FLAT CHAINS OVER A FINITE COEFFICIENT GROUP(¹)

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1. **Introduction.** Let G be a metric abelian group. Our purpose is to develop a theory of integration over chains with coefficients in G. The notion of chain should be broad enough to allow for general existence theorems for minimum problems of calculus of variations. At the same time, if G is a discrete group then the chains should be very nearly (in a measure theoretic sense) chains of class $C^{(1)}$.

When G is the additive group Z of integers such a theory was developed by Federer and the author [FF]. The "chains" in this paper are currents of certain kinds. A current (in the sense of DeRham) is a continuous linear function on a space of differential forms. Since the notion of current over an arbitrary coefficient group G is not defined, a different approach is needed. The one we shall follow is due to Whitney [W, p. 152] for chains with real coefficients.

Let $P_k(G)$ denote the group of polyhedral chains of dimension k in n-dimensional euclidean E^n, with coefficients in G. Let $M(P)$ be the elementary k-dimensional area of a polyhedral k-chain P, and let

$$W(P) = \inf_{Q,R} \{M(Q) + M(R) \colon P = Q + \partial R\}. \tag{1.1}$$

The *Whitney flat distance* between $P_1, P_2 \in P_k(G)$ is $W(P_1 - P_2)$. Let $C_k(G)$ be the W-completion of $P_k(G)$. The elements of $C_k(G)$ will be called *flat k-chains* over G. Every flat chain A has a boundary ∂A and a mass $M(A)$, defined below in §3. If G is a discrete group, then a flat k-chain A is called *rectifiable* if A is the M-limit of k-chains of class $C^{(1)}$. The idea of rectifiable chain can also be described in terms of local tangential properties almost everywhere in the sense of Hausdorff k-measure. See §9.

The main result of the paper is the following:

THEOREM. *Let G be a finite group. Then every flat chain A of finite mass is rectifiable.*

This will be proved in §10. The theorem is also true when $G = Z$, by [FF, 8.13] together with (5.5), (5.6) below. However, the method of proof is quite different. When $G = Z_2$, the integers mod 2, a result similar to our main theorem was proved by Ziemer [Z] for the special dimensions $k = 0, 1, n-1, n$.

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Since mass is lower semicontinuous, as a corollary we have a closure theorem:

THEOREM. *Let G be a finite group. If A_j is rectifiable for $j = 1, 2, \cdots$, $M(A_j)$ is bounded, and $W(A_j - A) \to 0$ as $j \to \infty$, then A is rectifiable.*

One of the main tools we shall use is a deformation theorem for flat chains having compact support and satisfying $M(A) + M(\partial A) < \infty$. See §7. The proof is just the same as for the corresponding result in [FF, §5], as soon as some rather routine preliminaries about intersections, Lipschitz maps, and homotopies have been established. Another tool is a structure theorem for sets of finite Hausdorff measure [F]. With any flat chain A of finite mass is associated a measure μ_A such that $\mu_A(X) = M(A \cap X)$ for every Borel set $X \subset E^n$. After bounding above the upper k-density $D_k^*\mu_A(x)$ and bounding away from 0 the lower k-density $D_{*k}\mu_A(x)$, in §8, we find a set X_A of finite Hausdorff k-measure whose complement is μ_A-null. Using the structure theorem it is shown that X_A is a (μ_A, k)-rectifiable set, in the sense of [F], and this implies that A is a rectifiable flat chain.

The finiteness of G is used only to show that $D^*\mu_A(x)$ is μ_A-almost everywhere finite; see (8.2). It is an interesting open problem to find a proof of this when G is a countable discrete group, and thereby extend the whole theory to such coefficient groups.

2. **Polyhedral chains.** Let E^n denote euclidean n-space. Its points will ordinarily be denoted by $x = (x^1, \cdots, x^n)$. The closure, interior, and frontier of a set $X \subset E^n$ are denoted respectively by $\mathrm{cl}\, X$, $\mathrm{int}\, X$, $\mathrm{fr}\, X$, and the complement of X by X^c.

Let k be an integer such that $0 \leqq k \leqq n$, and let $P_k(Z)$ denote the group of polyhedral chains of dimension k which have integer coefficients. The group $P_k(Z)$ may be defined precisely either as in [W, p. 153] or as a subgroup of the group of all currents in E^n of dimension k as in [FF, p. 463].

Let G be an abelian group provided with a translation invariant metric. Let $|g|$ denote the distance between $g \in G$ and the group identity 0. We shall assume that G is a complete metric space. Further assumptions about G will be needed in §7. For the detailed information about the local structure of flat chains of finite mass in §§9 and 10 it will be assumed that G is a finite group.

If H is a closed subgroup of G, then on G/H we may put the quotient metric

$$|\bar{g}| = \inf\{|g| : g \in \bar{g}\}.$$

If $G = Z$ is the additive group of integers, then for $|g|$ we take the usual absolute value. The groups $Z_p = Z/pZ$ are of special interest. We give Z_p the quotient metric.

Let $P_k(G) = G \otimes P_k(Z)$. It is the group of polyhedral k-chains in E^n with coefficients in G. The polyhedral chains of arbitrary dimension k, with the usual boundary operation ∂, form a chain complex $P_*(G)$. [For $k = 0$, we set $\partial P = 0$]. If $P \in P_k(G)$, then $P = \sum g_i \sigma_i$ (finite sum), where the σ_i are nonoverlapping oriented k-dimensional convex cells. Let $M(\sigma_i)$ be the elementary k area of σ_i. Then the *k-area* of P is

$$M(P) = \sum |g_i| M(\sigma_i).$$

If $k = 0$, $\sigma_i = \{x_i\}$ and $M(\sigma_i) = 1$. No orientation need be assigned the 0-cell σ_i.

If we take $M(P_1 - P_2)$ as the distance between P_1 and P_2, then $P_k(G)$ becomes a metric space. However, this metric is too strong for most purposes A more suitable one is provided by the *Whitney flat distance* $W(P_1 - P_2)$, where

$$W(P) = \inf_{Q,R} \{M(Q) + M(R) \colon P = Q + \partial R\}. \tag{1.1}$$

In this definition, $Q \in P_k(G)$, $R \in P_{k+1}(G)$. If $P = Q + \partial R$, then $\partial P = \partial Q$, and from the definition we see that $W(\partial P) \leqq W(P)$. From the definition and corresponding properties of M,

$$W(-P) = W(P), \quad W(P_1 + P_2) \leqq W(P_1) + W(P_2).$$

To show that W defines a metric we must show that $W(P) > 0$ if $P \neq 0$.

For definiteness, let us agree that a convex k-cell σ is open relative to the k-plane in which σ lies, $k \geqq 1$. If U is an open convex polytope in E^n (finite intersection of open half-spaces), then $\sigma \cap U$ is also a convex k-cell or is empty. If $P = \sum g_i \sigma_i$ is a polyhedral k-chain, then we set $P \cap U = \sum g_i(\sigma_i \cap U)$.

In particular, let $1 \leqq l \leqq n$ and consider the family of half-spaces $H_s = \{x : x^l < s\}$.

(2.1) Lemma. *If $\{P_j\}$ is a sequence in $P_k(G)$ such that $\sum_{j=1}^{\infty} W(P_j) < \infty$, then for almost all s, $\sum_{j=1}^{\infty} W(P_j \cap H_s) < \infty$.*

Proof. By definition of W, $P_j = Q_j + \partial R_j$ where $\sum [M(Q_j) + M(R_j)] < \infty$. Moreover,

$$P_j \cap H_s = Q_j \cap H_s + S_{js} + \partial(R_j \cap H_s),$$

where

$$S_{js} = (\partial R_j) \cap H_s - \partial(R_j \cap H_s).$$

By definition of W,

$$\begin{aligned} W(P_j \cap H_s) &\leqq M(Q_j \cap H_s + S_{js}) + M(R_j \cap H_s) \\ &\leqq M(Q_j) + M(S_{js}) + M(R_j). \end{aligned}$$

Let us use the elementary estimate

$$\int_{-\infty}^{\infty} M(S_{js}) ds \leqq M(R_j).$$

This is a very special case of a widely used estimate in the geometric theory of measure, which will be proved for flat chains below (Theorem 5.7). Then $\sum M(S_{js}) < \infty$ for almost every s, since $\sum M(R_j) < \infty$. For each such s, $\sum W(P_j \cap H_s) < \infty$. This proves (2.1).

Let us call an open n-dimensional interval I exceptional (relative to $\{P_j\}$) if $\sum W(P_j \cap I) = \infty$. From Lemma (2.1), I is exceptional only when its faces lie on hyperplanes taken from a certain null set.

Let $\Pi \subset E^n$ be a k-plane. If $P = \sum g_i\sigma_i$ is a polyhedral k-chain, its orthogonal projection onto Π is $P' = \sum g_i\sigma_i'$ where σ_i' is the projection of the oriented k-cell σ_i. Projection does not increase k-area and $(\partial P)' = \partial P'$. From these observations, if $P \in P_k(G)$ lies in a k-plane Π, then in the definition of W we may as well assume that Q and R lie in Π. But then $R = 0$, $Q = P$. Hence, $M(P) = W(P)$ if P lies in a k-plane.

(2.2) THEOREM. *If $P \neq 0$, then $W(P) > 0$.*

Proof. Let $P = \sum g_i\sigma_i \neq 0$. Suppose that $W(P) = 0$, and apply Lemma (2.1) with $P_j = P$ for $j = 1, 2, \cdots$. Let I be a nonexceptional n-dimensional interval such that $P \cap I = g_i(\sigma_i \cap I)$ for some i. Then

$$0 \neq |g_i| M(\sigma_i) = M(P \cap I) = W(P \cap I) = 0,$$

a contradiction.

(2.3) THEOREM. *M is a lower semicontinuous function on $P_k(G)$.*

Proof. We must show that $W(P_j - P) \to 0$ implies

$$M(P) \leqq \liminf_{j \to \infty} M(P_j). \tag{2.4}$$

It suffices to prove this for sequences such that $\sum W(P_j - P) < \infty$. Let I be nonexceptional with respect to the sequence $\{P_j - P\}$ and be such that $P^0 = P \cap I$ lies in some k-plane Π as in the preceding proof. Let P_j^0 be the projection of $P_j \cap I$ onto Π. Then

$$M(P_j^0 - P^0) = W(P_j^0 - P^0) \leqq W[(P_j \cap I) - P^0],$$

and the right side tends to 0. Hence

$$M(P^0) = \lim_{j \to \infty} M(P_j^0) \leqq \liminf_{j \to \infty} M(P_j \cap I). \tag{2.5}$$

Given $\varepsilon > 0$ there exists a finite disjoint collection $I_1, \cdots, I_p$ of such intervals such that $M[P \cap (I_1 \cup \cdots \cup I_p)^c] < \varepsilon$. From this fact and (2.5) follows formula (2.4).

