ON THE DEGREES OF INDEX SETS

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A class of recursively enumerable sets may be classified either as an object in itself — the range of a two-place function in the obvious way — or by means of the corresponding set of indices. The latter approach is not only more precise but also, as we show below, provides an alternative method for solving certain problems on recursively enumerable sets and their degrees of unsolvability. The main result of the present paper is the computation, for every recursively enumerable degree $\boldsymbol{a}$, of the degree (in fact, isomorphism-type) of the index-set corresponding to the recursively enumerable sets of degree $\boldsymbol{a}$: its degree is $\boldsymbol{a}^{(3)}$. It follows from a theorem of Sacks [10] that the degrees of such index-sets are exactly those which are $\geqq \mathbf{0}^{(3)}$ and recursively enumerable in $\mathbf{0}^{(3)}$. In particular, this proves Rogers' conjecture [9] that the index-set corresponding to $\mathbf{0}^{(1)}$ is of degree $\mathbf{0}^{(4)}$; partial results on this problem have been obtained by Rogers [9] and by Lacombe (unpublished). The most interesting immediate consequence of our result is a different proof of Sacks' theorem [11] that the recursively enumerable degrees are dense.

We refer the reader to Kleene [5] and Sacks [10] for our basic terminology and notation. A useful summary of many results which connect degrees with the arithmetical hierarchy is presented in [9], which is a good background to the present paper since without it the latter would not exist. For an assortment of results on classes of recursively enumerable sets the reader is referred to [2].

If e is a number and A is a set, then we define the partial function Θ_e^A by setting:

$$\Theta_e^A(n) = U(\mu y T_1^A(e, n, y));$$

we differ superficially from [5] by setting $U(0) = 2$. Also, for each e and s we define:

$$n \in R_e^s \equiv (\exists y)_{y \leqq s} T_1(e, n, y);$$

if, for each e, we now let R_e be the union of $R_e^0, R_e^1, \cdots$, then $R_0, R_1, \cdots$, is an enumeration of all recursively enumerable sets. If S is any recursively enumerable set, then each number e such that $R_e = S$ is called an *index* of S.

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DEFINITION 1. The *index-set* $G(\mathscr{A})$ of a class $\mathscr{A}$ of sets is defined by:

$$e \in G(\mathscr{A}) \equiv R_e \in \mathscr{A}.$$

In particular, the *index-set* $G(\boldsymbol{a})$ corresponding to a degree $\boldsymbol{a}$ is defined by:

$$e \in G(\boldsymbol{a}) \equiv R_e \text{ is of degree } \boldsymbol{a}.$$

Finally, we shall use the following convention: for any function γ, we say that $\lim_s \gamma(s)$ exists if and only if there is a number z and a number s^* such that $\gamma(s) = z$ for all $s > s^*$, in which case $\lim_s \gamma(s) = z$.

In the first section of the paper we compute upper bounds for the degrees of index-sets, and in the second section we prove Lacombe's result in the course of showing the existence of a certain representation needed for the main theorem. In the third section we give a new proof of Rogers' results; whereas Rogers adapted Friedberg's construction [4], the proof we give is direct and provides an alternative solution of Post's problem as well as an exact classification of the index-sets corresponding to all many-one degrees. The main theorem is contained in the fourth section and it may be useful here to indicate how we deduce from it that the recursively enumerable degrees are dense (in other words, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are recursively enumerable degrees such that $\boldsymbol{a} < \boldsymbol{b}$, then there is a recursively enumerable degree $\boldsymbol{c}$ such that $\boldsymbol{a} < \boldsymbol{c} < \boldsymbol{b}$). We show in §1 that there is a predicate Γ_0 of degree $\leqq \boldsymbol{a}$ such that for all e:

$$e \in G(\boldsymbol{a}) \equiv (\exists r)(s)(\exists t)\Gamma_0(e, r, s, t).$$

The main step in our classification of $G(\boldsymbol{b})$ then consists of showing that if Γ is any predicate of degree $\leqq \boldsymbol{b}$, then there is a recursive function γ such that for all e: $R_{\gamma(e)}$ is of degree $\boldsymbol{c}_e$ such that $\boldsymbol{a} \leqq \boldsymbol{c}_e \leqq \boldsymbol{b}$ and

$$\gamma(e) \in G(\boldsymbol{b}) \equiv (\exists r)(s)(\exists t)\Gamma(e, r, s, t).$$

Suppose then we consider the case when $\Gamma = \Gamma_0$ and let γ_0 be the corresponding recursive function. By the fixed-point theorem (which can be found, for example, in [13]) there is a number e^* such that $R_{e^*} = R_{\gamma_0(e^*)}$, and it follows immediately that $\gamma_0(e^*) \notin G(\boldsymbol{a})$ and $\gamma_0(e^*) \notin G(\boldsymbol{b})$ so that the degree $\boldsymbol{c}_{e^*}$ of $R_{\gamma_0(e^*)}$ is such that $\boldsymbol{a} < c_{e^*} < \boldsymbol{b}$. Notice that we have not been forced to arrange that $\boldsymbol{a} < \boldsymbol{c}_{e^*}$ and $\boldsymbol{c}_{e^*} < \boldsymbol{b}$ simultaneously; this is done instead for us by the fixed point theorem. We still have troubles using this approach, but they are clearer and so rather easier to overcome. A much simpler example of this procedure is used in §3 to provide an alternative solution to Post's problem (which is the particular case of the density-problem when $\boldsymbol{a} = \boldsymbol{0}$ and $\boldsymbol{b} = \boldsymbol{0}^{(1)}$ but the advantages of it are only really felt in the general case. For, in §4 we need to be capable of handling sets of degree up to $\boldsymbol{0}^{(4)}$, whereas in §3 we only deal with sets of degree up to $\boldsymbol{0}^{(3)}$.

