain meaningful for non-negative \( f \) not necessarily \( \lambda \)-essentially bounded and non-negative \( \sigma \)-finite measure \( v \). The iterates \( P^n \) of \( P \) are then given by

\[
P^n f(x) = \int p^{(n)}(x, y)f(y)\lambda(dy)
\]

and

\[
vP^n(A) = \int v(dx)\int_A p^{(n)}(x, y)\lambda(dy)
\]

where \( p^{(n)}(x, y) \) are defined inductively by

\[
(1.3) \quad p^{(n)}(x, y) = \int_{\mathbb{R}} p^{(n-1)}(x, z)p(z, y)dz.
\]

The function \( p(\cdot, \cdot) \) is called the density function of \( P \) with respect to \( \lambda \) and is only uniquely determined by \( P \) a.e. \((\lambda \times \lambda)\). All subsets of \( X \) discussed in this paper are elements of \( \mathcal{F} \) and all functions on \( X \) are \( \mathcal{F} \)-measurable functions. Unless otherwise indicated, for two sets \( A, B, A \subset B, A = B \) means that \( \lambda(A - B) = 0, \lambda(A \cap B) = 0 \) respectively, and for two functions \( f, g \) on \( X, f = g, f \leq g \) means that the equality and the inequality, respectively, are satisfied except on a \( \lambda \)-null set. For any set \( A, I_A \) is to represent the function which equals 1 on \( A \) and 0 on the complement \( A' \) of \( A \). \( I_A \) is the \( \lambda \)-measurable Markov operator defined by

\[
I_A f(x) = I_A(x)f(x),
\]

\[
vI_A(B) = v(A \cap B).
\]

For any set \( E \), define \( P_E \) by

\[
(1.4) \quad P_E = \sum_{n=0}^{\infty} P(I_E, P)^n
\]

where \( E' \) is the complement of \( E \). \( P_E \) operating on either non-negative functions or measures has well-defined meanings (cf. [8, §VI]). For a measure \( v \), and a function \( f \) we shall use the symbol \( \langle v, f \rangle \) to denote the integral \( \int f dv \). For any \( v \in \mathcal{A}^*(\lambda) \), the support of \( v \) is the set \( A \) such that \( v(X - A) = 0 \), and \( B \subset A \) with \( B \) being \( \lambda \) non-null implies that \( v(B) > 0 \), "non-null" and "null" shall mean \( \lambda \)-non-null and \( \lambda \)-null respectively.

Following E. Hopf and J. Feldman we call a set \( A \) a conservative set if for every non-null subset \( B \) of \( A, P_B 1_B = 1 \) on \( B \). The largest conservative set \( C \) is called the conservative part of \( X \). \( D = X - C \) is called the dissipative part of \( X \). \( P \) is conservative if \( X = C \), dissipative if \( X = D \). We say that a set \( A \) is closed if \( P1_A = 1 \) on \( A \). The collection of all closed subsets of \( C \) is a \( \sigma \)-algebra of subsets of \( C \) which we shall denote by \( \mathcal{C} \). An element \( A \) of \( \mathcal{C} \) is indecomposable if \( A \) is non-null and if the only closed subsets of \( A \) are null sets and \( A \) itself. A conservative operator \( P \) is ergodic if \( X \) is indecomposable or, equivalently, if the only elements of \( \mathcal{C} \) are \( X \) and the null set. In [8] it has been shown that, for a conservative \( \lambda \)-continuous
Markov operator $P$, the space $X$ may be decomposed into at most countably many indecomposably closed sets $C_1, C_2, \ldots$, and that to each $C_i$, there is a non-trivial $\sigma$-finite $P$-invariant measure $\mu_i$ which is equivalent to $\lambda_{C_i}$, and every $P$-invariant measure is of the form $\sum \lambda_{C_i} \mu_i$. Thus, if we consider $P$ acting on each $C_i$ only, $P$ is ergodic. In [8] we studied the convergence properties of the sequence $\{\sum_{n=1}^{N} p^n(z,x)/\sum_{n=1}^{N} p^n(z,y)\}$. It was proved that, for an ergodic conservative $P$, the sequence converges to the limit $f(x)/f(y)$ where $f$ is the derivative of an invariant measure with respect to $\lambda$. In this paper we shall proceed further to study the asymptotic behavior of sequences $\{p^{(n)}(x,y)\}$ and $\{P^n f\}$. As we know that $\sum_{n=0}^{\infty} p^n(x,y)$ converges on $X \times D$, therefore, $\lim_{n \to \infty} p^n(x,y) = 0$ on $X \times D$. The limiting behavior of $\{p^{(n)}(x,y)\}$ is relatively simple on the dissipative part.

In [8], it is well known that, if $X$ is discrete and if $P$ is conservative and ergodic, then $X$ may be partitioned into a finite number $\delta$ of cyclically moving sets where $\delta$ is the period of $P$, and $\{p^{\delta n}(x,y)\}$ converges as $n \to \infty$ [1]. Thus in §II, a theory of periods is developed for a $\lambda$-measurable, conservative and ergodic Markov operator. Much of the work here is inspired by the pioneer work of W. Doeblin. The theory of periods of a conservative ergodic $\lambda$-measurable Markov operator given here is modeled after Doeblin’s (which was perfected and completed by Chung [2]).

II. Periods of $\lambda$-measurable conservative ergodic Markov operators. We recall that the properties of a set in $S$ being transient, conservative, closed, etc., were defined with reference to a $\lambda$-measurable Markov operator $P$. If there are more than one Markov operator these terminologies will be prefixed by “$P$-” or “$Q$-” to distinguish that the properties are referred to operator $P$ or $Q$ respectively. In this section attention will be paid mainly to iterations $P^k$ of $P$. 

**LEMMA 2.1.** Let $k$ be a positive integer. Then, a set $R$ is $P$-conservative if and
only if $R$ is $P^k$-conservative; it follows that, if $P$ is conservative, so is $P^k$ and vice versa.

**Proof.** A non-null set $R$ is $P$-conservative if and only if, for every non-null set $S \subset R$, $\sum_{n=0}^{\infty} P^n 1_S$ is unbounded [5]. Since $\sum_{n=0}^{\infty} P^n 1_S \leq \sum_{n=0}^{\infty} P^n 1_S$, $R$ is $P$-conservative if $R$ is $P^k$-conservative. Conversely, if a non-null set $R$ is not $P^k$-conservative, then there is a non-null subset $S$ of $R$ for which $\sum_{n=0}^{\infty} P^n 1_S$ is bounded. It follows that $\sum_{n=0}^{\infty} P^n 1_S = \sum_{n=0}^{\infty} P^n 1_S$ is bounded so that $\sum_{n=0}^{\infty} P^n 1_S = \sum_{n=0}^{\infty} P^n 1_S$ is also bounded. Hence $R$ is also not $P$-conservative.

All through §II we shall assume that $P$ is conservative and ergodic. A $P^k$-closed set $E$ is said to be $P^k$-decomposable if and only if there is a non-null $P^k$-closed subset $B$ of $E$ such that $E - B$ is also non-null. Since $P^k$ is conservative, the collection of all $P^k$-closed sets is a $\sigma$-algebra; $C - B$ is then also $P^k$-closed. A $P^k$-closed set is $P^k$-indecomposable if it is not $P^k$-decomposable. Since $P$ is assumed to be ergodic, $X$ is $P$-indecomposable. In this section we shall study the decomposability of $X$ under iterates of $P$. For an arbitrary set $E$ we denote the set $[P^k 1_E = 1]$ by $A^k(E)$:

$$A^k(E) = [P^k 1_E = 1].$$

Then $E$ is $P^k$-closed if and only if $E \subset A^k(E)$. It is easy to see that

1. $A^k(E_1) \subset A^k(E_2)$ if $E_1 \subset E_2$,
2. $A^k(E_1) \cap A^k(E_2)$ is null if $E_1 \cap E_2$ is null,
3. if $\{E_n\}$ is a finite or infinite sequence of sets, then

$$\bigcup_{n} A^k(E_n) \subset A^k \left( \bigcup_{n} E_n \right).$$

Denote $A^1(E)$ by $A(E)$, then we have

$$A^2(E) = A(A(E)), \quad A^3(E) = A(A^2(E)), \cdots.$$  

**Lemma 2.2.** If $E$ is $P^k$-closed, then $A(E)$ is also and $A(E)$ is $P^k$-decomposable or $P^k$-indecomposable according as $E$ is $P^k$-decomposable or $P^k$-indecomposable. It follows that the lemma remains valid if we replace $A(E)$ by $A^j(E)$ where $j$ is an arbitrary positive integer.

**Proof.** If $E$ is $P^k$-closed then $E \subset A^k(E)$. Hence $A(E) \subset A(A^k(E)) = (A^k(A(E))$ and $A(E)$ is $P^k$-closed. If $E$ is $P^k$-decomposable, $E = B \cup C$ where $B$ and $C$ are non-null, disjoint and $P^k$-closed, then $A(B)$ and $A(C)$ are $P^k$-closed and disjoint. $A(B)$ and $A(C)$ are non-null because $A^k(B)$ and $A^k(C)$ are non-null. Hence $A(E)$ is also $P^k$-indecomposable.

Now suppose that $E$ is $P^k$-indecomposable, we shall show that $A(E)$ is also $P^k$-indecomposable. Let $F$ be a non-null $P^k$-closed subset of $A(E)$, we shall first show $A^{k-1}(F) \cap E$ is non-null. We have
\[ P^k 1_F = P_I^k P^{k-1} 1_F + P_I^k P^{k-1} 1_F. \]

Since \( F \subseteq A(E), P1_F = 0 \) on \( F \), hence \( P_I^k P^{k-1} 1_F = 0 \) on \( F \). Hence we have

(2.2) \[ P^k 1_F = P_I^k P^{k-1} 1_F = 1 \text{ on } F. \]

Since \( P1 = 1 \), it follows that if \( f > 0 \) a.e. \((\lambda)\) we also have \( Pf > 0 \) a.e. \((\lambda)\). Now \( 1 - I_E P^{k-1} 1_F \) is a non-negative function. If the set \([I_E P^{k-1} 1_F = 1]\) is null then \( P[1 - I_E P^{k-1} 1_F] = 1 - P_I E P^{k-1} 1_F > 0 \) a.e.\((\lambda)\) which contradicts (2.2). Hence \([I_E P^{k-1} 1_F = 1]\) is non-null, i.e., \( E \cap A^{k-1}(F) \) is non-null. Now suppose \( A(E) \) were \( P^k \)-decomposable and \( F_1, F_2 \) were two disjoint non-null \( P^k \)-closed subsets of \( A(E) \) then \( E \cap A^{k-1}(F_1) \) and \( E \cap A^{k-1}(F_2) \) would be two non-null, disjoint, \( P^k \)-closed subsets of \( E \) which is clearly impossible. Hence \( A(E) \) is also \( P^k \)-indecomposable.

**Lemma 2.3.** If \( P \) is conservative and ergodic, and if \( C_1, \ldots, C_n \) are \( P^k \)-closed, non-null and pairwise disjoint then \( n \leq k \).

