LIMIT THEOREMS FOR MARKOV PROCESSES(1)

BY
S. R. FOGUEL

Summary. Let $P(x, A)$ be the transition probability of a Markov Process that satisfies a "Doeblin Condition" and is irreducible (these notions are defined below). Then there are two possibilities:

1. The process has a finite invariant measure, $\lambda \neq 0$, and there exists an integer $k$ such that the limit of $P^{nk+j}(x, A)$ exists for every $x$, $A$ and $0 \leq j < k$.

2. There exists a sequence of sets $A_j$ with $\bigcup_{j=0}^{\infty} A_j = X$ such that $\lim_{n \to \infty} P^n(x, A_j) = 0$, $x \in X$.

1. Notation. Let $(X, \Sigma)$ be a measurable space. Let $P(x, A)$ be transition probabilities:

1.1. $P(x, A)$ is defined for $x \in X$ and $A \in \Sigma$ and $0 \leq P(x, A) \leq 1$.

1.2. For a fixed $x$ the set function $P(x, \cdot)$ is a measure on $\Sigma$.

1.3. For a fixed $A \in \Sigma$, the function $P(\cdot, A)$ is measurable.

By measure we shall mean a countably additive, positive, finite measure. When we deal with finitely additive bounded measures we shall write $\mu \in ba(X, \Sigma)$. On occasions we shall deal with $\sigma$-finite, countably additive positive measures.

Let us use the terminology of [2, p. 240]. It is well known that the transition probabilities induce an operator $P$ on $B(X, \Sigma)$ and on its conjugate space $ba(X, \Sigma)$ by:

1.4. If $f \in B(X, \Sigma)$, then $(Pf)(x) = \int f(y) P(x, dy)$.

1.5. If $\mu \in ba(X, \cdot)$, then $(\mu P)(A) = \int P(x, A) \mu(dx)$, where $\mu P$.

1.6. $\int(Pf)(x) \mu(dx) = \int f(x)(\mu P)(dx)$.

The iterates of these operators are given by the same expressions where $P$ is replaced by $P^n$:

$$P^n(x, A) = \int P^{n-k}(x, dy) P^k(y, A), \quad 0 \leq k < n.$$

Note that if $\mu$ is countably additive, so is $\mu P$.

2. The limit theorems. Throughout this section we assume:

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There exists a σ-finite measure \( \nu \) with

2.1. Doeblin's Condition: There exists an integer \( d \) such that if \( \nu(A) = 0 \) then \( \sup \{ P^d(x,A): x \in X \} < 1 \).

2.2. There exists a σ-finite measure \( \lambda \) that is stronger than \( \nu \) and subinvariant:

\[
\lambda(A) \geq \int P(x,A) \lambda(dx).
\]

2.3. The space \( X \) is a locally compact Hausdorff space and \( \Sigma \) consists of its Baire sets.

Thus by Theorem G on p. 52 of [4] every measure is regular.

**Definition.** The process will be called \( \nu \)-irreducible if:

2.4. \( \frac{1}{P^n(x,A)} = 0, \quad n = 1, 2, \ldots, \) for some \( x \), then \( \nu(A) = 0 \).

**Remarks.** Condition 2.1 is weaker than the classical Doeblin Condition (see [1, p. 192, hypothesis D]). There one assumes the conclusion whenever \( \nu(A) \leq \varepsilon \) for some fixed \( \varepsilon > 0 \); also uniformity in the sets \( A \) is assumed.

The σ-finite measure \( \nu \) can be replaced by a finite measure \( \nu_1 \) equivalent to it. Let \( \nu_2 = \sum \nu_1 P^n/2^n \) then \( \nu \ll \nu_2 \) and 2.1 holds with respect to \( \nu_2 \). We shall see below that if \( \mu \ll \tau \) then \( \mu P \ll \tau P \); thus \( \nu_1 P^n \ll \lambda P^n \ll \lambda \) and \( \nu_2 \ll \lambda \). Finally let us show that if the process is \( \nu \)-irreducible then it is \( \nu_2 \)-irreducible.