The classification of the set $G(\boldsymbol{0}^{(1)})$ constituted a major part of the author's

doctoral dissertation (Manchester, 1963) written under the direction of Dr. Robin Gandy. At an early stage in the work, the author announced a false result to a meeting of the Association of Symbolic Logic (Leeds, 1962), and he is grateful to Professor Georg Kreisel for informing him of Lacombe's result which it contradicted.

1. **Computing upper bounds.** We first need some discussion on the classification of index-sets. A special case of a set P being recursive in a set Q is that of P being many-one reducible to Q, in other words there is a recursive function ϕ such that $n \in P \equiv \phi(n) \in Q$ for all n. P and Q are said to be of the same many-one degree if they are many-one reducible to each other. A more specialised case still is when P is one-one reducible to Q, in other words when the function ϕ above is one-one. These concepts were introduced by Post [8]. It has been proved by Myhill [7] that if P and Q are one-one reducible to each other then they are recursively isomorphic, in other words there is a one-one recursive function ranging over the natural numbers which maps P onto Q. Hence a classification within the arithmetical hierarchy by means of one-one reducibility is in a sense the best possible. However, for any set P and class $\mathscr{A}$ of sets it is easy to show that if P is many-one reducible to $G(\mathscr{A})$ then P is one-one reducible to $G(\mathscr{A})$; a proof may be found on p. 133 of [9]. It is usually unnecessary, therefore, to distinguish between one-one and many-one reducibility when dealing with index-sets.

DEFINITION 2. For each degree $\boldsymbol{a}$ and number n, $\Sigma_n(\boldsymbol{a})$ is the class of all sets expressible by a predicate form with n alternating quantifiers the first of which is existential and $\Pi_n(\boldsymbol{a})$ is the class of all sets expressible by a predicate form with n alternating quantifiers the first of which is universal, where in each case the scope of the quantifiers is of degree $\leqq \boldsymbol{a}$. We shall refer to $\Sigma_n(\mathbf{0})$ and $\Pi_n(\mathbf{0})$ simply as Σ_n and Π_n for all n.

Clearly, if $\boldsymbol{a} < \boldsymbol{b}$ then $\Sigma_n(\boldsymbol{a}) \subset \Sigma_n(\boldsymbol{b})$ and $\Pi_n(\boldsymbol{a}) \subset \Pi_n(\boldsymbol{b})$ for all n. Hence, in particular, if $\boldsymbol{a}$ is a recursively enumerable degree then $\Sigma_n \subset \Sigma_n(\boldsymbol{a}) \subset \Sigma_{n+1}$ for all $n > 0$, since $\Sigma_n(\mathbf{0}^{(1)}) = \Sigma_{n+1}$ by Post's theorem (see [5]) for all $n > 0$. Also, it follows from well-known results in [5] that, for all n, the degree of each element of $\Sigma_{n+1}(\boldsymbol{a})$ is recursively enumerable in $\boldsymbol{a}^{(n)}$ and so $\leqq \boldsymbol{a}^{(n+1)}$. By the hierarchy-theorem [5], $\Sigma_n(\boldsymbol{a}) \not\subset \Pi_n(\boldsymbol{a})$ and $\Pi_n(\boldsymbol{a}) \not\subset \Sigma_n(\boldsymbol{a})$ for all $\boldsymbol{a}$ and all $n > 0$. In fact, t is shown in [5] that, for each $\boldsymbol{a}$ and $n > 0$, there exists an element S of $\Sigma_n(\boldsymbol{a})$ such that a set belongs to $\Sigma_n(\boldsymbol{a})$ if and only if it is one-one reducible to S (it follows immediately that $\bar{S}$ plays a similar role for $\Pi_n(\boldsymbol{a})$); of course, there are infinitely many such elements of $\Sigma_n(a)$, but by Myhill's theorem they are unique up to recursive isomorphism. The procedure for classifying an index-set G is then to show that G belongs to a particular $\Sigma_n(\boldsymbol{a})$ or $\Pi_n(\boldsymbol{a})$ and afterwards attempt to prove that every element of that class is one-one reducible to G. The major difficulty may be expected to lie in the latter part of the classification, in other words in obtaining optimum lower bounds. Upper bounds are usually compar-

atively straightforward using, if necessary, what Rogers calls the Tarski-Kuratowski algorithm. This consists of defining the set G under consideration in prenex form in the lower predicate calculus, in such a way that the scope of the quantifiers is a predicate whose degree is bounded in some useful way, and then collapsing like quantifiers (in all the cases considered by Rogers, the scope is most usefully chosen to be recursive but this is unsatisfactory for our present purposes). As this can only be done in finitely many ways given the original defining predicate, a minimal classification may be obtained relative to that predicate. We shall now prove two lemmas, the first of which provides an upper bound for all index-sets corresponding to many-one degrees. Our use of the Tarski-Kuratowski algorithm is very straightforward and so we not use the special symbolism introduced for this purpose in [9].

LEMMA 1. *If $\mathscr{A}$ is any many-one degree, then $G(\mathscr{A}) \in \Sigma_3$.*

Proof. If $\mathscr{A}$ contains no recursively enumerable set then $G(\mathscr{A})$ is the null set and trivially belongs to Σ_3. Orherwise, let A be a recursively enumerable set belonging to $\mathscr{A}$. Then we may immediately write:

$$\begin{aligned} e \in G(\mathscr{A}) \equiv (\exists f)(\exists g)(n)[&(\exists y)T_1(f,n,y) \& (\exists y)T_1(g,n,y) \\ &\& (z)(T_1(f,n,x) \to (n \in R_e \equiv U(z) \in A)) \\ &\& (z)(T_1(g,n,z) \to (n \in A \equiv U(z) \in R_e))]. \end{aligned}$$

Now, by removing $\to$ and by standard contraction we deduce that there is a recursive predicate Δ such that:

$$e \in G(\mathscr{A}) \equiv (\exists f)(\exists g)(n)(p)(\exists q)\Delta(f,g,n,p,q,e).$$

It easily follows that $G(\mathscr{A}) \in \Sigma_3$.

