LIMIT THEOREMS FOR MARKOV PROCESSES

BY
S. R. FOGUEL

Summary. Let $P(x,A)$ be the transition probability of a Markov Process that satisfies a "Doeblin Condition" and is irreducible (these notions are defined below). Then there are two possibilities:

1. The process has a finite invariant measure, $\lambda \neq 0$, and there exists an integer $k$ such that the limit of $P^{k+j}(x,A)$ exists for every $x, A$ and $0 \leq j < k$.

2. There exists a sequence of sets $A_j$ with $\bigcup_{j=0}^{\infty} A_j = X$ such that
   \[ \lim_{n \to \infty} P^n(x,A_j) = 0, \quad x \in X. \]

1. Notation. Let $(X, \Sigma)$ be a measurable space. Let $P(x,A)$ be transition probabilities:

1.1. $P(x,A)$ is defined for $x \in X$ and $A \in \Sigma$ and $0 \leq P(x,A) \leq 1$.

1.2. For a fixed $x$ the set function $P(x, \cdot)$ is a measure on $\Sigma$.

1.3. For a fixed $A \in \Sigma$, the function $P(\cdot, A)$ is measurable.

By measure we shall mean a countably additive, positive, finite measure. When we deal with finitely additive bounded measures we shall write $\mu \in ba(X, \Sigma)$. On occasions we shall deal with $\sigma$-finite, countably additive positive measures.

Let us use the terminology of [2, p. 240]. It is well known that the transition probabilities induce an operator $P$ on $B(X, \Sigma)$ and on its conjugate space $ba(X, \Sigma)$ by:

1.4. If $f \in B(X, \Sigma)$, then $(Pf)(x) = \int f(y) P(x,dy)$.

1.5. If $\mu \in ba(X, \cdot)$, then $(\mu P)(A) = \int P(x,A) \mu(dx)$, where

1.6. $\int (Pf)(x) \mu(dx) = \int f(x)(\mu P)(dx)$.

The iterates of these operators are given by the same expressions where $P$ is replaced by $P^n$:

\[ P^n(x,A) = \int P^{n-k}(x,dy) P^k(y,A), \quad 0 < k < n. \]

Note that if $\mu$ is countably additive, so is $\mu P$.

2. The limit theorems. Throughout this section we assume:

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There exists a $\sigma$-finite measure $\nu$ with

1. **Doeblin's Condition:** There exists an integer $d$ such that if $\nu(A) = 0$ then $\sup \{P^d(x, A) : x \in X\} < 1$.

2. **There exists a $\sigma$-finite measure $\lambda$ that is stronger than $\nu$ and subinvariant:**

   $$\lambda(A) \geq \int P(x, A) \lambda(dx).$$

3. **The space $X$ is a locally compact Hausdorff space and $\Sigma$ consists of its Baire sets.**

   Thus by Theorem G on p. 52 of [4] every measure is regular.

   **Definition.** The process will be called $\nu$-irreducible if:

   2.4. If $P^n(x, A) = 0$, $n = 1, 2, \ldots$, for some $x$, then $\nu(A) = 0$.

   **Remarks.** Condition 2.1 is weaker than the classical Doeblin Condition (see [1, p. 192, hypothesis D]). There one assumes the conclusion whenever $\nu(A) \leq \varepsilon$ for some fixed $\varepsilon > 0$; also uniformity in the sets $A$ is assumed.

   The $\sigma$-finite measure $\nu$ can be replaced by a finite measure $\nu_1$ equivalent to it. Let $\nu_2 = \sum \nu_1 P^n/2^n$ then $\nu \ll \nu_2$ and 2.1 holds with respect to $\nu_2$. We shall see below that if $\mu \ll \tau$ then $\mu P \ll \tau P$; thus $\nu_1 P^n \ll \lambda P^n \ll \lambda$ and $\nu_2 \ll \lambda$. Finally let us show that if the process is $\nu$-irreducible then it is $\nu_2$-irreducible.

   Note first that if $0 \leq f \in B(X, \Sigma)$ and $(P^n f)(x_0) = 0$, $n = 1, 2, \ldots$ for some $x_0$, then

   $$P^n(x_0, \{x : f(x) \geq \varepsilon\}) \leq \frac{1}{\varepsilon} (P^n f)(x_0) = 0.$$

   Thus $\int f d\nu = 0$. Apply this to $f(y) = P^k(y, A)$ to conclude:

   $$0 = P^n(x_0, A) = \int P^k(y, A) P^{n-k}(x_0, dy)$$

   implies

   $$\int P^k(y, A) \nu(dy) = 0.$$

   Hence $\nu_2(A) = 0$ whenever $P^n(x_0, A) = 0$ for all $n$.