Note first that if \( 0 \leq f \in B(X, \Sigma) \) and \( (P^nf)(x_0) = 0, \quad n = 1, 2, \ldots \) for some \( x_0 \), then

\[
P^n(x_0, \{x: f(x) \geq \varepsilon \}) \leq \frac{1}{\varepsilon} (P^n f)(x_0) = 0.
\]

Thus \( \int f \, d\nu = 0 \). Apply this to \( f(y) = P^k(y, A) \) to conclude:

\[
0 = P^n(x_0, A) = \int P^k(y, A) P^{n-k}(x_0, dy)
\]

implies

\[
\int P^k(y, A) \nu(dy) = 0.
\]

Hence \( \nu_2(A) = 0 \) whenever \( P^n(x_0, A) = 0 \) for all \( n \).

*Thus we shall assume, with no loss of generality, that \( \nu \) is finite and \( \nu P \ll \nu \).*

**Lemma 1.** Let \( \mu \) and \( \tau \) be two σ-finite measures. If \( \mu \ll \tau \), then \( \mu P \ll \tau P \).

**Proof.** Let \( d\mu = f \, d\tau \) and \( d\mu_k = \min(f,k) \, d\tau \). Then

\[
(\mu_k P)(A) = \int P(x,A) d\mu_k \leq k \int P(x,A) d\tau.
\]

Thus

\[
\mu_k P \ll \tau P \quad \text{and also} \quad \mu P \ll \tau P.
\]
Theorem 1. Let \( \mu \) be any measure. If \( \mu P^n = \tau_n + \sigma_n \), where \( \tau_n \ll \nu \) and \( \sigma_n \perp \nu \), then \( \lim \sigma_n(X) = 0 \).

Proof. Since \( \tau_{n+1} + \sigma_{n+1} = \tau_n P + \sigma_n P \) and \( \tau_n P \ll \nu \), then \( \sigma_{n+1} \ll \sigma_n P \).

Assume that \( \lim \sigma_n(X) \neq 0 \). Let \( \sigma \) be a weak * limit point of \( \sigma_n \), where \( \sigma \in ba \).

Let \( Y \in \Sigma \) be such that \( \nu(Y) = 0 \) and \( \sigma_n(X - Y) = 0 \). Given \( \varepsilon > 0 \), choose \( n \) so that

\[
| (\sigma P^d)(Y) - (\sigma_n P^d)(Y) | < \varepsilon,
\]

thus

\[
(\sigma P^d)(Y) \geq (\sigma_n P^d)(Y) - \varepsilon \geq \sigma_{n+1}(Y) - \varepsilon \geq \lim \sigma_n(X) - \varepsilon,
\]

and

\[
\lim \sigma_n(X) \leq (\sigma P^d)(Y) = \int P^d(x, Y) \sigma(dx) \leq \sup \{ P^d(x, Y) : x \in X \} \sigma(X) = \sup \{ P^d(x, Y) : x \in X \} \lim \sigma_n(X) < \lim \sigma_n(X)
\]

by 2.1. This contradiction proves that \( \lim \sigma_n(X) = 0 \).

We may, and shall, assume that \( \lambda \) is equivalent to \( \nu \):

Put \( \lambda = \lambda_1 + \lambda_2 \) where \( \lambda_1 \ll \nu \) and \( \lambda_2 \perp \nu \); then \( \lambda(A) \geq (\lambda_1 P)(A) + (\lambda_2 P)(A) \) for every \( A \in \Sigma \). Let \( X_1 \) be such that \( \nu(X - X_1) = 0 \) and \( \lambda_2(X_1) = 0 \) then \( (\lambda_1 P)(A) = (\lambda_1 P)(A \cap X_1) \leq \lambda(A \cap X_1) = \lambda_1(A) \). Thus \( \lambda_1 \) is subinvariant, too, and \( \lambda_1 \ll \nu \).