The next lemma provides upper bounds for all index-sets which correspond to degrees.

LEMMA 2. *If $\boldsymbol{a}$ is any degree, then $G(\boldsymbol{a}) \in \Sigma_3(\boldsymbol{a})$.*

Proof. If $\boldsymbol{a}$ is not a recursively enumerable degree then $G(\boldsymbol{a})$ is the null set and trivially belongs to $\Sigma_3(\boldsymbol{a})$. Otherwise, let A be a recursively enumerable set of degree $\boldsymbol{a}$. We may immediately write:

$$e \in G(\boldsymbol{a}) \equiv R_e \text{ is recursive in } A \ \& A \text{ is recursive in } R_e.$$

Since two recursively enumerable sets are recursive in each other if and only if their complements are recursively enumerable in each other, it follows that:

$$\begin{aligned} e \in G(\boldsymbol{a}) \equiv (\exists f)(\exists g)(n)[&(n \in \bar{R}_e \equiv (\exists y)T_1^A(f,n,y)) \& \\ &(n \in \bar{A} \equiv (\exists z)T_1^{R_e}(g,n,z))]. \end{aligned}$$

The next step is not so obvious:

$$e \in G(\boldsymbol{a}) \equiv (\exists f)(\exists g)(n)[(n \in \bar{R}_e \equiv (\exists y)T_1^A(f,n,y))$$
$$\& (n \in \bar{A} \equiv (\exists s)(\exists z)_{z \leq s}(T_1^{R_e^s}(g,n,z)$$
$$\& (u)_{u \leq z}(u \notin R_e^s \to (\exists v)T_1^A(f,u,v))))].$$

Now, by removing $\equiv$ and by standard contraction it will be found that there is a predicate Δ, expressible in both two-quantifier forms with a scope which is recursive in A, such that:

$$e \in G(\boldsymbol{a}) \equiv (\exists f)(\exists g)(n)\Delta(f,g,n,e).$$

It easily follows that $G(\boldsymbol{a}) \in \Sigma_3(\boldsymbol{a})$.

It remains now to compute lower bounds for the degrees of the index-sets in which we are interested. In the next section we introduce a system of representation which we shall need for this purpose.

2. **Representations for Σ_3 and $\Sigma_3(\boldsymbol{a})$.** Before we describe these representations, there is one point which needs a little discussion. As we have already mentioned, our procedure for computing lower bounds in each case involves showing that a certain set S is many-one reducible to the index-set $G(\mathscr{A})$ of a given class $\mathscr{A}$. Suppose we say that a sequence $\{C_k\}$ of recursively enumerable sets is itself recursively enumerable if there is a recursive function γ such that $C_k = R_{\gamma(k)}$ for all k (some discussion and alternative definitions may be found in [2] and [14]). Then our task is to show that there is a recursively enumerable sequence $\{C_k\}$ such that for all k: $k \in S \equiv C_k \in \mathscr{A}$. In each case, it is possible to compute an index of the corresponding function γ but, since it would be very tedious, we shall (as is usual) be content merely to describe an effective enumeration of $\{C_k\}$.

The following lemma provides the most useful representation for Σ_3; we shall use it in §3.

LEMMA 3. *If $S \in \Sigma_3$, then there is a uniformly recursively enumerable sequence $\{L_{kj}\}$ such that for all k:*

$$k \in S \equiv (\exists e)(f)_{f \geq e}\ (L_{kj} \textit{ is infinite}),$$
$$k \notin S \equiv (e)(L_{ke} \textit{ is finite}).$$

Proof. Since $S \in \Sigma_3$ there is a recursive predicate Γ such that for all k:

$$k \in S \equiv (\exists e)(u)(\exists v)\Gamma(k,e,u,v).$$

We first define a uniformly recursively enumerable sequence $\{S_{kj}\}$ by setting:

$$x \in S_{ke} \equiv (u)_{u \leq x}(\exists v)\Gamma(k,e,u,v).$$

It can be seen that for all k:

$$k \in S \equiv (\exists e)(S_{ke} \text{ is infinite}).$$

We now obtain the required sequence $\{L_{kj}\}$ by setting

$$L_{ke} = \bigcup_{f \leqq e} S_{kf}$$

for all k and e.

If $S \in \Sigma_3$, M is a recursively enumerable set and

$$M_k = \{2^e \cdot 3^x \mid e \in M \vee x \in L_{ke}\}$$

for all k, where $\{L_{kj}\}$ is the sequence corresponding to S whose existence is shown in Lemma 3, then it is not difficult to prove that for all k:

$$k \in S \rightarrow M_k \text{ is recursive,}$$

$$k \notin S \rightarrow M \text{ is recursive in } M_k.$$

It easily follows by letting M be of degree $\mathbf{0}^{(1)}$ that every element of Π_3 is one-one reducible to $G(\mathbf{0}^{(1)})$; this was first proved by Lacombe, We shall not give a more detailed proof of Lacombe's result here, since it is entirely superseded by the exact classification obtained in the next section and in any case we follow a similar but more general line of reasoning in the proof of Lemma 4. Incidentally, notice that Lacombe's result does not remain true if $\mathbf{0}^{(1)}$ is replaced by an arbitrary recursively enumerable degree. For, not only is $G(\mathbf{0})$ an element of Σ_3, but by a theorem in [10] (originally due to Friedberg) there is a recursively enumerable degree $\boldsymbol{a}$ such that $\mathbf{0} < \boldsymbol{a} < \mathbf{0}^{(1)}$ and $\boldsymbol{a}^{(1)} = \mathbf{0}^{(1)}$, whence it easily follows that $\Sigma_3(\boldsymbol{a}) = \Sigma_3$ and so $G(\boldsymbol{a}) \in \Sigma_3$.

