λ-CONTINUOUS MARKOV CHAINS. II(1)

BY
SHU-TEH C. MOY

Summary. Continuing the investigation in [8] we study a λ-continuous Markov operator P. It is shown that, if P is conservative and ergodic, P is indeed "periodic" as is the case when the state space is discrete; there is a positive integer δ, called the period of P, such that the state space may be decomposed into δ cyclically moving sets $C_0, \cdots, C_{\delta-1}$ and, for every positive integer n, $P^{n\delta}$ acting on each C_i alone is ergodic. It is also shown that P maps $L_q(\mu)$ into $L_q(\mu)$ where μ is the nontrivial invariant measure of P and $1 \leqq q \leqq \infty$. If μ is finite and normalized then it is shown that (1) if $f \in L_\infty(\lambda)$, then $\{P^{n\delta+k}f\}$ converges a.e. (λ) to $g_k = \sum_{i=0}^{\delta-1} c_{i+k} 1_{C_i}$ where $c_j = \delta \int_{C_j} f\, d\mu$ if $0 \leqq j \leqq \delta - 1$ and $c_j = c_i$ if $j = m\delta + i$, $0 \leqq i \leqq \delta - 1$, (2) $\{P^{n\delta+k}f\}$ converges in $L_q(\mu)$ to g_k if $f \in L_q(\mu)$, and (3) $\liminf_{n\to\infty} P^{n\delta+k} f = g_k$ a.e. (λ) if $f \in L_1(\mu)$ and $f \geqq 0$. If μ is infinite, then it is shown that (1) if $f \geqq 0$, $f \in L_q(\mu)$ for some $1 \leqq q < \infty$, then $\liminf_{n\to\infty} P^n f = 0$ a.e. (λ), (2) there exists a sequence $\{E_k\}$ of sets such that $X = \bigcup_{k=1}^{\infty} E_k$ and $\lim_{n\to\infty} P^{n\delta+i} 1_{E_k} = 0$ a.e. (λ) for $i = 0, 1, \cdots, \delta - 1$ and $k = 1, 2, \cdots$.

I. **Introduction.** Let X be a nonempty set, $\mathscr{X}$, a σ-algebra of subsets of X and λ, a σ-finite measure on $\mathscr{X}$. Let $p(x, y)$ be an $\mathscr{X} \times \mathscr{X}$ measurable function defined on $X \times X$ satisfying the following conditions:

1. $p(x, y) \geqq 0$ for ($\lambda \times \lambda$) almost all (x, y),
2. $\int p(x, y)\lambda(dy) \leqq 1$ for (λ) almost all x.

Let $L_\infty(\lambda)$ be the collection of all λ-essentially bounded functions and $\mathscr{A}(\lambda)$, the collection of all finite, real-valued, countably additive functions on $\mathscr{X}$ which are absolutely continuous with respect to λ. Let $\mathscr{A}^+(\lambda)$ be the collection of all nonnegative elements of $\mathscr{A}(\lambda)$. For any $f \in L_\infty(\lambda)$, Pf is defined by

$$(1.1) \qquad Pf(x) = \int p(x, y) f(y) \lambda(dy),$$

and for any $v \in \mathscr{A}(\lambda)$, vP is defined by

$$(1.2) \qquad vP(A) = \int v(dx) \int_A p(x, y)\lambda(dy).$$

The operator P here is a special kind of λ-measurable Markov operator of E,

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Hopf [7]. We call it a *λ-continuous Markov operator.* (1.1), (1.2) remain meaningful for non-negative f not necessarily λ-essentially bounded and non-negative σ-finite measure ν. The iterates P^n of P are then given by

$$P^n f(x) = \int p^{(n)}(x,y) f(y) \lambda(dy)$$

and

$$\nu P^n(A) = \int \nu(dx) \int_A p^{(n)}(x,y) \lambda(dy)$$

where $p^{(n)}(x,y)$ are defined inductively by

$$p^n(x,y) = \int p^{(n-1)}(x,z) p(z,y) \lambda(dz). \tag{1.3}$$

The function $p(\cdot, \cdot)$ is called the *density function* of P with respect to λ and is only uniquely determined by P a.e. $(\lambda \times \lambda)$. All subsets of X discussed in this paper are elements of $\mathscr{X}$ and all functions on X are $\mathscr{X}$-measurable functions. Unless otherwise indicated, for two sets A, B, $A \subset B$, $A = B$ means that $\lambda(A - B) = 0$, $\lambda(A \triangle B) = 0$ respectively, and for two functions f, g on X, $f = g$, $f \leqq g$ means that the equality and the inequality, respectively, are satisfied except on a λ-null set. For any set A, 1_A is to represent the function which equals 1 on A and 0 on the complement A' of A. I_A is the λ-measurable Markov operator defined by

$$I_A f(x) = 1_A(x) f(x),$$
$$\nu I_A(B) = \nu(A \cap B).$$

For any set E, define P_E by

$$P_E = \sum_{n=0}^{\infty} P(I_{E'} P)^n \tag{1.4}$$

where E' is the complement of E. P_E operating on either non-negative functions or measures has well-defined meanings (cf. [8, §VI]). For a measure ν, and a function f we shall use the symbol $\langle \nu, f \rangle$ to denote the integral $\int f \, d\nu$. For any $\nu \in \mathscr{A}^+(\lambda)$, the *support* of ν is the set A such that $\nu(X - A) = 0$, and $B \subset A$ with B being λ non-null implies that $\nu(B) > 0$, "non-null" and "null" shall mean λ-non-null and λ-null respectively.

Following E. Hopf and J. Feldman we call a set A a *conservative* set if for every non-null subset B of A, $P_B 1_B = 1$ on B. The largest conservative set C is called the *conservative part* of X. $D = X - C$ is called the *dissipative part* of X. P is *conservative* if $X = C$, dissipative if $X = D$. We say that a set A is *closed* if $P1_A = 1$ on A. The collection of all closed subsets of C is a σ-algebra of subsets of C which we shall denote by $\mathscr{C}$. An element A of $\mathscr{C}$ is *indecomposable* if A is non-null and if the only closed subsets of A are null sets and A itself. A conservative operator P is *ergodic* if X is indecomposable or, equivalently, if the only elements of $\mathscr{C}$ are X and the null set. In [8] it has been shown that, for a conservative λ-continuous

Markov operator P, the space X may be decomposed into at most countably many indecomposably closed sets $C_1, C_2, \cdots$, and that to each C_i there is a non-trivial σ-finite P-invariant measure μ_i which is equivalent to λI_{C_i}, and every P-invariant measure is of the form $\sum \alpha_i \mu_i$. Thus, if we consider P acting on each C_i only, P is ergodic. In [8] we studied the convergence properties of the sequence $\{\sum_{n=1}^{N} p^n(z,x)/\sum_{n=1}^{N} p^n(z,y)\}$. It was proved that, for an ergodic conservative P, the sequence converges to the limit $f(x)/f(y)$ where f is the derivative of an invariant measure with respect to λ. In this paper we shall proceed further to study the asymptotic behavior of sequences $\{p^{(n)}(x,y)\}$ and $\{P^n f\}$. As we know that $\sum_{n=0}^{\infty} p^n(x,y)$ converges on $X \times D$, therefore, $\lim_{n\to\infty} p^n(x,y) = 0$ on $X \times D$. The limiting behavior of $\{p^n(x,y)\}$ is relatively simple on the dissipative part. Thus we shall concentrate on conservative Markov operators.

It is well known that, if X is discrete and if P is conservative and ergodic, then X may be partitioned into a finite number δ of cyclically moving sets where δ is the period of P, and $\{p^{n\delta}(x,y)\}$ converges as $n \to \infty$ [1]. Thus in §II, a theory of periods is developed for a λ-measurable, conservative and ergodic Markov operator. Much of the work here is inspired by the pioneer work of W. Doeblin. The theory of periods of a conservative ergodic λ-measurable Markov operator given here is modeled after Doeblin's (which was perfected and completed by Chung [2]). Owing to the good manner in which the collection of all closed subsets conducts itself, the theory takes a much simpler form here than Doeblin's original. In §III, asymptotic properties of $\{p^n(x,y)\}$ and $\{P^n f\}$ are studied. The device $\rho(x)$ used here is similar to Doeblin's. Two very different cases arise as expected; namely, the case that the nontrivial invariant measure μ of P is finite and the case that μ is infinite. Theorems concerning the a.e. (λ) convergence of $\{P^n f\}$ when $f \in L_\infty(\lambda)$ may be considered as generalizations of convergence theorems of $\{P_{ij}^{(n)}\}$ of discrete state spaces. Theorems concerning the $L_q(\mu)$ convergence of $\{P^n f\}$ when $f \in L_q(\mu)$ are new even for discrete state spaces. There remains the open question whether there is also a.e.(λ) convergence for $\{P^n f\}$ when $f \in L_q(\mu)$ and P is aperiodic. I am only able to show that, if f is non-negative $\liminf_{n\to\infty} P^n f$ is equal to the $L_q(\mu)$ limit a.e. (λ) for the case of a finite μ, and $\liminf_{n\to\infty} P^n f = 0$ a.e. (λ) for the case of an infinite μ. Some results for the case that μ is finite are similar to those of S. Orey [9] which is based on a hypothesis of T. E. Harris.
I am indebted to K. L. Chung who introduced me to Doeblin's work.

II. **Periods of λ-measurable conservative ergodic Markov operators.** We recall that the properties of a set in $\mathscr{X}$ being transient, conservative, closed, etc., were defined with reference to a λ-measurable Markov operator P. If there are more than one Markov operator these terminologies will be prefixed by "P-" or "Q-" to distinguish that the properties are referred to operator P or Q respectively. In this section attention will be paid mainly to iterations P^k of P.

LEMMA 2.1. *Let k be a positive integer. Then, a set R is P-conservative if and* *only if R is P^k-conservative; it follows that, if P is conservative, so is P^k and vice versa.*

Proof. A non-null set R is P-conservative if and only if, for every non-null set $S \subset R$, $\sum_{n=0}^{\infty} P^n 1_S$ is unbounded [5]. Since $\sum_{n=0}^{\infty} P^{nk} 1_S \leqq \sum_{n=0}^{\infty} P^n 1_S$, R is P-conservative if R is P^k-conservative. Conversely, if a non-null set R is not P^k-conservative, then there is a non-null subset S of R for which $\sum_{n=0}^{\infty} P^{nk} 1_S$ is bounded. It follows that $\sum_{n=0}^{\infty} P^{nk+r} 1_S = P^r \sum_{n=0}^{\infty} P^{nk} 1_S$ is bounded so that $\sum_{n=0}^{\infty} p^n 1_S = \sum_{r=0}^{k-1} p^r \sum_{n=0}^{\infty} P^{nk} 1_S$ is also bounded. Hence R is also not P-conservative.