Finally since \( \nu \ll \lambda_1 \), then \( \nu \ll \lambda_1 \). If \( \lambda_1(A) = 0 \), then \( \lambda(A \cap X_1) = \lambda_1(A) = 0 \) and \( \nu(A) = \nu(A \cap X_1) = 0 \), too.

Let \( P \) be considered as an operator on \( L_2(X, \Sigma, \lambda) \) by extending it from \( B(X, \Sigma) \) as in [3, pp. 1–2]. For the next lemma we shall use the notation of [3, Theorem 1.1]. Thus there exists a subfield \( \Sigma_1 \) of \( \Sigma \) such that

2.5. If \( f \in L_2(\lambda) \) and \( \int_A f d\lambda = 0 \) for every \( A \in \Sigma_1 \), then weak \( \lim P^n f = 0 \) in \( L_2(\lambda) \) sense.

2.6. The sets \( A \in \Sigma_1 \) are defined in [3] as sets of finite \( \lambda \)-measure such that the functions \( P^n \chi_A \), \( P^n \chi_A \) are all characteristic functions a.e. where \( \chi_A \) denotes the characteristic function of \( A \).

Lemma 2. The \( \sigma \) field \( \Sigma_1 \) is generated by a countable collection of disjoint sets.

Proof. It is enough to show that each set \( A \in \Sigma_1 \) contains an atom.

Let us assume, to the contrary, that some set \( A \), in \( \Sigma_1 \), with \( \lambda(A) \neq 0 \) does not contain atoms of \( \Sigma_1 \). Let \( \chi_B = P^d \chi_A \) where \( P^d \) is the \( L_2(X, \Sigma, \lambda) \) adjoint of \( P^d \). Then by Theorem 1.1 of [3], \( P^d(x, B) = P^d(\chi_B) = \chi_A \) a.e.
Since $\lambda$ is a regular measure, there exists a compact subset $C_0$, $\lambda(C_0) \neq 0$, of $A$, such that $P_t(x, B) = 1$ for every $x \in C_0$. Let $A'$ be the set in $\Sigma_1$ which contains $C_0$ and has minimal $\lambda$-measure. Such a set is unique up to sets of measure zero and $A' \subset A$. Since $A'$ is not an atom it contains a set $A_1$, in $\Sigma_1$, with $\lambda(A_1) \leq \frac{1}{2} \lambda(A') \leq \frac{1}{2} \lambda(A)$. Now $\lambda(C_0 \cap A_1) \neq 0$ for otherwise $A' - A_1$ would be smaller than $A'$ and contain $C_0$. Let $\chi_{B_1} = P_t^* \chi_{A_1}$; then $P_t(x, B_1) = 1$, $x \in C_0 \cap A_1$ ae. Thus there exists a compact subset $C_1$ of $C_0$ such that $P_t(x, B_1) = 1$ for every $x \in C_1$ and $\lambda(B_1) = \lambda(A_1) \leq \frac{1}{2} \lambda(A)$. Using an induction argument, we find a decreasing sequence of sets $B_n \in \Sigma_1$, with $\lambda(B_n) \to 0$, and a decreasing sequence of compact sets $C_n$, such that $P_t(x, B_n) = 1$ for every $x \in C_n$. Let $x_0 \in C \cap C_n$; then

$$P_t(x_0, \cap B_n) = \lim P_t(x_0, B_n) = 1$$

while $\lambda(\cap B_n) = 0$, which contradicts 2.1.

Let $W \in \Sigma_1$ be an atom and let $P_W$ denote the set whose characteristic function is $P \chi_W$.

Call $W$ of the first kind if the sets $P^n W$ are a.e. disjoint. Otherwise $W$ will be called of the second kind. If $P^n W$ intersects $P^k W$ for $k < n$, then $P^n W = P^k W$ a.e. since they are atoms and hence $P^{n-k} W = W$ a.e. Define:

2.7. $X_1 = \bigcup \{W: W \in \Sigma_1 \text{ and is of the first kind}\}$.