**Proof.** Let \( G_m = \bigcup_{i=0}^{k-1} A_i(C_m) \), then

\[ A(G_m) = \bigcup_{i=0}^{k-1} A^{i+1}(C_m) \supset G_m, \]

hence each \( G_m \) is \( P \)-closed. \( G_m = X \) for \( m = 1, \ldots, n \). Hence

(2.3) \[ X = \bigcup_{m=1}^n G_m = \bigcup_{(i_1, i_2, \ldots, i_n)} [A^{i_1}(C_1) \cap A^{i_2}(C_2) \cap \cdots \cap A^{i_n}(C_n)]. \]

Where the union appearing in the right-hand side of (2.3) is taken over all \( n \)-tuple \((i_1, \ldots, i_n)\) where \( i_j \) may be \( 1, 2, \ldots, k \). There is at least one \( n \)-tuple \((i_1, i_2, \ldots, i_n)\) for which \( A^{i_1}(C_1) \cap \cdots \cap A^{i_n}(C_n) \) is non-null. Then \( i_1, i_2, \ldots, i_n \) are all distinct, for \( i_j = i_i \) would imply that \( A^{i_1}(C_1) \cap A^{i_2}(C_2) \) is null. Hence \( n \leq k \).

**Lemma 2.4.** Let \( P \) be conservative and ergodic and \( k \) be a positive integer. Let \( \mathcal{C}^{(k)} \) be the \( \sigma \)-algebra of \( P^k \)-closed subsets of \( X \). Then \( \mathcal{C}^{(k)} \) is generated by a finite number \( \delta = \delta(k) \) of distinct atoms with \( \delta \) dividing \( k \). Each atom in \( \mathcal{C}^{(k)} \) is also \( P^\delta \)-indecomposably closed. It follows that \( \mathcal{C}^{(k)} \) is identical with the \( \sigma \)-algebra \( \mathcal{C}^{(\delta)} \) of all \( P^\delta \)-closed sets.

**Proof.** By Lemma 2.3 \( \mathcal{C}^{(k)} \) must be generated by a finite number of atoms. Let \( C_1 \) be an atom of \( \mathcal{C}^{(k)} \). \( C_1 \) is a \( P^\delta \)-indecomposable closed set. Let \( C_2 = A(C_1), C_3 = A(C_2), \ldots \). By Lemma 2.2 every \( C_i \) is also \( P^\delta \)-indecomposably closed. Hence, if \( i \neq j \) we have either \( C_i \cap C_j \) null or \( C_i = C_j \). Since \( C_i \) is \( P^k \)-closed, \( C_i \subseteq A^k(C_i) = C_{i+k} \). Hence \( C_i = C_{i+k} = C_{i+2k} = \cdots \). It then follows that if \( d \) is a positive integer for which there is an \( i \) such that \( C_i = C_{i+d} \), then \( C_i = C_{i+d} \) for every positive integer \( i \). Let \( \delta \) be the smallest of all positive integers \( d \) for which
$C_1 = C_1 + d$. Clearly $\delta \leq k$. $\delta$ must divide $k$ for, if otherwise, then $k = n\delta + r$ where $r$ is a positive integer $< d$, $C_{1+n\delta} = C_1 = C_1 + n\delta + r$, hence $C_1 = C_1 + r$, which contradicts the defining property of $\delta$. Now for every $i$, $C_i = C_i + \delta = A^i(C_1)$, hence every $C_i$ is $P^\delta$-closed. Each $C_i$ is also $P^k$-indecomposable since it is $P^k$-indecomposable. $C_1, C_2, \ldots, C_\delta$ are all distinct. $\bigcup_{i=1}^\delta C_i$ is $P$-closed, therefore is equal to $X$. $\{C_1, C_2, \ldots, C_\delta\}$ consists of all atoms of $\mathcal{G}^{(k)}$ and also of $\mathcal{G}^{(\delta)}$. Hence $\mathcal{G}^{(k)} = \mathcal{G}^{(\delta)}$.

**Lemma 2.5.** For any positive integer $k$, let $\delta(k)$ be the positive integer of Lemma 2.4. Then, if $k_1, k_2$ are two positive integers such that $k_1$ divides $k_2$, then $\delta(k_1)$ is equal to the greatest common divisor $d$ of $k_1$ and $\delta(k_2)$.

**Proof.** By Lemma 2.4 $\delta(k_1)$ divides $k_1$. We shall show that $\delta(k_1)$ also divides $\delta(k_2)$. Then it follows that $\delta(k_1)$ divides $d$. Let $C_1$ be an atom of $\mathcal{G}^{(k_1)}$. $C_2 = A(C_1)$, $C_3 = A^2(C_1)$, $\ldots$. Then $C_1, \ldots, C_{\delta(k_1)}$ are the totality of distinct atoms of $\mathcal{G}^{(k_1)}$. Let us consider $P^{k_1}$ acting on $C_1$ only. It is ergodic, conservative and $P^{k_1} = (P^k)^l$ where $l = k_2/k_1$. By Lemma 2.4 $C_1$ is decomposed into $B_1, \ldots, B_j$, $P^{k_2}$-indecomposable sets with $B_2 = A^{k_1}(B_1)$, $B_3 = A^{k_1}(B_2)$, $\ldots$. Then each $C_i$ is decomposed into $j$ $P^{k_2}$-closed sets $A^{i-1}(B_1), \ldots, A^{i-1}(B_j)$. Hence $\mathcal{G}^{(k_2)}$ has a totality of $j \cdot \delta(k_1)$ distinct atoms, i.e., $\delta(k_2) = j \cdot \delta(k_1)$. To prove that $d$ divides $\delta(k_1)$, let $D_1$ be a $P^{k_2}$-indecomposable set. Let $D_2 = A(D_1)$, $D_3 = A^2(D_1)$, $\ldots$, then $D_1, \ldots, D_{\delta(k_2)}$ are all distinct whereas $D_{\delta(k_2) + i} = D_i$ for every couple of positive integers $n, i$. Let $q = \delta(k_2)/d$. Let $E_i = \bigcup_{n=0}^{q-1} D_{n+d+i}$. Then $A^d(E_i) = E_i$ so that $E_i$ is $P^d$-closed. Since $d$ divides $k_1$, $E_i$ is also $P^{k_1}$-closed. $E_1, \ldots, E_d$ are all distinct, $A(E_1) = E_{i+1} + X = \bigcup_{i=1}^d E_i$. If $E_1$ is $P^{k_1}$-indecomposable, so are all other $E_i$. If $E_1$ is $P^{k_1}$-decomposable so are all other $E_i$ and they may be decomposed into a same number of $P^{k_1}$-indecomposable sets. Hence $d$ divides $\delta(k_1)$. Since we have already proved the fact that $\delta(k_1)$ divides $d$, $d = \delta(k_1)$.

For a $\lambda$-measurable conservative ergodic Markov operator $P$ we define the period $\delta$ of $P$ by

\[ \delta = \sup \{\delta(k), k = 1, 2, \ldots\}. \]

The period $\delta$ of $P$ may or may not be finite. If $\delta = 1$, $P$ is said to be aperiodic. An aperiodic Markov operator is characterized by the property that all iterates of $P$ are ergodic. If the period $\delta$ of a Markov operator $P$ is finite then the restriction of $P^\delta$ to each $P^\delta$-indecomposable set is aperiodic. It is well known that if the state space $X$ is discrete then every conservative ergodic Markov operator has a finite period.

A sequence $\{C_n\}$ of sets in $X$ shall be called a consequent sequence if $C_1$ is non-null and $C_n = A(C_{n+1})$ for $n = 1, 2, \ldots$. Then all sets in the sequence are non-null. If $E$ is a $P^\delta$-indecomposable closed set and $d = \delta(k)$ then

\[ \{E, A^{d-1}(E), A^{d-2}(E), \ldots, E, A^{d-1}(E), A^{d-2}(E), \ldots, E, \ldots\} \]
is a consequent sequence. For a consequent sequence \( \{C_n\} \) we have \( C_n \subseteq \bigcup_{m=n+1}^{\infty} C_m \) for \( n = 1, 2, \ldots \) since \( \bigcup_{m=n+1}^{\infty} C_m \) is closed and, therefore, \( \bigcup_{m=n+1}^{\infty} C_m = X \). Hence for each \( C_n \), there is a \( C_m \) with \( m > n \) such that \( C_n \cap C_m \) is non-null (and therefore \( C_n \cap C_m \subseteq X(C_n \cap C_m) \)). To each \( \nu \in \mathcal{A}^+(\lambda) \), \( \nu \neq 0 \), we may attach a consequent sequence \( \{C_n(\nu)\} \) where \( C_1(\nu) = \text{supp} \nu \), \( C_2(\nu) = \text{supp} \nu P \), \( C_3(\nu) = \text{supp} \nu P^2 \), \ldots .

If \( \eta \) is absolutely continuous to \( \nu \) then \( C_n(\eta) \subseteq C_n(\nu) \) for every \( n \). We now define \( h(\nu) \) to be the greatest common divisor of all positive integers \( k \) for which there is an integer \( i_0 \) such that \( C_{n+i}(\nu) \neq 0 \). We note that \( h(\nu) \) divides \( h(\eta) \) if \( \eta \) is absolutely continuous to \( \nu P^n \) for some \( n \geq 0 \). Let

\[
(2.5) \quad H = \sup \{ h(\nu) : \nu \in \mathcal{A}^+(\lambda), \nu \neq 0 \}.
\]

\( H \) may be \( +\infty \) or a finite positive integer.

**Theorem 2.1.** \( H = \delta \).

**Proof.** Let \( k \) be an arbitrary positive integer and and \( E \) be a \( P^k \)-indecomposable closed set. Let \( \nu = \lambda_j E \). The sequence \( \{E, A^{(k)\nu-1}(E), \ldots , E, A^{(k)\nu-1}(E)\} \) is the consequent sequence of \( \nu \) and for this \( \nu \), \( h(\nu) = \delta(k) \). Hence \( H \geq \delta(k) \) for every positive integer \( k \). It follows that \( H \geq \delta \). To prove \( H \leq \delta \), let \( \nu \) be an arbitrary nonzero element of \( \mathcal{A}^+(\lambda) \) and let \( C_{n,i}(\nu) = \text{supp} \nu P^{n-1} \) for \( n = 1, 2, \ldots \) and \( h = h(\nu) \). Let \( E_i, i = 1, \ldots , h, \) be defined by

\[
(2.6) \quad E_i = \bigcup_{j=0}^{\infty} C_{i+jh}(\nu).
\]

Since \( C_{i+jh}(\nu) \subseteq A^h(C_{i+(j+1)h}) \), \( E_i \) are \( P^h \)-closed. If \( i_1 \neq i_2 \), \( E_{i_1} \cap E_{i_2} \) is null for if \( E_{i_1} \cap E_{i_2} \) is non-null, then, there are non-negative integers \( j_1, j_2 \) such that \( C_{i_1+jh}(\nu) \cap C_{i_2+jh}(\nu) \) is non-null. Then \( i_1 + j_1 h - (i_2 + j_2 h) = (i_1 - i_2) + (j_1 - j_2) h \) is divisible by \( h \). It follows that \( i_1 - i_2 \) is divisible by \( h \) which is impossible since \( |i_1 - i_2| < h \). Therefore, \( E_h \) constitute the totality of all \( P^h \)-indecomposable sets. Hence \( h = \delta(h) \leq \delta \). Hence \( H \leq \delta \).

For any nonzero measure \( \nu \in \mathcal{A}^+(\lambda) \) we shall define \( h'(\nu) \) to be the minimum of all positive integers \( k \) for which there is a positive integer \( N \) such that \( C_{N}(\nu) \cap C_{N+k}(\nu) \) is non-null. It is clear that \( h(\nu) \) divides \( h'(\nu) \). If \( \eta \) is absolutely continuous to \( \nu P^n \) for some \( n \geq 0 \) then \( h'(\eta) \geq h'(\nu) \). Let

\[
(2.7) \quad H' = \sup \{ h'(\nu) : \nu \in \mathcal{A}^+(\lambda), \nu \neq 0 \}.
\]

We always have \( H' \geq H \). For a general conservative ergodic \( \lambda \)-measurable Markov operator \( P \) it is possible to have \( H' > H \) as illustrated by the following example.