All through §II we shall assume that P is conservative and ergodic. *A* P^k-closed set E is said to be *P^k-decomposable* if and only if there is a non-null P^k-closed subset B of E such that $E - B$ is also non-null. Since P^k is conservative, the collection of all P^k-closed sets is a σ-algebra; $C - B$ is then also P^k-closed. *A* P^k-closed set is *P^k-indecomposable* if it is not P^k-decomposable. Since P is assumed to be ergodic, X is P-indecomposable. In this section we shall study the decomposability of X under iterates of P. For an arbitrary set E we denote the set $[P^k 1_E = 1]$ by $A^k(E)$:

$$A^k(E) = [P^k 1_E = 1]. \tag{2.1}$$

Then E is P^k-closed if and only if $E \subset A^k(E)$. It is easy to see that

1. $A^k(E_1) \subset A^k(E_2)$ if $E_1 \subset E_2$,
2. $A^k(E_1) \cap A^k(E_2)$ is null if $E_1 \cap E_2$ is null,
3. if $\{E_n\}$ is a finite or infinite sequence of sets, then

$$\bigcup_n A^k(E_n) \subset A^k\left(\bigcup_n E_n\right).$$

Denote $A^1(E)$ by $A(E)$, then we have

$$A^2(E) = A(A(E)),\ A^3(E) = A(A^2(E)), \cdots.$$

Lemma 2.2. *If E is P^k-closed, then $A(E)$ is also and $A(E)$ is P^k-decomposable or P^k-indecomposable according as E is P^k-decomposable or P^k-indecomposable. It follows that the lemma remains valid if we replace $A(E)$ by $A^j(E)$ where j is an arbitrary positive integer.*

Proof. If E is P^k-closed then $E \subset A^k(E)$. Hence $A(E) \subset A(A^k(E)) = (A^k(A(E))$ and $A(E)$ is P^k-closed. If E is P^k-decomposable, $E = B \cup C$ where B and C are non-null, disjoint and P^k-closed, then $A(B)$ and $A(C)$ are P^k-closed and disjoint. $A(B)$ and $A(C)$ are non-null because $A^k(B)$ and $A^k(C)$ are non-null. Hence $A(E)$ is also P^k-indecomposable.

Now suppose that E is P^k-indecomposable, we shall show that $A(E)$ is also P^k-indecomposable. Let F be a non-null P^k-closed subset of $A(E)$, we shall first show $A^{k-1}(F) \cap E$ is non-null. We have

$$P^k 1_F = PI_E P^{k-1} 1_F + PI_{E'} P^{k-1} 1_F.$$

Since $F \subset A(E)$, $P1_{E'} = 0$ on F, hence $PI_{E'}P^{k-1}1_F = 0$ on F. Hence we have

$$P^k 1_F = PI_E P^{k-1} 1_F = 1 \text{ on } F. \tag{2.2}$$

Since $P1 = 1$, it follows that if $f > 0$ a.e. (λ) we also have $Pf > 0$ a.e. (λ). Now $1 - I_E P^{k-1} 1_F$ is a non-negative function. If the set $[I_E P^{k-1} 1_F = 1]$ is null then $P[1 - I_E P^{k-1} 1_F] = 1 - PI_E P^{k-1} 1_F > 0$ a.e.(λ) which contradicts (2.2). Hence $[I_E P^{k-1} 1_F = 1]$ is non-null, i.e., $E \cap A^{k-1}(F)$ is non-null. Now suppose $A(E)$ were P^k-decomposable and F_1, F_2 were two disjoint non-null P^k-closed subsets of $A(E)$ then $E \cap A^{k-1}(F_1)$ and $E \cap A^{k-1}(F_2)$ would be two non-null, disjoint, P^k-closed subsets of E which is clearly impossible. Hence $A(E)$ is also P^k-indecomposable.

LEMMA 2.3. *If P is conservative and ergodic, and if $C_1, \cdots, C_n$ are P^k-closed, non-null and pairwise disjoint then $n \leqq k$.*

Proof. Let $G_m = \bigcup_{i=0}^{k-1} A^i(C_m)$, then

$$A(G_m) \supset \bigcup_{i=0}^{k-1} A^{i+1}(C_m) \supset G_m,$$

hence each G_m is P-closed. $G_m = X$ for $m = 1, \cdots, n$. Hence

$$X = \bigcap_{m=1}^{n} G_m = \bigcup_{(i_1, \ldots, i_n)} [A^{i_1}(C_1) \cap A^{i_2}(C_2) \cap \cdots \cap A^{i_n}(C_n)]. \tag{2.3}$$

Where the union appearing in the right-hand side of (2.3) is taken over all n-tuple $(i_1, \cdots, i_n)$ where i_j may be $1, 2, \cdots, k$. There is at least one n-tuple $(i_1, i_2, \cdots, i_n)$ for which $A^{i_1}(C_1) \cap \cdots \cap A^{i_n}(C_n)$ is non-null. Then $i_1, i_2, \cdots, i_n$ are all distinct, for $i_j = i_l$ would imply that $A^{i_j}(C_j) \cap A^{i_l}(C_l)$ is null. Hence $n \leqq k$.

LEMMA 2.4. *Let P be conservative and ergodic and k be a positive integer. Let $\mathscr{C}^{(k)}$ be the σ-algebra of P^k-closed subsets of X. Then $\mathscr{C}^{(k)}$ is generated by a finite number $\delta = \delta(k)$ of distinct atoms with δ dividing k. Each atom in $\mathscr{C}^{(k)}$ is also P^δ-indecomposably closed. It follows that $\mathscr{C}^{(k)}$ is identical with the σ-algebra $\mathscr{C}^{(\delta)}$ of all P^δ-closed sets.*

Proof. By Lemma 2.3 $\mathscr{C}^{(k)}$ must be generated by a finite number of atoms. Let C_1 be an atom of $\mathscr{C}^{(k)}$. C_1 is a P^k-indecomposable closed set. Let $C_2 = A(C_1)$, $C_3 = A(C_2), \cdots$. By Lemma 2.2 every C_i is also P^k- indecomposably closed. Hence, if $i \neq j$ we have either $C_i \cap C_j$ null or $C_i = C_j$. Since C_i is P^k-closed, $C_i \subset A^k(C_i) = C_{i+k}$. Hence $C_i = C_{i+k} = C_{i+2k} = \cdots$. It then follows that if d is a positive integer for which there is an i such that $C_i = C_{i+d}$, then $C_i = C_{i+d}$ for every positive integer i. Let δ be the smallest of all positive integers d for which

$C_1 = C_{1+d}$. Clearly $\delta \leqq k$. δ must divide k for, if otherwise, then $k = n\delta + r$ where r is a positive integer $< d$, $C_{1+n\delta} = C_1 = C_{1+n\delta+r}$, hence $C_1 = C_{1+r}$ which contradicts the defining property of δ. Now for every i, $C_i = C_{i+\delta} = A^\delta(C_i)$, hence every C_i is P^δ-closed. Each C_i is also P^δ-indecomposable since it is P^k-indecomposable. $C_1, C_2, \cdots, C_\delta$ are all distinct. $\bigcup_{i=1}^{\delta} C_i$ is P-closed, therefore is equal to X. $\{C_1, C_2, \cdots, C_\delta\}$ consists of all atoms of $\mathscr{C}^{(k)}$ and also of $\mathscr{C}^{(\delta)}$. Hence $\mathscr{C}^{(k)} = \mathscr{C}^{(\delta)}$.

LEMMA 2.5. *For any positive integer k, let $\delta(k)$ be the positive integer of Lemma 2.4. Then, if k_1, k_2 are two positive integers such that k_1 divides k_2, then $\delta(k_1)$ is equal to the greatest common divisor d of k_1 and $\delta(k_2)$.*

Proof. By Lemma 2.4 $\delta(k_1)$ divides k_1. We shall show that $\delta(k_1)$ also divides $\delta(k_2)$. Then it follows that $\delta(k_1)$ divides d. Let C_1 be an atom of $\mathscr{C}^{(k_1)}$. $C_2 = A(C_1)$, $C_2 = A^2(C_1), \cdots$. Then $C_1, \cdots, C_{\delta(k_1)}$ are the totality of distinct atoms of $\mathscr{C}^{(k_1)}$. Let us consider P^{k_1} acting on C_1 only. It is ergodic, conservative and $P^{k_2} = (P^{k_1})^l$ where $l = k_2/k_1$. By Lemma 2.4 C_1 is decomposed into $B_1, \cdots, B_j$, P^{k_2}-indecomposable sets with $B_2 = A^{k_1}(B_1)$, $B_3 = A^{k_1}(B_2), \cdots$. Then each C_i is decomposed into j P^{k_2}-closed sets $A^{i-1}(B_1), \cdots, A^{i-1}(B_j)$. Hence $\mathscr{C}^{(k_2)}$ has a totality of $j \cdot \delta(k_1)$ distinct atoms, i.e., $\delta(k_2) = j \cdot \delta(k_1)$. To prove that d divides $\delta(k_1)$, let D_1 be a P^{k_2}-indecomposable set. Let $D_2 = A(D_1)$, $D_3 = A(D_2), \cdots$, then $D_1, \cdots, D_{\delta(k_2)}$ are all distinct whereas $D_{n\delta(k_2)+i} = D_i$ for every couple of positive integers n, i. Let $q = \delta(k_2)/d$. Let $E_i = \bigcup_{n=0}^{q-1} D_{nd+i}$. Then $A^d(E_i) = E_i$ so that E_i is P^d-closed. Since d divides k_1, E_i is also P^{k_1}-closed. $E_1, \cdots, E_d$ are all distinct, $A(E_i) = E_{i+1}$ and $X = \bigcup_{i=1}^{d} E_i$. If E_1 is P^{k_1}-indecomposable, so are all other E_i. If E_1 is P^{k_1}-decomposable so are all other E_i and they may be decomposed into a same number of P^{k_1}-indecomposable sets. Hence d divides $\delta(k_1)$. Since we have already proved the fact that $\delta(k_1)$ divides d, $d = \delta(k_1)$.

For a λ-measurable conservative ergodic Markov operator P we define the period δ of P by

$$\delta = \sup[\delta(k),\ k = 1, 2, \cdots]. \tag{2.4}$$

The period δ of P may or may not be finite. If $\delta = 1$, P is said to be *aperiodic*. An aperiodic Markov operator is characterized by the property that all iterates of P are ergodic. If the period δ of a Markov operator P is finite then the restriction of P^δ to each P^δ-indecomposable set is aperiodic. It is well known that if the state space X is discrete then every conservative ergodic Markov operator has a finite period.

A sequence $\{C_n\}$ of sets in X shall be called a *consequent sequence* if C_1 is non-null and $C_n \subset A(C_{n+1})$ for $n = 1, 2, \cdots$. Then all sets in the sequence are non-null. If E is a P^k-indecomposable closed set and $d = \delta(k)$ then

$$\{E, A^{d-1}(E),\ A^{d-2}(E), \cdots, E,\ A^{d-1}(E),\ A^{d-2}(E), \cdots, E, \cdots\}$$

is a consequent sequence. For a consequent sequence $\{C_n\}$ we have $C_n \subset \bigcup_{m=n+1}^{\infty} C_m$ for $n = 1, 2, \cdots$ since $\bigcup_{m=n+1}^{\infty} C_m$ is closed and, therefore, $\bigcup_{m=n+1}^{\infty} C_m = X$. Hence for each C_n, there is a C_m with $m > n$ such that $C_n \cap C_m$ is non-null (and therefore $C_{n+k} \cap C_{m+k}$ is non-null for every positive integer k since $C_n \cap C_m \subset A^k(C_{n+k} \cap C_{m+k})$). To each $\nu \in \mathscr{A}^+(\lambda)$, $\nu \neq 0$, we may attach a consequent sequence $\{C_n(\nu)\}$ where $C_1(\nu) = \operatorname{supp} \nu$, $C_2(\nu) = \operatorname{supp} \nu P$, $C_3(\nu) = \operatorname{supp} \nu P^2, \cdots$. If η is absolutely continuous to ν then $C_n(\eta) \subset C_n(\nu)$ for every n. We now define $h(\nu)$ to be the greatest common divisor of all positive integers k for which there is an integer N such that $C_N(\nu) \cap C_{N+k}(\nu)$ is non-null. We note that $h(\nu)$ divides $h(\eta)$ if η is absolutely continuous to νP^n for some $n \geqq 0$. Let

$$H = \sup\{h(\nu):\ \nu \in \mathscr{A}^+(\lambda),\ \nu \neq 0\}. \tag{2.5}$$

H may be $+\infty$ or a finite positive integer.