The next lemma provides the representation for $\Sigma_3(\boldsymbol{a})$ that we shall use in §4. Let us say that a sequence $\{S_{x_1 \ldots x_n}\}$ is *uniformly of degree* $\leqq \boldsymbol{a}$ if the predicate Λ, where $\Lambda(x, x_1, \cdots, x_n) \equiv x \in S_{x_1 \ldots x_n}$ for all $x, x_1, \cdots, x_n$, is of degree $\leqq \boldsymbol{a}$.

LEMMA 4. *If $\boldsymbol{a}$ is a recursively enumerable degree and $s \in \Sigma_3(\boldsymbol{a})$, then there is a uniformly recursively enumerable sequence $\{L_{kj}\}$ which is uniformly of degree $\leqq \boldsymbol{a}$ and such that for all k, L_{ki} and L_{kj} are disjoint if $i \neq j$, and*

$$k \in S \rightarrow (\exists e)(L_{ke} \textit{ is of degree } \boldsymbol{a} \,\&\, (j)_{j<e}(L_{kj} \textit{ is recursive})),$$

$$k \notin S \rightarrow (e)(L_{ke} \textit{ is recursive}).$$

Proof. Let A be a fixed recursively enumerable set of degree $\boldsymbol{a}$, and for each s let A^s be the finite subset of A enumerated up to stage s in some fixed recursive enumeration of A. Since $S \in \Sigma_3(\boldsymbol{a})$, there is a number c such that for all k:

$$k \in S \equiv (\exists e)(u)(\exists v) T_3^A(c, k, e, u, v).$$

Let us define a uniformly recursively enumerable sequence $\{S_{kji}\}$ by setting:

$$x \in S_{keu} \equiv (v)_{v \leqq x}(\exists s)_{s \geqq x} \bar{T}_3^{A^s}(c, k, e, u, v).$$

We first claim that $\{S_{kji}\}$ is uniformly of degree $\leqq \boldsymbol{a}$. For, since A is recursively enumerable, the predicate Δ, where $\Delta(k, e, u, v, x) \equiv (\exists s)_{s \geqq x} \bar{T}_3^{A^s}(c, k, e, u, v)$ for all k, e, u, v and x, is recursive in A. Our second claim is that for all k:

$$k \in S \equiv (\exists e)(u)(S_{keu} \text{ is finite}).$$

To prove this, suppose first that $k \in S$. Then there is a number e such that for all u there exist numbers v_u and s_u such that $T_3^{A^s}(c, k, e, u, v_u)$ is true for all $s \geqq s_u$. If for each u, we set $x_u = \max(v_u, s_u)$ then it easily follows that if $x \geqq x_u$ then $x \notin S_{keu}$, and so S_{keu} is finite. Suppose on the other hand that $k \notin S$. Then it is fairly clear that for each e there is a number u such that $x \in S_{keu}$ for all x, which completes the proof of our second claim. Now, for each k, e and y, we define:

$$S^*_{key} = \bigcup_{u \leqq y} S_{keu}.$$

Clearly, $\{S^*_{kji}\}$ is uniformly recursively enumerable, uniformly of degree $\leqq \boldsymbol{a}$ and such that for all k:

$$k \in S \equiv (\exists e)(y)(S_{key} \text{ is finite}),$$

$$k \notin S \equiv (e)(\exists y)(u)_{u \geqq y}(x)(x \in S^*_{keu}).$$

Finally, for each k and e, we set:

$$L_{ke} = \{2^e \cdot 3^y \cdot 5^x \mid y \in A \vee x \in S^*_{key}\}.$$

It is clear that $\{L_{kj}\}$ is uniformly recursively enumerable, uniformly of degree $\leqq \boldsymbol{a}$ and such that for all k, L_{ki} and L_{kj} are disjoint if $i \neq j$. Now suppose that $k \notin S$. Then, for each e, there is a number y such that if $u \geqq y$, then $2^e \cdot 3^u \cdot 5^x \in L_{ke}$ for all x, if $u < y$ and $u \in A$, then $2^e \cdot 3^u \cdot 5^x \in L_{ke}$ for all x and if $u < y$ and $u \notin A$, then $2^e \cdot 3^u \cdot 5^x \in L_{ke}$ for only finitely many x. It easily follows that L_{ke} is recursive (though not uniformly so) for all e. Suppose on the other hand that $k \in S$. Then there is a least number e such that for all y:

$$y \notin A \equiv (\exists x)(2^e \cdot 3^y \cdot 5^x \notin L_{ke}).$$

It follows that A is recursive in L_{ke} and so, as $\{L_{kj}\}$ is uniformly of degree $\leqq \boldsymbol{a}$, L_{ke} is of degree $\boldsymbol{a}$. Clearly, by our remarks in the case $k \notin S$, if $j < e$, then L_{kj} is recursive, which completes the proof of this lemma.

The rest of the paper will be mainly devoted to proving (by means of these representations) that the upper bounds computed in §1 are in fact the best possible.