Let \( X \) be the set of all complex numbers of absolute value 1 and \( \lambda \) be the linear Lebesgue measure. Let \( \alpha = e^{ir\theta} \) where \( \theta \) is irrational and \( PF(x) = f(\alpha x) \). Then
$P^n$ is ergodic for every positive integer $n$, so that $P$ is aperiodic and $H = 1$ (cf. [6, p. 26]). Let $v$ have, as its support, the set $[e^{i2\pi y}: 0 \leq y \leq \varepsilon]$ where $\varepsilon$ is a positive number. Then $vP^n$ has the set $[e^{i2\pi y}: n\theta \leq y \leq n\theta + \varepsilon]$ as its support. Let $k$ be an arbitrary positive integer. Let $2\pi c$ be the minimum distance from the point 1 to $e^{i2\pi \theta_1}, e^{i4\pi \theta_2}, \ldots, e^{i2k\pi \theta}$. Then $c > 0$. Hence if $\varepsilon < c$ we have $h'(v) > k$. Hence $H' = \infty$.

**Lemma 2.6.** If $H'$ is finite, then $H' = H = \delta$ and for every consequent sequence $\{E_n\}$ there is a positive integer $N$ such that $E_n \cap E_{n+\delta}$ is non-null for every $n \geq N$.

**Proof.** If $H'$ is finite, then $H'$ is a positive integer and there is a nonzero measure $v_1 \in \mathcal{M}^+(\lambda)$ such that $h'(v_1) = H'$. Since $H' \geq H$, $H$ is finite and there is a nonzero measure $v_2 \in \mathcal{M}^+(\lambda)$ such that $h(v_2) = H$. Since $C_1(v_2) \subset X = \bigcup_{n=1}^{\infty} C_n(v_1)$, there is an $n$ such that $C_1(v_2) \cap C_n(v_1)$ is non-null. Let $v$ be a nonzero measure which has $C_1(v_2) \cap C_n(v_1)$ as its support, then $v$ is absolutely continuous to both $v_2$ and $vP^{n-1}$. Hence $h(v) \geq h(v_1)$, $h'(v) \geq h'(v_2)$. However, since $h(v) \leq H$, $h'(v) \leq H'$, $h(v) = H$, $h'(v) = H'$. Now consider the consequent sequence $\{C_n(v)\}$ of $v$. Let $k$ be a positive integer such that there is an $n$ for which $C_n(v) \cap C_{n+k}(v)$ is non-null. Then $H' \leq k$. Now, since $C_n(v) \cap C_{n+k}(v)$ is non-null we may choose a nonzero measure $\eta \in \mathcal{M}^+(\lambda)$ with $C_n(v) \cap C_{n+k}(v)$ as its support. Then $\eta$ is absolutely continuous to $vP^{n-1}$. Hence $h'(\eta) \geq h'(v)$. It follows that $h'(\eta) = H'$ and there is a positive integer $m$ such that $C_m(\eta) \cap C_m+H'(\eta)$ is non-null. But we have $C_m(\eta) \subset C_{m-1}+m(\eta) \cap C_{m-1}+m+h(\eta)$, $C_{m+H'}(\eta) \subset C_{m-1}+m+h(\eta) \cap C_{m-1}+m+h+H'(\eta)$. Hence $C_{m-1}+m(v) \cap C_{m-1}+m+h(\eta) \cap C_{m-1}+m+h+H'(\eta) \cap C_{m-1}+m+h+H'(\eta)$. It follows that $C_{m-1}+m+H'(\eta) \cap C_{m-1}+m+k(\eta)$ is non-null. Hence either $k - H' = 0$ or $k - H' \geq H'$. If $k - H' = 0$ then $k$ is divisible by $H'$. If $k - H' \geq H'$, repeating the same argument for $k - H'$ as for $k$ before, we conclude that $k - 2H'$ is either 0 or $\geq H'$. Repeating the same argument finitely many times we obtain the result $k - jH' = 0$. Hence $k$ is divisible by $H'$. This is true for all positive integers $k$ for which there is a positive integer $n$ such that $C_n(v) \cap C_n+H(v)$ is non-null. Hence $H'$ divides $H$. Hence $H' = H = \delta$. Now let $\{E_n\}$ be an arbitrary consequent sequence. Since $X = \bigcup_{n=1}^{\infty} E_n$, $C_1(v) \cap E_{n_0}$ is non-null for some positive integer $n_0$. Let $\zeta \in \mathcal{M}^+(\lambda)$ have $C_1(\zeta) \cap E_{n_0}$ as its support. Then $h'(\zeta) \geq h(\eta)$ so that $h'(\zeta) = h(\eta) = \delta$. There is a positive integer $l$ such that $C_l(\zeta) \cap C_{l+\delta}(\eta)$ is non-null. It follows that $C_{\zeta}(\eta) \cap C_{n+\delta}(\zeta)$ is non-null for all $n \geq l$. Now we have, for every positive integer $n$, $C_n(\zeta) \subset E_{n_0} \cap E_{n_0+1+n}$. Hence $C_n(\zeta) \cap C_{n+\delta}(\zeta)$ being non-null implies that $E_{n_0-1+n} \cap E_{n_0-1+n+\delta}$ is non-null. Let $N = n_0 - 1 + l$. Then $E_n \cap E_{n+\delta}$ is non-null for all positive integers $n \geq N$.

Now we shall proceed to show that the period of a conservative, ergodic, $\lambda$-continuous Markov operator is always finite. To do this we shall choose a definite version of $p(x, y)$ for $P$ to satisfy

1. $p(x, y) \geq 0$ for all $(x, y) \in X \times X$ and
2. $\int p(x, y)\lambda(dy) = 1$ for all $x \in X$. 

Then the iterates \( p^{(n)}(x,y) \) given by (1.3) also satisfy 1 and 2. For each \( x \in X, E \in \mathcal{F} \) let
\[
v_x(E) = \int_E p(x,y) \lambda(dy).
\]

For each \( x \in X, v_x \) is a probability measure absolutely continuous to \( \lambda \) and for each fixed \( E \in X, x \) varying over \( X, v_x(E) \) is a version of \( P_1 E \).

**Lemma 2.7.** For a \( \lambda \)-continuous, conservative, ergodic Markov operator \( P, H' \) (defined by (2.7)) is finite.

**Proof.** If \( H' \) were infinite, then there would be a sequence \( \{ \eta_k \} \) of nonzero measures in \( \mathcal{M}^+(\lambda) \) such that \( \lim_{k \to \infty} H'(\eta_k) = + \infty \). Let \( \{ C_n(\eta_k) \} \) be the consequent sequence of \( \eta_k \). Sets \( C_n(\eta_k) \) are only unique up to sets of \( \lambda \) measure zero. Now we shall make a definite choice of sets \( C_n(\eta_k) \) to satisfy the condition that if \( x \in C_n(\eta_k) \) then \( v_x(C_{n+1}(\eta_k)) = 1 \). This can always be accomplished by replacing the original \( C_n(\eta_k) \) by its intersection with the set \( \{ x : v_x(C_{n+1}(\eta_k)) = 1 \} \). Since \( P_1 C_n(\eta_k) = 1 \) a.e. (\( \lambda \)) on \( C_n(\eta_k) \), the intersection remains a support of \( \eta_k P^n \). Now sets \( C_n(\eta_k) \) have this property: if \( x \in C_n(\eta_k) \), then \( v_x \) is absolutely continuous to \( \eta_k P^n \). Hence, if \( x \in \bigcup_{n=1}^{\infty} C_n(\eta_k) \) then \( H'(v_x) \geq H'(\eta_k) \).

Now let \( X_k = \bigcup_{n=1}^{\infty} C_n(\eta_k) \) in the strict sense of set union. Then \( \lambda(\bigcap_{k=1}^{\infty} X_k) = 0 \) so that \( \lambda(X - \bigcap_{k=1}^{\infty} X_k) = 0 \). There must be a point \( x \in \bigcap_{k=1}^{\infty} X_k \). For this \( x, H'(v_x) \geq H'(\eta_k) \) for \( k = 1, 2, \ldots \), which is impossible since \( H'(v_x) \) is a finite integer and \( \lim_{k \to \infty} H'(\eta_k) = + \infty \).

Combining Lemmas 2.7, 2.6, we have the following:

**Theorem 2.2.** If a Markov operator \( P \) is conservative, ergodic and \( \lambda \)-continuous, then the period \( \delta \) of \( P \) is a finite positive integer and for any consequent sequence \( \{ C_n \} \) there is a positive integer \( N \) such that \( C_n \cap C_{n+\delta} \) is non-null for all \( n \geq N \).

**Theorem 2.3.** Let \( P \) be a \( \lambda \)-continuous, conservative, ergodic Markov operator. Let \( \delta \) be the period of \( P \) and \( \mu \) be a non-null invariant measure of \( P \). Let \( C_0, C_1, \ldots, C_{\delta-1} \) be the totality of distinct \( \mathcal{G}^{(\delta)} \) atoms with \( C_0 = A(C_1), C_1 = A(C_2), \ldots, C_{\delta-2} = A(C_{\delta-1}) \). Then each \( I_C \mu \) is an invariant measure of \( P^\delta \) and every invariant measure of \( P^\delta \) is of the form \( \sum_{i=0}^{\delta-1} \alpha_i I_C \eta \). Furthermore, we have \( \mu I_{C_0} P = \mu I_{C_1} P, \ldots, \mu I_{C_{\delta-2}} P = \mu I_{C_{\delta-1}}, \mu I_{C_{\delta-1}} P = \mu I_0 \) and \( \mu (C_0) = \mu (C_1) = \cdots = \mu (C_{\delta-1}) \). Hence if \( P \) has a finite invariant measure then all invariant measures of iterates of \( P \) are finite measures.

**Proof.** Since \( C_i \) is \( P^\delta \)-closed, \( I_C P^\delta = I_C P^\delta I_C \). Since \( P^\delta \) is conservative, \( X - C_i \) is \( P^\delta \)-closed. Hence \( I_{X-C_i} P^\delta I_C = 0 \) and \( P^\delta I_C = I_C P^\delta I_C + I_{X-C_i} P^\delta I_C = I_C P^\delta I_C = I_C P^\delta \). Thus we have \( \mu I_C P^\delta = \mu P^\delta I_C = \mu I_C \) and \( \mu I_C \) is \( P^\delta \)-invariant.
\[ \langle \mu I_{C_1}, f \rangle = \langle \mu I_{C_1} f \rangle = \langle \mu P, I_{C_1} f \rangle = \langle \mu I_{C_0} P, I_{C_1} f \rangle + \langle \mu I_{X-C_0} P, I_{C_1} f \rangle. \]

Since the support of \( \mu I_{C_0} P \) is \( C_0 \) and the support of \( \mu I_{X-C_0} P \) is \( X-C_1 \), we have

\[ \langle \mu I_{C_0} P, I_{C_1} f \rangle = \langle \mu I_{C_0} P, f \rangle \]

and

\[ \langle \mu I_{X-C_0} P, I_{C_1} f \rangle = 0. \]

Hence

\[ \langle \mu I_{C_1}, f \rangle = \langle \mu I_{C_0} P, f \rangle. \]

Since (2.8) is true for every \( f \in L_\alpha(\mu) \), \( \mu I_{C_0} P = \mu I_{C_1} \). By the same argument, we have \( \mu I_{C_1} P = \mu I_{C_2}, \ldots, \mu I_{C_\delta-1} P = \mu I_{C_0} \). Substituting 1 for \( f \) in (2.8) we then obtain \( \mu(C_0) = \mu(C_1) \). Similarly \( \mu(C_1) = \mu(C_2), \ldots, \mu(C_{\delta-1}) = \mu(C_0) \).