Theorem 2.1. $H = \delta$.

Proof. Let k be an arbitrary positive integer and and E be a P^k-indecomposable closed set. Let $\nu = \lambda I_E$. The sequence $\{E, A^{\delta(k)-1}(E), \cdots, E, A^{\delta(k)-1}(E), \cdots\}$ is the consequent sequence of ν and for this ν, $h(\nu) = \delta(k)$. Hence $H \geqq \delta(k)$ for every positive integer k. It follows that $H \geqq \delta$. To prove $H \leqq \delta$, let ν be an arbitrary nonzero element of $\mathscr{A}^+(\lambda)$ and let $C_n(\nu) = \operatorname{supp} \nu P^{n-1}$ for $n = 1, 2, \cdots$ and $h = h(\nu)$. Let E_i, $i = 1, \cdots, h$, be defined by

$$E_i = \bigcup_{j=0}^{\infty} C_{i+jh}(\nu). \tag{2.6}$$

Since $C_{i+jh}(\nu) \subset A^h(C_{i+(j+1)h})$, E_i are P^h-closed. If $i_1 \neq i_2$, $E_{i_1} \cap E_{i_2}$ is null for if $E_{i_1} \cap E_{i_2}$ is non-null, then, there are non-negative integers j_1, j_2 such that $C_{i_1+j_1h}(\nu) \cap C_{i_2+j_2h}(\nu)$ is non-null. Then $i_1 + j_1h - (i_2 + j_2h) = (i_1 - i_2) + (j_1 - j_2)h$ is divisible by h. It follows that $i_1 - i_2$ is divisible by h which is impossible since $|i_1 - i_2| < h$. $E_1, \cdots, E_h$ constitute the totality of all P^h-indecomposable sets. Hence $h = \delta(h) \leqq \delta$. Hence $H \leqq \delta$.

For any nonzero measure $\nu \in \mathscr{A}^+(\lambda)$ we shall define $h'(\nu)$ to be the *minimum* of all positive integers k for which there is a positive integer N such that $C_N(\nu) \cap C_{N+k}(\nu)$ is non-null. It is clear that $h(\nu)$ divides $h'(\nu)$. If η is absolutely continuous to νP^n for some $n \geqq 0$ then $h'(\eta) \geqq h'(\nu)$. Let

$$H' = \sup\{h'(\nu): \nu \in \mathscr{A}^+(\lambda), \nu \neq 0\}. \tag{2.7}$$

We always have $H' \geqq H$. For a general conservative ergodic λ-measurable Markov operator P it is possible to have $H' > H$ as illustrated by the following example. Let X be the set of all complex numbers of absolute value 1 and λ be the linear Lebesgue measure. Let $\alpha = e^{i2\pi\theta}$ where θ is irrational and $Pf(x) = f(\alpha x)$. Then

P^n is ergodic for every positive integer n, so that P is aperiodic and $H = 1$ (cf. [6, p. 26]). Let ν have, as its support, the set $[e^{i2\pi y}: 0 \leqq y \leqq \varepsilon]$ where ε is a positive number. Then νP^n has the set $[e^{i2\pi y}: n\theta \leqq y \leqq n\theta + \varepsilon]$ as its support. Let k be an arbitrary positive integer. Let $2\pi c$ be the minimum distance from the point 1 to $e^{i2\pi\theta}, e^{i4\pi\theta}, \cdots, e^{i2k\pi\theta}$. Then $c > 0$. Hence if $\varepsilon < c$ we have $h'(\nu) > k$. Hence $H' = \infty$.

LEMMA 2.6. *If H' is finite, then $H' = H = \delta$ and for every consequent sequence $\{E_n\}$ there is a positive integer N such that $E_n \cap E_{n+\delta}$ is non-null for every $n \geqq N$.*

Proof. If H' is finite, then H' is a positive integer and there is a nonzero measure $\nu_1 \in \mathscr{A}^+(\lambda)$ such that $h'(\nu_1) = H'$. Since $H' \geqq H$, H is finite and there is a nonzero measure $\nu_2 \in \mathscr{A}^+(\lambda)$ such that $h(\nu_2) = H$. Since $C_1(\nu_2) \subset X = \bigcup_{n=1}^{\infty} C_n(\nu_1)$, there is an n such that $C_1(\nu_2) \cap C_n(\nu_1)$ is non-null. Let ν be a nonzero measure which has $C_1(\nu_2) \cap C_n(\nu_1)$ as its support, then ν is absolutely continuous to both ν_2 and $\nu_1 P^{n-1}$. Hence $h(\nu) \geqq h(\nu_1)$, $h'(\nu) \geqq h'(\nu_2)$. However, since $h(\nu) \leqq H$, $h'(\nu) \leqq H'$, $h(\nu) = H$, $h'(\nu) = H'$. Now consider the consequent sequence $\{C_n(\nu)\}$ of ν. Let k be a positive integer such that there is an n for which $C_n(\nu) \cap C_{n+k}(\nu)$ is non-null. Then $H' \leqq k$. Now, since $C_n(\nu) \cap C_{n+k}(\nu)$ is non-null we may choose a nonzero measure $\eta \in \mathscr{A}^+(\lambda)$ with $C_n(\nu) \cap C_{n+k}(\nu)$ as its support. Then η is absolutely continuous to νP^{n-1}. Hence $h'(\eta) \geqq h'(\nu)$. It follows that $h'(\eta) = H'$ and there is a positive integer m such that $C_m(\eta) \cap C_{m+H'}(\eta)$ is non-null. But we have $C_m(\eta) \subset C_{n-1+m}(\nu) \cap C_{n-1+m+k}(\nu)$, $C_{m+H'}(\eta) \subset C_{n-1+m+H'}(\nu) \cap C_{n-1+m+H'+k}(\nu)$. Hence $C_{n-1+m}(\nu) \cap C_{n-1+m+k}(\nu) \cap C_{n-1+m+H'}(\nu) \cap C_{n-1+m+H'+k}(\nu)$ is non-null. It follows that $C_{n-1+m+H'}(\nu) \cap C_{n-1+m+k}(\nu)$ is non-null. Hence either $k - H' = 0$ or $k - H' \geqq H'$. If $k - H' = 0$ then k is divisible by H'. If $k - H' \geqq H'$, repeating the same argument for $k - H'$ as for k before, we conclude that $k - 2H'$ is either 0 or $\geqq H'$. Repeating the same argument finitely many times we obtain the result $k - jH' = 0$. Hence k is divisible by H'. This is true for all positive integers k for which there is a positive integer n such that $C_n(\nu) \cap C_{n+k}(\nu)$ is non-null. Hence H' divides H. Hence $H' = H = \delta$. Now let $\{E_n\}$ be an arbitrary consequent sequence. Since $X = \bigcup_{n=1}^{\infty} E_n$, $C_1(\nu) \cap E_{n_0}$ is non-null for some positive integer n_0. Let $\zeta \in \mathscr{A}^+(\lambda)$ have $C_1(\nu) \cap E_{n_0}$ as its support. Then $h'(\zeta) \geqq h(\nu)$ so that $h'(\zeta) = h(\nu) = \delta$. There is a positive integer l such that $C_l(\zeta) \cap C_{l+\delta}(\zeta)$ is non-null. It follows that $C_n(\zeta) \cap C_{n+\delta}(\zeta)$ is non-null for all $n \geqq l$. Now we have, for every positive integer n, $C_n(\zeta) \subset E_{n_0-1+n}$. Hence $C_n(\zeta) \cap C_{n+\delta}(\zeta)$ being non-null implies that $E_{n_0-1+n} \cap E_{n_0-1+n+\delta}$ is non-null. Let $N = n_0 - 1 + l$. Then $E_n \cap E_{n+\delta}$ is non-null for all positive integers $n \geqq N$.

Now we shall proceed to show that the period of a conservative, ergodic, λ-continuous Markov operator is always finite. To do this we shall choose a definite version of $p(x, y)$ for P to satisfy

1. $p(x, y) \geqq 0$ for *all* $(x, y) \in X \times X$ and
2. $\int p(x, y)\lambda(dy) = 1$ for *all* $x \in X$.

Then the iterates $p^{(n)}(x,y)$ given by (1.3) also satisfy 1 and 2 For each $x \in X$, $E \in \mathscr{X}$ let

$$\nu_x(E) = \int_E p(x,y)\lambda(dy).$$

For each $x \in X$, ν_x is a probability measure absolutely continuous to λ and for each fixed $E \in X$, x varying over X, $\nu_x(E)$ is a version of $P1_E$.

LEMMA 2.7. *For a λ-continuous, conservative, ergodic Markov operator P, H' (defined by (2.7)) is finite.*

Proof. If H' were infinite, then there would be a sequence $\{\eta_k\}$ of nonzero measures in $\mathscr{A}^+(\lambda)$ such that $\lim_{k\to\infty} h'(\eta_k) = +\infty$. Let $\{C_n(\eta_k)\}$ be the consequent sequence of η_k. Sets $C_n(\eta_k)$ are only unique up to sets of λ measure zero. Now we shall make a definite choice of sets $C_n(\eta_k)$ to satisfy the condition that if $x \in C_n(\eta_k)$ then $\nu_x(C_{n+1}(\eta_k)) = 1$. This can always be accomplished by replacing the original $C_n(\eta_k)$ by its intersection with the set $[x : \nu_x(C_{n+1}(\eta_k)) = 1]$. Since $P1_{C_{n+1}(\eta_k)} = 1$ a.e. (λ) on $C_n(\eta_k)$, the intersection remains a support of $\eta_k P^n$. Now sets $C_n(\eta_k)$ have this property: if $x \in C_n(\eta_k)$, then ν_x is absolutely continuous to $\eta_k P^n$. Hence, if $x \in \bigcup_{n=1}^{\infty} C_n(\eta_k)$ then $h'(\nu_x) \geqq h'(\eta_k)$.

Now let $X_k = \bigcup_{n=1}^{\infty} C_n(\eta_k)$ in the strict sense of set union. Then $\lambda(X - X_k) = 0$ so that $\lambda(X - \bigcap_{k=1}^{\infty} X_k) = 0$. There must be a point $x \in \bigcap_{k=1}^{\infty} X_k$. For this x, $h'(\nu_x) \geqq h'(\eta_x)$ for $k = 1,2,\cdots$, which is impossible since $h'(\nu_x)$ is a finite integer and $\lim_{k\to\infty} h'(\nu_k) = +\infty$.