2.8. $X_2 = \bigcup \{W: W \in \Sigma_1 \text{ and is of the second kind}\}$.

2.9. $X_3 = X - X_1 \cup X_2$.

**Lemma 3.** If the process is v-irreducible then either $X = X_3$ or $X = X_2$ and there exists an integer $k$ such that $\Sigma_1 = \{W, P_W, \ldots, P^{k-1} W\}$ where $P^k W = W$. In this case the measure $\lambda$ is finite.

**Proof.** If $W \in \Sigma_1$ and $W \subset X_1$, then $\int W P^n \chi_W d\lambda = 0$, $n = 1, 2, \ldots$. Thus $P^n(x, W) = 0$, $x \in W$ a.e. Thus, since the process is v-irreducible, $v(W) = 0$ and also $\lambda(W) = 0$. Therefore $X_1$ is empty. Now let us assume that $X_2$ contains the nonempty set $W$. Then $P^n(x, W) = 0$ a.e. for $x \in X - W \cup \ldots \cup P^{k-1} W$ and thus this difference is empty.

**Theorem 2.** Let $\mu$ be any measure. Let $A \in \Sigma$ and $\lambda(A) < \infty$.

(a) If $A \subset X_3$ then $\lim (\mu P^n)(A) = 0$.

(b) If $A \subset W \subset X_2$ where $W \in \Sigma_1$ and $P^k W = W$, then the limit of $(\mu P^{n+k+j})(A)$ exists as $n \to \infty$ and $0 \leq j < k$.

(c) If $A \subset W \subset X_1$ where $W \in \Sigma_1$, then $\lim (\mu P^n)(A) = 0$.

**Proof.** By Theorem 1 it is enough to prove these results for a measure $\mu \ll \lambda$. We may assume that $d\mu = f d\lambda$ where $f \in L_2(\lambda)$ since any measure $\mu$ which is weaker than $\lambda$ can be approximated by such measures. Thus:

If $A \subset X_3$, then $(\mu P^n)(A) = \int P^n \chi_A d\lambda \to 0$ since $\chi_A$ is orthogonal to the sets in $\Sigma_1$ and 2.5.
If \( A \subset W \subset X_1 \) where \( W \in \Sigma_1 \), then \( \lim \int P^n \chi_A f d\lambda \leq \lim \int P^n \chi_W f d\lambda = 0 \) since \( \int_W P^n \chi_W d\lambda = 0 \) and Theorem 2.1 of [3] applies.

Finally let \( A \subset W \subset X_2 \) where \( W \in \Sigma_1 \) and \( P^k W = W \). Then

\[
\lim_{n \to \infty} (\mu P^{n+k+j})(A) = \lim_{n \to \infty} \int P^{n+k+j} \chi_A \cdot f d\lambda = \frac{\lambda(A)}{\lambda(W)} \int P^j \chi_W \cdot f d\lambda
\]

since the function \( g = \chi_A - (\lambda(W)^{-1}/\lambda(A)) \chi_W \) is orthogonal to all sets in \( \Sigma_1 \) and thus weak limit of \( P^n g \) is 0 or

\[
\lim \int P^{n+k+j} \chi_A \cdot f d\lambda = \lim \frac{\lambda(A)}{\lambda(W)} \int P^{n+k+j} \chi_W \cdot f d\lambda = \frac{\lambda(A)}{\lambda(W)} \int P^j \chi_W \cdot f d\lambda.
\]

**Corollary.** If the process is \( \nu \)-irreducible, then either

(a) \( \lim_{n \to \infty} P^n(x, A) = 0 \) for every \( x \in X \) and every set \( A \) with \( \lambda(A) < \infty, \pi \) or:

(b) The limit of \( P^{n+k+j}(x, A) \) exists for every \( x \in X, A \in \Sigma \) and \( 0 \leq j < k \).