3. **Many-one degrees and Post's problem.** Let $\mathscr{R}$ be the class of all recursive sets excluding the empty set and its complement) and $\mathscr{K}$ be the class of all creative sets. It was shown by Rogers [9] that both $G(\mathscr{R})$ and $G(\mathscr{K})$ are of the highest

recursive isomorphism type possible for elements of Σ_3; in this section we shall extend his result to $G(\mathscr{A})$ where $\mathscr{A}$ is any many-one degree that contains an infinite recursively enumerable set whose complement is nonempty. Since we have proved in Lemma 1 that $G(\mathscr{A}) \in \Sigma_3$, it remains to prove the following theorem.

THEOREM 1. *If $S \in \Sigma_3$ and $\mathscr{A}$ is any many-one degree that contains an infinite recursively enumerable set whose complement is nonempty, then S is one-one reducible to $G(\mathscr{A})$.*

Proof. It is easy to show (and well-known) that $\mathscr{R}$ is a many-one degree satisfying the conditions of the theorem, but the case when $\mathscr{A} = \mathscr{R}$ has already been accounted for by Rogers so we shall assume that $\mathscr{A} \neq \mathscr{R}$. Let A be an arbitrary fixed infinite recursively enumerable set belonging to $\mathscr{A}$. We shall define a uniformly recursively enumerable sequence $\{B_k\}$ such that, if $k \in S$, then B_k differs from A by a finite set and, if $k \notin S$, then A is not recursive in B_k. Instead of directly enforcing that A is not recursive in B_k we shall arrange that a *simple* set H of the same degree as A is not recursive in B_k; this shortens the proof. The existence of such a simple set is known from Dekker's work [1]. We cannot use H throughout the proof instead of A since we cannot in general assume that $\mathscr{A}$ contains a simple set.

Let $\{L_{kj}\}$ be a sequence of the sort shown to correspond to S in Lemma 3, and for all k, j and s let L_{kj}^s be the finite subset of L_{kj} enumerated up to stage s in the recursive enumeration of the sequence $\{L_{kj}\}$ *induced by* a fixed recursive enumeration of the sequence $\{S_{kj}\}$ in Lemma 3. Let λ_{kj}^s be the greatest element of L_{kj}^s and, if L_{kj} is finite, let λ_{kj} be the largest element of L_{kj}. Then if $k \in S$, there is a number e_k such that if $e < e_k$ then $\lim_s \lambda_{ke}^s$ exists and is equal to λ_{ke}, and if $e \geqq e_k$ then $\lim_s \lambda_{ke}^s$ does not exist. On the other hand, if $k \notin S$, then $\lim_s \lambda_{ke}^s$ exists and is equal to λ_{ke} for all e. Notice also that if $f \leqq e$, then $\lambda_{kf}^s \leqq \lambda_{ke}^s$ for all k, e and s.

We shall define B_k stage by stage simultaneously with some functions and sets which are necessary for its definition. For each s, let A^s and H^s be the finite subsets of A and H that have been enumerated up to stage s in some fixed recursive enumerations of A and H. Also, we shall let B_k^s denote the finite set of numbers which have been put into B_k^0 through our procedure (below) up to stage s. First, we define B_k^0 to be the null set. Then for all k, e, n and s we set:

$$\begin{aligned}
\varepsilon_k^0(e,n) &= 0, \\
\varepsilon_k^{+1}(e,n) &= \begin{cases} \mu y_{y \leqq s} T_1^{B_k^s}(e,n,y) & \text{if } (\exists y)_{y \leqq s} T_1^{B_k^s}(e,n,y), \\ 0 & \text{otherwise}, \end{cases} \\
\theta_k^s(e,n) &= U(\varepsilon_k^s(e,n)).
\end{aligned}$$

It can be seen that if $\Theta_e^{B_k}(n)$ is defined for all n then $\Theta_e^{B_k}(n) = \lim_s \theta_k^s(e,n)$ for all n. Hence, H is recursive in B_k only if there is a number e such that for all n: $n \in H \equiv \lim_s \theta_k^s(e,n) = 0$ and $n \notin H \equiv \lim_s \theta_k^s(e,n) = 1$. (Notice that since we are

letting $U(0) = 2$, $\theta_k^s(e,n)$ takes the values 0 and 1 only if $\varepsilon_k^s(e,n) > 0$.) We shall arrange that if $k \notin S$ then there is no such number e. For this we need the sets F_{ke}^s and numbers τ_{ke}^s defined for each k, e and s by:

$$n \in F_{ke}^s \equiv \theta_k^s(e,n) = 1 \ \& \ (m)_{m<n}(\theta_k^s(e,m) = 1 \to m \notin H^s),$$

$$\tau_{ke}^s = \max\{\varepsilon_k^r(f,m) \mid f \leqq e \ \& \ r \leqq s \ \& \ m \in F_{kf}^r\}.$$

Now, for each s, if there is a number e such that $\lambda_{ke}^s \leqq z \leqq \tau_{ke}^s$, then we say that z is *restrained from B_k by e at stage s*. Finally, we let B_k^{s+1} consist of B_k^s together with those elements of A^s which are not restrained from B_k at stage s. This completes the construction.

Let k be a fixed element of $\bar{S}$, so that L_{ke} is finite for all e. It follows that there is a stage s_e such that $\lambda_{ke}^s = \lambda_{ke}$ for all $s \geqq s_e$, and every element of B_k less than λ_{ke} belongs to $B_k^{s_e}$. We may now define for each e a recursively enumerable set F_e by setting:

$$n \in F_e \equiv (\exists s)_{s > s_e}(n \in F_{ke}^s).$$

We cannot (and need not) in general suppose that $\{F_e\}$ is uniformly recursively enumerable.