Combining Lemmas 2.7, 2.6, we have the following:

THEOREM 2.2. *If a Markov operator P is conservative, ergodic and λ-continuous, then the period δ of P is a finite positive integer and for any consequent sequence $\{C_n\}$ there is a positive integer N such that $C_n \cap C_{n+\delta}$ is non-null for all $n \geqq N$.*

THEOREM 2.3. *Let P be a λ-continuous, conservative, ergodic Markov operator. Let δ be the period of P and μ be a non-null invariant measure of P. Let $C_0, C_1, \cdots, C_{\delta-1}$ be the totality of distinct $\mathscr{C}^{(\delta)}$ atoms with $C_0 = A(C_1)$, $C_1 = A(C_2), \cdots, C_{\delta-2} = A(C_{\delta-1})$. Then each μI_{C_i} is an invariant measure of $P^{n\delta}$ and every invariant measure of $P^{n\delta}$ is of the form $\sum_{i=0}^{\delta-1} \alpha_i \mu I_{C_i}$. Furthermore, we have $\mu I_{C_0} P = \mu I_{C_1}, \cdots, \mu I_{C_{\delta-2}} P = \mu I_{C_{\delta-1}}$, $\mu I_{C_{\delta-1}} P = \mu I_{C_0}$ and $\mu(C_0) = \mu(C_1) = \cdots = \mu(C_{\delta-1})$. Hence if P has a finite invariant measure then all invariant measures of iterates of P are finite measures.*

Proof. Since C_i is P^δ-closed, $I_{C_i} P^\delta = I_{C_i} P^\delta I_C$. Since P^δ is conservative, $X - C_i$ is P^δ-closed. Hence $I_{X-C_i} P^\delta I_{C_i} = 0$ and $P^\delta I_{C_i} = I_{C_i} P^\delta I_{C_i} + I_{X-C_i} P^\delta I_C$ $= I_{C_i} P^\delta I_{C_i} = I_{C_i} P^\delta$. Thus we have $\mu I_{C_i} P^\delta = \mu P^\delta I_{C_i} = \mu I_{C_i}$ and μI_{C_i} is P^δ-invariant. Now, for $f \in L_\infty(\mu)$,

$$\begin{aligned}\langle \mu I_{C_1}, f\rangle &= \langle \mu, I_{C_1} f\rangle = \langle \mu P, I_{C_1} f\rangle \\ &= \langle \mu I_{C_0} P, I_{C_1} f\rangle + \langle \mu I_{X-C_0} P, I_{C_1} f\rangle.\end{aligned}$$

Since the support of $\mu I_{C_0} P$ is C_0 and the support of $\mu I_{X-C_0} P$ is $X - C_1$, we have

$$\langle \mu I_{C_0} P, I_{C_1} f\rangle = \langle \mu I_{C_0} P, f\rangle$$

and

$$\langle \mu I_{X-C_0} P, I_{C_1} f\rangle = 0.$$

Hence

(2.8) $$\langle \mu I_{C_1}, f\rangle = \langle \mu I_{C_0} P, f\rangle.$$

Since (2.8) is true for every $f \in L_\infty(\mu)$, $\mu I_{C_0} P = \mu I_{C_1}$. By the same argument, we have $\mu I_{C_1} P = \mu I_{C_2}, \cdots, \mu I_{C_{\delta-1}} P = \mu I_{C_0}$. Substituting 1 for f in (2.8) we then obtain $\mu(C_0) = \mu(C_1)$. Similarly $\mu(C_1) = \mu(C_2), \cdots, \mu(C_{\delta-1}) = \mu(C_0)$.

Now every C_i is also a $\mathscr{C}^{(n\delta)}$ atom for every positive integer n. Hence $P^{n\delta}$ acting on C_i only is conservative and ergodic. It follows that for any $P^{n\delta}$-invariant measure v, vI_{C_i} must be a constant multiple of μI_{C_i}. Hence v is of the form $\sum_{i=0}^{\delta-1} \alpha_i \mu I_{C_i}$.

III. **Asymptotic properties of $[p^n(x,y)]$ for a λ-continuous, conservative, ergodic Markov operator.** All through this section, the Markov operator P is assumed to be *λ-continuous*, *conservative* and *ergodic*. Then P possesses a nontrivial σ-finite invariant measure μ which is unique up to a constant multiple [8]. μ is equivalent to λ. Hence "a.e. (λ)" is the same as "a.e. (μ)" and $L_\infty(\lambda)$ and $L_\infty(\mu)$ are the same space.

LEMMA 3.1. *If $f \in L_q(\mu)$, $1 \leqq q < \infty$, then Pf, given by*

$$Pf = Pf^+ - Pf^-,$$

belongs to $L_q(\mu)$ also. Furthermore, we have

$$\|Pf\|_q \leqq \|f\|_q$$

where $\| \quad \|_q$ denotes the $L_q(\mu)$ norm.

Proof. We only need to prove for the case $1 \leqq q < \infty$. For any non-negative function f, by Jensen's inequality, for (λ) almost all x

(3.1) $$|Pf(x)|^q \leqq \int p(x,y)|f(y)|^q \lambda(dy).$$

Hence

(3.2) $$\begin{aligned}\int \mu(dx)|Pf(x)|^q &\leqq \int \mu(dx)\left\{\int p(x,y)|f(y)|^q\lambda(dy)\right\} \\ &= \int \mu(dy)|f(y)|^q.\end{aligned}$$

Hence $f \in L_q(\mu)$ implies that $Pf \in L_q(\mu)$ and $\| Pf \|_q \leqq \| f \|_q$. Then for the general case that f may take on both positive and negative values and $f \in L_q(\mu)$, Pf^+, Pf^- are in $L_q(\mu)$ and, therefore, Pf is well defined and is in $L_q(\mu)$. Jensen's inequality again implies (3.1) and from which (3.2) and the equality $\| Pf \|_q \leqq \| f \|_q$ follow immediately.

Lemma 3.2. *If f is non-negative and $f \in L_q(\mu)$ where $1 \leqq q \leqq +\infty$, then $\liminf_{n\to\infty} P^n f$ is equal to a finite constant a.e. (λ). If, in addition, the invariant measure μ is infinite and $q < +\infty$, then $\liminf_{n\to\infty} P^n f = 0$ a.e. (λ).*

Proof. Since f is non-negative, we have, by Fatou's lemma,

$$\liminf_{n\to\infty} \int p(x,y) P^n f(y) \lambda(dy) \geqq \int p(x,y) \liminf_{n\to\infty} P^n f(y) \lambda(dy).$$

Hence $\liminf_{n\to\infty} P^n f \leqq \int P \liminf_{n\to\infty} P^n f$ so that $\liminf_{n\to\infty} P^n f$ is an excessive function. (A non-negative function g is excessive if $Pf \leqq f$. For the properties of excessive functions see [8, §VI].) Since excessive functions for a conservative, ergodic Markov operator are constant functions $\liminf_{n\to\infty} P^n f =$ constant a.e. (λ). Since $\inf_{k \geqq n} P^k f \leqq P^k f$ and $\| P^k f \|_q \leqq \| f \|_q$ by Lemma 3.1, we have also $\inf_{k\geqq n} P^k f \in L_q(\mu)$ and $\| \inf_{k \geqq n} P^k f \|_q \leqq \| f \|_q$. Hence $\| \liminf_{n\to\infty} P^n f \|_q \leqq \| f \|_q$. Since $\mu(X) = \infty$ and $\liminf_{n\to\infty} P^n f$ is a constant function, we must have $\liminf_{n\to\infty} P^n f = 0$ a.e. (λ).

Now we shall proceed to study asymptotic properties of sequences $\{P^n f\}$. We shall again, as in §II, choose a definite version of the density function $p(x,y)$ of P to satisfy

1. $p(x,y) \geqq$ for *all* $(x,y) \in X \times X$ and
2. $\int p(x,y)\lambda(x,y) = 1$ for all $x \in X$.

The iterates $p^{(n)}(x,y)$ will be given inductively by (1.3). They also satisfy 1 and 2. For every positive integer n, every $x \in X$ and $E \in \mathscr{X}$ define

$$\nu_x^{(n)}(E) = \int_E p^{(n)}(x,y)\lambda(dy). \tag{3.3}$$

$\nu_x^{(n)}$ are probability measures and $\nu_x^{(n+1)} = \nu_x^{(n)} P$. Since P is ergodic the union of the supports of $\nu_x^{(n)}$, $n = 1, 2, \cdots$, is X. Now for every non-negative f, $P^n f(x)$ shall be given definitely by

$$P^n f(x) = \int \nu_x^{(n)}(dy) f(y) = \int p^{(n)}(x,y) f(y) \lambda(dy). \tag{3.4}$$

Let f be a *fixed* non-negative function which belongs to $L_q(\mu)$ for some q satisfying $1 \leqq q \leqq +\infty$. By Lemma 3.2 there is a non-negative number a such that

$$\liminf_{n\to\infty} P^n f(x) = a$$

for (λ) almost all x. Hence for (λ) almost all x there is an increasing sequence $\{n_i\}$

(the sequence depends on x) of positive integers such that $\lim_{i\to\infty} P^{n_i} f(x) = a$. Let $\rho(x)$ be the supremum of all non-negative integers k with the property that there is an increasing sequence $\{n_i\}$ of positive integers such that

$$\lim_{i\to\infty} P^{(n_i+j)} f(x) = a \quad \text{for} \quad j = 0, \cdots, k.$$

$\rho(x)$ is defined for (λ) almost all x and $0 \leqq \rho(x) \leqq +\infty$. We shall show that $\rho(x) = +\infty$ for (λ) almost all x.

LEMMA 3.3. *Let η be a probability measure, and let $\{g_n\}$ be a sequence of η-integrable non-negative functions. If* $\liminf_{n\to\infty} g_n \geqq a$ *a.e.* (η) *and* $\lim_{n\to\infty} \int g_n d\eta = a$, *then there is an increasing sequence $\{n_i\}$ of positive integers such that $\{g_{n_i}\}$ converges a.e. (η) to a.*

Proof. If $a = 0$, then $\{g_n\}$ converges to 0 in $L_1(\eta)$, hence, there is a subsequence $\{g_{n_i}\}$ converging a.e. (η) to 0. Suppose $a > 0$. We shall find an increasing sequence $\{n_i\}$ of positive integers such that

$$\eta(F_i) < \frac{2+a}{2^i} \quad \text{for } i \text{ sufficiently large} \tag{3.5}$$

where

$$F_i = \left[x : g_{n_i}(x) \geqq a + \frac{1}{2^i} \right].$$

(3.5) implies

$$\limsup_{i\to\infty} g_{n_i} \leqq a \text{ a.e. } (\eta). \tag{3.6}$$

(3.6) and the fact that $\liminf_{i\to\infty} g_{n_i} \geqq a$ imply $\lim_{i\to\infty} g_{n_i} = a$ a.e. (η).

Now there is an increasing sequence $\{n_i\}$ of positive integers satisfying the following two conditions for every i:

1. $\int g_{n_i} d < a + 1/4^i$,
2. $\eta[g_{n_i} \leqq a - 1/4^i] < 1/4^i$.

Then, if $a - 1/4^i \geqq 0$, we have

$$\begin{aligned}
a+\frac{1}{4^i} > \int g_{n_i} d\eta &= \int_{[g_{n_i} \geqq a+1/2^i]} g_{n_i} d\eta + \int_{[a+1/2^i > g_{n_i} > a-1/4^i]} g_{n_i} d\eta + \int_{[g_{n_i} \leqq a-1/4^i]} g_{n_i} d\eta \\
&\geqq \left(a + \frac{1}{2^i}\right) \eta(F_i) + \left(a - \frac{1}{4^i}\right) \eta \left[a + \frac{1}{2^i} > g_{n_i} > a - \frac{1}{4^i} \right] \\
&\geqq \left(a + \frac{1}{2^i}\right) \eta(F_i) + \left(a - \frac{1}{4^i}\right) \left\{ \eta \left[a + \frac{1}{2^i} > g_{n_i} \right] - \frac{1}{4^i} \right\} \\
&= \left(a + \frac{1}{2^i}\right) \eta(F_i) + \left(a - \frac{1}{4^i}\right) \left[1 - \eta(F_i) - \frac{1}{4^i} \right] \\
&= \frac{2+1}{4^i} \eta(F_i) + \left(a - \frac{1}{4^i}\right) \left(1 - \frac{1}{4^i}\right).
\end{aligned}$$

Hence $\eta(F_i) < (2 + a)/2^i$.