   Thus we shall assume, with no loss of generality, that $\nu$ is finite and $\nu P \ll \nu$.

   **Lemma 1.** Let $\mu$ and $\tau$ be two $\sigma$-finite measures. If $\mu \ll \tau$, then $\mu P \ll \tau P$.

   **Proof.** Let $d\mu = f d\tau$ and $d\mu_k = \min(f, k) d\tau$. Then

   $$\mu_k P(A) = \int P(x, A) d\mu_k \leq k \int P(x, A) d\tau.$$

   Thus

   $$\mu_k P \ll \tau P$$

   and also $\mu P \ll \tau P$. 

THEOREM 1. Let $\mu$ be any measure. If $\mu P^n = \tau_n + \sigma_n$, where $\tau_n \ll \nu$ and $\sigma_n \perp \nu$, then $\lim \sigma_n(X) = 0$.

**Proof.** Since $\tau_{n+1} + \sigma_{n+1} = \tau_n + \sigma_n$ and $\tau_n \ll \nu$, then $\sigma_{n+1} \ll \sigma_n$.

Assume that $\lim \sigma_n(X) \neq 0$. Let $\sigma$ be a weak * limit point of $\sigma_n$, where $\sigma \in \mathfrak{ba}$. Let $Y \in \Sigma$ be such that $\nu(Y) = 0$ and $\sigma_n(X - Y) = 0$. Given $\varepsilon > 0$, choose $n$ so that

$$\left| \left( \sigma P^d \right)(Y) - \left( \sigma_n P^d \right)(Y) \right| < \varepsilon,$$

thus

$$\left( \sigma P^d \right)(Y) \geq \left( \sigma_n P^d \right)(Y) - \varepsilon \geq \sigma_{n+d}(Y) - \varepsilon \geq \lim \sigma_n(X) - \varepsilon,$$

and

$$\lim \sigma_n(X) \leq \left( \sigma P^d \right)(Y) = \int P^d(x, Y) \sigma(dx) \leq \sup \{ P^d(x, Y) : x \in X \} \sigma(X) = \sup \{ P^d(x, Y) : x \in X \} \lim \sigma_n(X) < \lim \sigma_n(X)$$

by 2.1. This contradiction proves that $\lim \sigma_n(X) = 0$.

We may, and shall, assume that $\lambda$ is equivalent to $\nu$.

Put $\lambda = \lambda_1 + \lambda_2$ where $\lambda_1 \ll \nu$ and $\lambda_2 \perp \nu$; then $\lambda(A) \geq (\lambda_1 P)(A) + (\lambda_2 P)(A)$ for every $A \in \Sigma$. Let $X_1$ be such that $\nu(X - X_1) = 0$ and $\lambda_2(X_1) = 0$ then $(\lambda_1 P)(A) = (\lambda_1 P)(A \cap X_1) \leq \lambda_1(A \cap X_1) = \lambda_1(A)$. Thus $\lambda_1$ is subinvariant, too, and $\lambda_1 \ll \nu$.

Finally since $\nu \ll \lambda$, then $\nu \ll \lambda_1$. If $\lambda_1(A) = 0$, then $\nu(A \cap X_1) = \lambda_1(A) = 0$ and $\nu(A) = \nu(A \cap X_1) = 0$, too.

Let $P$ be considered as an operator on $L_2(X, \Sigma, \lambda)$ by extending it from $B(X, \Sigma)$ as in [3, pp. 1–2]. For the next lemma we shall use the notation of [3, Theorem 1.1]. Thus there exists a subfield $\Sigma_1$ of $\Sigma$ such that

2.5. If $f \in L_2(\lambda)$ and $\int_A f d\lambda = 0$ for every $A \in \Sigma_1$, then weak $\lim P^* f = 0$ in $L_2(\lambda)$ sense.

2.6. The sets $A$ in $\Sigma_1$ are defined in [3] as sets of finite $\lambda$-measure such that the functions $P^* \chi_A$, $P^n \chi_A$ are all characteristic functions a.e. where $\chi_A$ denotes the characteristic function of $A$.