**Proof.** It is enough to note that we get (a) when \( X = X_3 \) and (b) when \( X = X_2 \) since every set \( A \in \Sigma \) can be written as

\[
A = (A \cap W) \cup (A \cap P W) \cup \cdots \cup (A \cap P^k W)
\]

and the previous theorem applies to \( A \cap P^l W \).

**Theorem 3.** If the process is \( \nu \)-irreducible and \( X = X_2 \), then for any measure \( \mu \) and every \( j \) there are constants \( \gamma_1 \cdots \gamma_k \) such that

\[
\lim_{n \to \infty} (\mu P^{n+k+j})(A) = \sum_{i=0}^{k-1} \gamma_i \lambda(A \cap P^l W)
\]

for all \( A \).

**Proof.** It is enough to consider \( \mu P^n \). Let \( \tau(A) = \lim (\mu P^n)(A) \) where the limit exists for any \( A \in \Sigma \). Then, by Corollary III. 7.4. of [2] the set function \( \tau \) is countably additive and clearly \( \tau = \tau P^k \).

From Theorem 1 it follows that \( \tau \leq \lambda \). Let \( \tau = \tau^0 + \cdots + \tau^{k-1} \) where \( \tau^{(i)} \) is the restriction of \( \tau \) to \( P^i W \). Thus \( \tau^{(i)} P^k = \tau^{(i)} \) and so \( \tau^{(i)} + \tau^{(i)} P + \cdots + \tau^{(i)} P^{k-1} \) is invariant under \( P \). It is easy to see that the invariant measure is unique (Theorem 1 and the \( \nu \)-irreducibility) hence this sum is equal to \( \gamma \lambda(A) \) for some constant \( \gamma \).

Now \( \tau^{(i)} P^j \) is zero on any subset of \( W_j \):

If \( A \subset W_i \), then \( P^j \chi_A \subset W_i = \emptyset \) a.e. \( \lambda \), hence a.e. \( \tau \), for \( 0 < j \leq k - 1 \). Thus \( \tau^{(i)}(A) = \gamma \lambda(A \cap W_i) \).

3. Existence of a subinvariant measure for irreducible processes. In this section we use a small modification of Harris' argument to find a subinvariant measure.
In [5] Harris constructs a \( \sigma \)-finite invariant measure for infinitely recurrent process. Here we find only subinvariant measure under weaker conditions. Throughout this section we assume:

3.1. For every \( x \), \( P(x, X) = 1 \).
3.2. The \( \sigma \)-field \( \Sigma \) is the Borel extension of a countable family of sets.
3.3. The process is \( \nu \)-irreducible where \( \nu \) is a given \( \sigma \)-finite measure.

Notice that \( X \) is not assumed to be a topological space and 2.1 is not assumed. Let us just mention those parts of [5] that require a modification in this case.

**Theorem 4.** The process has a \( \sigma \)-finite subinvariant measure that is stronger than \( \nu \).

Let \( P_A \) be defined as in [5]. Lemma 1 of [5] should be restated:

A. Let \( A \) be a measurable set with \( 0 < \nu(A) < \infty \). If \( \lambda_A \) is a bounded sub-invariant measure for \( P_A \), then the measure \( \lambda \):

\[
3.4. \quad \lambda(E) = \int_A \lambda_A(dx)P_A(x, E)
\]

is subinvariant for \( P \) and is \( \sigma \)-finite.

The proof is almost identical to Harris’. First if \( E \subset A \), then \( \lambda(E) \leq \lambda_A(E) \). Also

\[
\int \lambda(dy)P(y, E) = \int_A \lambda(dy)P(y, E) + \int_{X-A} \left[ \int_A \lambda_A(dx)P_A(x, dy) \right] P(y, E)
\]

\[
\leq \int_A \lambda_A(dx) \left[ P(x, E) + \int_{X-A} P_A(x, dy)P(y, E) \right] = \int_A \lambda_A(dx)P_A(x, E)
\]

\[
= \lambda(E).
\]

The proof that \( \lambda \) is \( \sigma \)-finite is the same as in [5] and also \( \lambda(A) \neq 0 \), for we will see that \( P_A(x, A) > 0 \) for every \( x \in A \).