3. **Flat chains.** Let $C_k(G)$ be the W-completion of $P_k(G)$. The elements of the group $C_k(G)$ are called *flat chains* of dimension k (or *flat k-chains*) and will

be denoted by $A, B, C, \cdots$. The distance between two flat k-chains A and B is still denoted by $W(A-B)$.

If $\{P_j\}$ is a fundamental sequence of polyhedral k-chains, then

$$W(\partial P_i - \partial P_j) = W[\partial(P_i - P_j)] \leqq W(P_i - P_j).$$

Hence $\{\partial P_j\}$ is a fundamental sequence of polyhedral $(k-1)$-chains. If P_j tends to A, then ∂P_j tends to a flat $(k-1)$-chain ∂A called the *boundary* of A. Moreover, $W(\partial A) \leqq W(A)$. With this definition of boundary, the flat chains of arbitrary dimension $k = 0, 1, \cdots, n$ form a chain complex $C_*(G)$ and the boundary operator ∂ is continuous.

By *mass* of a flat chain A let us mean the smallest number λ such that there is a sequence $\{P_j\}$ of polyhedral chains tending to A with $M(P_j)$ tending to λ If $M(P_j) \to \infty$ for every such sequence, then $M(A) = \infty$. By Theorem (2.3), $M(P)$ agrees with its previous definition if P is a polyhedral chain. From the definition, mass is the largest lower semicontinuous extension from $P_k(G)$ to $C_k(G)$ of the elementary k-area for polyhedra. Let

$$M_k(G) = \{A \in C_k(G) : M(A) < \infty\}.$$

Since $M(A+B) \leqq M(A) + M(B)$ and $M(-A) = M(A), M_k(G)$ is a subgroup of $C_k(G)$. The number $M(A-B)$ is the *M-distance* between $A, B \in M_k(G)$. When $k = n$, $W(A) = M(A)$ since this is true for polyhedral n-chains. In general, $W(A) \leqq M(A)$. When $k < n$ the M-metric is in fact much stronger than the W-metric. [When we mean M-convergence this will be explicitly indicated.]

(3.1) THEOREM. *For every flat chain A,*

$$W(A) = \inf_{B,C} \{M(B) + M(C) : A = B + \partial C\}.$$

Proof. Given $B \in C_k(G)$, $C \in C_{k+1}(G)$, with $A = B + \partial C$, let $\{Q_j\}$, $\{R_j\}$ be sequences in $P_k(G)$ tending respectively to B, C, such that $M(Q_j) \to M(B)$, $M(R_j) \to M(C)$. Then $P_j = Q_j + \partial R_j$ tends to A and $W(P_j) \leqq M(Q_j) + M(R_j)$. Hence, $W(A) \leqq M(B) + M(C)$ for every such B, C.

On the other hand, let $\{P_j\}$ be a sequence in $P_k(G)$ tending to A such that $\Sigma W(P_{j+1} - P_j) < \infty$. Choose polyhedral chains Q_j, R_j such that

$$P_{j+1} - P_j = Q_j + \partial R_j, \quad \Sigma\, [M(Q_j) + M(R_j)] < \infty,$$

and polyhedral chains Q^j, R^j such that

$$P_j = Q^j + \partial R^j, \qquad M(Q^j) + M(R^j) < W(P_j) + \varepsilon_j,$$

where $\varepsilon_j \to 0$. Let

$$B_j = Q^j + \sum_{i=j}^{\infty} Q_i, \quad C_j = R^j + \sum_{i=j}^{\infty} R_i.$$

Then $A = B_j + \partial C_j$ for each $j = 1,2,\cdots$, and $M(B_j) + M(C_j) \leqq W(A) + \varepsilon_j'$, where $\varepsilon_j' \to 0$. This proves (3.1).

It is often important to know not merely that A has finite mass but that both A and ∂A have finite mass. Let

$$N(A) = M(A) + M(\partial A),$$

and

$$N_k(G) = \{A \in C_k(G) \colon N(A) < \infty\}.$$

Since $\partial^2 A = 0$, $N(\partial A) \leqq N(A)$. The flat chains of arbitrary dimensions satisfying $N(A) < \infty$ form a chain complex $N_*(G)$.

SUPPORTS. Let A be a flat chain and X a closed set. Then A is *supported by* X if, for every open set U containing X, there is a sequence $\{P_j\}$ of polyhedral chains tending to A such that each P_j lies in U (i.e. the cells of each P_j are contained in U). If A is supported by X, then clearly so is ∂A.

If there is a smallest set X which supports A, then X is called the *support of* A and is denoted by $\operatorname{spt} A$.

Let us show that $\operatorname{spt} A$ exists in case A is supported by some compact set X_0. Let F be the collection of all closed $X \subset X_0$ such that A is supported by X. To show that F has a smallest element, it suffices to prove: (1) if F' is any subcollection of F which is totally ordered by $\subset$, then $X' = \bigcap_{X \in F'} X$ belongs to F; and (2) $X \in F$, $Y \in F$ imply $X \cap Y \in F$. But (1) follows from the fact that if U is any open set containing X', then U contains some $X \in F'$ (otherwise the collection of compact sets $X - U, X \in F'$, would have the finite intersection property, and hence nonempty intersection). To prove (2), there exist sequences $\{P_j\},\{Q_j\}$ tending to A such that $\operatorname{spt} P_j \subset \delta_j$-nbd. of X, $\operatorname{spt} Q_j \subset \delta_j$-nbd. of Y, where $\delta_j \to 0$, and $\sum W(P_j - Q_j) < \infty$. Given an open set U containing $X \cap Y$, let Z be a figure composed of nonoverlapping intervals $I_1,\cdots,I_p$ nonexceptional with respect to the sequence $\{P_j - Q_j\}$ such that $X_0 - U \subset Z$ and $\operatorname{cl} Z \cap X \cap Y$ is empty. We may choose these intervals small enough that either $I_k \cap X$ or $I_k \cap Y$ is empty for each $k = 1,\cdots,p$. If $I_k \cap X$ is empty, then $P_j \cap I_k = 0$ for large enough j; and if $I_k \cap Y$ is empty, then $P_j \cap I_k \to 0$ since $(P_j - Q_j) \cap I_k \to 0$ and $Q_j \cap I_k = 0$ for large enough j. Hence

$$P_j \cap Z = \sum_k P_j \cap I_k \to 0.$$

The sequence $\{P_j \cap Z^c\}$ tends to A, and $P_j \cap Z^c$ lies in U for large enough j.

In the next section we show that if A is any flat chain of finite mass then spt A exists. However, we have not settled the question whether $\operatorname{spt} A$ exists for every $A \in C_k(G)$.

Let

$$N_*^0(G) = \{A \in N_*(G) \colon A \text{ has compact support}\}.$$

If G is the additive group of real numbers, then $N_*^0(G)$ may be identified with the chain complex of normal currents in E^n in the sense of [FF]. A mass preserving isomorphism between these two chain complexes is provided by requiring that each polyhedron with real coefficients correspond to itself. This follows from [FF, §7]. The chain complex $N_*^0(Z)$ corresponds to the subcomplex consisting of the integral currents.

Let $P = \sum n_i\sigma_i$, the n_i being integers, For $g \in G$ let $gP = \sum n_i g\sigma_i$. If $P_j \to T \in C_k(Z)$, then gP_j tends to a limit denoted by gT.

4. **Flat chains of finite mass.** Let A be a flat chain such that $M(A) < \infty$. With A is associated a measure μ_A, and with every μ_A-measurable set $X \subset E^n$ a flat chain $A \cap X$ such that $\mu_A(X) = M(A \cap X)$. This can be done in the following way. Let $\{P_j\}$ be a sequence of polyhedral chains tending to A such that $M(P_j) \to M(A)$. For $j = 1, 2, \cdots$ let μ_j be the measure determined by the property $\mu_j(I) = M(P_j \cap I)$ if I is any open n-dimensional interval. Then $\mu_j(E^n) = M(P_j)$ is bounded. By taking subsequences we may assume that $\sum W(P_{j+1} - P_j) < \infty$ and that μ_j tends weakly to a limit. The limit measure is μ_A. It will turn out that μ_A is the same for all such sequences $\{P_j\}$.

In this section let us call an n-dimensional interval I exceptional if either $\sum W[(P_{j+1} - P_j) \cap I] = \infty$ or $\mu_A(\operatorname{fr} I) > 0$. If I is nonexceptional, then the sequence $\{P_j \cap I\}$ tends to a limit, denoted by $A \cap I$. Moreover, since $\mu_A(\operatorname{fr} I) = 0$, $\mu_j(I) \to \mu_A(I)$ and $\mu_j(I^c) \to \mu_A(I^c)$.

(4.1) LEMMA. *If I is nonexceptional, then* $M(A \cap I) = \mu_A(I)$, $M(A - A \cap I) = \mu_A(I^c)$.

Proof. Since $P_j \cap I \to A \cap I$ and $P_j - P_j \cap I \to A - A \cap I$, we have by lower semicontinuity of mass $M(A \cap I) \leqq \mu_A(I)$ and $M(A - A \cap I) \leqq \mu_A(I^c)$. But

$$\mu_A(I) + \mu_A(I^c) = \mu_A(E^n) = M(A) \leqq M(A \cap I) + M(A - A \cap I),$$

which proves (4.1).

Next if $X = I_1 \cup \cdots \cup I_p$, where the intervals I_l are nonoverlapping and nonexceptional, then we set $A \cap X = \sum A \cap I_l$. Applying Lemma (4.1) repeatedly, we find that

$$M(A \cap X) = \sum_{l=1}^{p} \mu_A(I_l) = \mu_A(X),$$

$$M(A - A \cap X) = \mu_A(X^c).$$

Any μ_A-measurable set X is the limit in μ_A-measure of a sequence $\{X_j\}$ of figures of the above type. Then $A \cap X$ is defined as the M-limit of the sequence $\{A \cap X_j\}$. The set function $A \cap \cdot$ so defined (with values in $M_k(G)$) is countably additive, and $M(A \cap X) = \mu_A(X)$ for every μ_A-measurable X.

If $\{\tilde{P}_j\}$ is another sequence of polyhedral chains which has the above properties and tends to A, then by taking subsequences we may assume that $\sum W(P_j - \tilde{P}_j) < \infty$. Except for intervals I whose faces lie on hyperplanes of a certain null set, $P_j \cap I$ and $\tilde{P}_j \cap I$ tend to the same limit by (2.1). It follows that μ_A and $A \cap \cdot$ do not depend on the particular sequence $\{P_j\}$ used in their definition.

NOTE. If $\tilde{P}_j \to A, M(\tilde{P}_j)$ is bounded (but not necessarily $M(\tilde{P}_j) \to M(A)$), then $\tilde{\mu}_j$ tends weakly to a measure $\tilde{\mu}$ for a subsequence. By lower semicontinuity of mass,

$$\tilde{\mu}(I) = \lim_{j \to \infty} \mu_j(I) \geqq \tilde{\mu}_A(I)$$

for each I such that $\tilde{\mu}(\mathrm{fr}\, I) = 0$ and $\tilde{P}_j \cap I \to A \cap I$. Since the intervals with these two properties are dense, $\tilde{\mu} \geqq \mu_A$.