The following lemma is a slight improvement of Lemma 3.3. The proof is trivial.

LEMMA 3.3′. *Let η be a probability measure, and let $\{g_n^{(j)}\}$, $j = 0, 1, \cdots, k$, be $k + 1$ sequences of η-integrable, non-negative functions. If $\liminf_{n\to\infty} g_n^{(j)} \geqq a$ a.e. (η) and $\lim_{n\to\infty} \int g_n^{(j)} d\eta = a$ for $j = 0, 1, \cdots, k$, then there is an increasing sequence $\{n_i\}$ of positive integers such that $\lim_{i\to\infty} g_{n_i}^{(j)} = a$ a.e. (η) for $j = 0, 1, \cdots, k$.*

In what follows f shall be a *fixed* non-negative function in $L_q(\mu)$, and a is equal to $\liminf_{n\to\infty} P^n f$ a.e.(λ). Since $P^m f$, $m = 1, 2, \cdots$, are also in $L_q(\mu)$, there is a set E_0 of 0 λ-measure such that, for every $x \notin E_0$, we have, simultaneously,

1. $\liminf_{n\to\infty} P^n f(x) = a$,
2. $P^m f(x)$ is finite for $m = 1, 2, \cdots$. 2 is the same as,

2′. f is $v_x^{(m)}$-integrable for $m = 1, 2, \cdots$, where $v_x^{(m)}$ is given by (3.3).

LEMMA 3.4. *Let x_0 be a point of $X - E_0$. If $\{n_i\}$ is an increasing sequence of positive integers such that $\lim_{i\to\infty} P^{n_i + j} f(x_0) = a$ for $j = 0, 1, \cdots, k$, then for every positive integer m there is a subsequence $\{n_i'\}$ of $\{n_i\}$ such that*

$$\lim_{i\to\infty} P^{(n_i' - m) + j} f(x) = a \quad \text{for } j = 0, 1, \cdots, k$$

for (λ) almost all x on the support of the probability measure $v_{x_0}^{(m)}$.

Proof. Since

$$P^{n_i + j} f(x_0) = \int P^{(n_i - m) + j} f dv_{x_0}^{(m)},$$

we have

$$\lim_{i\to\infty} \int P^{(n_i - m) + j} f dv_{x_0}^{(m)} = a \quad \text{for } j = 0, 1, \cdots, k.$$

Since $\liminf_{i\to\infty} P^{(n_i - m) + j} f \geqq a$ a.e. $(v_{x_0}^{(m)})$, Lemma 3.3′ is applicable. Hence there exists a subsequence $\{n_i'\}$ of $\{n_i\}$ such that for (λ) almost all x on the support of $v_{x_0}^{(m)}$ we have

$$\lim_{i\to\infty} P^{(n_i' - m) + j} f(x) = a \quad \text{for } j = 0, 1, \cdots, k.$$

LEMMA 3.5. *If, for some $x \in X - E_0$, $\rho(x) \geqq k$, then $\rho(x) \geqq k$ for (λ) almost all x.*

Proof. If $\rho(x_0) \geqq k$ where $x_0 \in X - E_0$, then there is an increasing sequence $\{n_i\}$ of positive integers such that $\lim_{i\to\infty} P^{n_i + j} f(x_0) = a$ for $j = 0, 1, \cdots, k$. By Lemma 3.3, $\rho(x) \geqq k$ for (λ) almost all x belonging to the support of the measure $v_{x_0}^{(m)}$. Let the support of $v_{x_0}^{(m)}$ be C_m. $\{C_m\}$ is a consequent sequence. Hence $\lambda(X - \bigcup_{m=1}^{\infty} C_m) = 0$. Now $\rho(x) \geqq k$ for (λ) almost all x in $\bigcup_{m=1}^{\infty} C_m$. Hence the lemma is proved .

LEMMA 3.6. *If P is aperiodic, then for every non-negative integer k, there is an $x_0 \in X - E_0$ for which $\rho(x_0) \geqq k$.*

Proof. The lemma is obviously true for $k = 0$. Suppose the lemma is true for k. There is an $x_0 \in X - E_0$ and an increasing sequence $\{n_i\}$ of positive integers for which

$$\lim_{i \to \infty} P^{n_i + j} f(x_0) = a \quad \text{for } j = 0, 1, \cdots, k.$$

Let C_m be the support of the measure $\nu_{x_0}^{(m)}$. $\{C_m\}$ is a consequent sequence. Since P is aperiodic, by Theorem 2.2, there is a positive integer N such that $C_N \cap C_{N+1}$ is non-null. By Lemma 3.4 there is a subsequence $\{n'_i\}$ of $\{n_i\}$ for which we have, simultaneously, $\lim_{i\to\infty} P^{(n'_i - N)+j} f(x) = a$, $j = 0, 1, \cdots, k$, for (λ) almost all x in C_N and $\lim_{i\to\infty} P^{(n'_i - N - 1)+j} f(x) = a$, $j = 0, 1, \cdots, k$ for (λ) almost all x in C_{N+1}. Since $C_N \cap C_{N+1}$ is non-null, there is a point y in $C_N \cap C_{N+1}$ and $y \notin E_0$ such that

$$\lim_{i \to \infty} P^{(n'_i - N)+j} f(y) = a \quad \text{and} \quad \lim_{i \to \infty} P^{(n'_i - N - 1)+j} f(y) = a$$

for $j = 0, 1, \cdots, k$. Hence we have

$$\lim_{i \to \infty} P^{(n'_i - N - 1)+j} f(y) = a \quad \text{for} \quad j = 0, 1, \cdots, k+1.$$

Therefore $\rho(y) \geqq k + 1$ and the lemma is proved.

LEMMA 3.7. *If P is aperiodic, then for (λ) almost all x and for every positive integer k, there is an increasing sequence $\{n_i\}$ of positive integers for which*

$$\lim_{n \to \infty} P^{n_i + j} f(x) = a \quad \textit{for} \quad j = 0, 1, \cdots, k.$$

In other words, $\rho(x) = \infty$, for (λ) almost all x.

Proof. It follows from Lemma 3.5 and Lemma 3.6 that for every positive integer k, $\rho(x) \geqq k$ for (λ) almost all x. Hence $\rho(x) = \infty$ for (λ) almost all x.

LEMMA 3.8. *If P is aperiodic and λ is a finite measure, then, for every positive number ε, there is a set A with $\lambda(X - A) < \varepsilon$ and an increasing sequence $\{n_i\}$ of positive integers such that the sequence of functions:*

$$P^{n_0} f,\ P^{n_1} f,\ P^{n_1 + 1} f,\ P^{n_2} f,\ P^{n_2 + 1} f,\ P^{n_2 + 2} f, \cdots$$

converges uniformly to a on A where $a = \liminf_{n\to\infty} P^n f$.

Proof. Let x_0 be a point of $X - E_0$ for which $\rho(x_0) = \infty$, and let C_n be the support of $\nu_{x_0}^{(n)}$. Then $\lambda(X - \bigcup_{n=1}^{\infty} C_n) = 0$ and, hence, there is a positive integer b such that $\lambda(X - \bigcup_{n=1}^{b} C_n) < \varepsilon/2$. Let $B = \bigcup_{n=1}^{b} C_n$.

Since $\rho(x_0) = \infty$, for every positive integer k, there is an increasing sequence $\{n_i^{(k)}\}$ of positive integers such that

$$\lim_{i\to\infty} P^{n_i^{(k)}+1}(x_0) = a, \cdots, \lim_{i\to\infty} P^{n_i^{(k)}+k} f(x_0) = a.$$

Applying Lemma 3.4 repeatedly for b times, we obtain a subsequence $\{m_i^{(k)}\}$ of $\{n_i^{(k)}\}$ such that, for every integer m, $1 \leqq m \leqq b$,

$$\lim_{i\to\infty} P^{\{m_i^{(k)}-m\}+1} f(x) = a, \cdots, \lim_{i\to\infty} P^{\{m_i^{(k)}-m\}+k} f(x) = a$$

for (λ) almost all x on C_m. Let $k \geqq b$. Then

$$\lim_{i\to\infty} P^{m^{(k)}} f(x) = a, \quad \lim_{i\to\infty} P^{m^{(k)}+1} f(x) = a, \cdots, \quad \lim_{i\to\infty} P^{m_i^{(k)}+(k-b)} f(x) = a$$

for (λ) almost all x on B. Let $l_i^k = m_i^{(k+b)}$. Then for every non-negative integer k, the sequence $\{l_i^{(k)}\}$ has the property that

$$\lim_{i\to\infty} P^{l_i^{(k)}} f(x) = a, \quad \lim_{i\to\infty} P^{l_i^{(k)}} f(x) = a, \cdots, \quad \lim_{i\to\infty} P^{l_i^{(k)}+k} f(x) = a$$

for (λ) almost all x on B. Now, for every non-negative integer k, let n_k be a member of the sequence $\{l_i^{(k)}\}$ such that

$$\lambda\left\{ B \cap \left[\left(|P^{n_k} f - a| > \frac{1}{2^k} \right) \cup \left(|P^{n_k+1} f - a| > \frac{1}{2^k} \right) \cup \cdots \cup \left(|P^{n_k+k} f - a| > \frac{1}{2^k} \right) \right] \right\} < \frac{1}{2^k}.$$

Then the sequence of functions:

$$P^{n_0}f, P^{n_1}f, P^{n_1+1}f, P^{n_2}f, P^{n_2+1}f, P^{n_2+2}f, \cdots \tag{3.7}$$

converges to a a.e. (λ) on B. By Egoroff's theorem, there is a subset A of B such that $\lambda(B - A) < \varepsilon/2$ and the sequence (3.7) converges uniformly to a on A.

The following lemma follows immediately from Lemma 3.8.

Lemma 3.9. *If P is aperiodic, then, for every positive number ε, there is an increasing sequence $\{n_i\}$ of positive integers such that the set*

$$E = \bigcap_{i=1}^{\infty} \bigcap_{k=0}^{i} [P^{n_i+k} f < a + \varepsilon] \tag{3.8}$$

has positive λ measure.