**Lemma 2.** The $\sigma$ field $\Sigma_1$ is generated by a countable collection of disjoint sets.

**Proof.** It is enough to show that each set $A \in \Sigma_1$ contains an atom.

Let us assume, to the contrary, that some set $A$, in $\Sigma_1$, with $\lambda(A) \neq 0$ does not contain atoms of $\Sigma_1$. Let $\chi_B = P^* \chi_A$ where $P^* \chi_A$ is the $L_2(X, \Sigma, \lambda)$ adjoint of $P^d$. Then by Theorem 1.1 of [3], $P^d(x, B) = P^d(\chi_B) = \chi_A$ a.e.
Since \( \lambda \) is a regular measure, there exists a compact subset \( C_0 \), \( \lambda(C_0) \neq 0 \), of \( A \), such that \( P^d(x, B) = 1 \) for every \( x \in C_0 \). Let \( A' \) be the set in \( \Sigma_1 \) which contains \( C_0 \) and has minimal \( \lambda \)-measure. Such a set is unique up to sets of measure zero and \( A' \subset A \). Since \( A' \) is not an atom it contains a set \( A_1 \), in \( \Sigma_1 \), with \( \lambda(A_1) \leq \frac{1}{2} \lambda(A') \leq \frac{1}{2} \lambda(A) \). Now \( \lambda(C_0 \cap A_1) \neq 0 \) for otherwise \( A' - A_1 \) would be smaller than \( A' \) and contain \( C_0 \). Let \( \chi_{A_1} = P^d \chi_{A_1} \); then \( P^d(x, B_1) = 1 \), \( x \in C_0 \cap A_1 \) a.e. Thus there exists a compact subset \( C_1 \) of \( C_0 \) such that \( P^d(x, B_1) = 1 \) for every \( x \in C_1 \) and \( \lambda(B_1) = \lambda(A_1) \leq \frac{1}{2} \lambda(A) \). Using an induction argument, we find a decreasing sequence of sets \( B_n \in \Sigma_1 \), with \( \lambda(B_n) \to 0 \), and a decreasing sequence of compact sets \( C_n \), such that \( P^d(x, B_n) = 1 \) for every \( x \in C_n \). Let \( x_0 \in C_0 \cap C_n \); then

\[
P^d(x_0, \cap B_n) = \lim P^d(x_0, B_n) = 1
\]

while \( \lambda(\cap B_n) = 0 \), which contradicts 2.1.

Let \( W \in \Sigma_1 \) be an atom and let \( PW \) denote the set whose characteristic function is \( P\chi_W \).

Call \( W \) of the first kind if the sets \( P^k W \) are a.e. disjoint. Otherwise \( W \) will be called of the second kind. If \( P^k W \) intersects \( P^k W \) for \( k < n \), then \( P^k W = P^k W \) a.e. since they are atoms and hence \( P^{n-k} W = W \) a.e. Define:

2.7. \( X_1 = \bigcup \{W: W \in \Sigma_1 \text{ and is of the first kind}\} \).
2.8. \( X_2 = \bigcup \{W: W \in \Sigma_1 \text{ and is of the second kind}\} \).
2.9. \( X_3 = X - X_1 \cup X_2 \).

**Lemma 3.** If the process is \( \nu \)-irreducible then either \( X = X_3 \) or \( X = X_2 \) and there exists an integer \( k \) such that \( \Sigma_1 = \{W, PW, \ldots, P^{k-1} W\} \) where \( P^k W = W \). In this case the measure \( \lambda \) is finite.

**Proof.** If \( W \in \Sigma_1 \) and \( W \subset X_1 \), then \( \int W P^\nu \chi_W d\lambda = 0 \), \( n = 1, 2, \ldots \). Thus \( P^k(x, W) = 0 \), \( x \in W \) a.e. Thus, since the process is \( \nu \)-irreducible, \( \nu(W) = 0 \) and also \( \lambda(W) = 0 \). Therefore \( X_1 \) is empty. Now let us assume that \( X_2 \) contains the nonempty set \( W \). Then \( P^k(x, W) = 0 \) a.e. for \( x \in X - W \cup \cdots \cup P^{k-1} W \) and thus this difference is empty.