(4.2) LEMMA. *Let $P_j \to A$ and $M(P_j) \to M(A) < \infty$. Then $P_j \cap X \to A \cap X$ for every set X such that $\mu_A(\mathrm{fr}\, X) = 0$.*

Proof. By taking subsequences it suffices to prove this for sequences such that $\sum W(P_{j+1} - P_j) < \infty$. Given $\varepsilon > 0$ there is a figure $Y \subset X$ which is the finite union of nonexceptional intervals, such that $\mu_A(X - Y) < \varepsilon/2$. Then

$$W(A \cap X - P_j \cap X) \leqq W(A \cap Y - P_j \cap Y) + M[A \cap (X - Y)] + M[P_j \cap (X - Y)].$$

But $P_j \cap Y \to A \cap Y$, and $\mu_j(X - Y) \to \mu_A(X - Y)$ since $\mu_A[\mathrm{fr}(X - Y)] = 0$. Hence $W(A \cap X - P_j \cap X) < \varepsilon$ for all sufficiently large j, which proves (4.2).

NOTE. If $\{A_j\}$ is any sequence of flat chains tending to A such that $M(A_j) \to M(A) < \infty$ and X is as in (4.2), then $A_j \cap X \to A \cap X$. The proof is the same as for sequences of polyhedra provided one uses (5.7) below instead of the special case of it used in §2.

The measure μ_A has a support, which is the smallest closed set whose complement is μ_A-null.

(4.3) THEOREM. *If $M(A) < \infty$, then* $\mathrm{spt}\, A = \mathrm{spt}\, \mu_A$.

Proof. Let $\{P_j\}$ be as in the definition of μ_A, and let U be any open set containing $\mathrm{spt}\, \mu_A$. Then $M(P_j \cap U^c) = \mu_j(U^c) \to 0$, and hence $P_j \cap U \to A$. For each cell σ of P_j, let us replace $\sigma \cap U$ by a good enough polyhedral approximation to it contained in $\sigma \cap U$. This gives a sequence $\{P_j'\}$ lying in U and tending to A. Hence A has support contained in $\mathrm{spt}\, \mu_A$.

On the other hand, let A have support contained in X. If $x \in \mathrm{spt}\, \mu_A - X$, then there exists a sequence $\{\tilde{P}_j\}$ tending to A and an interval I containing x such that $\tilde{P}_j \cap I = 0$, $(P_j - \tilde{P}_j) \cap I = P_j \cap I \to 0$, $P_j \cap I \to A \cap I$. Then $A \cap I = 0$, contrary to the fact that $x \in \mathrm{spt}\, \mu_A$. Hence, $\mathrm{spt}\, \mu_A \subset X$, which proves (4.3).

5. **Lipschitz maps.** Let $U \subset E^n$ be open and let $C_*(G, U)$ be the subcomplex of $C_*(G)$ consisting of all A such that A is the limit of a sequence of polyhedral chains lying in U. Let f be Lipschitz from U into an open set $V \subset E^m$. Then f induces a chain map $f_\#$ from $C_*(G;U)$ into $C_*(G;V)$ in the following way. If σ is an oriented convex k-cell then $f_\#\sigma$ is an integral flat k-chain, defined in [FF, §3] by approximating f by smooth functions (and in [W, p. 297] by approximating f by piecewise-linear functions). If $P = \sum g_i\sigma_i$ is a polyhedral chain lying in U, then we set $f_\# P = \sum g_i f_\#\sigma_i$. Let λ be a Lipschitz constant for f. Then

$$M[f_\# P] \leqq \lambda^k M(P), \tag{5.1}$$

since this estimate is true when P is a k-cell. If $P = Q + \partial R$, then $f_\# P = f_\# Q + \partial f_\# R$, and hence

$$W(f_\# P) \leqq \lambda^k M(Q) + \lambda^{k+1} M(R).$$

Since this is true for every such Q and R,

$$W(f_\# P) \leqq \max(\lambda^k, \lambda^{k+1})\, W(P). \tag{5.2}$$

A flat chain of the form $f_\# P$ is called a *Lipschitz chain*, and a *chain of class* $C^{(1)}$ if f is of class $C^{(1)}$.

Let $\{P_j\}$ be a sequence of polyhedral chains lying in U and tending to A. Applying (5.2) to $P_i - P_j$ and using the fact that $f_\#(P_i - P_j) = f_\# P_i - f_\# P_j$, we see that the sequence $\{f_\# P_i\}$ is fundamental. Its limit is $f_\# A$. This defines $f_\#$, which is a chain map. If A has support contained in $X \subset U$, then $f_\# A$ has support contained in $\mathrm{cl} f(X)$. The estimates (5.1), (5.2) remain true if instead of P we put A. If ϕ is Lipschitz from V into W, then $(\phi \circ f)_\# P = \phi_\#[f_\# P]$ for every polyhedral chain P lying in U. The same is then true for every $A \in C_*(G;U)$, i.e., $(\phi \circ f)_\# = \phi_\# \circ f_\#$.

It will be useful to reduce certain statements to the case of flat chains with compact supports. For this purpose let us consider the following functions f_r. Let $\|x\| = \max\{|x^1|, \cdots, |x^n|\}$, and for each $r > 0$ let

$$\begin{aligned} f_r(x) &= x, && \text{if } \|x\| \leqq r, \\ &= \|x\|^{-1} r x && \text{if } \|x\| > r. \end{aligned}$$

Since f_r does not increase euclidean distance, by (5.1) $M(f_{r\#}A) \leqq M(A)$. If P is a polyhedral chain, then $f_{r\#}P$ is a polyhedral chain. If A has support in the n-cube $K_r = f_r(E^n)$, then it is easy to show that there is a sequence $\{P_j\}$ lying in K_r and tending to A; hence $f_{r\#}A = A$.

(5.3) Lemma. *For every $A \in C_k(G)$, $f_{r\#}A \to A$ as $r \to \infty$. If $M(A) < \infty$, then the convergence is in the M-metric.*

Proof. First suppose that $M(A) < \infty$. Since $f_{r\#}(A \cap K_r) = A \cap K_r$,

$$f_{r\#}A - A = f_{r\#}(A \cap K_r^c) - A \cap K_r^c,$$
$$M[f_{r\#}A - A] \leqq 2M(A \cap K_r^c),$$

which tends to 0 as $r \to \infty$. In the general case, $A = B - \partial C$ where $M(B) < \infty$, $M(C) < \infty$. Since $f_{r\#}$ is a chain map, $f_{r\#}A = f_{r\#}B + \partial f_{r\#}C$. and we apply to B and C what has already been proved.

(5.4) Lemma: *Let $P \in P_k(G)$ satisfy $\partial P = 0$, $W(P) < \varepsilon$, and $\operatorname{spt} P \subset K_r$. Then there exists a polyhedral $(k+1)$-chain R such that $P = \partial R$, $\operatorname{spt} R \subset K_r$, and*

$$M(R) \leqq \left(1 + \frac{rn^{1/2}}{k+1}\right)\varepsilon.$$

Proof. By definition of W, $P = Q + \partial R_1$ where $M(Q) + M(R_1) < \varepsilon$. By replacing Q, R_1 by $f_{r\#}Q$, $f_{r\#}R_1$, we may assume that $\operatorname{spt} Q \subset K_r$, $\operatorname{spt} R_1 \subset K_r$. Let $R = R_1 + 0Q$, where $0Q$ is the cone on Q with vertex 0. This proves (5.4).

In the next section we define the cone $0A$ when A has compact support. Then (5.4) generalizes to flat cycles with compact support.

Using these lemmas let us deduce two results which are useful later.

(5.5) Theorem. *Let $M(A) < \infty$. Then given $\varepsilon > 0$ there exists A_ε such that $M(A - A_\varepsilon) < \varepsilon, \operatorname{spt} A_\varepsilon$ is compact, and ∂A_ε is a polyhedral $(k-1)$-chain.*

Proof. By (5.3) we may assume that $\operatorname{spt} A \subset K_r$ for some r. Let $P_j \to A$, $\operatorname{spt} P_j \subset K_r$, $\sum W(P_{j+1} - P_j) < \infty$. By (5.4), $\partial(P_{j+1} - P_j) = \partial R_j$ where $\sum M(R_j) < \infty$. Let

$$A_\varepsilon = A - \sum_{i=j}^{\infty} R_i,$$

for j large enough that $\sum_{i=j}^{\infty} M(R_i) < \varepsilon$. Then $\partial A_\varepsilon = \partial P_j$ and $M(A - A_\varepsilon) < \varepsilon$.

Since $A_\varepsilon \in N_k^0(G)$, (5.5) implies that $N_k^0(G)$ is M-dense in $M_k(G)$.

(5.6) Theorem. *For every $A \in C_k(G)$ there exists a sequence $\{\tilde{P}_j\}$ of polyhedral k-chains tending to A such that $M(\tilde{P}_j) \to M(A), M(\partial \tilde{P}_j) \to M(\partial A)$. If ∂A is a polyhedral $(k-1)$-chain, then we may arrange that $\partial \tilde{P}_j = \partial A$ for every $j = 1, 2, \cdots$.*

Proof. Let us proceed in three steps.

Step 1. Let A have compact support, and let ∂A be polyhedral. Choose r large enough that $\operatorname{spt} A \subset K_r$. Let $\{P_j\}$ be a sequence of polyhedral chains tending to A such that $M(P_j) \to M(A)$ and $\operatorname{spt} P_j \subset K_r$. Since $W(\partial P_j - \partial A) \leqq W(P_j - A)$, which tends to 0, there exist by (5.4) polyhedral chains R_j lying in K_r such that

$$\partial P_j - \partial A = \partial R_j, \qquad M(R_j) \to 0.$$

Let $\tilde{P}_j = P_j - R_j$.

Step 2. Let $\operatorname{spt} A$ be compact. Choose K_r and $\{P_j\}$ as in Step 1. Applying Step 1 to the cycle ∂A, there is a sequence $\{Q_j\}$ of polyhedral $(k-1)$-cycles tending to ∂A such that $M(Q_j) \to M(\partial A)$ and $\operatorname{spt} Q_j \subset K_r$. Since $\partial P_j \to \partial A$, $W(\partial P_j - Q_j) \to 0$. By (5.4) there exist polyhedral chains R_j such that

$$\partial P_j - Q_j = \partial R_j, \qquad M(R_j) \to 0.$$

Again let $\tilde{P}_j = P_j - R_j$.