Lemma 3.10. *If E is a set of positive λ measure, then $\lim_{n\to\infty}(I_{E'}P)^n 1 = 0$ a.e. (λ) where $E' = X - E$.*

Proof. Let $\nu \in \mathscr{A}^+(\lambda)$. Then

$$\begin{aligned}
\nu(X) &= \nu P^n(X) = \langle \nu P^n, 1\rangle \\
&= \left\langle \nu \sum_{k=0}^{n-1} (I_{E'}P)^k I_A P^{n-k}, 1\right\rangle + \langle \nu(I_{E'}P)^n, 1\rangle \\
&= \left\langle \nu \sum_{k=0}^{n-1} (I_{E'}P)^k I_A, 1\right\rangle + \nu(I_{E'}P)^n(X) \\
&= \left\langle \nu \sum_{k=0}^{n-1} (I_{E'}P)^k 1_A\right\rangle + \nu(I_{E'}P)^n(X).
\end{aligned}$$

Since E is conservative and P is ergodic, we have

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} (I_{E'}P)^k 1_E = 1 \text{ a.e. } (\lambda).$$

Hence

$$\lim_{n\to\infty} \nu(I_{E'}P)^n(X) = 0. \tag{3.9}$$

Setting $\nu = \nu_x^{(1)}$ in (3.9), we obtain $\lim_{n\to\infty} P(I_{E'}P)^n 1(x) = 0$. Hence $\lim_{n\to\infty}(I_{E'}P)^n 1 = 0$ a.e. (λ).

We recall that the invariant measure μ for P may be finite or infinite. We shall first study the case that μ is finite. *μ is then always normalized to be a probability measure.*

LEMMA 3.11. *If the invariant measure μ of P is finite, then, for every $\nu \in \mathscr{A}^+(\lambda)$, the measures ν, νP, $\nu P^2, \cdots$ are uniformly absolutely continuous with respect to μ.*

Proof. Let Q be the μ-reverse of P. Q is a μ-measurable Markov operator characterized by the following equality

$$\int (Pg)h \, d\mu = \int g(Qh) \, d\mu \tag{3.10}$$

where g, h are non-negative functions (cf. [8, §VI]). Let $g = d\nu/d\mu$, then $Q^n g = d\nu P^n/d\mu$. Construct the infinite product space $\Omega = \prod_{n=0}^{\infty} X_n$ and the product σ-algebra $\mathscr{F} = \prod_{n=0}^{\infty} \mathscr{X}_n$ of subsets of Ω where $X_n = X$, $\mathscr{X}_n = \mathscr{X}$ for $n = 0, 1, 2, \cdots$. A probability measure $\boldsymbol{\mu}$ on $\mathscr{F}$ is then defined by

$$\begin{aligned}
&\boldsymbol{\mu}[X_0 \in A_0, X_1 \in A_1, \cdots, X_n \in A_n] \\
&\quad = \int_{A_0} \mu(dx_0) \int_{A_1} \lambda(dx_1) \cdots \int_{A_n} \lambda(dx_n) p(x_0, x_1) p(x_1, x_2) \cdots p(x_{n-1}, x_n)
\end{aligned}$$

where $A_i \in \mathscr{X}$ for $i = 0, 1, \cdots, n$. Coordinates $X_0, X_1, \cdots$, considered as random variables defined on Ω, constitute a stationary Markov process. $Q^n g(X_n)$ is then the conditional expectation of $g(X_0)$ given random variables $X_n, X_{n+1}, \cdots$. By the well-known martingale convergence theorem $\{Q^n g(X_n)\}$ converges in $L_1(\mathbf{\mu})$ and $Q^n g(X_n)$ are uniformly $\mathbf{\mu}$-integrable. Since the process is stationary, every X_n has μ as its distribution, hence

$$\int_{[Q^n g \geqq K]} Q^n g d\mu = \int_{[Q^n g(X_n) \geqq K]} Q^n g(X_n) d\mathbf{\mu}.$$

It follows that the functions $Q^n g$ are uniformly μ-integrable. Hence the measures νP^n are uniformly absolutely continuous with respect to μ.

THEOREM 3.1. *If P is a λ-continuous, conservative, ergodic and aperiodic Markov operator whose invariant μ is finite (μ is then normalized), then, for every $f \in L_\infty(\lambda)$, $\{P^n f\}$ converges a.e. (λ) to $\int f d\mu$.*

Proof. If the theorem is true for non-negative functions then, applying the result to f^+, f^-, we obtain the same conclusion for a function f which takes on both positive and negative values. So we shall only prove the theorem for a non-negative f. Let us assume $f \leqq 1$ a.e. (λ).

By Lemma 3.2 $\liminf_{n\to\infty} P^n f$ is equal to a constant a a.e (μ). Let ε be an arbitrary positive number. By Lemma 3.9, there is an increasing sequence $\{n_i\}$ of positive integers such that the set E given by (3.8) has positive λ measure. Let x_0 be an arbitrary point of X and $\nu_{x_0}^{(m)}$ be given by (3.3). Then

$$\begin{aligned} P^{m+n_i+i} f(x_0) &= \int P^{n_i+i} f d\nu_{x_0}^{(m)} \\ &= \int \left[\sum_{k=0}^{i-1} (I_{E'} P)^k I_E P^{n_i+i-k} f + (I_{E'} P)^i P^{n_i} f \right] d\nu_{x_0}^{(m)} \\ &\leqq (a+\varepsilon) \int_{k=0}^{i-1} (I_{E'} P)^k 1_E d\nu_{x_0}^{(m)} + \int (I_{E'} P)^i 1 d\nu_{x_0}^{(m)} \\ &\leqq (a+\varepsilon) + \int (I_{E'} P)^i 1 d\nu_{x_0}^{(m)}. \end{aligned}$$

By Lemma 3.10 $\lim_{i\to\infty} (I_{E'} P)^i 1 = 0$ a.e. (μ). Hence, for every positive integer δ, there is an integer i_0 and a set A with $\mu(X - A) < \delta$ such that $(I_{E'} P)^{i_0} 1 < \varepsilon$ on A. The number δ is chosen to satisfy the condition that $\nu_{x_0}^{(m)}(F) < \varepsilon$ for $m = 1, 2, \cdots$ whenever $\mu(F) < \delta$. This can be done because $\nu_{x_0}^{(1)}, \nu_{x_0}^{(2)}, \cdots$ are uniformly absolutely continuous with respect to μ (Lemma 3.11). Hence for any positive integer m,

$$\begin{aligned} P^{m+n_{i_0}+i_0} f(x_0) &\leqq (a+\varepsilon) + \int_A (I_{E'} P)^{i_0} 1 d\nu_{x_0}^{(m)} + \nu_{x_0}^{(m)}(X - A) \\ &\leqq a + 3\varepsilon. \end{aligned}$$

Hence we have

$$\limsup_{n\to\infty} P^n f(x_0) \leqq a + 3\varepsilon.$$

Since ε is an arbitrary positive number,

$$\limsup_{n\to\infty} P^n f(x_0) \leqq a. \tag{3.11}$$

(3.11) holds for every $x_0 \in X$, hence $\lim_{n\to\infty} P^n f = a$ a.e. (λ). Since μ is the normalized invariant measure of P, $\int P^n f\, d\mu = \int f\, d\mu$ for $n = 1, 2, \cdots$. Now $\lim_{n\to\infty} \int P^n f\, d\mu = a$, hence $\int f\, d\mu = a$ and the proof of the theorem is then complete.

THEOREM 3.2. *If P is a λ-continuous, conservative, ergodic and aperiodic Markov operator whose invariant measure μ is finite, and if $f \in L_q(\mu)$, where $1 \leqq q < \infty$, then the sequence $\{P^n f\}$ converges in $L_q(\mu)$ to $\int f\, d\mu$.*

Proof. If $g \in L_\infty(\mu)$, by Theorem 3.1, $\{P^n g\}$ converges a.e. (μ) to $\int g\, d\mu$. Hence $\{P^n g\}$ converges to $\int g\, d\mu$ in $L_q(\mu)$. Since $L_\infty(\mu)$ is dense in $L_q(\mu)$ in the sense of $L_q(\mu)$ norm, we have, for every $f \in L_q(\mu)$ and every $\varepsilon > 0$, a $g \in L_\infty(\mu)$ such that $\|f - g\|_q < \varepsilon/2$ and $\left|\int f\, d\mu - \int g\, d\mu\right| < \varepsilon/2$. By Lemma 3.1, $\|P^n(f-g)\|_q \leqq \|f-g\|_q$, hence

$$\begin{aligned}\left\|P^n f - \int f\, d\mu\right\|_q &\leqq \|P^n(f-g)\|_q + \left\|P^n g - \int g\, d\mu\right\|_q + \left|\int f\, d\mu - \int g\, d\mu\right| \\ &\leqq \frac{\varepsilon}{2} + \left\|P^n g - \int g\, d\mu\right\|_q + \frac{\varepsilon}{2}.\end{aligned}$$

Therefore $\limsup_{n\to\infty} \left\|P^n f - \int f\, d\mu\right\|_q \leqq \varepsilon$ and the conclusion of the theorem follows.

THEOREM 3.3. *If P is a λ-continuous, conservative, ergodic and aperiodic Markov operator whose invariant measure μ is finite and if $f \geqq 0$ and $f \in L_1(\mu)$, then $\liminf_{n\to\infty} P^n f = \int f\, d\mu$ a.e. (μ).*

Proof. Let x be a fixed point of X and $v_x^{(m)}$ be given by (3.3). Let ε be an arbitrary positive number. Since $v_x^{(m)}$, $m = 1, 2, \cdots$, are uniformly absolutely continuous to μ by Lemma 3.11, there is a positive number δ such that $\mu(E) < \delta$ implies $v_x^{(m)}(E) < \varepsilon$ for $m = 1, 2, \cdots$. Now, by Theorem 3.2, $\{P^n f\}$ converges in $L_1(\mu)$ to $\int f\, d\mu$, hence there is an integer n_0 such that $\mu\left[P^{n_0} f < \int f\, d\mu - \varepsilon\right] < \delta$. Hence for any positive integer m

$$\begin{aligned}P^{m+n_0} f(x) &= \int v_x^{(m)}(dy) P^{n_0} f(y) \\ &\geqq \int_{[P^{n_0} f \geqq \int f\, d\mu - \varepsilon]} v_x^{(m)}(dy) P^{n_0} f(y) \\ &\geqq v_x^m\left[P^{n_0} f \geqq \int f\, d\mu - \varepsilon\right]\left(\int f\, d\mu - \varepsilon\right) \\ &\geqq (1-\varepsilon)\left(\int f\, d\mu - \varepsilon\right).\end{aligned}$$

Hence $\liminf_{n\to\infty} P^n f(x) \geqq \int f\,d\mu$. But by Fatou's lemma

$$\int \liminf_{n\to\infty} P^n f\,d\mu \leqq \liminf_{n\to\infty} \int P^n f\,d\mu = \int f\,d\mu.$$

Hence

$$\liminf_{n\to\infty} P^n f = \int f\,d\mu \text{ a.e. } (\mu).$$

THEOREM 3.4. *Let P and μ be as in Theorem 3.3. Then, for (λ) almost all x $\{p^{(n)}(x,\cdot)\}$ converges in $L_1(\lambda)$ to $d\mu/d\lambda$, and $\{p^{(n)}(x,y)\}$ converges to $d\mu(y)/d\lambda$ in $L_1(\nu\times\lambda)$ for any $\nu\in\mathscr{A}^+(\lambda)$. We also have $\liminf_{n\to\infty} p^{(n)}(x,y) = d\mu/d\lambda(y)$ for $(\lambda\times\lambda)$ almost all (x,y).*