LEMMA 5. *For each e, if $n \in F_e$, then $\Theta_e^{B_k}(n) = 1$.*

Proof. Fix e and n. If $n \in F_e$, then there is a stage $s' > s_e$ such that $n \in F_{ke}^{s'}$ and so of course $\theta_k^{s'}(e,n) = 1$. Therefore, $\varepsilon_k^{s'}(e,n) > 0$ as $U(0) = 2$. Suppose now that $s \geqq s'$. Then $\tau_{ke}^s \geqq \varepsilon_k^{s'}(e,n)$ so that as $s \geqq s' > s_e$ no number $\leqq \varepsilon_k^{s'}(e,n)$ enters B_k as stage s. By induction it follows that $\theta_k^s(e,n) = \theta_k^{s'}(e,n) = 1$, and so $\Theta_e^{B^k}(n) = \lim_s \theta_k^s(e,n) = 1$.

LEMMA 6. *F_e is finite for all e.*

Proof. F_e is recursively enumerable and so (as H is simple) cannot be an infinite subset of $\bar{H}$. Consider then a number e for which there is a number n^* such that $n^* \in H \cap F_e$. By Lemma 5 there is a stage $s^* > s_e$ such that $n^* \in H^s$ and $\theta_k^s(e,n^*) = 1$ for all $s > s^*$. It follows that if $s > s^*$ then F_{ke} contains no numbers exceeding n^* and so the only elements of F_e which exceed n^* are those which belong to F_{ke} for some s such that $s_e < s \leqq s^*$. There are only finitely many such numbers and so in this case F_e again cannot be infinite, which proves the lemma.

LEMMA 7. *For each e, $\Theta_e^{B_k}$ is not the representing function of H.*

Proof. Suppose, for the sake of a *reductio ad absurdum*, that there is a number e such that $\Theta_e^{B_k}$ is the representing function of H. Now, F_e is a finite subset of $\bar{H}$ by Lemmas 5 and 6 but $\bar{H}$ is infinite since H is simple. So let $\bar{n}$ be an element of $\bar{H}$ which does not belong to F_e. Since $\bar{n} \in \bar{H}$ there must be a stage $\bar{s} > s_e$ such that

$\theta_k^s(e, \bar{n}) = 1$ for all $s > \bar{s}$, whence as $\bar{n} \notin F_e$ there must at every stage $s > \bar{s}$ be a number $n < \bar{n}$ such that $\theta_k^s(e, n) = 1$ and $n \in H^s$. Since $\lim_s \theta_k^s(e, n)$ exists for all n it easily follows that there is an element n of H such that $\Theta_e^{B_k}(n) = 1$, which contradicts our initial assumption.

This completes our proof that if $k \notin S$ then H is not recursive in B_k, and so A is not recursive in B_k; it follows immediately that $B_k \notin \mathscr{A}$ if $k \notin S$.

Now let k be a fixed element of S. In this case there is a number e_k such that L_{ke} is finite if $e < e_k$ and L_{ke} is infinite if $e \geqq e_k$. It follows that if $e < e_k$ then F_e is finite (the proof of this is exactly the same as that for all e when $k \in \bar{S}$).

LEMMA 8. *If L_{ke} is finite, then $\lim_s \tau_{ke}^s$ exists.*

Proof. If L_{ke} is finite, then L_{kf} is finite for all $f < e$ (this follows from the way we defined $\{L_{kj}\}$ in Lemma 3). Hence, F_f is finite for all $f \leqq e$ by Lemma 6. Also if $n \in F_f$ then $\lim_s \varepsilon_k^s(f, n)$ exists by Lemma 5. It easily follows that $\lim_s \tau_{ke}^s$ exists.

It follows from Lemma 8 that only finitely many numbers are restrained from B_k by numbers $e < e_k$. Let the largest such number be n_k. Then, if $n \in A$ with $n > n_k$, there will be a stage s such that $n \in A^s$, $\lambda_{ke}^s > n$ for all $e \geqq e_k$ and so $n \in B_k^s$. Hence, B_k differs from A by a finite set and so B_k belongs to $\mathscr{A}$. But this means that we have proved that for all k:

$$k \in S \equiv B_k \in \mathscr{A}.$$

The theorem follows immediately now from our remarks in §1 and §2.

THEOREM 2. *If $\mathscr{A}$ is any many-one degree that contains an infinite recursively enumerable set whose complement is nonempty, then the degree of $G(\mathscr{A})$ is $\mathbf{0}^{(3)}$. Also $G(\mathscr{A}) \in \Sigma_3 - \Pi_3$.*

Proof. It follows from Lemma 1 that the degree of $G(\mathscr{A})$ is $\leqq \mathbf{0}^{(3)}$ and it follows from Theorem 1 that the degree of $G(\mathscr{A})$ is $\geqq \mathbf{0}^{(3)}$. The rest follows from remarks in §1.

In fact, we have proved that if $\mathscr{A}_1$ and $\mathscr{A}_2$ are any two many-one degrees which satisfy the conditions of the theorem then $G(\mathscr{A}_1)$ and $G(\mathscr{A}_2)$ are recursively isomorphic and of the highest isomorphism-type possible for elements of Σ_3. Notice that it also follows from the proof of Theorem 1 that if $\boldsymbol{a}$ is any recursively enumerable degree then every element of Σ_3 is one-one reducible to $G(\boldsymbol{a})$ so that $G(\boldsymbol{a}) \notin \Pi_3$. One consequence of this is that $G(\mathbf{0}^{(1)})$ is not recursively isomorphic to (and so certainly not equal to) $\overline{G(\mathbf{0})}$, which was proved by Rogers and (if proved directly as above) provides an alternative method for solving Post's problem. This solution can be made fully effective by observing that as $G(\mathbf{0}) \in \Sigma_3$, it follows from Theorem 1 that there is a recursive function γ such that for all

$$k\colon k \in G(\mathbf{0}) \equiv \gamma(k) \in G(\mathbf{0}^{(1)}).$$

By the fixed-point theorem there is a number k^* such that $R_{k^*} = R_{\gamma(k^*)}$ and it is clear that then $k^* \notin G(\mathbf{0})$ and $k^* \notin G(\mathbf{0}^{(1)})$ so that R_{k^*} is neither recursive nor of degree $\mathbf{0}^{(1)}$. In fact, of course, it follows from Theorem 1 that if A is any nonrecursive recursively enumerable set then there is a nonrecursive recursively enumerable set in which A is not recursive. In the next section, however, we shall prove much stronger results than these.