**Theorem 2.** Let \( \mu \) be any measure. Let \( A \in \Sigma \) and \( \lambda(A) < \infty \).

(a) If \( A \subset X_3 \) then \( \lim (\mu P^n)(A) = 0 \).
(b) If \( A \subset W \subset X_2 \) where \( W \in \Sigma_1 \) and \( P^k W = W \), then the limit of \( (\mu P^{n+j})(A) \) exists as \( n \to \infty \) and \( 0 \leq j < k \).
(c) If \( A \subset W \subset X_1 \) where \( W \in \Sigma_1 \), then \( \lim (\mu P^n)(A) = 0 \).

**Proof.** By Theorem 1 it is enough to prove these results for a measure \( \mu \leq \lambda \).

We may assume that \( d\mu = f d\lambda \) where \( f \in L_2(\lambda) \) since any measure \( \mu \) which is weaker than \( \lambda \) can be approximated by such measures. Thus:

If \( A \subset X_3 \), then \( (\mu P)^n(A) = \int P^n \chi_A f d\lambda \to 0 \) since \( \chi_A \) is orthogonal to the sets in \( \Sigma_1 \) and 2.5.
If $A \subseteq W \subseteq X_1$ where $W \in \Sigma_1$, then $\lim \int P^n_{\chi_A} f d\lambda = \lim \int P^n_{\chi_W} f d\lambda = 0$ since $\int_W P^n_{\chi_W} d\lambda = 0$ and Theorem 2.1 of [3] applies.

Finally let $A \subseteq W \subseteq X_2$ where $W \in \Sigma_1$ and $P^k W = W$. Then

$$\lim_{n \to \infty} (\mu P^{n+j})(A) = \lim_{n \to \infty} \int P^{n+j} \chi_A \cdot f d\lambda = \frac{A}{\lambda(W)} \int P^j \chi_W \cdot f d\lambda$$

since the function $g = \chi_A - (\lambda(W)^{-1} / \lambda(A)) \chi_W$ is orthogonal to all sets in $\Sigma_1$ and thus weak $\lim P^n g = 0$ or

$$\lim \int P^{n+j} \chi_A \cdot f d\lambda = \lim \frac{\lambda(A)}{\lambda(W)} \int P^{n+j} \chi_W \cdot f d\lambda = \frac{\lambda(A)}{\lambda(W)} \int P^j \chi_W \cdot f d\lambda.$$

**Corollary.** If the process is $\nu$-irreducible, then either

(a) $\lim_{n \to \infty} P^n(x, A) = 0$ for every $x \in X$ and every set $A$ with $\lambda(A) < \infty$, or

(b) The limit of $P^{n+j}(x, A)$ exists for every $x \in X$, $A \in \Sigma$ and $0 \leq j < k$.

**Proof.** It is enough to note that we get (a) when $X = X_3$ and (b) when $X = X_2$ since every set $A \in \Sigma$ can be written as

$$A = (A \cap W) \cup (A \cap PW) \cup \cdots \cup (A \cap P^{k-1}_W)$$

and the previous theorem applies to $A \cap P^i W$.

**Theorem 3.** If the process is $\nu$-irreducible and $X = X_2$, then for any measure $\mu$ and every $j$ there are constants $\gamma_1 \cdots \gamma_k$ such that

$$\lim_{n \to \infty} (\mu P^{n+j})(A) = \sum_{i=0}^{k-1} \gamma_i \lambda(A \cap P^i W)$$

for all $A$.

**Proof.** It is enough to consider $\mu P^k$. Let $\tau(A) = \lim (\mu P^n)(A)$ where the limit exists for any $A \in \Sigma$. Then, by Corollary III. 7.4. of [2] the set function $\tau$ is countably additive and clearly $\tau = \tau P^k$.

From Theorem 1 it follows that $\tau \in \lambda$. Let $\tau = \tau^0 + \cdots + \tau^{(k-1)}$ where $\tau^{(i)}$ is the restriction of $\tau$ to $P^i W$. Thus $\tau^{(i)} P^k = \tau^{(i)}$ and so $\tau^{(i)} P^k + \cdots + \tau^{(0)} P^{k-1}$ is invariant under $P$. It is easy to see that the invariant measure is unique (Theorem 1 and the $\nu$-irreducibility) hence this sum is equal to $\gamma_\lambda \lambda$ for some constant $\gamma$. Now $\tau^{(i)} P^1$ is zero on any subset of $W_i$.

If $A \subseteq W_i$, then $P^i \chi_A \cap W_i = \emptyset$ a.e. $\lambda$, hence a.e. $\tau$, for $0 < j \leq k - 1$. Thus $\tau^{(i)}(A) = \gamma_i \lambda(A \cap W_i)$.