Proof. Define $\bar p^{(n)}(x,y)$ by

$$\bar p^{(n)}(x,y) = p^{(n)}(x,y)\frac{d\lambda}{d\mu}(y) \tag{3.12}$$

and $\bar p(x,y) = \bar p^{(1)}(x,y)$. Then $\bar p^{(n)}(x,y)$ is the density function of P^n with respect to the invariant measure μ, and we have for (μ) almost all x

$$\int \bar p(x,y)\mu(dy) = 1,$$

and also for (μ) almost all y,

$$\int \bar p(x,y)\mu(dx) = 1.$$

$\bar p(\cdot,\cdot)$ is "doubly stochastic." Let Q be the μ-reverse of P. Then (3.10) implies that for every non-negative function h

$$Qh(y) = \int \bar p(x,y)h(x)\mu(dx).$$

Thus, Q is μ-continuous. Let $q^{(n)}(x,y)$ be the density function of Q^n with respect to μ. Then

$$q^{(n)}(x,y) = \bar p^{(n)}(y,x).$$

Since P is conservative, so is Q [5, Theorem 3.1]. Since a Q-closed set is also P-closed [8, Lemma 7.2], Q is ergodic. Since the same relationship holds between P^n and Q^n as P and Q, Q is also aperiodic. Now, let x be fixed and let us consider $\bar p(x,\cdot)$ as a function of the second variable alone. Thus for (μ) almost all x, $\bar p(x,\cdot)$ is an element of $L_1(\mu)$ with its μ-integral equal to 1. We also have

$$Q^n\bar p(x,\cdot) = \bar p^{(n+1)}(x,\cdot).$$

Applying Theorem 3.3 to Q and $\bar p(x,\cdot)$ we have

$$\liminf \bar p^{(n)}(x,y) = 1$$

for $(\mu \times \mu)$ almost all (x, y). Hence it follows that

$$\liminf p^{(n)}(x, y) = \frac{d\mu}{d\lambda}(y)$$

for $(\lambda \times \lambda)$ almost all (x, y). Furthermore, applying Theorem 3.2, we have, for (μ) almost all x, $\{\bar{p}^{(n)}(x, \cdot)\}$ converges in $L_1(\mu)$ to 1. Now

$$\int |\bar{p}^{(n)}(x, y) - 1| \mu(dy) = \int \left| p^{(n)}(x, y) - \frac{d\mu}{d\lambda}(y) \right| \lambda(dy),$$

hence $\{p^{(n)}(x, \cdot)\}$ converges in $L_1(\lambda)$ to $d\mu/d\lambda$. Now, let

$$g_n(x) = \int \left| p^{(n)}(x, y) - \frac{d\mu}{d\lambda}(y) \right| \lambda(dy) = \int |\bar{p}^{(n)}(x, y) - 1| \mu(dy),$$

$\{g_n(x)\}$ converges to 0 a.e. (μ). We also have

$$g_n(x) \leqq \int \bar{p}^{(n)}(x, y)\mu(dy) + 1 = 2.$$

Hence $\{g_n(x)\}$ converges to 0 in $L_1(\nu)$ for any $\nu \in \mathscr{A}^+(\lambda)$. Hence

$$\iint \left| p^{(n)}(x, y) - \frac{d\mu}{d\lambda}(y) \right| \lambda(dy)\nu(dx) \to 0$$

and $\{p^{(n)}(x, y)\}$ converges to $d\mu(y)/d\lambda$ in $L_1(\nu \times \lambda)$.

THEOREM 3.5. *Let P be a λ-continuous, conservative and ergodic Markov operator whose nontrivial invariant measure μ is finite (μ is normalized as usual). Let the period of P be δ and $C_0, C_1, \cdots, C_{\delta-1}$ be the totality of distinct $\mathscr{C}^{(\delta)}$ atoms with $C_0 = A(C_1), \cdots, C_{\delta-2} = A(C_{\delta-1})$. Let $f \in L_1(\mu)$ and $c_0, c_1, \cdots$ be defined by*

$$c_i = \delta \int_{C_i} f d\mu \qquad \text{for } i = 0, \cdots, \delta - 1,$$

$$c_i = c_j \quad \text{if} \quad i \geqq \delta, \quad i = n\delta + j, \qquad 0 \leqq j \leqq \delta - 1.$$

Then

1. *if f also belongs to $L_\infty(\mu)$, then for every non-negative integer k the sequence $\{P^{n\delta+k}f\}$ converges to $\sum_{i=0}^{\delta-1} c_{i+k} 1_{C_i}$ a.e. (λ),*
2. *if f belongs to $L_q(\mu)$ where $1 \leqq q < \infty$, then for every non-negative integer k the sequence $\{P^{n\delta+k}f\}$ converges in $L_q(\mu)$ to $\sum_{i=0}^{\delta-1} c_{i+k} 1_{C_i}$,*
3. *if $f \geqq 0$, then for every non-negative integer k,*

$$\liminf_{n \to \infty} P^{n\delta+k} f = \sum_{i=0}^{\delta-1} c_{i+k} 1_{C_i} \text{ a.e. } (\lambda).$$

Proof. By Theorem 2.3, μI_{C_i} is P^δ-invariant, $\mu(C_i) = 1/\delta$, and $\mu I_{C_i} P^k = \mu I_{C_{i+k-j\delta}}$ where j is the largest non-negative integer for which $j\delta \leqq i + k$. Hence

$$\int_{C_i} P^k f \, d\mu = \int f \, d(\mu I_{C_i} P^k) = \int f \, d\mu I_{C_{i+k-j\delta}} \, f \, d\mu = c_{i+k}.$$

Now P^δ acting on C_i is aperiodic. For any $f \in L_\infty(\lambda)$, applying Theorem 3.1, we arrive at the conclusion that the sequence $\{P^{n\delta} f\}$ converges a.e. (λ) on C_i to the limit $c_i = \delta \int_{C_i} f \, d\mu$. Hence the sequence converges a.e. (λ) to $\sum_{i=0}^{\delta-1} c_i I_{C_i}$. Replacing f by $P^k f$ in the sequence, we conclude that the sequence $\{P^{n\delta+k} f\}$ converges a.e. (λ) to $\sum_{i=0}^{\delta-1} d_i 1_{C_i}$ where $d_i = \delta \int_{C_i} P^k f \, d\mu = c_{i+k}$. In a similar manner, 2 may be derived from Theorem 3.2 and 3 may be derived from Theorem 3.3.

THEOREM 3.6. *Let P be a λ-continuous, conservative and ergodic Markov operator whose nontrivial invariant measure μ is finite (μ is normalized as usual). Let the period of P be δ and $C_0, C_1, \cdots, C_{\delta-1}$ be the totality of distinct, indecomposable P^δ-closed sets with $C_0 = A(C_1), \cdots, C_{\delta-2} = A(C_{\delta-1})$. For $j > \delta - 1$, let $C_j = C_{j-n\delta}$ where n is the largest non-negative integer such that $n\delta \leqq j$. For every non-negative integer k, define function g_k on $X \times X$ by*

$$g_k(x, y) = \delta \sum_{i=0}^{\delta-1} 1_{C_i \times C_{i+k}}(x, y) \frac{d\mu}{d\lambda}(y).$$

Then the sequence $\{p^{(n\delta+k)}(\cdot, \cdot)\}$ converges in $L_1(\nu \times \lambda)$ to g_k for every $\nu \in \mathscr{A}^+(\lambda)$. We also have

$$\liminf_{n \to \infty} p^{(n\delta+k)}(x, y) = g_k(x, y) \quad \textit{for } (\lambda \times \lambda) \textit{ almost all } (x, y).$$

Proof. As in the proof of Theorem 3.4 we define $\bar{p}^{(n)}(x, y)$ by (3.12) and $\bar{p}(x, y) = \bar{p}^{(1)}(x, y)$. Then for (λ) almost all x, $\bar{p}^{(n)}(x, \cdot) \in L_1(\mu)$ with $L_1(\mu)$ norm equal to 1. Furthermore, since $C_i = A^k(C_{i+k})$ we have $P^k 1_{C_{i+k}} \geqq 1_{C_i}$. Hence

$$\sum_{i=0}^{\delta-1} P^k 1_{C_{i+k}} \geqq \sum_{i=0}^{\delta-1} 1_{C_i}. \tag{3.13}$$

However, equality holds in (3.13) since both sides of (3.13) are equal to 1. Hence $P^k 1_{C_{i+k}} = 1_{C_i}$, therefore, $P^k 1_{C_{i+k}} = 1_{C_i} P^k 1_{C_{i+k}}$ and $1_{C_i} P^k 1_{X - C_{i+k}} = 0$. Thus for every $f \in L_\infty(\lambda)$, $P^k I_{C_{i+k}} f = I_{C_i} P^k I_{C_{i+k}} f = I_{C_i} P^k f$. In terms of the density function $\bar{p}^{(k)}(x, y)$, we then have

$$1_{C_i}(x) \bar{p}^{(k)}(x, y) = 1_{C_i}(x) \bar{p}^{(k)}(x, y) 1_{C_{i+k}}(y) = \bar{p}^{(k)}(x, y) 1_{C_{i+k}}(y)$$

for $(\lambda \times \lambda)$ almost all (x, y). Hence for (λ) almost all $x \in C_i$ $\bar{p}^{(k)}(x, \cdot) = \bar{p}^{(k)}(x, \cdot) 1_{C_{i+k}}$. Now we consider the μ-reverse Q of P as in the proof of Theorem 3.4. Since a set is P^n-closed if and only if it is Q closed, Q also has δ as its period and $\{C_0, \cdots, C_{\delta-1}\}$ is also the collection of all indecomposable Q^δ-closed sets.

Applying Theorem 3.5 to Q and $\bar{p}^{(k)}(x,\cdot)$ we have the sequence $\{Q^{n\delta}\bar{p}^{(k)}(x,\cdot)\} = \{\bar{p}^{(n\delta+k)}(x,\cdot)\}$ converging in $L_1(\mu)$ to $\delta\cdot 1_{C_{i+k}}$ for (λ) almost all $x\in C_i$ and $\liminf_{n\to\infty}\bar{p}^{(n\delta+k)}(x,y)=\delta$ for $(\lambda\times\lambda)$ almost all $(x,y)\in C_i\times C_{i+k}$. Hence $\liminf_{n\to\infty}\bar{p}^{(n\delta+k)}(x,y)=\delta\sum_{i=0}^{\delta-1}1_{C_i\times C_{i+k}}(x,y)$ and $\liminf_{n\to\infty}p^{(n\delta+k)}(x,y)=g_k(x,y)$ follows immediately for $(\lambda\times\lambda)$ almost all (x,y). Moreover, if we define h_n by

$$h_n(x)=\int\left|\,\bar{p}^{(n\delta+k)}(x,y)-\sum_{i=0}^{\delta-1}\delta 1_{C_i\times C_{i+k}}(x,y)\,\right|\mu(dy),$$

then $h_n(x)\to 0$ for (λ) almost all x. We also have, for (λ) almost all x

$$h_n(x)\leqq\int\bar{p}^{(n\delta+k)}(x,y)\mu(dy)+\int\delta\sum_{i=0}^{\delta-1}1_{C_i\times C_{i+k}}(x,y)\mu(dy)\leqq 2.$$

Hence for any $\nu\in\mathscr{A}^+(\lambda)$, $\int h_n(x)\nu(dx)\to 0$, i.e.,

$$\lim_{n\to\infty}\iint\left|\,\bar{p}^{(n\delta+k)}(x,y)-\sum_{i=0}^{\delta-1}\delta 1_{C_i\times C_{i+k}}(x,y)\,\right|\mu(dy)\nu(dx)=0. \tag{3.14}$$

The $L_1(\nu\times\lambda)$ convergence of $\{p^{n\delta+k}(\cdot,\cdot)\}$ to g_k then follows from (3.14).