4. **The main result and the density-problem.** We shall now compute the isomorphism type of $G(\boldsymbol{b})$ for every recursively enumerable degree $\boldsymbol{b}$. In order to obtain this result alone a slightly simpler construction would be sufficient, but we shall also be deriving Sacks' theorem that the recursively enumerable degrees are dense; since all the hard work is necessarily done in the process of classification, this should cause no confusion. We showed in Lemma 2 that $G(\boldsymbol{b}) \in \Sigma_3(\boldsymbol{b})$ and so it remains to prove the following theorem.

THEOREM 3. *If $\boldsymbol{b}$ is any recursively enumerable degree and $S \in \Sigma_3(\boldsymbol{b})$, then S is one-one reducible to $G(\boldsymbol{b})$.*

Proof. It has already been proved (by Rogers) that every element of Σ_3 is one-one reducible to $G(\mathbf{0})$ and so we shall assume that $\boldsymbol{b} \neq \mathbf{0}$. Let $\boldsymbol{a}$ then be an arbitrary recursively enumerable degree $< \boldsymbol{b}$, and let A and B be fixed recursively enumerable sets whose degrees are $\boldsymbol{a}$ and $\boldsymbol{b}$. Our only purpose in introducing $\boldsymbol{a}$ and A into the construction is to derive the theorem of Sacks mentioned above. We shall define a uniformly recursively enumerable sequence $\{C_k\}$ such that A is recursive in C_k for all k, and such that, if $k \in S$, then C_k is of degree $\boldsymbol{b}$ and, if $k \notin S$, then B is not recursive in C_k. For convenience, we shall assume that B is a recursively enumerable set whose complement is *retraceable*. The existence of such a set in any recursively enumerable degree follows from Dekker's work [1] and is proved in [3]; the property that B now has which is useful here is that B is recursive in every infinite subset of $\bar{B}$ (this is also proved in [3]; further discussion of recursively enumerable sets with retraceable complements may be found in [14]). It follows that $\bar{B}$ has no infinite subset which is recursively enumerable in A, since otherwise it would have an infinite subset which is recursive in A and in which B is not recursive.

Let $\{L_{kj}\}$ be a sequence of the sort shown to correspond to S in Lemma 4, and for all k, j and s let L_{kj}^s be the finite subset of L_{kj} enumerated up to stage s in some fixed recursive enumeration of the sequence $\{L_{kj}\}$. For each k, we shall define C_k to be of the form $A_0 \cup C_{k0} \cup C_{k1} \cup \cdots$, where

$$A_0 = \{7^{x+1} \mid x \in A\}$$

and $C_{ke} \subset L_{ke}$ for all e; clearly, A is recursive in C_k as A_0 is disjoint from the sequence $\{L_{kj}\}$. We shall also arrange that, if $k \in S$ and $\bar{e}$ is then the least e such that L_{ke} is of degree $\boldsymbol{b}$, then $\bar{C}_k \cap L_{k\bar{e}}$ is finite, and if $k \notin S$, then $\bar{C}_k \cap L_{ke}$ is finite for all e. This should help to explain some of the definitions below. Our problem

is to transform the representation $\{L_{kj}\}$ in such a way that B is recursive in C_k if and only if $k \in S$. We do this by extending and combining the methods of the previous section with an essentially combinatorial procedure for dealing with situations in which a certain partial recursive functional may interfere infinitely often with an opposed requirement of lower priority; this happens when the functional is itself disturbed by an opposed requirement of higher priority such as, in the present case, the requirement that $A_0 \subset C_k$ (at any rate, when A_0 is not recursive). Different methods have been introduced by Sacks for tackling this sort of situation, notably in [11] to prove that the recursively enumerable degrees are dense.

One again, we shall define C_k stage by stage simultaneously with some functions and sets which are necessary for its definition. For each s, let A_0^s and B^s be the finite subsets of A_0 and B that have been enumerated up to stage s in some fixed recursive enumerations of A_0 and B. Also, we shall let C_k^s denote the finite set of numbers which have been put into C_k through our procedure up to stage s. First, we let C_k^0 be the null set, and then for all k, e, n and s we set:

$$\begin{aligned}
\varepsilon_k^0(e,n) &= 0,\\
\varepsilon_k^{s+1}(e,n) &= \begin{cases} \mu y_{y \leqq s} T_1^{C_k^s}(e,n,y) \text{ if } (\exists y)_{y \leqq s} T_1^{C_k^s}(e,n,y), \\ 0 \text{ otherwise,} \end{cases}\\
\theta_k^s(e,n) &= U(\varepsilon_k^s(e,n)).
\end{aligned}$$