3. Existence of a subinvariant measure for irreducible processes. In this section we use a small modification of Harris' argument to find a subinvariant measure.
In [5] Harris constructs a σ-finite invariant measure for infinitely recurrent process. Here we find only subinvariant measure under weaker conditions. Throughout this section we assume:

3.1. For every $x$, $P(x, X) = 1$.
3.2. The σ-field $\Sigma$ is the Borel extension of a countable family of sets.
3.3. The process is $\nu$-irreducible where $\nu$ is a given σ-finite measure.
Notice that $X$ is not assumed to be a topological space and 2.1 is not assumed.

Let us just mention those parts of [5] that require a modification in this case.

**Theorem 4.** The process has a σ-finite subinvariant measure that is stronger than $\nu$.

Let $P_A$ by defined as in [5]. Lemma 1 of [5] should be restated:

A. Let $A$ be a measurable set with $0 < \nu(A) < \infty$. If $\lambda_A$ is a bounded sub-

invariant measure for $P_A$, then the measure $\lambda$:

3.4. $\lambda(E) = \int_A \lambda_A(dx)P_A(x, E)$

is subinvariant for $P$ and is σ-finite.

The proof is almost identical to Harris' First if $E \subset A$, then $\lambda(E) \leq \lambda_A(E)$. Also

$$\int \lambda(dy) P(y, E) = \int_A \lambda(dy) P(y, E) + \int_{X-A} \left[ \int_A \lambda_A(dx) P_A(x, dy) \right] P(y, E)$$

$$\leq \int_A \lambda_A(dx) \left[ P(x, E) + \int_{X-A} P_A(x, dy) P(y, E) \right] = \int_A \lambda_A(dx) P_A(x, E)$$

$$= \lambda(E).$$

The proof that $\lambda$ is σ-finite is the same as in [5] and also $\lambda(A) \neq 0$, for we will see that $P_A(x, A) > 0$ for every $x \in A$.

Lemma 2 and Lemma 3 of [5] are unchanged. Thus $P_A^1 + \cdots + P_A^\infty \geq P^1 + \cdots + P^n$ see [5, Equation 4.17]. Now if $P_A(x, A) = 0$, $x \in A$, then

$$P_A^i(x, A) = \int_A P_A(x, dy) P_A^{i-1}(y, A) = 0.$$

Thus $P^i(x, A) = 0$, $i = 1, 2, \cdots$, contrary to 3.3. Let us define

3.5. $Q(x, E) = \frac{P_A(x, E)}{P_A(x, A)}$, $x \in A$, $E \subset A$.

Then clearly $Q^i \geq P_A^i$.

Put

3.6. $R(x, E) = \frac{Q^1(x, E) + \cdots + Q^k(x, E)}{k}$, $x \in A$, $E \subset A$. 
where $k$ is defined as in [5]. Then Lemmas 4 and 5 of [5] will show us that there exists a measure $\lambda_A$ with

$$\lambda_A(E) = \int_A \lambda_A(dx)Q(x,E), \quad E \subset A.$$  

Finally, it follows from 3.7 that

$$\lambda_A(E) = \int_A \lambda_A(dx)Q(x,E) \leq \int_A \lambda_A(dx)P_A(x,E), \quad E \subset A.$$  

It remains to show that $\lambda$ is stronger than $\nu$. Now if $\lambda(E) = 0$, then

$$\int \lambda(dx)P^n(x,E) \leq \lambda(E) = 0.$$  

Hence $P^n(x,E) = 0$ a.e., $n = 1, 2, \ldots$. Since $\lambda \neq 0$, there exists an $x_0 \in X$ with $P^n(x_0,E) = 0$, $n = 1, 2, \ldots$; hence $\nu(E) = 0$.

4. Existence of an invariant measure. Throughout this section we assume:

4.1. For every $x$, $P(x,X) = 1$.

4.2. There exists a $\sigma$-finite measure $\nu$, and an increasing sequence of sets $X_n$, in $\Sigma$, such that:

- $\bigcup X_n = X$.
- $\nu(X_n) < \infty$.
- If $A \in \Sigma$ and $A \subset X_k$, then for every $\varepsilon > 0$ there exists an integer $n = n(A,\varepsilon)$ such that

$$\sup \{P^n(x,A) : x \in X\} \leq \nu(A) + \varepsilon.$$  

LEMMA 4. Let $\mu \in ba(X,\Sigma)$ be invariant. If $A \subset X_k$, then $\mu(A) \leq \nu(A)$.