Step 3. If $\operatorname{spt} A$ is not compact, for each $r > 0$ let $A_r = f_{r\#} A$. Then $A_r \to A, M(A_r) \leqq M(A), M(\partial A_r) \leqq M(\partial A)$. Since mass is lower semicontinuous, $M(A_r) \to M(A), M(\partial A_r) \to M(\partial A)$ as $r \to \infty$. We appeal to Step 2.

NOTE. In §7 we shall find that, at least in some instances, one can in addition arrange the convergence of $\operatorname{spt} \tilde{P}_j$ to $\operatorname{spt} A$ and $\operatorname{spt} \partial \tilde{P}_j$ to $\operatorname{spt} \partial A$. See the remarks following (7.7).

Using (5.6) let us next prove an inequality corresponding to the Eilenberg inequality for Hausdorff measure.

(5.7) THEOREM. *Let F be a real-valued Lipschitz function on E^n, ξ a Lipschitz constant for F, and $X_s = \{x : F(x) < s\}$, $Y_s = \{x : F(x) = s\}$. Let $A \in N_k(G)$ and*

$$B_s = \partial(A \cap X_s) - (\partial A) \cap X_s.$$

Then for almost all s, $A \cap X_s \in N_{k-1}(G)$ and $\operatorname{spt} B_s \subset Y_s$. Moreover, if $-\infty \leqq a < b \leqq +\infty$, then

$$\int_a^b M(B_s)\,ds \leqq \xi M[A \cap (X_b - X_a)].$$

Proof. If A is an oriented k-cell σ, this is a special case of [FF, 3.10]. By addition the theorem is true for polyhedral k-chains. Let $\tilde{P}_j$ be as in (5.6), and

$$B_{js} = \partial(\tilde{P}_j \cap X_s) - (\partial \tilde{P}_j) \cap X_s.$$

Let $\Lambda = \{s : \mu_A(Y_s) = \mu_{\partial A}(Y_s) = 0\}$. Then Λ^c is a countable set and by (4.2)

$$\tilde{P}_j \cap X_s \to A \cap X_s, \ (\partial \tilde{P}_j) \cap X_s \to (\partial A) \cap X_s$$

for $s \in \Lambda$. Since ∂ is continuous $B_{js} \to B_s$ for each $s \in \Lambda$. Since $\operatorname{spt} B_{js} \subset Y_s$, $\operatorname{spt} B_s \subset Y_s$, for $s \in \Lambda$. Moreover, for $s \in \Lambda$, $A \cap X_t \to A \cap X_s$ and $(\partial A) \cap X_t \to (\partial A) \cap X_s$ as $t \to s$ (in fact, in the M-metric). Hence $B_t \to B_s$ as $t \to s$. Since mass is lower semicontinuous, $M(B_s)$ is a lower semicontinuous function of s except perhaps on the countable set Λ^c. Whenever $M(B_s) < \infty$, $A \cap X_s \in N_{k-1}(G)$. It suffices to assume that $a, b \in \Lambda$. Since for each $j = 1, 2, \cdots$

$$\int_a^b M(B_{js})ds \leqq \xi M[\tilde{P}_j \cap (X_b - X_a)],$$

Theorem (5.7) now follows from Fatou's lemma and lower semicontinuity of M.

An important particular case of (5.7) occurs when $(\partial A) \cap X_s = 0$ for $a \leqq s \leqq b$. Then $B_s = \partial(A \cap X_s)$.

6. **Cartesian products, linear homotopies.** It is not possible to give a satisfactory definition of the cartesian product $A \times B$ of two arbitrary flat chains. However, this can be done in either of the following two cases: (1) either $N(A) < \infty$ or $N(B) < \infty$; or (2) both $M(A) < \infty$ and $M(B) < \infty$. The proof is based on the following inequality (6.1).

Let P be a polyhedral k-chain in E^n and Q a polyhedral l-chain in E^m. Their cartesian product $P \times Q$ is a polyhedral $(k+l)$-chain $E^{n+m} = E^n \times E^m$. If $(k+l) \leqq n$, then $M(P \times Q) = M(P)M(Q)$; otherwise, $P \times Q = 0$. Moreover,

$$\partial(P \times Q) = (\partial P) \times Q + (-1)^k P \times \partial Q.$$

Let us show that

$$W(P \times Q) \leqq N(P)W(Q). \tag{6.1}$$

Let $Q = R + \partial S$, where R and S are polyhedral chains. Then

$$\begin{aligned} P \times \partial S &= \pm[(\partial P) \times S - \partial(P \times S)], \\ W(P \times \partial S) &\leqq M[(\partial P) \times S] + M(P \times S) \leqq N(P)M(S), \\ W(P \times Q) &\leqq W(P \times R) + W(P \times \partial S) \\ &\leqq M(P)M(R) + N(P)M(S) \\ &\leqq N(P)[M(R) + M(S)]. \end{aligned}$$

Since this is true for every such R, S, we get (6.1).

The cartesian product $A \times B$ is now defined for $A \in N_k(G)$, $B \in C_l(G)$ as follows. First of all, if P_j tends to A then for any polyhedral l-chain Q,

$$W[(P_i - P_j) \times Q] \leqq W(P_i - P_j)N(Q).$$

The limit of the fundamental sequence $\{P_j \times Q\}$ is $A \times Q$. By (5.6) we may arrange that $N(P_j) \to N(A)$, and then (6.1) still holds with A in place of P. If Q_j tends to B, then

$$W[A \times (Q_i - Q_j)] \leqq N(A)W(Q_i - Q_j),$$

and $A \times B$ is the limit of the fundamental sequence $\{A \times Q_j\}$. The operation $\times$ is bilinear and

$$\partial(A \times B) = (\partial A) \times B + (-1)^k A \times (\partial B). \tag{6.2}$$

Moreover, $M(A \times B) \leqq M(A)M(B)$ and $W(A \times B) \leqq N(A)W(B)$.

If $M(A) < \infty$ and $M(B) < \infty$, then by (5.5) there is a sequence $\{A_j\}$ such that $N(A_j) < \infty$ for each $j = 1, 2, \cdots$ and $M(A_j - A)$ tends to 0. Then

$$M[(A_i - A_j) \times B] \leqq M(A_i - A_j)M(B),$$

and $A \times B$ is the M-limit of the sequence $\{A_j \times B\}$.

Linear homotopies. Let f and g be Lipschitz from an open set U into a bounded subset of E^m, and let $h(t,x) = (1-t)f(x) + tg(x)$, $0 \leqq t \leqq 1$. Let I be the interval $[0,1]$, regarded as a polyhedral 1-chain. Then

$$g_{\#}A - f_{\#}A = \partial h_{\#}(I \times A) + h_{\#}(I \times \partial A), \tag{6.3}$$

since this is true when A is a polyhedral chain. By the same discussion as in [FF, pp. 466, 468] one can prove the following estimates. Let $|Df(x)| \leqq \psi(x)$, $|Dg(x)| \leqq \psi(x)$, where ψ is continuous on U and Df denotes the differential. If $M(A) < \infty$, then

$$M[f_{\#}A] \leqq \int_U \psi^k d\mu_A, \tag{6.4}$$

$$M[h_{\#}(I \times A)] \leqq \int_U |g - f| \psi^k d\mu_A, \tag{6.5.}$$

$$M[g_{\#}A - f_{\#}A] \leqq 2 \int_{U_0} \psi^k d\mu_A, \tag{6.6}$$

where $U_0 = \{x \in U : f(x) \neq g(x)\}$. In particular, if $f(x) = g(x)$ for every $x \in \operatorname{spt} A$, then $f_{\#}A = g_{\#}A$ when $M(A) < \infty$.

(6.7) Lemma. *Let X be a compact Lipschitz neighborhood retract such that X has zero Hausdorff k-measure. If $M(A) < \infty$ and $\operatorname{spt} A \subset X$, then $A = 0$.*

Proof. By assumption there exist U open with $X \subset U$, and f Lipschitz from U onto X such that $f(x) = x$ for $x \in X$. If σ is any convex k-cell lying in U, then $f_{\#}\sigma = 0$ [FF, p. 500]. Let $\{P_j\}$ be a sequence of polyhedral k-chains lying in U and tending to A. Then $f_{\#}P_j = 0$, $f_{\#}A = A$, and $f_{\#}P_j \to f_{\#}A$. Hence $A = 0$.

Cones. Let $x_0 \in E^n$ and let $h(t,x) = (1-t)x_0 + tx$. If A has compact support, then $h_{\#}(I \times A)$ is defined. It is called the *cone* on A with vertex x_0 and is denoted by $x_0 A$. If $k > 0$, then from (6.3) and (6.7)

$$A = \partial x_0 A + x_0 \partial A.$$

From (6.5), if $|x - x_0| \leqq r$ for every $x \in \operatorname{spt} A$, then [FF, p. 509]

$$M(x_0 A) \leqq \frac{r}{k+1} M(A).$$

7. **A deformation theorem.** Except for Lemma (7.8) which is new, all of the results of this section have counterparts in [FF]. In most instances the proofs are virtually the same as in [FF]. However, the proof of (7.2) differs from that for its counterpart [FF, 3.14].

(7.1) LEMMA. *Let A be a flat 0-chain such that $M(A) < \infty$ and $\operatorname{spt} A$ is a finite set. Then A is polyhedral; i.e. $A = \sum g_i\{x_i\}$ where $x_1, \cdots, x_m$ are the points of $\operatorname{spt} A$.*

From the proof of (4.3) there is a sequence P_j of polyhedral 0-chains tending to A such that $M(P_j) \to M(A)$ and $\operatorname{spt} P_j \subset \delta_j$-neighborhood of $\operatorname{spt} A$, where $\delta_j \to 0$. The proof of (7.1) is then left to the reader.

By ε-cubical grid χ let us mean a cell complex subdividing E^n, composed of of cubes of side length 2ε. Let X_k be the k-skeleton of χ, $X_0 \subset X_1 \subset \cdots \subset X_n = E^n$.

(7.2) LEMMA. *Let $A \in N_k^0(G)$ be such that $\operatorname{spt} A \subset X_k, \operatorname{spt} \partial A \subset X_{k-1}$. Then A is a polyhedral chain of the grid χ.*

Proof. (by induction on k). For $k = 0$ this follows from (7.1). Assume that the lemma is true in dimension $k-1$. Let σ be a k-cube of X_k. By (5.7) there is a sequence of k-cubes $\sigma_1 \subset \sigma_2 \subset \cdots$ with union σ such that $N(A \cap \sigma_i) < \infty$ for each i and $\operatorname{spt} \partial(A \cap \sigma_i) \subset \dot{\sigma}_i$, where $\dot{\sigma}_i$ is the frontier of σ_i relative to σ. By the induction hypothesis, $\partial(A \cap \sigma_i)$ is a polyhedral $(k-1)$-chain, and hence from topology $\partial(A \cap \sigma_i) = g_i \partial \sigma_i$ for some $g_i \in G$. By (5.6) there is a sequence P_j tending to $A \cap \sigma_i$ such that $\partial P_j = g_i \partial \sigma_i$. Let P_j' be the projection of P_j onto the k-plane of σ. Then $P_j' \to A \cap \sigma_i$ and from topology $P_j' = g_i \sigma_i$. Hence $A \cap \sigma_i = g_i \sigma_i$. It follows that $g_i = g$ is independent of i, and then that $A \cap \sigma = g\sigma$. Since this is true for each such σ, A is a polyhedral chain of χ.