Now every \( C_i \) is also a \( \mathcal{G}(n^\beta) \) atom for every positive integer \( n \). Hence \( P^n \) acting on \( C_i \) only is conservative and ergodic. It follows that for any \( P^n \)-invariant measure \( \nu, \nu I_{C_i} \) must be a constant multiple of \( \mu I_{C_i} \). Hence \( \nu \) is of the form \( \sum_{i=0}^{\delta-1} \alpha_i I_{C_i} \).

III. Asymptotic properties of \( [p^n(x, y)] \) for a \( \lambda \)-continuous, conservative, ergodic Markov operator. All through this section, the Markov operator \( P \) is assumed to be \( \lambda \)-continuous, conservative and ergodic. Then \( P \) possesses a nontrivial \( \sigma \)-finite invariant measure \( \mu \) which is unique up to a constant multiple [8]. \( \mu \) is equivalent to \( \lambda \). Hence "a.e. (\( \lambda \))" is the same as "a.e. (\( \mu \))" and \( L_\alpha(\lambda) \) and \( L_\alpha(\mu) \) are the same space.

Lemma 3.1. If \( f \in L_q(\mu), 1 \leq q < \infty \), then \( Pf \), given by

\[ Pf = Pf^+ - Pf^- , \]

belongs to \( L_q(\mu) \) also. Furthermore, we have

\[ \| Pf \|_q \leq \| f \|_q \]

where \( \| \|_q \) denotes the \( L_q(\mu) \) norm.

Proof. We only need to prove for the case \( 1 \leq q < \infty \). For any non-negative function \( f \), by Jensen's inequality, for (\( \lambda \)) almost all \( x \)

\[ |Pf(x)|^q \leq \int p(x, y) |f(y)|^q \lambda(dy). \]

Hence

\[ \int \mu(dx) |Pf(x)|^q \leq \int \mu(dx) \left\{ \int p(x, y) |f(y)|^q \lambda(dy) \right\} \]

\[ = \int \mu(dy) |f(y)|^q. \]
Hence \( f \in L_q(\mu) \) implies that \( Pf \in L_q(\mu) \) and \( \| Pf \|_q \leq \| f \|_q \). Then for the general case that \( f \) may take on both positive and negative values and \( f \in L_q(\mu), Pf^+, Pf^- \) are in \( L_q(\mu) \) and, therefore, \( Pf \) is well defined and is in \( L_q(\mu) \). Jensen’s inequality again implies (3.1) and from which (3.2) and the equality \( \| Pf \|_q \leq \| f \|_q \) follow immediately.

**Lemma 3.2.** If \( f \) is non-negative and \( f \in L_q(\mu) \) where \( 1 \leq q \leq +\infty \), then \( \lim\inf_{n \to \infty} Pf \) is equal to a finite constant a.e. \((\lambda)\). If, in addition, the invariant measure \( \mu \) is infinite and \( q < +\infty \), then \( \lim\inf_{n \to \infty} Pf = 0 \) a.e. \((\lambda)\).

**Proof.** Since \( f \) is non-negative, we have, by Fatou’s lemma,

\[
\lim\inf_{n \to \infty} \int p(x, y) Pf(y) \lambda(dy) = \int p(x, y) \lim\inf_{n \to \infty} Pf(y) \lambda(dy).
\]

Hence \( \lim\inf_{n \to \infty} Pf \leq \int \lim\inf_{n \to \infty} Pf \), so that \( \lim\inf_{n \to \infty} Pf \) is an excessive function. (A non-negative function \( g \) is excessive if \( Pf \leq f \). For the properties of excessive functions see [8, §IV].) Since excessive functions for a conservative, ergodic Markov operator are constant functions \( \lim\inf_{n \to \infty} Pf = \) constant a.e. \((\lambda)\). Since excessive functions for a conservative, ergodic Markov operator are constant functions, \( \lim\inf_{n \to \infty} Pf = \) constant a.e. \((\lambda)\).

Now we shall proceed to study asymptotic properties of sequences \( \{P^nf\} \). We shall again, as in §II, choose a definite version of the density function \( p(x, y) \) of \( P \) to satisfy

1. \( p(x, y) \geq 0 \) for all \((x, y) \in X \times X \) and
2. \( \int p(x, y) \lambda(dy) = 1 \) for all \( x \in X \).

The iterates \( p^{(n)}(x, y) \) will be given inductively by (1.3). They also satisfy 1 and 2. For every positive integer \( n \), every \( x \in X \) and \( E \in \mathcal{E} \) define

\[
(3.3) \quad v_x^{(n)}(E) = \int_E p^{(n)}(x, y) \lambda(dy).
\]

\( v_x^{(n)} \) are probability measures and \( v_x^{(n+1)} = v_x^{(n)} P \). Since \( P \) is ergodic the union of the supports of \( v_x^{(n)}, n = 1, 2, \ldots \), is \( X \). Now for every non-negative \( f \), \( P^nf(x) \) shall be given definitely by

\[
(3.4) \quad P^nf(x) = \int v_x^{(n)}(dy) f(y) = \int p^{(n)}(x, y) f(y) \lambda(dy).
\]

Let \( f \) be a fixed non-negative function which belongs to \( L_q(\mu) \) for some \( q \) satisfying \( 1 \leq q \leq +\infty \). By Lemma 3.2 there is a non-negative number \( a \) such that

\[
\lim\inf_{n \to \infty} P^nf(x) = a
\]

for \((\lambda)\) almost all \( x \). Hence for \((\lambda)\) almost all \( x \) there is an increasing sequence \( \{n_i\} \).
(the sequence depends on \(x\)) of positive integers such that \(\lim_{i \to \infty} P^i f(x) = a\). Let \(\rho(x)\) be the supremum of all non-negative integers \(k\) with the property that there is an increasing sequence \(\{n_i\}\) of positive integers such that

\[
\lim_{i \to \infty} P^{(n_j + j)} f(x) = a \quad \text{for} \quad j = 0, \ldots, k.
\]

\(\rho(x)\) is defined for (\(\lambda\)) almost all \(x\) and \(0 \leq \rho(x) \leq +\infty\). We shall show that \(\rho(x) = +\infty\) for (\(\lambda\)) almost all \(x\).

**Lemma 3.3.** Let \(\eta\) be a probability measure, and let \(\{g_n\}\) be a sequence of \(\eta\)-integrable non-negative functions. If \(\liminf_{n \to \infty} g_n(\eta) \geq a\) a.e. (\(\eta\)) and \(\lim_{n \to \infty} \int g_n(\eta) = a\), then there is an increasing sequence \(\{n_i\}\) of positive integers such that \(\{g_n\}\) converges a.e. (\(\eta\)) to \(a\).

**Proof.** If \(a = 0\), then \(\{g_n\}\) converges to 0 in \(L_1(\eta)\), hence, there is a subsequence \(\{g'_n\}\) converging a.e. (\(\eta\)) to 0. Suppose \(a > 0\). We shall find an increasing sequence \(\{n_i\}\) of positive integers such that

\[
\eta(F_i) \leq \frac{2 + a}{2^i} \quad \text{for} \quad i \text{ sufficiently large}
\]

where

\[
F_i = \left\{ x : g_{n_i}(x) \geq a + \frac{1}{2^i} \right\}.
\]

(3.5) implies

\[
\limsup_{i \to \infty} g_{n_i} \leq a \quad \text{a.e. (\(\eta\)).}
\]

(3.6) and the fact that \(\liminf_{i \to \infty} g_{n_i} \geq a\) imply \(\lim_{i \to \infty} g_{n_i} = a\) a.e. (\(\eta\)).

Now there is an increasing sequence \(\{n_i\}\) of positive integers satisfying the following two conditions for every \(i\):

1. \(\int g_{n_i} d\eta < a + 1/4^i\),
2. \(\eta[n_{i-1}] < 1/4^i\).

Then, if \(a - 1/4^i \geq 0\), we have

\[
a + \frac{1}{4^i} > \int g_{n_i} d\eta = \int_{\{\eta_{n_i} > a + 1/4^i\}} g_{n_i} d\eta + \int_{\{\eta_{n_i} > a - 1/4^i\}} g_{n_i} d\eta + \int_{\{\eta_{n_i} < a - 1/4^i\}} g_{n_i} d\eta
\]

\[
\geq \left( a + \frac{1}{2^i} \right) \eta(F_i) + \left( a - \frac{1}{4^i} \right) \eta \left[ a + \frac{1}{2^i} > g_{n_i} > a - \frac{1}{4^i} \right]
\]

\[
\geq \left( a + \frac{1}{2^i} \right) \eta(F_i) + \left( a - \frac{1}{4^i} \right) \left\{ \eta \left[ a + \frac{1}{2^i} > g_{n_i} \right] - \frac{1}{4^i} \right\}
\]

\[
= \left( a + \frac{1}{2^i} \right) \eta(F_i) + \left( a - \frac{1}{4^i} \right) \left[ 1 - \eta(F_i) - \frac{1}{4^i} \right]
\]

\[
= \frac{2 + \frac{1}{4^i}}{\eta(F_i)} + \left( a - \frac{1}{4^i} \right) \left( 1 - \frac{1}{4^i} \right).
\]
Hence \( n(F_j) < (2 + a)/2^j \).

The following lemma is a slight improvement of Lemma 3.3. The proof is trivial.

**Lemma 3.3'.** Let \( \eta \) be a probability measure, and let \( \{g_n^{(j)}\}, j = 0, 1, \ldots, k, \) be \( k + 1 \) sequences of \( \eta \)-integrable, non-negative functions. If \( \liminf_{n \to \infty} g_n^{(j)} \geq a \) a.e. \( (\eta) \) and \( \lim_{n \to \infty} \int g_n^{(j)} d\eta = a \) for \( j = 0, 1, \ldots, k, \) then there is an increasing sequence \( \{n_i\} \) of positive integers such that \( \lim_{i \to \infty} g_{n_i}^{(j)} = a \) a.e. \( (\eta) \) for \( j = 0, 1, \ldots, k. \)

In what follows \( f \) shall be a fixed non-negative function in \( L_q(\mu) \), and \( a \) is equal to \( \liminf_{n \to \infty} P^n f \) a.e. \( (\lambda) \). Since \( P^n f, m = 1, 2, \ldots, \) are also in \( L_q(\mu) \), there is a set \( E_0 \) of 0 \( \lambda \)-measure such that, for every \( x \notin E_0 \), we have, simultaneously,

1. \( \liminf_{n \to \infty} P^n f(x) = a \),
2. \( P^n f(x) \) is finite for \( m = 1, 2, \ldots \). 2 is the same as,
2'. \( f \) is \( v_x^{(m)} \)-integrable for \( m = 1, 2, \ldots \), where \( v_x^{(m)} \) is given by (3.3).