Lemma 2 and Lemma 3 of [5] are unchanged. Thus \( P_A^1 + \cdots + P_A^n \geq P_A^1 + \cdots + P_A^n \) see [5, Equation 4.17]. Now if \( P_A(x, A) = 0 \), \( x \in A \), then

\[
P_A^i(x, A) = \int_A P_A(x, dy)P_A^{i-1}(y, A) = 0.
\]

Thus \( P_A^i(x, A) = 0 \), \( i = 1, 2, \ldots \), contrary to 3.3. Let us define

\[
3.5. \quad Q(x, E) = \frac{P_A(x, E)}{P_A(x, A)}, \quad x \in A, \ E \subset A.
\]

Then clearly \( Q_i \geq P_A^i \).

Put

\[
3.6. \quad R(x, E) = \frac{Q_1(x, E) + \cdots + Q_k(x, E)}{k}, \quad x \in A, \ E \subset A,
\]

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where \( k \) is defined as in [5]. Then Lemmas 4 and 5 of [5] will show us that there exists a measure \( \lambda_A \) with

\[
3.7. \quad \lambda_A(E) = \int_A \lambda_A(dx)Q(x,E), \quad E \subset A.
\]

Finally, it follows from 3.7 that

\[
3.8. \quad \lambda_A(E) = \int_A \lambda_A(dx)Q(x,E) \geq \int_A \lambda_A(dx)P_A(x,E), \quad E \subset A.
\]

It remains to show that \( \lambda \) is stronger than \( v \). Now if \( \lambda(E) = 0 \), then

\[
\int \lambda(dx)P^0(x,E) \leq \lambda(E) = 0.
\]

Hence \( P^0(x,E) = 0 \) a.e., \( n = 1, 2, \cdots \). Since \( \lambda \neq 0 \), there exists an \( x_0 \in X \) with \( P^0(x_0,E) = 0 \), \( n = 1, 2, \cdots \); hence \( v(E) = 0 \).

4. **Existence of an invariant measure.** Throughout this section we assume:

4.1. For every \( x \), \( P(x,X) = 1 \).

4.2. There exists a \( \sigma \)-finite measure \( v \), and an increasing sequence of sets \( X_n \), in \( \Sigma \), such that:
   a. \( \bigcup X_n = X \).
   b. \( v(X_n) < \infty \).
   c. If \( A \in \Sigma \) and \( A \subset X_k \), then for every \( \varepsilon > 0 \) there exists an integer \( n = n(A,\varepsilon) \) such that

   \[
   \sup \{ P^n(x,A) : x \in X \} \leq v(A) + \varepsilon.
   \]

**Lemma 4.** Let \( \mu \in ba(X,\Sigma) \) be invariant. If \( A \subset X_k \), then \( \mu(A) \leq v(A) \).

**Proof.** Let \( n = n(A,\varepsilon) \); then

\[
\mu(A) = \int P^n(x,A) \mu(dx) \leq (v(A) + \varepsilon) \int \mu(dx) = v(A) + \varepsilon.
\]

**Definition.** Let \( S \) be the collection of invariant measures with unit total measure.

If \( \mu \in S \), then \( \mu \leq v \) on subsets of \( X_k \) by Lemma 4. Since both are countably additive, \( \mu \leq v \). Thus \( d\mu = fdv \) where \( 0 \leq f \leq 1 \) and \( f \in L_1(v) \).