(7.3) DEFORMATION THEOREM. *There exists a positive number $c = c(k,n)$ with the following property. Given $A \in N_k^0(G)$ and $\varepsilon > 0$ there exist an ε-cubical grid χ, a polyhedral k-chain P of χ and $B \in N_k^0(G)$, $C \in N_{k+1}^0(G)$, such that:*

(1) $A = P + B + \partial C$;

(2) $M(P) \leqq c[M(A) + \varepsilon M(\partial A)]$,
$M(\partial P) \leqq cM(\partial A)$, $M(B) \leqq c\varepsilon M(\partial A)$,
$M(C) \leqq c\varepsilon M(A)$;

(3) $(\operatorname{spt} P) \cup (\operatorname{spt} C) \subset 2n\varepsilon$-nbd. *of* $\operatorname{spt} A$,
$(\operatorname{spt} \partial P) \cup (\operatorname{spt} B) \subset 2n\varepsilon$-nbd. *of* $\operatorname{spt} \partial A$;

(4) *if A is a Lipschitz chain, then B and C are Lipschitz chains;*
if A is a polyhedral chain, then B and C are polyhedral chains.

Note that (2) implies $W(A - P) < c\varepsilon N(A)$. This theorem can be proved by precisely the same construction used for the corresponding result [FF, 5.5] about normal currents. Therefore, we omit the proof. For our purposes it would actually suffice to know (7.3) when A is a Lipschitz chain (or even when A is polyhedral). The discussion on [FF, §4] of u-admissible maps could then be avoided. Moreover, (7.3) can be deduced from the case when A is polyhedral by passage to the limit, at least when G is a countable discrete group with property (H), stated below. This follows from (7.4) and an extension of (5.6). See the remark following the proof of (7.7). The counterpart of the statement in (4) about polyhedral chains is not explicitly stated in [FF, 5.5]. However, it follows from the observation [FF, p. 478] that the deformations used in the proof are merely composites of successive central projections of cubes onto their boundaries.

Let us now list some important consequences of (7.3). For this purpose the following additional assumptions are made about G:

(H) For every $M > 0$, $\{g : |g| \leqq M\}$ is compact. If G is discrete, then $|g| \geqq 1$ for every $g \in G$.

(7.4) LEMMA. *Let G satisfy* (H). *Let $q > 0$, $K \subset E^n$ be compact, and let Γ be a set of flat k-chains such that $N(A) \leqq q$ and $\operatorname{spt} A \subset K$ for every $A \in \Gamma$. Then Γ is totally bounded.*

Proof. Let K' be a n-cube containing K in its interior. For each $\varepsilon > 0$ let $\Gamma'_\varepsilon = \{P : N(P) \leqq 2cq, \operatorname{spt} P \subset K'$, P is a polyhedral chain of some ε-cubical grid$\}$. Then Γ'_ε is totally bounded, and for small ε, Γ is contained in the $c\varepsilon q$-neighborhood of Γ'_ε. This implies that Γ is totally bounded.

(7.5) COROLLARY. *Using the notation of* (7.4), $\{A \in N_k(G) : N(A) \leqq q,\ \operatorname{spt} A \subset K\}$ *is compact.*

Proof. This set of flat chains is closed, and hence compact by (7.4).

(7.6) (*Isoperimetric inequality*). *Let G be a discrete group satisfying* (H). *Then there exists a positive number $b = b(k,n)$ with the following property. If $A \in N_k^0(G)$ and $\partial A = 0$, then there exists $C \in N_{k+1}^0(G)$ such that $A = \partial C$ and $M(C) \leqq bM(A)^{(k+1)/k}$.*

Proof. Choose ε so that $\varepsilon^k = cM(A)$. Let $A = P + \partial C$ as in (7.3). Then $M(P) \leqq \varepsilon^k$, while since $|g| \geqq 1$ for every $g \neq 0$, the least mass of any nonzero polyhedral k-chain of an ε-grid is $(2\varepsilon)^k$. Hence $P = 0$. Then $A = \partial C$ and $M(C) \leqq bM(A)^{(k+1)/k}$, where $b = c^{(k+1)/k}$. This proves (7.6).

More generally, a relative isoperimetric inequality like [FF, 6.1] can be proved.

From (4) of (7.3) and the proof of (7.6) we may assume that C is a Lipschitz (polyhedral) chain if A is a Lipschitz (polyhedral) chain. By (3) of (7.3) we may assume that $\operatorname{spt} C \subset 2n\varepsilon$-nbd. of $\operatorname{spt} A$, where ε is small if $M(A)$ is small.

The next lemma is included for sake of completeness, but will not be used in later sections.

(7.7) LEMMA. *Let G be a discrete group satisfying* (H), *and let X be compact. Then given $\delta > 0$ and a sequence $\{Q_j\}$ of polyhedral cycles such that*

$\operatorname{spt} Q_j \subset X$, $j = 1, 2, \cdots, W(Q_j) \to 0$, *there is a sequence* $\{R_j\}$ *of polyhedral chains such that* $\partial R_j = Q_j$, $M(R_j) \to 0$, *and* $\operatorname{spt} R_j \subset \delta$-nbd. *of* X *for all sufficiently large* j.

Proof. By (5.4), $Q_j = \partial R_j'$ where $M(R_j') \to 0$. Let Y be a figure (finite union of open n-cubes) such that $X \subset Y$ and $\operatorname{cl} Y \subset (\delta/2)$-nbd. of X. Let $f(x) = \| \ \|$-distance from x to Y, where $\| \ \|$ is as in §5. By (5.7) there exist $s_j \in (0, \delta/2)$ such that, writing $R_j'' = R_j' \cap \{x : f(x) < s_j\}$, $M(\partial R_j'' - Q_j) \to 0$. Moreover, R_j'' is a polyhedral chain. By (7.6) $\partial R_j'' - Q_j = \partial S_j$, where $M(S_j) \to 0$, $\operatorname{spt} S_j \subset (\delta/2)$-nbd. of $\{x : f(x) = s_j\}$ for sufficiently large j, and S_j is polyhedral. Let $R_j = R_j'' - S_j$. This proves (7.7).

Let $A \in N_k^0(G)$. In the proof of (5.6) we may assume that $\operatorname{spt} P_j \subset \delta_j$-nbd. of $\operatorname{spt} A$, where $\delta_j \to 0$. Using (7.7) we can then arrange in the proof of (5.6) that

$$\operatorname{spt} R_j \subset \delta_j'\text{-nbd. of } \operatorname{spt} A$$

$$\operatorname{spt} Q_j \subset \delta_j'\text{-nbd. of } \operatorname{spt} \partial A,$$

where $\delta_j' \to 0$. Therefore, *if* G *is a discrete group satisfying* (H), *then in* (5.6) *we may arrange that* $\operatorname{spt} P_j$ *tends to* $\operatorname{spt} A$ *and* $\operatorname{spt} \partial P_j$ *tends to* $\operatorname{spt} \partial A$ *in the Hausdorff distance on the space of compact subsets of* E^n.

(7.8) LEMMA. *Let* G *be compact. Then given a compact set* K *and* $\varepsilon > 0$ *there exists a positive number* $\gamma = \gamma(\varepsilon, K, k, n, G)$ *with the following property. If* $A \in N_k^0(G)$, $\operatorname{spt} A \subset K$, *and* $\partial A = 0$, *then there exists* $\tilde{C} \in N_{k+1}^0(G)$ *such that* $\partial \tilde{C} = A$ *and* $M(\tilde{C}) \leqq \gamma + c\varepsilon M(A)$.

Proof. Let $A = P + \partial C$ as in (7.3). Then P is a polyhedral chain of an ε-grid, and lies in the $2n\varepsilon$-neighborhood K' of the compact set K. Since G is compact $M(P)$ is bounded by some number γ'. Let $\gamma = \gamma' r/(k+1)$ where $r = \max\{|x| : x \in K'\}$, and let $\tilde{C} = 0P + C$. This proves (7.8).

Finally, let X and Y be local Lipschitz neighborhood retracts with $Y \subset X$. Let $N_*^0(G, X)$ be the chain complex of those $A \in N_*^0(G)$ with $\operatorname{spt} A \subset X$, and let $N_*^0(G, Y)$ be the subcomplex of those $A \in N_*^0(G)$ with $\operatorname{spt} A \subset Y$. Then the homology groups of $N_*^0(G, X)/N_*^0(G, Y)$ are isomorphic with the singular homology groups of (X, Y), with coefficients in G. This follows from the deformation theorem just as in [FF, 5.11].

8. **Estimates on densities.** Let A be a flat k-chain of finite mass, $k > 0$. Let $S(x, r) = \{y : |y - x| \leqq r\}$, and for brevity set

$$\mu(x, r) = \mu_A[S(x, r)].$$

Let $\alpha(k)$ denote the k-measure of a spherical k-disk of radius 1. The lower and upper k-densities of A at a point x are

$$D_{*k}(x) = \liminf_{r \to 0^+} \frac{\mu(x,r)}{\alpha(k)r^k}, \quad D_k^*(x) = \limsup_{r \to 0^+} \frac{\mu(x,r)}{\alpha(k)r^k}.$$

In order to obtain estimates on these densities, let us first show that near μ_A-almost every point, A has nearly minimum mass. To be more precise, let

$$\nu(x,r) = \inf\{M(B) : \partial B = \partial[A \cap S(x,r)]\}.$$

Obviously, $\nu(x,r) \leqq \mu(x,r)$. Equality holds if and only if $A \cap S(x,r)$ is a solution of the Plateau problem (i.e. problem of least mass with given boundary).

(8.1) Lemma. *For μ_A-almost every x,*

$$\lim_{r \to 0^+} \frac{\nu(x,r)}{\mu(x,r)} = 1.$$

Proof. Suppose not. Then there exist $\delta > 0$, a set X of positive μ_A-measure and a Vitali covering of X by spherical n-balls $S(x,r)$ such that $\nu(x,r) < (1-\delta)\mu(x,r)$. Let $\varepsilon > 0$. By a covering theorem of Besicovitch [B] there exists a countable disjoint family of such balls $S(x_i, r_i)$ of diameter $< \varepsilon$ covering μ_A-almost all of X. Let $A_i = A \cap S(x_i, r_i)$, and let B_i be such that

$$\partial B_i = \partial A_i, \qquad \operatorname{spt} B_i \subset S(x_i, r_i),$$
$$M(B_i) < (1-\delta) M(A_i).$$

Then $B_i - A_i$ is a cycle, and bounds the cone $x_i(B_i - A_i)$. Hence

$$W(B_i - A_i) \leqq M[x_i(B_i - A_i)] \leqq \frac{2\varepsilon}{k+1} \mu(x_i, r_i).$$

Let

$$A(\varepsilon) = A - \sum_i A_i + \sum_i B_i = A \cap \Big(\bigcup_i S(x_i, r_i)\Big)^c - \sum_i B_i.$$

Then

$$M[A(\varepsilon)] \leqq M(A) - \sum_i M(A_i) + \sum_i M(B_i)$$
$$< M(A) - \delta \sum_i \mu(x_i, r_i) < M(A) - \delta \mu_A(X).$$

Since

$$W[A(\varepsilon) - A] \leqq \sum_i W(B_i - A_i) \leqq \frac{2\varepsilon}{k+1} M(A),$$

$A(\varepsilon) \to A$ as $\varepsilon \to 0^+$. This contradicts lower semicontinuity of mass.