**Lemma 3.4.** Let \( x_0 \) be a point of \( X - E_0 \). If \( \{n_i\} \) is an increasing sequence of positive integers such that \( \lim_{i \to \infty} P^{n_i+j} f(x_0) = a \) for \( j = 0, 1, \ldots, k, \) then for every positive integer \( m \) there is a subsequence \( \{n_{i_j}\} \) of \( \{n_i\} \) such that

\[
\lim_{i \to \infty} P^{n_{i_j}+j} f(x) = a \quad \text{for } j = 0, 1, \ldots, k
\]

for \( (\lambda) \) almost all \( x \) on the support of the probability measure \( v_x^{(m)} \).

**Proof.** Since

\[
P^{n_i+j} f(x_0) = \int P^{n_i-m+j} f d\nu_x^{(m)},
\]

we have

\[
\lim_{i \to \infty} \int P^{n_i-m+j} f d\nu_x^{(m)} = a \quad \text{for } j = 0, 1, \ldots, k.
\]

Since \( \liminf_{i \to \infty} P^{n_i-m+j} f \geq a \) a.e. \( (v_x^{(m)}) \), Lemma 3.3' is applicable. Hence there exists a subsequence \( \{n_{i_j}\} \) of \( \{n_i\} \) such that for \( (\lambda) \) almost all \( x \) on the support of \( v_x^{(m)} \) we have

\[
\lim_{i \to \infty} P^{n_{i_j}+j} f(x) = a \quad \text{for } j = 0, 1, \ldots, k.
\]

**Lemma 3.5.** If, for some \( x \in X - E_0 \), \( \rho(x) \geq k \), then \( \rho(x) \geq k \) for \( (\lambda) \) almost all \( x \).

**Proof.** If \( \rho(x_0) \geq k \) where \( x_0 \in X - E_0 \), then there is an increasing sequence \( \{n_i\} \) of positive integers such that \( \lim_{i \to \infty} P^{n_i+j} f(x_0) = a \) for \( j = 0, 1, \ldots, k. \) By Lemma 3.3, \( \rho(x) \geq k \) for \( (\lambda) \) almost all \( x \) belonging to the support of the measure \( v_x^{(m)} \). Let the support of \( v_x^{(m)} \) be \( C_m \). \{\{C_m\}\} is a consequent sequence. Hence \( \lambda(X - \bigcup_{m=1}^{\infty} C_m) = 0. \) Now \( \rho(x) \geq k \) for \( (\lambda) \) almost all \( x \) in \( \bigcup_{m=1}^{\infty} C_m. \) Hence the lemma is proved.
Lemma 3.6. If $P$ is aperiodic, then for every non-negative integer $k$, there is an $x_0 \in X - E_0$ for which $\rho(x_0) \geq k$.

Proof. The lemma is obviously true for $k = 0$. Suppose the lemma is true for $k$. There is an $x_0 \in X - E_0$ and an increasing sequence $\{n_i\}$ of positive integers for which

$$\lim_{i \to \infty} P^{n_i + j} f(x_0) = a \quad \text{for} \quad j = 0, 1, \ldots, k.$$ 

Let $C_m$ be the support of the measure $\nu_{x_0}^{(m)}$. $\{C_m\}$ is a consequent sequence. Since $P$ is aperiodic, by Theorem 2.2, there is a positive integer $N$ such that $C_N \cap C_{N+1}$ is non-null. By Lemma 3.4 there is a subsequence $\{n'_i\}$ of $\{n_i\}$ for which we have, simultaneously, $\lim_{i \to \infty} P^{n'_i + j} f(x) = a$, $j = 0, 1, \ldots, k$, for $(\lambda)$ almost all $x$ in $C_N$ and $\lim_{i \to \infty} P^{n'_i - N - 1 + j} f(x) = a$, $j = 0, 1, \ldots, k$ for $(\lambda)$ almost all $x$ in $C_{N+1}$. Since $C_N \cap C_{N+1}$ is non-null, there is a point $y$ in $C_N \cap C_{N+1}$ and $y \notin E_0$ such that

$$\lim_{i \to \infty} P^{n'_i - N + j} f(y) = a \quad \text{and} \quad \lim_{i \to \infty} P^{n'_i - (N+1) + j} f(y) = a$$

for $j = 0, 1, \ldots, k$. Hence we have

$$\lim_{i \to \infty} P^{n'_i - N + j} f(y) = a \quad \text{for} \quad j = 0, 1, \ldots, k + 1.$$ 

Therefore $\rho(y) \geq k + 1$ and the lemma is proved.

Lemma 3.7. If $P$ is aperiodic, then for $(\lambda)$ almost all $x$ and for every positive integer $k$, there is an increasing sequence $\{n_i\}$ of positive integers for which

$$\lim_{n \to \infty} P^{n + j} f(x) = a \quad \text{for} \quad j = 0, 1, \ldots, k.$$ 

In other words, $\rho(x) = \infty$, for $(\lambda)$ almost all $x$.

Proof. It follows from Lemma 3.5 and Lemma 3.6 that for every positive integer $k$, $\rho(x) \geq k$ for $(\lambda)$ almost all $x$. Hence $\rho(x) = \infty$ for $(\lambda)$ almost all $x$.

Lemma 3.8. If $P$ is aperiodic and $\lambda$ is a finite measure, then, for every positive number $\varepsilon$, there is a set $A$ with $\lambda(X - A) < \varepsilon$ and an increasing sequence $\{n_i\}$ of positive integers such that the sequence of functions:

$$P^{n_0} f, P^{n_1} f, P^{n_1 + 1} f, P^{n_2} f, P^{n_2 + 1} f, P^{n_3} f, P^{n_3 + 2} f, \ldots$$

converges uniformly to $a$ on $A$ where $a = \lim_{n \to \infty} P^nf$.

Proof. Let $x_0$ be a point of $X - E_0$ for which $\rho(x_0) = \infty$, and let $C_n$ be the support of $\nu_{x_0}^{(n)}$. Then $\lambda(X - \bigcup_{n=1}^{\infty} C_n) = 0$ and, hence, there is a positive integer $b$ such that $\lambda(X - \bigcup_{n=1}^{b} C_n) < \varepsilon/2$. Let $B = \bigcup_{n=1}^{b} C_n$. 

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Since \( p(x_0) = \infty \), for every positive integer \( k \), there is an increasing sequence \( \{n_i^{(k)}\} \) of positive integers such that

\[
\lim_{i \to \infty} P^{n_i^{(k)} + 1}(x_0) = a, \ldots, \lim_{i \to \infty} P^{n_i^{(k)} + k}(x_0) = a.
\]

Applying Lemma 3.4 repeatedly for \( b \) times, we obtain a subsequence \( \{m_i^{(k)}\} \) of \( \{n_i^{(k)}\} \) such that, for every integer \( m \), \( 1 \leq m \leq b \),

\[
\lim_{i \to \infty} P^{m_i^{(k)} - m + 1}(x) = a, \ldots, \lim_{i \to \infty} P^{m_i^{(k)} - m + k}(x) = a
\]

for \((\lambda)\) almost all \( x \) on \( C_m \). Let \( k \geq b \). Then

\[
\lim_{i \to \infty} P^{m_i^{(k)}}(x) = a, \quad \lim_{i \to \infty} P^{m_i^{(k)} + 1}(x) = a, \ldots, \lim_{i \to \infty} P^{m_i^{(k)} + (k - b)}(x) = a
\]

for \((\lambda)\) almost all \( x \) on \( B \). Let \( l_i^k = m_i^{(k + b)} \). Then for every non-negative integer \( k \), the sequence \( \{l_i^{(k)}\} \) has the property that

\[
\lim_{i \to \infty} P^{l_i^{(k)}}(x) = a, \quad \lim_{i \to \infty} P^{l_i^{(k)} + 1}(x) = a, \ldots, \lim_{i \to \infty} P^{l_i^{(k)} + k}(x) = a
\]

for \((\lambda)\) almost all \( x \) on \( B \). Now, for every non-negative integer \( k \), let \( n_k \) be a member of the sequence \( \{l_i^{(k)}\} \) such that

\[
\lambda \left( B \cap \left( \left| P^{n_k} f - a \right| > \frac{1}{2^k} \right) \cup \left( \left| P^{n_k + 1} f - a \right| > \frac{1}{2^k} \right) \cup \cdots \right. \\
\left. \cup \left( \left| P^{n_k + k} f - a \right| > \frac{1}{2^k} \right) \right) < \frac{1}{2^k}.
\]

Then the sequence of functions:

\[
P^{n_0} f, P^{n_1} f, P^{n_1 + 1} f, P^{n_2} f, P^{n_2 + 1} f, P^{n_2 + 2} f, \ldots
\]

converges to \( a \) a.e. \((\lambda)\) on \( B \). By Egoroff's theorem, there is a subset \( A \) of \( B \) such that \( \lambda(B - A) < \varepsilon/2 \) and the sequence (3.7) converges uniformly to \( a \) on \( A \).

The following lemma follows immediately from Lemma 3.8.

**Lemma 3.9.** If \( P \) is aperiodic, then, for every positive number \( \varepsilon \), there is an increasing sequence \( \{n_i\} \) of positive integers such that the set

\[
E = \bigcap_{i=1}^{\infty} \bigcap_{k=0}^{i} \left[ P^{n_i + k} f < a + \varepsilon \right]
\]

has positive \( \lambda \) measure.

**Lemma 3.10.** If \( E \) is a set of positive \( \lambda \) measure, then \( \lim_{n \to \infty} (I_{E'} P)^* 1 = 0 \) a.e. \((\lambda)\) where \( E' = X - E \).

**Proof.** Let \( v \in \mathcal{A}^+(\lambda) \). Then
\[ v(X) = vP^n(X) = \langle vP^n, 1 \rangle \]

\[ = \langle v \sum_{k=0}^{n-1} (I_{E,P})^k I_A, 1 \rangle + \langle (I_{E,P})^n, 1 \rangle \]

\[ = \langle v \sum_{k=0}^{n-1} (I_{E,P})^k I_A, 1 \rangle + v(I_{E,P})^n(X) \]

\[ = \langle v \sum_{k=0}^{n-1} (I_{E,P})^k I_A \rangle + v(I_{E,P})^n(X). \]

Since \( E \) is conservative and \( P \) is ergodic, we have

\[ \lim_{n \to \infty} \sum_{k=0}^{n-1} (I_{E,P})^k I_E = 1 \text{ a.e. (}\lambda) \]

Hence

(3.9) \[ \lim_{n \to \infty} v(I_{E,P})^n(X) = 0. \]

Setting \( v = v_x^{(1)} \) in (3.9), we obtain \( \lim_{n \to \infty} P(I_{E,P})^n 1(x) = 0 \). Hence \( \lim_{n \to \infty} (I_{E,P})^n = 0 \text{ a.e. (}\lambda) \).

We recall that the invariant measure \( \mu \) for \( P \) may be finite or infinite. We shall first study the case that \( \mu \) is finite. \( \mu \) is then always normalized to be a probability measure.

**Lemma 3.11.** If the invariant measure \( \mu \) of \( P \) is finite, then, for every \( v \in \mathcal{A}^+(\lambda) \), the measures \( v, vP, vP^2, \ldots \) are uniformly absolutely continuous with respect to \( \mu \).