Now

\[
4.3. \quad (\mu P)(A) = \int P(x,A) \mu(dx) = \int P(x,A)f(x)v(dx).
\]

**Lemma 5.** Let \( d\mu_1 = f_1dv \), \( d\mu_2 = f_2dv \) where \( \mu_1 \) and \( \mu_2 \) are in \( S \). If \( d\mu = \max(f_1,f_2)dv \), then \( \mu \) is invariant, too.
Proof. Put $Y_1 = \{x : f_1(x) \geq f_2(x)\}, Y_2 = X - Y_1$. Then

$$\int_A \max(f_1, f_2) \, dv = \int_{A \cap Y_1} f_1 \, dv + \int_{A \cap Y_2} f_2 \, dv$$

$$= \int P(x, A \cap Y_1)f_1(x) \, dv(x) + \int P(x, A \cap Y_2)f_2(x) \, dv(x)$$

$$\leq \int \max(f_1(x), f_2(x))P(x, A) \, dv(x).$$

We used 4.2 and the invariance of $\mu_1$ and $\mu_2$. Thus $\mu(A) \leq (\mu P)(A)$ for every $A \in \Sigma$. But $(\mu P)(X) = \int P(x, X)\mu(dx) = \mu(X) < \infty$; hence $\mu(A) = (\mu P)(A)$.

Consider the collection of functions $f$ such that $f \, dv \in S$. Since $0 \leq f \leq 1$, the supremum of this collection in $L_1(v)$ is the supremum of a sequence $f_n$ in this collection (Theorem IV, 11.7 of [2]). Let $g = \sup f_n$ and $d\lambda = gdv$. If $S = \emptyset$, then take $g = 0$. Let $g_n = \max(f_1, \ldots, f_n)$, then $g = \lim g_n$ and by Lemma 5 and 4.3:

$$\int P(x, A)g_n(x) \, dv(dx) = \int g_n(x) \, dv(dx).$$

Passing to a limit, we see that $\lambda$ is an invariant measure. Also $\lambda \leq v$ since $0 \leq g \leq 1$; thus it is countably additive and finite on $X_n$.

Theorem 5. There exists a $\sigma$-finite measure $\lambda$ with

a. $\lambda \leq v$.

b. $\lambda$ is invariant under $P$.

c. If $\mu \in S$, then $\mu \leq \lambda$.

d. Let $A$ be contained in some $X_k$ and $\lambda(A) = 0$. For every $\tau \in ba$

$$\lim \frac{1}{n}(\tau(A) + (\tau P)(A) + \cdots + (\tau P^{n-1})(A)) = 0.$$ 

Proof. Parts a, b and c were proved above. Let $\tau_n = (\tau + \tau P + \cdots + \tau P^{n-1})/n$ and assume that for some subsequence $n_i$, $\tau_{n_i}(A) \geq \delta > 0$. Since $\tau_n$ form a bounded sequence in $B(X, \Sigma)^* = ba$, there exists a weak * limit point $\mu$ to the sequence $\tau_n$. Thus $\mu \geq 0$, $\mu(X) \leq 1$ and $\mu(A) \geq \delta > 0$. It is easily seen that $\mu P = \mu$. Let $\mu = \mu_1 + \mu_2$ where $\mu_1$ is a measure (c.a.) and $\mu_2$ is purely finitely additive (see [7, p. 52]). Then $\mu \leq v$ on subsets of $X_k$ by Lemma 4. Hence $\mu_2(X_k) = 0$, for the restriction of $\mu_2$ to $X_k$ is countably additive. It remains to show that $\mu_1$ is invariant which will contradict part c. Now

$$\mu_1 + \mu_2 = \mu = \mu P = \mu_1 P + \mu_2 P.$$ 

Let $\mu_2 P = \sigma_1 + \sigma_2$ where $\sigma_1$ is c.a. and $\sigma_2$ is purely finitely additive. Then $\mu_1 = \mu_1 P + \sigma_1$ but $\mu_1(X) = (\mu_1 P)(X) + \sigma_1(X) = \mu_1(X) + \sigma_1(X)$ and $\sigma_1 = 0$. 