In the next lemma choose $\varepsilon_0 < 1/4ck$, where c is the constant in (7.3). Let $\zeta = 4\gamma/\alpha(k)$ where γ is as in (7.8) with $\varepsilon = \varepsilon_0, K = S(0,1)$.

(8.2) Lemma. *Let G be compact, $A \in N_k(G)$, and $x \notin \operatorname{spt} \partial A$. If $r_1 \leqq \operatorname{dist}(x, \operatorname{spt} \partial A)$ and $\mu(x,r) \leqq 2\nu(x,r)$ for $0 < r \leqq r_1$, then*

$$D_k^*(x) \leqq \max\{\zeta, \mu(x, r_1)/\alpha(k) r_1^k\}.$$

Proof. Let $l(x,r) = M[\partial(A \cap S(x,r))]$. By (5.7), $l(x,r) \leqq \mu'(x,r)$ for almost every $r \in (0,r_1)$, where $'$ means d/dr. Moreover, by (7.8)

$$\frac{\nu(x,r)}{r^k} \leqq \gamma + \frac{c\varepsilon_0 l(x,r)}{r^{k-1}}$$

for $0 < r \leqq r_1$. Suppose that $\zeta < D_k^*(x)$. Given η such that $\zeta < \eta < D_k^*(x)$ choose r_0 such that $0 < r_0 < r_1$ and $\mu(x,r_0) > \eta\alpha(k)r_0^k$. Let J be the largest interval contained in $[r_0, r_1]$ such that $\mu(x,r) \geqq \eta\alpha(k)r^k$ for every $r \in J$. Since $\zeta < \eta$, $\gamma = \alpha(k)\zeta/4$, and $l \leqq \mu'$, we find that

$$k\mu(x,r) < r\mu'(x,r),$$

and consequently $(r^{-k}\mu)' > 0$, for almost every $r \in J$. Since μ is increasing,

$$0 < \int_r^s (\rho^{-k}\mu)' d\rho \leqq (\rho^{-k}\mu)\Big|_r^s,$$

and hence $r^{-k}\mu$ is increasing on J. This implies that $J = [r_0, r_1]$, and since η is arbitrary (8.2) follows.

(8.3) *Let G be a discrete group satisfying* (H). *Then there exists $\lambda = \lambda(k,n) > 0$ with the following property. If $A \in N_k(G)$, $x \notin \operatorname{spt} \partial A$, and* $\lim_{r\to 0^+} \nu(x,r)/\mu(x,r) = 1$, *then $D_{*k}(x) \geqq \lambda$.*

Proof. By the isoperimetric inequality (7.6),

$$\nu(x,r) \leqq b[l(x,r)]^{k/(k-1)}$$

for $r < \operatorname{dist}(x, \operatorname{spt} \partial A)$, where $b = b(k-1,n)$ and $l(x,r)$ is as above. If r_1 is as (8.2), then for $r < r_1$

$$\mu(x,r) \leqq 2b[\mu'(x,r)]^{k/(k-1)},$$

$$(\mu^{1/k})' = \frac{1}{k}\mu^{-(k-1)/k}\mu' \geqq \frac{(2b)^{(1-k)/k}}{k}.$$

Let $\lambda = k^{-k}(2b)^{1-k}/\alpha(k)$. By (6.7), $\mu(x,0) = 0$ and we have

$$\mu^{1/k}(x,r) \geqq \int_0^r (\mu^{1/k})' d\rho \geqq [\lambda\alpha(k)]^{1/k} r,$$

for $0 < r < r_1$. This proves (8.3).

If G is a finite group, then from these lemmas $\lambda \leqq D_{*k}(x) \leqq D_k^*(x) < \infty$ for μ_A-almost every $x \notin \operatorname{spt} \partial A$.

9. **Rectifiable chains.** In the remainder of the paper let us assume that G is a finite group. Except for those places where (8.2) is used, one can equally well

let G be a countable discrete group such that $\{g: |g| \leqq M\}$ is finite for each M (assumption (H)). Without loss of generality, we may assume that $|g| \geqq 1$ for every $g \neq 0$.

DEFINITION. A flat chain A is *rectifiable* if for every $\varepsilon > 0$ there exists a Lipschitz chain B such that $M(A - B) < \varepsilon$.

Let $R_k(G)$ denote the set of all rectifiable k-chains in E^n. Then $R_k(G)$ is a group, which is M-closed. The group $R_k(Z)$ corresponds precisely to the group of rectifiable currents in [FF, 3.7], and $R_k(Z_2)$ to the group of rectifiable classes in [Z, §3].

If $k = 0$, then $M(A) < \infty$ implies that A is a polyhedral 0-chain. In the remainder of the section we let $k > 0$.

(9.1) LEMMA. *Let $A \in R_{k-1}(G)$ have compact support. Then the cone $0A$ is rectifiable.*

Proof. Let $\operatorname{spt} A \subset S(0,r)$. In the definition above we may assume that $\operatorname{spt} B \subset S(0,r)$. Moreover, $0B$ is a Lipschitz chain, and $0A - 0B = 0(A - B)$. Then

$$M(0A - 0B) \leqq \frac{r}{k} M(A - B),$$

and the right side can be made arbitrarily small.

Following [F] let us call a set $X \subset E^n$ (μ, k)-rectifiable if for every $\varepsilon > 0$ there exists a Borel set $F \subset E^k$ and f Lipschitz from F into E^n such that $\mu(X - f(F)) < \varepsilon$. In place of E^k we may equally well assume that F is contained in a finite union of k-planes. If A is a rectifiable k-chain, then E^n is (μ_A, k)-rectifiable. To show this, let $B = f_\# P$ be a Lipschitz chain such that $M(A - B) < \varepsilon$, and let $Y = f(\operatorname{spt} P)$. Let $P_j \to A$, $M(P_j) \to M(A)$, and $Q_j \to B$, $\operatorname{spt} Q_j \subset (\delta_j\text{-neighborhood of } Y)$, where P_j, Q_j are polyhedral chains and $\delta_j \to 0$. Then

$$(A - B) \cap I = \lim_j (P_j - Q_j) \cap I = \lim_j P_j \cap I = A \cap I$$

for a dense set of intervals $I \subset Y^c$. It follows that $\mu_{A-B}(Z) = \mu_A(Z)$ when $Z \subset Y^c$. In particular,

$$\mu_A(Y^c) = \mu_{A-B}(Y^c) \leqq M(A - B) \leqq \varepsilon,$$

showing that E^n is (μ_A, k)-rectifiable.

The main object of this section is to show that, conversely, if E^n is (μ_A, k)-rectifiable, then A is rectifiable, under an additional restriction on A (see (9.4) below).

Let us adopt the following notation: Π denotes a k-plane containing 0, and for $\delta > 0$, $\Pi^\delta = \{x : \operatorname{dist}(x, \Pi) > \delta\}$. Let $S_0 = S(0,1)$, $D = \Pi \cap S_0$. The k-disk D is assigned an orientation.

(9.2) LEMMA. *Given* $\varepsilon > 0$, $q > 0$ *there exists* $\delta > 0$ *with the following property. If* $\operatorname{spt} A \subset S_0$, $\operatorname{spt} \partial A \subset \operatorname{fr} S_0$, $N(A) \leqq q$, $\mu_A(\Pi^\delta) < \delta$, $\mu_{\partial A}(\Pi^\delta) < \delta$, *then there exists* $g \in G$ *such that* $W(A - gD) < \varepsilon$.

Proof. Suppose not. Then there exist $\varepsilon_0 > 0$, $q_0 > 0$ and a sequence $\{A_j\}$ such that $W(A_j - gD) \geqq \varepsilon_0$ for every $g \in G$, while $N(A_j) \leqq q_0$ and

$$\mu_{A_j}(U) \to 0, \quad \mu_{\partial A_j}(V) \to 0$$

for every closed set U such that $U \cap D$ is empty and closed set V such that $V \cap \dot{D}$ is empty, $\dot{D}$ being the frontier of D relative to Π. By (7.5) a subsequence tends to a limit A_0, and $\operatorname{spt} A_0 \subset D$, $\operatorname{spt} \partial A_0 \subset \dot{D}$. From (5.7) and (7.2), there exists $g_0 \in G$ such that $A_0 \cap \sigma = g_0 \sigma$ for any k-cube $\sigma \subset D$. Hence $A_0 = g_0 D$, a contradiction since $W(A_j - A_0) \to 0$ as $j \to \infty$ through a subsequence.

For any $A \in N_k(G)$, let

$$X_A = \left\{ x \in \operatorname{spt} A - \operatorname{spt} \partial A ; \lambda \leqq D_{*k}(x) \leqq D_k^*(x) < \infty, \lim_{r \to 0^+} \nu(x,r)/\mu(x,r) = 1 \right\}.$$

In the next lemma let us for brevity set $A_r = A \cap S(x,r)$, $Y_r = Y \cap S(x,r)$.

(9.3) LEMMA. *Let* $x \in X_A$, *and* Y *be an oriented (regular, proper)* k*-submanifold of* E^n *of class* $C^{(1)}$ *such that*

$$\lim_{r \to 0^+} r^{-k} \mu_A[S(x,r) - Y] = 0.$$

hen there exists $g \in G$ *such that*

$$\lim_{r \to 0^+} r^{-k} M(A_r - gY_r) = 0.$$

Proof. We may take $x = 0$. By making a suitable diffeomorphism of a neighborhood of 0 we may assume that Y is a k-plane Π. Given $r > 0$ let

$$\tau_r(y) = r^{-1} y, \qquad A_r^* = \tau_{r\#} A_r.$$

We must find $g \in G$ such that $M(A_r^* - gD) \to 0$, where $D = \Pi \cap S_0$ as above. Let us divide the proof into two steps.

Step 1. Let $0 < \eta < \alpha(k)/2$. Let us find, for each sufficiently small $r_0, g \in G$ such that $W(A_r^* - gD) < \eta$ for all r in some neighborhood (s,t) of r_0.