**Proof.** Let \( Q \) be the \( \mu \)-reverse of \( P \). \( Q \) is a \( \mu \)-measurable Markov operator characterized by the following equality

\[ \int (Pg)h \, d\mu = \int g(Qh) \, d\mu \]

where \( g, h \) are non-negative functions (cf. [8, §VI]). Let \( g = dv/d\mu \), then \( Q^*g = dvP^n/d\mu \). Construct the infinite product space \( \Omega = \prod_{n=0}^{\infty} X_n \) and the product \( \sigma \)-algebra \( \mathcal{F} = \prod_{n=0}^{\infty} \mathcal{F}_n \) of subsets of \( \Omega \) where \( X_n = X, \mathcal{F}_n = \mathcal{F} \) for \( n = 0,1,2,\ldots \). A probability measure \( \mu \) on \( \mathcal{F} \) is then defined by

\[ \mu[X_0 \in A_0, X_1 \in A_1, \ldots, X_n \in A_n] \]

\[ = \int_{A_0} \mu(dx_0) \int_{A_1} \lambda(dx_1) \ldots \int_{A_n} \lambda(dx_n) p(x_0, x_1)p(x_1, x_2) \ldots p(x_{n-1}, x_n) \]

where \( A_i \in \mathcal{F} \) for \( i = 0,1,\ldots,n \). Coordinates \( X_0, X_1, \ldots \), considered as random variables defined on \( \Omega \), constitute a stationary Markov process. \( Q^*g(X_n) \) is then
the conditional expectation of \( g(X_0) \) given random variables \( X_n, X_{n+1}, \cdots \). By the well-known martingale convergence theorem \( \{Q^n g(X_n)\} \) converges in \( L_1(\mu) \) and \( Q^n g(X_n) \) are uniformly \( \mu \)-integrable. Since the process is stationary, every \( X_n \) has \( \mu \) as its distribution, hence

\[
\int_{\{Q^n g \geq \varepsilon\}} Q^n g d\mu = \int_{\{Q^n g(X_n) \geq \varepsilon\}} Q^n g(X_n) d\mu.
\]

It follows that the functions \( Q^n g \) are uniformly \( \mu \)-integrable. Hence the measures \( \nu P^n \) are uniformly absolutely continuous with respect to \( \mu \).

**Theorem 3.1.** If \( P \) is a \( \lambda \)-continuous, conservative, ergodic and aperiodic Markov operator whose invariant \( \mu \) is finite (\( \mu \) is then normalized), then, for every \( f \in L_\infty(\mu) \), \( \{P^n f\} \) converges a.e. (\( \lambda \)) to \( \int f d\mu \).

**Proof.** If the theorem is true for non-negative functions then, applying the result to \( f^+ \), \( f^- \), we obtain the same conclusion for a function \( f \) which takes on both positive and negative values. So we shall only prove the theorem for a non-negative \( f \). Let us assume \( f \neq 0 \) a.e. (\( \lambda \)).

By Lemma 3.2 \( \lim_{n \to \infty} P^n f \) is equal to a constant \( a \) a.e. (\( \mu \)). Let \( \varepsilon \) be an arbitrary positive number. By Lemma 3.9, there is an increasing sequence \( \{n_i\} \) of positive integers such that the set \( E \) given by (3.8) has positive \( \lambda \) measure. Let \( x_0 \) be an arbitrary point of \( X \) and \( \nu_{x_0}^{(m)} \) be given by (3.3). Then

\[
P^{m+n_i+i} f(x_0) = \int P^{n_i+i} f d\nu_{x_0}^{(m)}
\]

\[
= \int \left[ \sum_{k=0}^{i-1} (I_{E'} P)^k I_E P^{n_i+i-k} f + (I_{E'} P)^i P^{n_i} f \right] d\nu_{x_0}^{(m)}
\]

\[
\leq (a + \varepsilon) \int_{k=0}^{i-1} (I_{E'} P)^k I_E d\nu_{x_0}^{(m)} + \int (I_{E'} P)^i 1 d\nu_{x_0}^{(m)}
\]

\[
\leq (a + \varepsilon) + \int (I_{E'} P)^i 1 d\nu_{x_0}^{(m)}.
\]

By Lemma 3.10 \( \lim_{i \to \infty} (I_{E'} P)^i 1 = 0 \) a.e. (\( \mu \)). Hence, for every positive integer \( \delta \), there is an integer \( i_0 \) and a set \( A \) with \( \mu(X - A) < \delta \) such that \( (I_{E'} P)^{i_0} 1 < \varepsilon \) on \( A \). The number \( \delta \) is chosen to satisfy the condition that \( \nu_{x_0}^{(m)}(F) < \varepsilon \) for \( m = 1, 2, \cdots \) whenever \( \mu(F) < \delta \). This can be done because \( \nu_{x_0}^{(1)}, \nu_{x_0}^{(2)}, \cdots \) are uniformly absolutely continuous with respect to \( \mu \) (Lemma 3.11). Hence for any positive integer \( m \),

\[
P^{m+n_i+i_0} f(x_0) \leq (a + \varepsilon) + \int_A (I_{E'} P)^{i_0} 1 d\nu_{x_0}^{(m)} + \nu_{x_0}^{(m)}(X - A)
\]

\[
\leq a + 3\varepsilon.
\]

Hence we have

\[
\limsup_{n \to \infty} P^n f(x_0) \leq a + 3\varepsilon.
\]
Since $\varepsilon$ is an arbitrary positive number,

$$
\limsup_{n \to \infty} P^n f(x_0) \leq a.
$$

(3.11) holds for every $x_0 \in X$, hence $\lim_{n \to \infty} P^n f = a$ a.e. (\lambda). Since $\mu$ is the normalized invariant measure of $P$, $\int P^n f \, d\mu = \int f \, d\mu$ for $n = 1, 2, \ldots$. Now $\lim_{n \to \infty} \int P^n f \, d\mu = a$, hence $\int f \, d\mu = a$ and the proof of the theorem is then complete.

**Theorem 3.2.** If $P$ is a $\lambda$-continuous, conservative, ergodic and aperiodic Markov operator whose invariant measure $\mu$ is finite, and if $f \in L_q(\mu)$, where $1 \leq q < \infty$, then the sequence $\{P^n f\}$ converges in $L_q$ to $\int f \, d\mu$.

**Proof.** If $g \in L_\infty(\mu)$, by Theorem 3.1, $\{P^n g\}$ converges a.e. ($\mu$) to $\int g \, d\mu$. Hence $\{P^n g\}$ converges to $\int g \, d\mu$ in $L_q(\mu)$. Since $L_\infty(\mu)$ is dense in $L_q(\mu)$ in the sense of $L_q$ norm, we have, for every $f \in L_q(\mu)$ and every $\varepsilon > 0$, a $g \in L_\infty(\mu)$ such that $\|f - g\|_q < \varepsilon/2$ and $|\int f \, d\mu - \int g \, d\mu| < \varepsilon/2$. By Lemma 3.1, $\|P^n(f - g)\|_q \leq \|f - g\|_q$, hence

$$
\begin{align*}
\|P^n f - \int f \, d\mu\|_q &\leq \|P^n(f - g)\|_q + \|P^n g - \int g \, d\mu\|_q + \left|\int f \, d\mu - \int g \, d\mu\right| \\
&\leq \frac{\varepsilon}{2} + \|P^n g - \int g \, d\mu\|_q + \frac{\varepsilon}{2}.
\end{align*}
$$

Therefore $\limsup_{n \to \infty} \|P^n f - \int f \, d\mu\|_q \leq \varepsilon$ and the conclusion of the theorem follows.

**Theorem 3.3.** If $P$ is a $\lambda$-continuous, conservative, ergodic and aperiodic Markov operator whose invariant measure $\mu$ is finite and if $f \equiv 0$ and $f \in L_1(\mu)$, then $\liminf_{n \to \infty} P^n f = \int f \, d\mu$ a.e. ($\mu$).

**Proof.** Let $x$ be a fixed point of $X$ and $\psi_x^{(m)}$ be given by (3.3). Let $\varepsilon$ be an arbitrary positive number. Since $\psi_x^{(m)}$, $m = 1, 2, \ldots$, are uniformly absolutely continuous to $\mu$ by Lemma 3.11, there is a positive number $\delta$ such that $\mu(E) < \delta$ implies $\psi_x^{(m)}(E) < \varepsilon$ for $m = 1, 2, \ldots$. Now, by Theorem 3.2, $\{P^n f\}$ converges in $L_1(\mu)$ to $\int f \, d\mu$, hence there is an integer $n_0$ such that $\mu[P^n f < \int f \, d\mu - \varepsilon] < \delta$. Hence for any positive integer $m$

$$
P^{m+n_0} f(x) = \int \psi_x^{(m)}(dy) P^{n_0} f(y)
$$

$$
\leq \int_{\{P^n f < \int f \, d\mu - \varepsilon\}} \psi_x^{(m)}(dy) P^{n_0} f(y)
$$

$$
\leq \psi_x^{(m)} \left[ P^n f \geq \int f \, d\mu - \varepsilon \right] \left(\int f \, d\mu - \varepsilon\right)
$$

$$
\leq (1 - \varepsilon) \left(\int f \, d\mu - \varepsilon\right).
$$
Hence \( \liminf_{n \to \infty} P^n f(x) \geq \int f \, d\mu \). But by Fatou’s lemma

\[
\int \liminf_{n \to \infty} P^n f \, d\mu \leq \liminf_{n \to \infty} \int P^n f \, d\mu = \int f \, d\mu.
\]

Hence

\[
\liminf_{n \to \infty} P^n f = \int f \, d\mu \text{ a.e.} \ (\mu).
\]

**Theorem 3.4.** Let \( P \) and \( \mu \) be as in Theorem 3.3. Then, for (\( \lambda \)) almost all \( x \) \{\( p^{(n)}(x, \cdot) \)\} converges in \( L_1(\lambda) \) to \( d\mu/d\lambda \), and \{\( p^{(n)}(x, y) \)\} converges to \( d\mu(y)/d\lambda \) in \( L_1(\nu \times \lambda) \) for any \( \nu \in \mathcal{M}^+(\lambda) \). We also have \( \liminf_{n \to \infty} p^{(n)}(x, y) = d\mu/d\lambda(y) \) for (\( \lambda \times \lambda \)) almost all \((x, y)\).