Remark. Part d can be replaced by: If $A$ is contained in $X_k$, then $\lambda(A) = 0$ if and only if

\[ d^1: \lim (P(x, A) + \cdots + P^n(x, A))/n = 0 \text{ for every } x \in X. \]

$d^1$ follows from $d$ when we take $\tau$ to be a unit mass at $x$. Conversely given $d^1$ then for any $\mu \in S$

\[ \mu(A) = \frac{1}{n} \int (P(x, A) + \cdots + P^n(x, A)) \mu(dx) \to 0. \]

Thus $\lambda(A) = 0$, too.

An example. Let $\nu$ be a $\sigma$-finite measure and $P(x, A) = \int_A f(x, \xi) \nu(d\xi)$ where $0 \leq f(x, \xi)$ and $\int_X f(x, \xi) \nu(d\xi) = 1$. It is easy to see that

\[ P^n(x, A) = \int f^n(x, \xi) \nu(d\xi), \quad f^n(x, \xi) = \int f^{n-k}(x, y) f^k(y, \xi) \nu(dy). \]

Put

\[ g_n(\xi) = \sup \{ f^n(x, \xi) : x \in X \} \leq \infty. \]

Lemma 6. For every $\xi \in X$, $g_{n+1}(\xi) \leq g_n(\xi)$.

Proof.

\[ f^{n+1}(x, \xi) = \int f(x, y) f^n(y, \xi) \nu(dy) \leq g_n(\xi) \int f(x, y) \nu(dy) = g_n(\xi). \]

Hence $g_{n+1}(\xi) \leq g_n(\xi)$.

Let $g(\xi) = \lim g_n(\xi)$.

Theorem 6. Condition 4.2 holds with respect to a measure $\nu_1$ equivalent to $\nu$ if $g(\xi) < \infty$ for every $\xi \in X$.

Proof. Let $Y_k = \{ \xi : g(\xi) < k \}$, then $Y_k \subset Y_{k+1}$ and $\bigcup_{k=1}^\infty Y_k = X$. Define $\nu_1$ by: $\nu_1(A) = k \nu(A)$ if $A \subset Y_k - Y_{k-1}$. Then $\nu_1 \sim \nu$. If $f^n_1(x, \xi)$ is the Radon-Nikodym derivative of $P^n(x, A)$ with respect to $\nu_1$, then $f^n_1(x, \xi) = (1/k) f^n(x, \xi)$ whenever $\xi \in Y_k - Y_{k-1}$. Hence if $g_n$ and $g$ were defined for $f^n_1$ in the same way that $g_n$ and $g$ were defined for $f^n$, then $g^1_n(\xi) = (1/k) g_n(\xi)$, $g^1(\xi) = (1/k) g(\xi)$ for $\xi \in Y_k - Y_{k-1}$. Thus $g^1(\xi) < 1$ for every $\xi \in X$. Also $\nu_1$ is $\sigma$-finite: if $\bigcup Z_k = X$ where $Z_k \subset Z_{k+1}$ and $\nu(Z_k) < \infty$; then $\nu_1(Z_k \cap Y_k) < k \nu(Z_k) < \infty$ and $\bigcup (Z_k \cap Y_k) = X$.

Finally let $V_k = \{ \xi : g^1_k(\xi) < 1 \}$; then $V_k \subset V_{k+1}$ by Lemma 6 and with $X_k = Z_k \cap Y_k \cap V_k$ we get 4.2.

Let us conclude with a comparison between our results and Orey's [6]. In [6], Theorem 3 corresponds to part (b) of the corollary of Theorem 2. There it is assumed that the process is infinitely recurrent. We have to add a "Dooblin Condition," namely 2.1, but instead of assuming that whenever $\nu(A) > 0$, $P \{ \text{entering } A \text{ at some time } | X_0 = x \} = 1$, we only assumed that this quantity
is not zero. Part (a) of Theorem 3 furnishes, under our conditions, a positive answer to the problem posed by Orey in [6 end of §3, p. 816].

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THE HEBREW UNIVERSITY OF JERUSALEM,

JERUSALEM, ISRAEL