Let $\Omega = \max\{|g| : g \in G\}$; and let $\varepsilon > 0$ be small enough that $\varepsilon < 1$ and if we set $\theta = \varepsilon^{1/2}$, then

$$\Omega \alpha(k) \left[\left(\frac{1+\theta}{1-2\theta} \right)^k - 1 \right] + (2\varepsilon + \varepsilon^{1/2}) \left(\frac{1+\theta}{1-2\theta} \right)^k < \frac{\eta}{2}.$$

Let $q = 2D_k^*(0)\alpha(k)\,[1 + 2\theta^{-1}(1+\theta)^k]$, and δ as in (9.2). Let r_0 be small enough that $2r_0 < \operatorname{dist}(0, \operatorname{spt} \partial A)$ and for $0 < r < 2r_0$,

$$M(A_r) \leqq 2D_k^*(0)\alpha(k)r^k,$$

$$2M(A_r \cap \Pi^c) < \delta(1+\theta)^{-k}r^k,$$

$$\mu(0,r) - \nu(0,r) < \varepsilon r^k.$$

Let $r = r_0(1+\theta)$. From (5.7)

$$\int_{r_0}^{r} M(\partial A_s)ds \leqq M(A_r),$$

$$\int_{r_0}^{r} M[(\partial A_s) \cap \Pi^c]ds \leqq M(A_r \cap \Pi^c).$$

Hence there exists $s \in (r_0, r)$ such that

$$M(\partial A_s) \leqq 2r_0^{-1}\theta^{-1}M(A_r) \leqq 4D_k^*(0)\alpha(k)\theta^{-1}(1+\theta)^k r_0^{k-1},$$

$$M[(\partial A_s) \cap \Pi^c] \leqq 2r_0^{-1}\theta^{-1}M(A_r \cap \Pi^c) < \delta r_0^{k-1}.$$

Moreover,

$$M(A_s) \leqq 2D_k^*(0)\alpha(k)s^k,$$

$$M(A_s \cap \Pi^c) < \delta s^k.$$

Since A_s^* satisfies the hypotheses of (9.2), there exists $g \in G$ and B^*, C^* such that

$$A_s^* = gD + B^* + \partial C^*,$$

$$M(B^*) + M(C^*) < \varepsilon.$$

[For present purposes we can replace Π^δ by Π^c in (9.2.)] In particular, since $\varepsilon < \eta/2$,

$$W(A_s^* - gD) < \eta/2.$$

Let $B = (\tau_s^{-1})_\# B^*$, $C = (\tau_s^{-1})_\# C^*$. Then

$$A_s = gD_s + B + \partial C,$$

$$s^{-k}M(B) + s^{-k-1}M(C) < \varepsilon.$$

For $t < s$

$$A_t = gD_t + B_t + E_t + \partial C_t,$$

where as usual $B_t = B \cap S(0,t)$, $C_t = C \cap S(0,t)$, while $E_t = (\partial C)_t - \partial C_t$. By (5.7)

$$\int_{s-2\theta s}^{s-\theta s} M(E_t)dt \leqq M(C).$$

Hence, there exists $t \in (s-2\theta s, s-\theta s)$ such that

$$M(E_t) \leqq \theta^{-1}s^{-1}M(C) \leqq \varepsilon^{1/2}s^k$$

since $\theta = \varepsilon^{1/2}$. Since $s - \theta s < r_0, (t,s)$ is a neighborhood of r_0. If $t < r < s$, then

$$M(A_s - A_r) = \mu(0,s) - \mu(0,r) \leqq \mu(0,s) - \mu(0,t)$$

Let us estimate the term on the right side. Since

$$\partial A_s = \partial[g(D_s - D_t) + B - B_t - E_t + A_t],$$

we find that

$$\mu(0,s) - \varepsilon s^k < \nu(0,s) \leqq |g| M(D_s - D_t) + M(B) + M(B_t) + M(E_t) + \mu(0,t),$$

$$\mu(0,s) - \mu(0,t) \leqq \Omega\alpha(k)(s^k - t^k) + (2\varepsilon + \varepsilon^{1/2})s^k < t^k\eta/2.$$

Hence, $M(A_s - A_r) < \eta t^k/2$. Let $f(y) = y$ if $|y| \leqq 1$, $f(y) = |y|^{-1}y$ if $|y| > 1$. Then

$$A_s^* - A_r^* = f_\# \tau_{r\#}(A_s - A_r),$$

$$M(A_s^* - A_r^*) \leqq M[\tau_{r\#}(A_s - A_r)] \leqq r^{-k}t^k\eta/2 < \eta/2.$$

Therefore,

$$W(A_r^* - gD) \leqq M(A_s^* - A_r^*) + W(A_s^* - gD) < \eta.$$

Step 2. For each r_0, the element g in Step 1 is unique. For if $W(A_r^* - g'D) < \eta$ and $g' \neq g$, then

$$\alpha(k) \leqq |g' - g| M(D) = W[(g'-g)D] < 2\eta$$

contrary to choice of η. The same reasoning shows that g does not in fact depend on r_0. Hence, $W(A_r - gD) \to 0$ as $r \to 0^+$.

Let p denote the orthogonal projection of E^n onto $\Pi, p(y) = y'$ if $y = y' + y''$, $y' \in \Pi$, $y'' \in \Pi^\perp$. By assumption, $M(A_r^* \cap \Pi^c) \to 0$, and since $p_\#(A_r^* \cap \Pi) = A_r^* \cap \Pi$,

$$M(A_r^* - p_\# A_r^*) = M[A_r^* \cap \Pi^c - p_\#(A_r^* \cap \Pi^c)] \to 0.$$

Moreover,

$$M(p_\# A_r^* - gD) = W(p_\# A_r^* - gD) \leqq W(A_r^* - gD),$$

which tends to 0. Hence, $M(A_r^* - gD) \to 0$, completing the proof of (9.3).

(9.4) THEOREM. *Let G be a finite group. If $A \in N_k(G)$, $\mu_A(\operatorname{spt}\partial A) = 0$, and E^n is a (μ_A, k)-rectifiable set, then A is rectifiable.*

Proof. By §8, $\mu_A(X_A^c) = 0$. Since E^n is (μ_A, k)-rectifiable, for μ_A-almost every $x \in X_A$ there exists a k-submanifold $Y(x)$ such that the hypotheses of (9.3) are satisfied.

Using a covering theorem [B], given $\varepsilon > 0$ there exists disjoint spherical balls $S(x_i, r_i)$, $i = 1, \cdots, p$ and $g_i \in G$ such that

$$M[A(x_i, r_i) - g_i Y(x_i, r_i)] < \frac{\varepsilon}{2M(A)} \mu_A(x_i, r_i),$$

$$M(A - \Sigma A(x_i, r_i)) < \frac{\varepsilon}{2},$$

where $A(x, r) = A \cap S(x, r)$, $Y(x, r) = Y(x) \cap S(x, r)$, $x_i \in X_A$. Then $\Sigma g_i Y(x_i, r_i)$ is rectifiable and

$$M(A - \Sigma g_i Y(x_i, r_i)) < \varepsilon.$$

Since the group $R_k(G)$ of rectifiable k-chains is M-closed, $A \in R_k(G)$.

10. **Main theorem.** In this final section we shall prove the following main result, announced in the introduction.

(10.1) THEOREM. *Let G be a finite group. Then every flat chain A of finite mass is rectifiable.*

We may assume that $k > 0$, since if $k = 0$ then A is a polyhedral 0-chain (see beginning of §9). Since the group $R_k(G)$ of rectifiable k-chains is M-closed, it suffices by (5.5) to prove the theorem when ∂A is a polyhedral $(k-1)$-chain. By (9.4) it then suffices to prove that E^n is a (μ_A, k)-rectifiable set.

If Π is a k-plane containing 0, let X_Π denote the orthogonal projection of a set $X \subset E^n$ onto Π. Let H_k denote Hausdorff k-measure. By (8.2), (8,3), and results of Federer [F, §4 and 8.7], to prove that E^n is (μ_A, k)-rectifiable it suffices to show that there is no set X such that $\mu_A(X) > 0$ and $H_k(X_\Pi) = 0$ for almost all Π.

In order to prove that there is no such X, let us first prove four lemmas. As in §9 let $S_0 = S(0,1)$, $C_\rho = C \cap S(0, \rho)$. Let $B_\rho = \partial C_\rho - (\partial C)_\rho$, as in (5.7).

(10.2) LEMMA. *Let $C \in N_k(G)$, $\operatorname{spt} C \subset S_0$. Let $\delta(C) = M(C) - \int_0^1 M(B_\rho) d\rho$. Then*

$$W(C - 0\partial C) \leqq [2M(C)\delta(C)]^{1/2}/(k+1).$$

Proof. Suppose first that C is a polyhedral chain. For any $x \neq 0$ lying in a cell σ of C, let $\theta(x)$ be the angle between x and the k-plane $\Pi(x)$ through 0 parallel to σ [$\cos\theta(x) = |v(x)|$, where $v(x)$ is the component in $\Pi(x)$ of the unit vector $|x|^{-1}x$]. It is elementary that

$$M(0C) \leqq \frac{1}{k+1} \int_{S_0} |x| \sin\theta(x) d\mu_C$$

and replacing $|x|$ by 1, that

$$M(0C) \leqq \frac{1}{k+1} \int_0^1 d\rho \int_{|x|=\rho} \sin\theta(x)\sec\theta(x) d\mu_{B_\rho}.$$

Since $\sin\theta\sec\theta = \tan\theta$, we get from Schwarz's inequality,

$$M(0C) \leqq \frac{1}{k+1}\left(\int_0^1 d\rho \int_{|x|=\rho} (\sec\theta + 1) d\mu_{B_\rho}\right)^{1/2}\left(\int_0^1 d\rho \int_{|x|=\rho} (\sec\theta - 1) d\mu_{B_\rho}\right)^{1/2}$$

and thus using (5.7)

$$(*) \qquad M(OC) \leqq \frac{1}{k+1}[2M(C)]^{1/2}[\delta(C)]^{1/2}.$$

If C is not polyhedral, then by (5.6) there is a sequence $\{C_j\}$ of polyhedral chains lying in S_0 such that $C_j \to C$, $M(C_j) \to M(C)$, $M(\partial C_j) \to M(\partial C)$. Reasoning as in the proof of (5.7), $\delta(C) \geqq \liminf_j \delta(C_j)$, and hence (*) is still correct. Since $\partial(0C) = C - 0\partial C, W(C - 0\partial C) \leqq M(0C)$. Therefore, (10.2) follows from (*).

In the next lemma let $\tau(x,r)$ be the linear transformation such that

$$\tau(x,r)(y) = r^{-1}(y - x),$$

and $A^*(x,r) = \tau(x,r)_\#[A \cap S(x,r)]$. Note that if $r < \operatorname{dist}(x, \operatorname{spt}\partial A)$, then $\operatorname{spt} A^*(x,r) \subset S_0, \operatorname{spt}\partial A^*(x,r) \subset \operatorname{fr} S_0$.