Proof. Let $n = n(A,\varepsilon)$; then

$$\mu(A) = \int P^n(x,A) \mu(dx) \leq (\nu(A) + \varepsilon) \int \mu(dx) = \nu(A) + \varepsilon.$$  

DEFINITION. Let $S$ be the collection of invariant measures with unit total measure.

If $\mu \in S$, then $\mu \leq \nu$ on subsets of $X_k$ by Lemma 4. Since both are countably additive, $\mu \leq \nu$. Thus $d\mu = f dv$ where $0 \leq f \leq 1$ and $f \in L_1(\nu)$.

Now

$$4.3 \quad (\mu P)(A) = \int P(x,A) \mu(dx) = \int P(x,A)f(x)\nu(dx).$$  

LEMMA 5. Let $d\mu_1 = f_1 dv$, $d\mu_2 = f_2 dv$ where $\mu_1$ and $\mu_2$ are in $S$. If $d\mu = \max(f_1,f_2)dv$, then $\mu$ is invariant, too.
Proof. Put $Y_1 = \{x : f_1(x) \geq f_2(x)\}$, $Y_2 = X - Y_1$. Then

$$
\int_A \max(f_1, f_2) dv = \int_{A \cap Y_1} f_1 dv + \int_{A \cap Y_2} f_2 dv
= \int P(x, A \cap Y_1) f_1(x) v(dx) + \int P(x, A \cap Y_2) f_2(x) v(dx)
\leq \int \max(f_1(x), f_2(x)) P(x, A) v(dx).
$$

We used 4.2 and the invariance of $\mu_1$ and $\mu_2$. Thus $\mu(A) \leq (\mu P)(A)$ for every $A \in \Sigma$. But $(\mu P)(X) = \int P(x, X) \mu(dx) = \mu(X) < \infty$; hence $\mu(A) = (\mu P)(A)$.

Consider the collection of functions $f$ such that $f dv \in S$. Since $0 \leq f \leq 1$, the supremum of this collection in $L_1(v)$ is the supremum of a sequence $f_n$ in this collection (Theorem IV, 11.7 of [2]). Let $g = \sup f_n$ and $d \lambda = gdv$. If $S = \emptyset$, then take $g = 0$. Let $g_n = \max(f_1, \ldots, f_n)$, then $g = \lim g_n$ and by Lemma 5 and 4.3:

$$
\int P(x, A) g_n(x) v(dx) = \int g_n(x) v(dx).
$$

Passing to a limit, we see that $\lambda$ is an invariant measure. Also $\lambda \leq v$ since $0 \leq g \leq 1$; thus it is countably additive and finite on $X_n$.

**Theorem 5.** There exists a $\sigma$-finite measure $\lambda$ with

a. $\lambda \leq v$.

b. $\lambda$ is invariant under $P$.

c. If $\mu \in S$, then $\mu \leq \lambda$.

d. Let $A$ be contained in some $X_k$ and $\lambda(A) = 0$. For every $\tau \in \mathfrak{ba}$

$$
\lim \frac{1}{n} (\tau(A) + (\tau P)(A) + \cdots + (\tau P^{n-1})(A)) = 0.
$$

Proof. Parts a, b and c were proved above. Let $\tau_n = (\tau + \tau P + \cdots + \tau P^{n-1})/n$ and assume that for some subsequence $n_i, \tau_{n_i}(A) \geq \delta > 0$. Since $\tau_n$ form a bounded sequence in $B(X, \Sigma)^* = \mathfrak{ba}$, there exists a weak * limit point $\mu$ to the sequence $\tau_n$. Thus $\mu \geq 0$, $\mu(X) \leq 1$ and $\mu(A) \geq \delta > 0$. It is easily seen that $\mu P = \mu$. Let $\mu = \mu_1 + \mu_2$ where $\mu_1$ is a measure (c.a.) and $\mu_2$ is purely finitely additive (see [7, p. 52]). Then $\mu \leq v$ on subsets of $X_k$ by Lemma 4. Hence $\mu_2(X_k) = 0$, for the restriction of $\mu_2$ to $X_k$ is countably additive. It remains to show that $\mu_1$ is invariant which will contradict part c. Now

$$
\mu_1 + \mu_2 = \mu = \mu P = \mu_1 P + \mu_2 P.
$$

Let $\mu_2 P = \sigma_1 + \sigma_2$ where $\sigma_1$ is c.a. and $\sigma_2$ is purely finitely additive. Then $\mu_1 = \mu_1 P + \sigma_1$ but $\mu_1(X) = (\mu_1 P)(X) + \sigma_1(X) = \mu_1(X) + \sigma_1(X)$ and $\sigma_1 = 0$. 