Now we turn to study the case that the invariant measure μ is *infinite*. We shall need the following

LEMMA 3.12. *If a set E has the property that there exists an increasing sequence $\{n_k\}$ of positive integers for which the sequence of functions*:

$$P^{n_0}1_E,\ P^{n_1}1_E,\ P^{n_1+1}1_E,\cdots,\ P^{n_k}1_E,\ P^{n_k+1}1_E,\cdots,\ P^{n_k+k}1_E,\cdots \tag{3.14}$$

converges to 0 *uniformly on E, then* $\limsup_{n\to\infty}P^n1_E=0$ *a.e.* (λ).

Proof. Let ε be an arbitrary positive number. Then there is a positive integer k_1 such that $P^{n_{k_1}}1_E$ and all the terms in the sequence (3.14) which follow $P^{n_{k_1}}1_E$ are $<\varepsilon$ on E. Let k_2 be an integer such that $k_2>n_{k_1}$. Then $n_{k_2}>n_{k_1}$, hence

$$P^{n_{k_2}}1_E<\varepsilon,\ P^{n_{k_2}+n_{k_1}}<\varepsilon \text{ on } E.$$

Let $k_3>n_{k_1}+n_{k_2}$, then $n_{k_3}>n_{k_1}$ and

$$P^{n_{k_3}}1_E<\varepsilon,\ P^{n_{k_3}+n_{k_1}}1_E<\varepsilon,\ P^{n_{k_3}+n_{k_2}+n_{k_1}}1_E<\varepsilon \text{ on } E,$$

$\cdots$, etc. In this manner, we obtain a sequence $\{n_{k_i}\}$ of positive integers. We shall rename it $\{m_i\}$. This sequence has the property that, for every positive integer i,

$$P^{m_i}1_E<\varepsilon,\ P^{m_i+m_{i-1}}1_E<\varepsilon,\cdots,\ P^{m_i+m_{i-1}+\cdots+m_1}1_E<\varepsilon \tag{3.15}$$

on E.

Now suppose $\limsup_{n\to\infty}P^n1_E$ is not equal to 0 a.e. (λ). Then $\liminf_{n\to\infty}P^n1_{E'}$ is not equal to 1 a.e. (λ) where $E'=X-E$. Since, by Lemma 3.2, $\liminf_{n\to\infty}P^n1_{E'}$ is a constant function, $\liminf_{n\to\infty}P^n1_{E'}=a$ a.e.(λ) for some $a<1$. Let $b=1-a$

and $\varepsilon < b/2$. Let i_0 be an integer such that $i_0(b-2\varepsilon) > 1$. By Lemma 3.8, there is a point x of X and a positive integer N such that

$$P^N 1_{E'}(x) < a+\varepsilon,\ P^{N+m_1}1_{E'}(x) < a+\varepsilon, \cdots,\ P^{N+m_1+\cdots+m_{i_0}}1_{E'}(x) < a+\varepsilon.$$

Then

$$P^N 1_E(x) > b-\varepsilon,\ P^{N+m_1}1_E(x) > b-\varepsilon, \cdots,\ P^{N+m_1+\cdots+m_{i_0}}1_E(x) < b-\varepsilon. \tag{3.16}$$

Now let

$$p_1(x,y) = \int_{E'} p^{(N)}(x,y_1)p^{(m_1)}(y_1,y)\lambda(dy_1);$$

$$p_2(x,y) = \int_{E'}\int_{E'} p^{(N)}(x,y_1)p^{(m_1)}(y_1,y_2)p^{(m_2)}(y_2,y)\lambda(dy_1)\lambda(dy_2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_{i_0}(x,y) = \int_{E'}\cdots\int_{E'} p^{(N)}(x,y_1)p^{(m_1)}(y_1,y_2)\cdots p^{(m_{i_0})}(y_{i_0},y)\lambda(dy_1)\lambda(dy_2)\cdots\lambda(dy_{i_0}),$$

and

$$K_0(x,E) = P^N 1_E(x),$$

$$K_1(x,E) = \int_E p_1(x,y)\lambda(dy) = P^N I_{E'}P^{m_1}1_E(x),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$K_{i_0}(x,E) = \int_E p_{i_0}(x,y)\lambda(dy) = P^N I_{E'}P^{m_1}I_{E'}\cdots P^{m_{i_0-1}}I_{E'}P^{m_{i_0}}1_E(x).$$

Then

$$K_0(x,E) + K_1(x,E) + \cdots + K_{i_0}(x,E) \leqq 1. \tag{3.17}$$

(3.17) may be proved by an elementary method similar to the one used in the proof of Lemma 6.1 of [8], or by constructing the infinite product space Ω and the infinite product σ-algebra $\mathscr{F}$ as in the proof of Lemma 3.11 and then defining a probability measure η on $\mathscr{F}$ by

$$\eta[X_1 \in A_1, \cdots, X_n \in A_n]$$
$$= \int_{A'}\cdots\int_{A_n} p(x,x_1)(x_1,x_2)\cdots p(x_{n-1},x_n)\lambda(dx_1)\cdots\lambda(dx_n).$$

Then the left-hand side of (3.17) is

$$\eta[X_n \in E \text{ for some } n \text{ equal to one of } N,\ N+m_1, \cdots, N+m_1+\cdots+m_{i_0}].$$

Now, for $1 \leqq k \leqq i_0$

$$P^{N+m_1+\dots+m_k}1_E(x)$$
$$= P^N I_E P^{m_1+\dots+m_k}1_E(x) + P^N I_{E'} P^{m_1} I_E P^{m_2+\dots+m_k}1_E(x) + \cdots$$
$$+ P^N I_{E'} P^{m_1} I_{E'} \cdots P^{m_{k-2}} I_{E'} P^{m_{k-1}} I_E P^{m_k} 1_E(x) + K_k(x,E).$$

Applying (3.15), we have

$$P^{N+m_1+\dots+m_k}1_E(x) \leqq [K_0(x,E) + \cdots + K_{k-1}(x,E)]\varepsilon + K_k(x,E)$$
$$\leqq \varepsilon + K_k(x,E).$$

Hence $K_k(x,E) \geqq b - 2\varepsilon$ by (3.16). Thus we obtain the inequality

$$K_1(x,E) + \cdots + K_{i_0}(x,E) \geqq i_0(b - 2\varepsilon) > 1$$

which contradicts (3.17). Thus the conclusion of Lemma 3.12 is proved.

THEOREM 3.7. *If P is a λ-continuous, conservative, ergodic and aperiodic Markov operator whose invariant measure μ is infinite, and if E is a set of finite μ measure, then, for every positive number ε, there is a set $E_\varepsilon \subset E$ such that $\mu(E_\varepsilon) < \varepsilon$ and $\lim_{n\to\infty} P^n 1_{E-E_\varepsilon} = 0$ a.e. (λ).*

Proof. Since E is a set of finite μ measure, $\liminf_{n\to\infty} P^n 1_E = 0$ a.e. (λ) by Lemma 3.2. By Lemma 3.8, for any positive number δ there is a set A with $\lambda(X - A) < \delta$ and an increasing sequence $\{n_i\}$ of positive integers such that the sequence of functions:

$$P^{n_0}1_E,\ P^{n_1}1_E,\ P^{n_1+1}1_E,\ P^{n_2}1_E,\ P^{n_2+1}1_E,\ P^{n_2+2}1_E,\ P^{n_3}1_E, \cdots \tag{3.18}$$

converges to 0 uniformly on A. We choose δ to satisfy the condition that $\mu(E \cap B) < \varepsilon$ whenever $\lambda(B) < \delta$. This is possible because μI_E is absolutely continuous to λ. Take E_ε to be $E - A$, then the sequence (3.18) converges to 0 uniformly on $E - E_\varepsilon$. Since $1_{E-E_\varepsilon} \leqq 1_E$, the sequence of functions:

$$P^{n_0}1_{E-E_\varepsilon},\ P^{n_1}1_{E-E_\varepsilon},\ P^{n_1+1}1_{E-E_\varepsilon},\ P^{n_2}1_{E-E_\varepsilon}, \cdots$$

converges to 0 uniformly on $E - E_\varepsilon$. Applying Lemma 3.12, we have $\lim_{n\to\infty} P^n 1_{E-E_\varepsilon} = 0$ a.e. (λ).

THEOREM 3.8. *If P is a λ-continuous, conservative, ergodic and aperiodic Markov operator whose invariant measure μ is infinite, then there is an increasing sequence $\{E_k\}$ of sets such that $\bigcup_{k=1}^\infty E_k = X$ and $\lim_{n\to\infty} P^n 1_{E_k} = 0$ a.e. (λ) for every k.*

Proof. Since μ is σ-finite, there exists an increasing sequence $\{F_k\}$ of sets such that $\bigcup_{k=1}^\infty F_k = X$ and $\mu(F_k) < \infty$ for every k. By Theorem 3.7, for each k, there is a set $E_k \subset F_k$ such that $\mu(F_k - E_k) < 1/2^k$ and $\lim_{n\to\infty} P^n 1_{E_k} = 0$ a.e. (λ). We nay assume the sequence $\{E_k\}$ to be monotonically increasing. Then

$$\mu\left(X - \bigcup_{k=1}^{\infty} E_k\right) = \mu\left(X - \bigcup_{k=N}^{\infty} E_k\right) \leqq \mu\left[\bigcup_{k=1}^{\infty} (F_k - E_k)\right] \leqq \frac{1}{2^{N-1}}.$$

Hence $\mu(X - \bigcup_{k=1}^{\infty} E_k) = 0$ and the theorem is proved.

THEOREM 3.9. *Let P be a λ-continuous, conservative and ergodic Markov operator whose invariant measure μ is infinite, then there is an increasing sequence $\{E_k\}$ of sets such that*

$$\bigcup_{k=1}^{\infty} E_k = X \text{ and } \lim_{n\to\infty} P^n 1_{E_k} = 0 \text{ a.e. } (\lambda) \text{ for } k = 1, 2, \cdots.$$

Proof. Since P has a finite period δ, the space X is partitioned into δ sets: $C_0, C_1, \cdots, C_{\delta-1}$, of which each is a $\mathscr{C}^{(\delta)}$ atom. Then P^δ, acting on C_i alone, is aperiodic and has μI_{C_i} as its invariant measure. By Theorem 2.3, μI_{C_i} is also infinite. Applying Theorem 3.8, we obtain an increasing sequence $\{E_{i,k}, k=1,2,\cdots\}$ of sets such that $C_i = \bigcup_{k=1}^{\infty} E_{i,k}$ and $\lim_{n\to\infty} P^{n\delta} 1_{E_{i,k}} = 0$, a.e. (λ). Let $E_k = \bigcup_{i=0}^{\delta-1} E_{i,k}$. Then $\{E_k\}$ is an increasing sequence of sets such that $X = \bigcup_{k=1}^{\infty} E_k$ and $\lim_{n\to\infty} P^{n\delta} 1_{E_k} = 0$ a.e. (λ) for every k. Now

$$P^{n\delta+i} 1_{E_k}(x) = \int P^{n\delta} 1_{E_k} d\nu_x^{(i)},$$

hence $\lim_{n\to\infty} P^{n\delta+i} 1_{E_k} = 0$ a.e. (λ) for $i = 0, 1, \cdots, \delta - 1$ and the conclusion of the theorem follows immediately.

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SYRACUSE UNIVERSITY,
SYRACUSE, NEW YORK