(10.3) Lemma. *Let $C \in N_k(G)$ minimize mass among all C' with $\partial C' = \partial C$. Let $x_0 \in \operatorname{spt} C - \operatorname{spt}\partial C$. Let $\{\rho_\nu\}$ be any sequence tending to* 0 *such that $C^*(x_0, \rho_\nu) \in N_k(G)$ for $\nu = 1, 2, \cdots$ and $C^*(x_0, \rho_\nu)$ tends to a limit C^*. Then C^* is the cone $0\partial C^*$.*

Proof. Since C minimizes mass, well-known reasoning based on cone construction and (5.7) shows that $M[C^*(x_0,\rho_\nu)]$ is bounded and $\delta[C^*(x_0,\rho_\nu)] \to 0$ as $\nu \to \infty$; see for instance [FF, 9.26]. Then (10.3) follows from (10.2).

Let X_A be as in §9. Then $0 < H_k(X_A) < \infty$; in fact, $\lambda H_k(X) \leqq \mu_A(X)$ for any $X \subset X_A$. See [F, 3.1].

(10.4) Lemma. *Let $P_j \to A, M(P_j) \to M(A)$, and U be any set such that $\mu_A(\operatorname{fr} U) = 0$. Then for every k-plane Π containing* 0,

$$\limsup_{j\to\infty} H_k[((\operatorname{spt} P_j) \cap U)_\Pi] \leqq H_k[(X_A \cap U)_\Pi].$$

Proof. Given $\varepsilon > 0$, let $K_\varepsilon \subset X_A \cap \operatorname{int} U$ be compact, with $\mu_A(U - K_\varepsilon) < \varepsilon/2$. Choose U_ε open such that $K_\varepsilon \subset U_\varepsilon \subset U, \mu_A(\operatorname{fr} U_\varepsilon) = 0$, and

$$H_k[(U_\varepsilon)_\Pi] < H_k[(K_\varepsilon)_\Pi] + \varepsilon/2.$$

Observe that

$$(\operatorname{spt} P_j) \cap U \subset U_\varepsilon \cup [(\operatorname{spt} P_j) \cap (U - U_\varepsilon)],$$

$$H_k[((\operatorname{spt} P_j) \cap U)_\Pi] \leqq H_k[(U_\varepsilon)_\Pi] + \mu_j(U - U_\varepsilon),$$

$$\limsup_{j\to\infty} H_k[((\operatorname{spt} P_j) \cap U)_\Pi] \leqq H_k[(K_\varepsilon)_\Pi] + \mu_A(U - U_\varepsilon) + \varepsilon/2.$$

Since $K_\varepsilon \subset X_A \cap U$ and $U - U_\varepsilon \subset U - K_\varepsilon$, the right side is no more than $H_k[(X_A \cap U)_\Pi] + \varepsilon$. This proves (10.4).

On the other hand, if A is close to a k-disk, then we have the following lower estimate:

(10.5) LEMMA. *Let $W(A - gD) < \varepsilon$, where $D = \Pi \cap S_0$ (with an orientation). If $\{Q_\nu\}$ is any sequence of polyhedral chains tending to A, then*

$$\Omega \liminf_{\nu \to \infty} H_k[(\operatorname{spt} Q_\nu)_\Pi] \geqq |g| \alpha(k) - \varepsilon,$$

where $\Omega = \max\{|g| : g \in G\}$.

Proof. Let p denote orthogonal projection of E^n onto $\Pi, p(x) = x'$, where $x = x' + x''$ with $x' \in \Pi, x'' \in \Pi^\perp$. Then

$$\Omega H_k[(\operatorname{spt} Q_\nu)_\Pi] \geqq M(p_\# Q_\nu).$$

Since $p_\# D = D$, we have for large enough ν,

$$M(p_\# Q_\nu - gD) = W(p_\# Q_\nu - gD) \leqq W(Q_\nu - gD) < \varepsilon,$$

and hence $M(p_\# Q_\nu) \geqq |g| M(D) - \varepsilon$.

Proof of (10.1). Let us proceed by induction on k. For $k = 0$, A is a polyhedral 0-chain and X_A a finite set. Suppose (10.1) true in dimension $k - 1$. To obtain a contradiction, let us assume that there exists $A \in N_k(G)$ with ∂A a polyhedral chain for which there is a set $X \subset X_A$ such that $\mu_A(X) > 0$, $H_k(X_\Pi) = 0$ for almost every k-plane Π through 0.

By [F, §3], (8.2), and (8.3), there exists $X_1 \subset X$ such that $\mu_A(X - X_1) = 0$ and

$$\lim_{r \to 0} r^{-k} \mu_A[X^c(x,r)] = 0$$

for every $x \in X_1$, where we have set $Z(x,r) = Z \cap S(x,r)$, in this case with $Z = X^c$. Since projection does not increase Hausdorff measure,

$$H_k[(X_A - X)(x,r)_\Pi] \leqq H_k[(X_A - X)(x,r)] \leqq \lambda^{-1} \mu_A[X^c(x,r)].$$

Hence, when $x \in X_1$,

$$\lim_{r \to 0} r^{-k} H_k[X_A(x,r)_\Pi] = 0,$$

for almost all Π.

In what follows, we use the notation

$$A(x,r) = A \cap S(x,r), \quad A^*(x,r) = \tau(x,r)_\# A(x,r).$$

If $0 < r < \operatorname{dist}(x, \operatorname{spt} \partial A)$, then from (5.7)

$$\operatorname{spt} \partial A(x,r) \subset \operatorname{fr} S(x,r), \quad \operatorname{spt} \partial A^*(x,r) \subset \operatorname{fr} S_0$$

for almost all r. Let $x \in X_1$. By (5.7) and the finiteness of $D_k^*(x)$, there is a sequence $\{r_m\}$ tending to 0 such that $\mu_A[\mathrm{fr}\, S(x, r_m)] = 0$ for $m = 1, 2, \cdots$ and, writing $A_m^* = A^*(x, r_m)$, $N(A_m^*)$ is bounded and by (7.4), A_m^* tends to a limit C for a subsequence (still denoted by $\{r_m\}$). Since $\partial A_m^* \to \partial C$ and $\mathrm{spt}\, A_m^* \subset S_0$, by (5.4) we have

$$\partial(A_m^* - C) = \partial D_m, \text{ where } M(D_m) \to 0.$$

Since (see (8.1)) $r_m^{-k}[\mu(x, r_m) - \nu(x, r_m)] \to 0$, C minimizes mass and $M(A_m^*) \to M(C)$. In particular, $M(C) \geqq D_k^*(x) \geqq \lambda$.

In the same way, let

$$C_\nu^* = C^*(0, \rho_\nu) = \tau(0, \rho_\nu)_\# C,$$

where the sequence $\{\rho_\nu\}$ tending to 0 is chosen so that:

(1) $\mu_C[\mathrm{fr}\, S(0, \rho_\nu)] = \mu_A[\mathrm{fr}\, S(x, r_m \rho_\nu)] = 0$ for $m, \nu = 1, 2, \cdots$,

(2) $N(C_\nu^*)$ is bounded, $\mathrm{spt}\, \partial C_\nu^* \subset \mathrm{fr}\, S_0$, and C_ν^* tends to a limit C^*.

Then $N(C^*) < \infty$, $\mathrm{spt}\, \partial C^* \subset \mathrm{fr}\, S_0$, and $M(C^*) \geqq \lambda$. By (10.3), $C^* = 0 \partial C^*$. Since $M(\partial C^*) < \infty$, by the induction hypothesis ∂C^* is a rectifiable $(k-1)$-chain. By (9.1) the cone C^* is rectifiable. Therefore, E^n is a(μ_{C^*}, k)-rectifiable set. Let y be some point such that $|y| < 1$, the lower k-density of C^* at y is at least λ [actually the k-density exists since C^* minimizes mass], and there is an oriented $C^{(1)}$ k-submanifold $Y(y)$ such that

$$\lim_{t \to 0} t^{-k} \mu_{C^*}[S(y, t) - Y(y)] = 0.$$

For each $\nu = 1, 2, \cdots$,

$$A^*(x, r_m) \cap S(0, \rho_\nu) \to C \cap S(0, \rho_\nu)$$

as $m \to \infty$, and hence $A^*(x, r_m \rho_\nu) \to C_\nu^*$. Let $\rho_\nu' = r_{m_\nu} \rho_\nu$, where the subsequence $\{m_\nu\}$ is chosen so that $A^*(x, \rho_\nu') \to C^*$ as $\nu \to \infty$.

Let Π_0 be parallel to the tangent k-plane to $Y(y)$ at y, and $D_0 = \Pi_0 \cap S_0$. Let

$$C^{**}(y, t) = \tau(y, t)_\# C^*(y, t).$$

By (9.3) there exists $g \in G$ such that

$$\lim_{t \to 0^+} W[C^{**}(y, t) - g D_0] = 0.$$

Since the lower density of C^* at y is positive, $g \neq 0$. Let $0 < \varepsilon < |g| \alpha(k)/2$. There exist $t_0 > 0$ and a neighborhood U of Π_0 in the space of k-planes such that

$$W[C^{**}(y, t_0) - g D] < \varepsilon$$

for every $\Pi \in U$ and $\mu_{C^*}[\mathrm{fr}\, S(y, t_0)] = 0$. Choose some $\Pi_1 \in U$ such that

$$\lim_{r \to 0^+} r^{-k} H_k[X_A(x, r)]_{\Pi_1} = 0.$$

Let $\{P_j\}$ be a sequence of polyhedral chains tending to A such that $M(P_j) \to M(A)$. For each $\nu, P_j(x, \rho_\nu) \to A(x, \rho_\nu')$, and hence $P_j(x, \rho_\nu) \to A^*(x, \rho_\nu')$ as $j \to \infty$. Choose a subsequence $\{j_\nu\}$ such that $P_\nu^* \to C^*$, where $P_\nu^* = P_{j_\nu}^*(x, \rho_\nu')$. Moreover, using (10.4) we choose j_ν large enough that

$$H_k[(\operatorname{spt} P_{j_\nu}(x, \rho_\nu'))_{\Pi_1}] \leqq 2H_k[X_A(x, \rho_\nu')_{\Pi_1}].$$

Then $H_k[(\operatorname{spt} P_\nu^*)_{\Pi_1}] \to 0$. Let

$$Q_\nu = \tau(y, t_0)_\# P_\nu^*(y, t_0).$$

Then $Q_\nu \to C^{**}(y, t_0)$ and $H_k[(\operatorname{spt} Q_\nu)_{\Pi_1}] \to 0$. Since $\varepsilon < |g|\alpha(k)/2$, by (10.5) $\liminf_\nu H_k[(\operatorname{spt} Q_\nu)_{\Pi_1}] > 0$. This is a contradiction.

Therefore, no such set X exists, which proves that A is rectifiable.

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