**Proof.** Define \( \tilde{p}^{(n)}(x, y) \) by

\[
(3.12) \quad \tilde{p}^{(n)}(x, y) = p^{(n)}(x, y) \frac{d\lambda}{d\mu}(y)
\]

and \( \tilde{p}(x, y) = \tilde{p}^{(1)}(x, y) \). Then \( \tilde{p}^{(n)}(x, y) \) is the density function of \( P^n \) with respect to the invariant measure \( \mu \), and we have for (\( \mu \)) almost all \( x \)

\[
\int \tilde{p}(x, y) \mu(dy) = 1,
\]

and also for (\( \mu \)) almost all \( y \),

\[
\int \tilde{p}(x, y) \mu(dx) = 1.
\]

\( \tilde{p}(\cdot, \cdot) \) is “doubly stochastic.” Let \( Q \) be the \( \mu \)-reverse of \( P \). Then (3.10) implies that for every non-negative function \( h \)

\[
Qh(y) = \int \tilde{p}(x, y)h(x) \mu(dx).
\]

Thus, \( Q \) is \( \mu \)-continuous. Let \( q^{(n)}(x, y) \) be the density function of \( Q^n \) with respect to \( \mu \). Then

\[
q^{(n)}(x, y) = \tilde{p}^{(n)}(y, x).
\]

Since \( P \) is conservative, so is \( Q \) [5, Theorem 3.1]. Since a \( Q \)-closed set is also \( P \)-closed [8, Lemma 7.2], \( Q \) is ergodic. Since the same relationship holds between \( P^n \) and \( Q^n \) as \( P \) and \( Q \), \( Q \) is also aperiodic. Now, let \( x \) be fixed and let us consider \( \tilde{p}(x, \cdot) \) as a function of the second variable alone. Thus for (\( \mu \)) almost all \( x \), \( \tilde{p}(x, \cdot) \) is an element of \( L_1(\mu) \) with its \( \mu \)-integral equal to 1. We also have

\[
Q^n \tilde{p}(x, \cdot) = \tilde{p}^{(n+1)}(x, \cdot).
\]

Applying Theorem 3.3 to \( Q \) and \( \tilde{p}(x, \cdot) \) we have

\[
\liminf \tilde{p}^{(n)}(x, y) = 1
\]
for \((\mu \times \mu)\) almost all \((x, y)\). Hence it follows that

\[
\lim \inf p^{(n)}(x, y) = \frac{d\mu}{d\lambda}(y)
\]

for \((\lambda \times \lambda)\) almost all \((x, y)\). Furthermore, applying Theorem 3.2, we have, for \((\mu)\) almost all all \(x\), \(\{\hat{p}^{(n)}(x, \cdot)\}\) converges in \(L_1(\mu)\) to 1. Now

\[
\int |\hat{p}^{(n)}(x, y) - 1| \mu(dy) = \int \left| p^{(n)}(x, y) - \frac{d\mu}{d\lambda}(y) \right| \lambda(dy),
\]

hence \(\{p^{(n)}(x, \cdot)\}\) converges in \(L_1(\lambda)\) to \(d\mu/d\lambda\). Now, let

\[
g_n(x) = \int \left| p^{(n)}(x, y) - \frac{d\mu}{d\lambda}(y) \right| \lambda(dy) = \int |\hat{p}^{(n)}(x, y) - 1| \mu(dy),
\]

\(\{g_n(x)\}\) converges to 0 a.e. \((\mu)\). We also have

\[
g_n(x) \leq \int \hat{p}^{(n)}(x, y) \mu(dy) + 1 = 2.
\]

Hence \(\{g_n(x)\}\) converges to 0 in \(L_1(\nu)\) for any \(\nu \in \mathcal{M}^+(\lambda)\). Hence

\[
\int \int \left| p^{(n)}(x, y) - \frac{d\mu}{d\lambda}(y) \right| \lambda(dy) \nu(dx) \to 0
\]

and \(\{p^{(n)}(x, y)\}\) converges to \(d\mu(y)/d\lambda\) in \(L_1(\nu \times \lambda)\).

**Theorem 3.5.** Let \(P\) be a \(\lambda\)-continuous, conservative and ergodic Markov operator whose nontrivial invariant measure \(\mu\) is finite \((\mu\) is normalized as usual). Let the period of \(P\) be \(\delta\) and \(C_0, C_1, \ldots, C_{\delta - 1}\) be the totality of distinct \(G^{(0)}\) atoms with \(C_0 = A(C_1), \ldots, C_{\delta - 2} = A(C_{\delta - 1})\). Let \(f \in L_1(\mu)\) and \(c_0, c_1, \ldots\) be defined by

\[
c_i = \delta \int_{C_i} f \, d\mu \quad \text{for } i = 0, \ldots, \delta - 1,
\]

\[
c_i = c_j \quad \text{if } i \geq \delta, \quad 0 \leq j \leq \delta - 1.
\]

Then

1. if \(f\) also belongs to \(L_\infty(\mu)\), then for every non-negative integer \(k\) the sequence \(\{P^{n+k}f\}\) converges to \(\sum_{i=0}^{\delta-1} c_{i+k}1_{C_i}\) a.e. \((\lambda)\),

2. if \(f\) belongs to \(L_q(\mu)\) where \(1 \leq q < \infty\), then for every non-negative integer \(k\) the sequence \(\{P^{n+k}f\}\) converges in \(L_q(\mu)\) to \(\sum_{i=0}^{\delta-1} c_{i+k}1_{C_i}\),

3. if \(f \geq 0\), then for every non-negative integer \(k\),

\[
\lim \inf_{n \to \infty} P^{n+k}f = \sum_{i=0}^{\delta-1} c_{i+k}1_{C_i} \quad \text{a.e. } (\lambda).
\]
Proof. By Theorem 2.3, \( \mu I_{C_i} \) is \( P^\delta \)-invariant, \( \mu(C_i) = 1/\delta \), and \( \mu I_{C_i} P^k = \mu I_{C_{i+k} - j\delta} \) where \( j \) is the largest non-negative integer for which \( j\delta \leq i + k \). Hence

\[
\int_{C_i} P^k f \, d\mu = \int f (\mu I_{C_i} P^k) = \int f \, d\mu I_{C_{i+k} - j\delta} \, f \, d\mu = c_{i+k}.
\]

Now \( P^k \) acting on \( C_i \) is aperiodic. For any \( f \in L_\infty (\lambda) \), applying Theorem 3.1, we arrive at the conclusion that the sequence \( \{P^\delta f\} \) converges a.e. (\( \lambda \)) on \( C_i \) to the limit \( c_i = \delta \int_{C_i} f \, d\mu \). Hence the sequence converges a.e. (\( \lambda \)) to \( \sum_{i=0}^{\delta-1} c_i I_{C_i} \). Replacing \( f \) by \( P^k f \) in the sequence, we conclude that the sequence \( \{P^{n\delta+k} f\} \) converges a.e. (\( \lambda \)) to \( \sum_{i=0}^{\delta-1} d_i 1_{C_i} \) where \( d_i = \delta \int_{C_i} P^k f \, d\mu = c_{i+k} \). In a similar manner, 2 may be derived from Theorem 3.2 and 3 may be derived from Theorem 3.3.

Theorem 3.6. Let \( P \) be a \( \lambda \)-continuous, conservative and ergodic Markov operator whose nontrivial invariant measure \( \mu \) is finite (\( \mu \) is normalized as usual). Let the period of \( P \) be \( \delta \) and \( C_0, C_1, \ldots, C_{\delta-1} \) be the totality of distinct, indecomposable \( P^\delta \)-closed sets with \( C_0 = A(C_1), \ldots, C_{\delta-2} = A(C_{\delta-1}) \). For \( j > \delta - 1 \), let \( C_j = C_{j-n\delta} \) where \( n \) is the largest non-negative integer such that \( n\delta \leq j \). For every non-negative integer \( k \), define function \( g_k \) on \( X \times X \) by

\[
g_k(x, y) = \delta \sum_{i=0}^{\delta-1} 1_{C_i \times C_{i+k}}(x, y) \, \frac{d\mu}{d\lambda}(y).
\]

Then the sequence \( \{P^{n\delta+k}f(\cdot, \cdot)\} \) converges in \( L_1(\nu \times \lambda) \) to \( g_k \) for every \( \nu \in 2^X \). We also have

\[
\lim \inf \nu_{n \to \infty} P^{n\delta+k}(x, y) = g_k(x, y) \quad \text{for (} \lambda \times \lambda \text{) almost all (} x, y \text{)}.
\]

Proof. As in the proof of Theorem 3.4 we define \( \tilde{p}^{(n)}(x, y) \) by (3.12) and \( \tilde{p}(x, y) = \tilde{p}^{(1)}(x, y) \). Then for (\( \lambda \)) almost all \( x \), \( \tilde{p}^{(n)}(x, \cdot) \in L_1(\mu) \) with \( L_1(\mu) \) norm equal to 1. Furthermore, since \( C_i = A^\delta(C_{i+k}) \) we have \( P^\delta 1_{C_{i+k}} \geq 1_{C_i} \). Hence

\[
\sum_{i=0}^{\delta-1} P^k 1_{C_{i+k}} \geq \sum_{i=0}^{\delta-1} 1_{C_i}.
\]

However, equality holds in (3.13) since both sides of (3.13) are equal to 1. Hence \( P^k 1_{C_{i+k}} = 1_{C_i} \), therefore, \( P^k 1_{C_{i+k}} = 1_{C_i} P^k 1_{C_{i+k}} \) and \( 1_{C_i} P^k 1_{X-C_{i+k}} = 0 \). Thus for every \( f \in L_\infty(\lambda) \), \( P^k I_{C_{i+k}} f = I_{C_i} P^k I_{C_{i+k}} f = I_{C_i} P^k f \). In terms of the density function \( \tilde{p}^{(k)}(x, y) \), we then have

\[
1_{C_i}(x) \tilde{p}^{(k)}(x, y) = 1_{C_i}(x) \tilde{p}^{(k)}(x, y) 1_{C_{i+k}}(y) = \tilde{p}^{(k)}(x, y) 1_{C_{i+k}}(y)
\]

for (\( \lambda \times \lambda \)) almost all \( (x, y) \). Hence for (\( \lambda \)) almost all \( x \in C_i \), \( \tilde{p}^{(k)}(x, \cdot) = \tilde{p}^{(k)}(x, \cdot) 1_{C_{i+k}} \). Now we consider the \( \mu \)-reverse \( Q \) of \( P \) as in the proof of Theorem 3.4. Since a set is \( P^{\delta} \)-closed if and only if it is \( Q \)-closed, \( Q \) also has \( \delta \) as its period and \( \{C_0, \ldots, C_{\delta-1}\} \) is also the collection of all indecomposable \( Q^\delta \)-closed sets.
Applying Theorem 3.5 to $Q$ and $\bar{p}^{(k)}(x, \cdot)$ we have the sequence $\{Q^{n_k}p^{(k)}(x, \cdot)\}$ converging in $L_1(\mu)$ to $\delta \cdot 1_{C_i+k}$ for $(\lambda)$ almost all $x \in C_i$ and $\liminf_{n \to \infty} \bar{p}^{(n_k+k)}(x,y) = \delta$ for $(\lambda \times \lambda)$ almost all $(x,y) \in C_i \times C_i+k$. Hence $\liminf_{n \to \infty} \bar{p}^{(n_k+k)}(x,y) = \delta \sum_{i=0}^{\delta-1} 1_{C_i \times C_i+k}(x,y)$ and $\liminf_{n \to \infty} \bar{p}^{(n_k+k)}(x,y) = g_k(x,y)$ follows immediately for $(\lambda \times \lambda)$ almost all $(x,y)$. Moreover, if we define $h_n$ by

$$h_n(x) = \int \left| \bar{p}^{(n_k+k)}(x,y) - \delta \sum_{i=0}^{\delta-1} 1_{C_i \times C_i+k}(x,y) \right| \mu(dy),$$

then $h_n(x) \to 0$ for $(\lambda)$ almost all $x$. We also have, for $(\lambda)$ almost all $x$

$$h_n(x) \leq \int \bar{p}^{(n_k+k)}(x,y)\mu(dy) + \int \delta \sum_{i=0}^{\delta-1} 1_{C_i \times C_i+k}(x,y)\mu(dy) \leq 2.$$  

Hence for any $v \in \mathcal{M}^+(\mathcal{X})$, $\int h_n(x)v(dx) \to 0$, i.e.,

$$(3.14) \lim_{n \to \infty} \int \left| \bar{p}^{(n_k+k)}(x,y) - \delta \sum_{i=0}^{\delta-1} 1_{C_i \times C_i+k}(x,y) \right| \mu(dy)v(dx) = 0.$$  

The $L_1(\lambda \times \lambda)$ convergence of $\{p^{n_k+k}(\cdot, \cdot)\}$ to $g_k$ then follows from (3.14).