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Remark. Part d can be replaced by: If $A$ is contained in $X_k$, then $\lambda(A) = 0$ if and only if

\[ d^1: \lim\left(\frac{P(x,A) + \cdots + P^n(x,A)}{n}\right) = 0 \text{ for every } x \in X. \]

$d^1$ follows from $d$ when we take $\tau$ to be a unit mass at $x$. Conversely given $d^1$ then for any $\mu \in S$ 

\[ \mu(A) = \frac{1}{n} \int (P(x,A) + \cdots + P^n(x,A)) \mu(dx) \to 0. \]

Thus $\lambda(A) = 0$, too.

An example. Let $v$ be a $\sigma$-finite measure and $P(x,A) = \int f(x,\xi) v(d\xi)$ where $0 \leq f(x,\xi)$ and $\int f(x,\xi) v(d\xi) = 1$. It is easy to see that 

\[ P^n(x,A) = \int f^n(x,\xi) v(d\xi), \quad f^n(x,\xi) = \int f^{n-k}(x,y) f^k(y,\xi) v(dy). \]

Put 

\[ g_n(\xi) = \sup \{ f^n(x,\xi): x \in X \} \leq \infty. \]

Lemma 6. For every $\xi \in X$, $g_{n+1}(\xi) \leq g_n(\xi)$.

Proof.

\[ f^{n+1}(x,\xi) = \int f(x,y) f^n(y,\xi) v(dy) \leq g_n(\xi) \int f(x,y) v(dy) = g_n(\xi). \]

Hence $g_{n+1}(\xi) \leq g_n(\xi)$.

Let $g(\xi) = \lim g_n(\xi)$.

Theorem 6. Condition 4.2 holds with respect to a measure $v_1$ equivalent to $v$ if $g(\xi) < \infty$ for every $\xi \in X$.

Proof. Let $Y_k = \{ \xi: g(\xi) < k \}$, then $Y_k \subset Y_{k+1}$ and $\bigcup_{k=1}^{\infty} Y_k = X$. Define $v_1$ by: $v_1(A) = kv(A)$ if $A \subset Y_k - Y_{k-1}$. Then $v_1 \sim v$. If $f^1_1(x,\xi)$ is the Radon-Nikodym derivative of $P(x,A)$ with respect to $v_1$, then $f^1_1(x,\xi) = (1/k) f^n(x,\xi)$ whenever $\xi \in Y_k - Y_{k-1}$. Hence if $g_n$ and $g$ were defined for $f^n$, then $g^1_n(\xi) = (1/k) g_n(\xi)$, $g^1(\xi) = (1/k) g(\xi)$ for $\xi \in Y_k - Y_{k-1}$. Thus $g^1(\xi) < 1$ for every $\xi \in X$. Also $v_1$ is $\sigma$-finite: if $\bigcup Z_k = X$ where $Z_k \subset Z_{k+1}$ and $v(Z_k) < \infty$; then $v_1(Z_k \cap Y_k) < kv(Z_k) < \infty$ and $\bigcup (Z_k \cap Y_k) = X$. Finally let $V_k = \{ \xi: g^1_k(\xi) < 1 \}$; then $V_k \subset V_{k+1}$ by Lemma 6 and with $X_k = Z_k \cap Y_k \cap V_k$ we get 4.2.

Let us conclude with a comparison between our results and Orey's [6]. In [6], Theorem 3 corresponds to part (b) of the corollary of Theorem 2. There it is assumed that the process is infinitely recurrent. We have to add a "Doeblin Condition," namely 2.1, but instead of assuming that whenever $v(A) > 0$, $P[\text{entering } A \text{ at some time } |X_0 = x] = 1$, we only assumed that this quantity
is not zero. Part (a) of Theorem 3 furnishes, under our conditions, a positive answer to the problem posed by Orey in [6 end of §3, p. 816].

BIBLIOGRAPHY


THE HEBREW UNIVERSITY OF JERUSALEM,
JERUSALEM, ISRAEL