Now we turn to study the case that the invariant measure $\mu$ is infinite. We shall need the following

**Lemma 3.12.** If a set $E$ has the property that there exists an increasing sequence $\{n_k\}$ of positive integers for which the sequence of functions:

$$(3.14) P^{n_0}1_E, P^{n_1}1_E, P^{n_1+n_2}1_E, \ldots, P^{n_k}1_E, P^{n_k+n_{k+1}}1_E, \ldots, P^{n_k+k}1_E, \ldots$$

converges to 0 uniformly on $E$, then $\limsup_{n \to \infty} P^n1_E = 0$ a.e. $(\lambda)$.

**Proof.** Let $\varepsilon$ be an arbitrary positive number. Then there is a positive integer $k_1$ such that $P^{n_k+1}1_E$ and all the terms in the sequence (3.14) which follow $P^{n_k+1}1_E$ are $< \varepsilon$ on $E$. Let $k_2$ be an integer such that $k_2 > n_k$. Then $n_{k_2} > n_k$, hence

$$P^{n_{k_2}1_E} < \varepsilon, \quad P^{n_{k_2}+n_k}1_E < \varepsilon \quad \text{on } E.$$  

Let $k_3 > n_{k_1} + n_{k_2}$, then $n_{k_3} > n_{k_1}$ and

$$P^{n_{k_3}1_E} < \varepsilon, \quad P^{n_{k_3}+n_{k_1}+n_{k_2}+n_{k_2}+n_{k_1}}1_E < \varepsilon \quad \text{on } E,$$  

\[ \vdots \text{etc.} \]  

In this manner, we obtain a sequence $\{n_{k_i}\}$ of positive integers. We shall rename it $\{m_i\}$. This sequence has the property that, for every positive integer $i$,

$$(3.15) P^{m_1}1_E < \varepsilon, \quad P^{m_1+m_2+1}1_E < \varepsilon, \quad P^{m_1+m_2+\ldots+m_{i-1}+m_i}1_E < \varepsilon \quad \text{on } E.$$  

Now suppose $\limsup_{n \to \infty} P^n1_E$ is not equal to 0 a.e. $(\lambda)$. Then $\liminf_{n \to \infty} P^n1_{E'}$ is not equal to 1 a.e. $(\lambda)$ where $E' = X - E$. Since, by Lemma 3.2, $\liminf_{n \to \infty} P^n1_{E'}$ is a constant function, $\liminf_{n \to \infty} P^n1_{E'} = a$ a.e.$(\lambda)$ for some $a < 1$. Let $b = 1 - a$
and \( \varepsilon < b/2 \). Let \( i_0 \) be an integer such that \( i_0(b - 2\varepsilon) > 1 \). By Lemma 3.8, there is a point \( x \) of \( X \) and a positive integer \( N \) such that

\[
P^{N+1}_E(x) < a + \varepsilon, \quad P^{N+m_1+\cdots+m_{i_0}}_E(x) < a + \varepsilon.
\]

Then

(3.16) \( P^{N+1}_E(x) > b - \varepsilon, \quad P^{N+m_1+\cdots+m_{i_0}}_E(x) > b - \varepsilon. \)

Now let

\[
p_1(x, y) = \int_E p^{(N)}(x, y_1) p^{(m_1)}(y_1, y) \lambda(dy_1);
\]

\[
p_2(x, y) = \int_E \int_E p^{(N)}(x, y_1) p^{(m_1)}(y_1, y_2) p^{(m_2)}(y_2, y) \lambda(dy_1) \lambda(dy_2),
\]

\[
p_{i_0}(x, y) = \int_E \cdots \int_E p^{(N)}(x, y_1) p^{(m_1)}(y_1, y_2) \cdots p^{(m_{i_0})}(y_{i_0}, y) \lambda(dy_1) \lambda(dy_2) \cdots \lambda(dy_{i_0}),
\]

and

\[
K_0(x, E) = P^{N+1}_E(x),
\]

\[
K_1(x, E) = \int_E p_1(x, y) \lambda(dy) = P^N_E P^{m_1}_E(x),
\]

\[
K_{i_0}(x, E) = \int_E p_{i_0}(x, y) \lambda(dy) = P^N_E P^{m_1}_E \cdots P^{m_{i_0-1}}_E P^{m_{i_0}}_E(x).
\]

Then

(3.17) \( K_0(x, E) + K_1(x, E) + \cdots + K_{i_0}(x, E) \leq 1. \)

(3.17) may be proved by an elementary method similar to the one used in the proof of Lemma 6.1 of [8], or by constructing the infinite product space \( \Omega \) and the infinite product \( \sigma \)-algebra \( \mathcal{F} \) as in the proof of Lemma 3.11 and then defining a probability measure \( \eta \) on \( \mathcal{F} \) by

\[
\eta[X_1 \in A_1, \ldots, X_n \in A_n]
\]

\[
= \int_{A_1} \cdots \int_{A_n} p(x, x_1)(x_1, x_2) \cdots p(x_{n-1}, x_n) \lambda(dx_1) \cdots \lambda(dx_n).
\]

Then the left-hand side of (3.17) is

\[
\eta[X_n \in E \text{ for some } n \text{ equal to one of } N, N + m_1, \ldots, N + m_1 + \cdots + m_{i_0}].
\]

Now, for \( 1 \leq k \leq i_0 \)
Applying (3.15), we have
\[ P^{N+m_1+\ldots+m_k}1_E(x) \leq [K_0(x,E) + \cdots + K_{k-1}(x,E)]\varepsilon + K_k(x,E) \]
\[ \leq \varepsilon + K_k(x,E). \]
Hence \( K_k(x,E) \geq b - 2\varepsilon \) by (3.16). Thus we obtain the inequality
\[ K_1(x,E) + \cdots + K_{i_0}(x,E) \geq i_0(b - 2\varepsilon) > 1 \]
which contradicts (3.17). Thus the conclusion of Lemma 3.12 is proved.

**Theorem 3.7.** If \( P \) is a \( \lambda \)-continuous, conservative, ergodic and aperiodic Markov operator whose invariant measure \( \mu \) is infinite, and if \( E \) is a set of finite \( \mu \) measure, then, for every positive number \( \varepsilon \), there is a set \( E_\varepsilon \subseteq E \) such that
\[ \mu(E_\varepsilon) < \varepsilon \text{ and } \lim_{n \to \infty} P^n1_{E_\varepsilon^c} = 0 \text{ a.e. (}\lambda\text{).} \]

**Proof.** Since \( E \) is a set of finite \( \mu \) measure, \( \liminf_{n \to \infty} P^n1_E = 0 \) a.e. (\( \lambda \)) by Lemma 3.2. By Lemma 3.8, for any positive number \( \delta \) there is a set \( A \) with \( \lambda(X - A) < \delta \) and an increasing sequence \( \{n_i\} \) of positive integers such that the sequence of functions:
\[ (3.18) \quad P^{n_0}1_E, P^{n_1}1_E, P^{n_1+1}1_E, P^{n_2}1_E, P^{n_2+1}1_E, P^{n_3}1_E, \ldots \]
converges to 0 uniformly on \( A \). We choose \( \delta \) to satisfy the condition that \( \mu(E \cap B) < \varepsilon \) whenever \( \lambda(B) < \delta \). This is possible because \( \mu_E \) is absolutely continuous to \( \lambda \). Take \( E_\varepsilon \) to be \( E - A \), then the sequence (3.18) converges to 0 uniformly on \( E - E_\varepsilon \). Since \( 1_{E_\varepsilon^c} \leq 1_E \), the sequence of functions:
\[ P^{n_0}1_{E_\varepsilon^c}, P^{n_1}1_{E_\varepsilon^c}, P^{n_1+1}1_{E_\varepsilon^c}, P^{n_2}1_{E_\varepsilon^c}, P^{n_2+1}1_{E_\varepsilon^c}, P^{n_3}1_{E_\varepsilon^c}, \ldots \]
converges to 0 uniformly on \( E - E_\varepsilon \). Applying Lemma 3.12, we have
\[ \lim_{n \to \infty} P^n1_{E_\varepsilon^c} = 0 \text{ a.e. (}\lambda\text{).} \]

**Theorem 3.8.** If \( P \) is a \( \lambda \)-continuous, conservative, ergodic and aperiodic Markov operator whose invariant measure \( \mu \) is infinite, then there is an increasing sequence \( \{E_k\} \) of sets such that \( \bigcup_{k=1}^\infty E_k = X \) and \( \lim_{n \to \infty} P^n1_{E_k} = 0 \) a.e. (\( \lambda \)) for every \( k \).

**Proof.** Since \( \mu \) is \( \sigma \)-finite, there exists an increasing sequence \( \{F_k\} \) of sets such that \( \bigcup_{k=1}^\infty F_k = X \) and \( \mu(F_k) < \infty \) for every \( k \). By Theorem 3.7, for each \( k \), there is a set \( E_k \subseteq F_k \) such that \( \mu(F_k - E_k) < 1/2^k \) and \( \lim_{n \to \infty} P^n1_{E_k} = 0 \) a.e. (\( \lambda \)). We may assume the sequence \( \{E_k\} \) to be monotonically increasing. Then
\[ \mu\left(X - \bigcup_{k=1}^{\infty} E_k\right) = \mu\left(X - \bigcup_{k=1}^{N} E_k\right) \leq \mu\left[ \bigcup_{k=1}^{\infty} (F_k - E_k)\right] \leq \frac{1}{2^{N-1}}. \]

Hence \( \mu(X - \bigcup_{k=1}^{\infty} E_k) = 0 \) and the theorem is proved.

**Theorem 3.9.** Let \( P \) be a \( \lambda \)-continuous, conservative and ergodic Markov operator whose invariant measure \( \mu \) is infinite, then there is an increasing sequence \( \{E_k\} \) of sets such that

\[ \bigcup_{k=1}^{\infty} E_k = X \text{ and } \lim_{n \to \infty} P^n 1_{E_k} = 0 \text{ a.e. (}\lambda\text{)} \text{ for } k = 1, 2, \cdots. \]

**Proof.** Since \( P \) has a finite period \( \delta \), the space \( X \) is partitioned into \( \delta \) sets: \( C_0, C_1, \cdots, C_{\delta-1} \), of which each is a \( \mathcal{G}^{(\delta)} \) atom. Then \( P^\delta \), acting on \( C_i \) alone, is aperiodic and has \( \mu|C_i \) as its invariant measure. By Theorem 2.3, \( \mu|C_i \) is also infinite. Applying Theorem 3.8, we obtain an increasing sequence \( \{E_{i,k}, k = 1, 2, \cdots\} \) of sets such that \( C_i = \bigcup_{k=1}^{\infty} E_{i,k} \) and \( \lim_{n \to \infty} P^{n\delta} 1_{E_{i,k}} = 0 \), a.e. (\( \lambda \)). Let \( E_k = \bigcup_{i=0}^{\delta-1} E_{i,k} \). Then \( \{E_k\} \) is an increasing sequence of sets such that \( X = \bigcup_{k=1}^{\infty} E_k \) and \( \lim_{n \to \infty} P^{n\delta} 1_{E_k} = 0 \) a.e. (\( \lambda \)) for every \( k \). Now

\[ P^{n\delta+i} 1_{E_k}(x) = \int P^{n\delta} 1_{E_k} d\lambda(x), \]

hence \( \lim_{n \to \infty} P^{n\delta+i} 1_{E_k} = 0 \) a.e. (\( \lambda \)) for \( i = 0, 1, \cdots, \delta - 1 \) and the conclusion of the theorem follows immediately.

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