Obviously, if $\Theta_e^{C_k}(n)$ is defined for all n, then $\Theta_e^{C_k}(n) = \lim_s \theta_k^s(e,n)$ for all n. Hence, B is recursive in C_k only if there is a number e such that $\lim_s \theta_k^s(e,n)$ exists for all n and describes the representing function of B. (Notice once again that since we are letting $U(0) = 2$, $\theta_k^s(e,n)$ takes the values 0 and 1 only when $\varepsilon_k^s(e,n) > 0$.) We shall arrange that if $k \notin S$ then there is no such number e. To this end we define a set F_{ke}^s for each k, e and s by:

$$\begin{aligned}
n \in F_{ke}^s &\equiv \theta_k^s(e,n)\\
&= 1 \ \& \ (m)_{m<n}[\theta_k^s(e,m) = 1 \rightarrow (m \notin B^s \vee (\exists r)_{n<r \leqq s}(\varepsilon_k^r(e,m) \neq \varepsilon_k^{r-1}(e,m)))].
\end{aligned}$$

Now, for each k, e and s, and for every $n \in F_{ke}^{s+1}$ we say that a number $z \leqq \varepsilon_k^{s+1}(e,n)$, which does not belong to C_k^s, *begins to be restrained from C_k by Θ_e through n* at stage $s+1$ if at stage s it has not restrained from C_k by Θ_f for any $f < e$ (through any number) or by Θ_e through any number $\leqq n$; such a number then *ceases to be restrained from C_k by Θ_e through n* at a later stage $s'+1$ if and only if $\varepsilon_k^{s'}(e,n) = \varepsilon_k^{+1}(e,n)$ but $\varepsilon_k^{s'+1}(e,n) \neq \varepsilon_k^{s'}(e,n)$. Finally, we let C_k^{s+1} consist of C_k^s together with all elements of A_0^{s+1}, and (for each e) all elements of L_{ke}^{s+1} which are not restrained from C_k by Θ_f for any $f \leqq e$ at stage $s+1$. This completes the definition of C_k.

Let k be a fixed element of $\bar{S}$. It is clear from the definition of C_k that $\bar{C}_k \cap L_{ke}$ is finite unless infinitely many elements of L_{ke} are permanently withheld from C_k through our system of restraints. In order to make this more precise, let us define a sequence $\{D_e\}$ by setting:

$$z \in D_e \equiv (\exists s)(r)_{r>s}(\exists f)_{f \leqq e}(\exists m)$$
$$(z \text{ is restrained from } C_k \text{ by } \Theta_f \text{ through } m \text{ at stage } r).$$

Then our assertion is that $\bar{C}_k \cap L_{ke}$ is finite if D_e is finite for all e. Now, we shall prove (amongst other things) that D_e *is* finite for all e. Let P_e^0 be the null set for all e. Then for each e and s we set:

$$\begin{aligned} n \in P_e^{s+1} \equiv\ & n \in F_{ke}^{s+1} \,\&\, (z)_{z \leqq \varepsilon_k^{s+1}(e,n)}([z \in A_0 \to z \in A_0^s] \\ & \&\, (j)_{j<e}[z \in L_{kj} \to (z \in C_k^s \vee z \in D_j)] \\ & \&\, [(j)_{j<e}(z \in \bar{A}_0 \cap \bar{L}_{kj}) \to (z \in C_k^s \vee z \in D_{e-1} \vee z \text{ is re-} \\ & \text{strained from } C_k \text{ by } \Theta_e \text{ through } n \text{ or through some} \\ & \text{number } m < n \text{ such that } m \in P_e^{s+1} \text{ at stage } s+1)]). \end{aligned}$$

Finally, we define P_e for each e by:

$$n \in P_e \equiv (\exists s)(n \in P_e^s).$$

Clearly, P_0 is recursively enumerable in A and if $e > 0$ and D_j is finite for all $j < e$ then, as L_{kj} is recursive for all $j < e$, P_e is recursively enumerable in A. We cannot (and need not) in general suppose that $\{P_e\}$ is uniformly recursively enumerable in A, since we cannot assume that the sequence $\{L_{kj}\}$ is uniformly recursive.

LEMMA 9. *For each e, if* $n \in P_e$, *then* $\Theta_e^{C_k}(n) = 1$.

Proof. Let e be fixed. If $n \in P_e$ then there is a stage s such that $n \in P_e^s$. We claim that, for any n and s, if $n \in P_e^s$, then $\varepsilon_k^r(e,n) = \varepsilon_k^s(e,n)$ for all $r > s$; clearly, since $n \in P_e^s$ implies that $n \in F_{ke}$ and so $\theta_k^s(e,n) = 1$, it then follows that $\Theta_e^{C_k}(n) = 1$. We shall prove our claim by induction on n with s fixed but arbitrary. Suppose then that our claim is true for this fixed s and all $m < n$; suppose also that $\bar{r} \geqq s$ and $\varepsilon_k^r(e,n) = \varepsilon_k(e,n)$ for all r such that $s \leqq r \leqq \bar{r}$. We wish to prove that $\varepsilon_k^{\bar{r}+1}(e,n) = \varepsilon_k^s(e,n)$. Notice that $\varepsilon_k^{\bar{r}}(e,n) > 0$, since $\theta_k(e,n) = 1$ implies that $\varepsilon_k(e,n) > 0$, and so $\varepsilon_k^{\bar{r}+1}(e,n)$ can only differ from $\varepsilon_k^{\bar{r}}(e,n)$ if some number $z \leqq \varepsilon_k^r(e,n)$ is put into C_k at stage $\bar{r}$. We now show, assuming our induction hypothesis, that this is impossible. First, if $z \leqq \varepsilon_k^{\bar{r}}(e,n)$ and $z \in A_0$, then $z \in A_0^{s-1}$ (since $\varepsilon_k^{\bar{r}}(e,n) = \varepsilon_k^s(e,n)$ and $n \in P_e^s$) so that $z \in C_k^{s-1}$ and z does not actually enter C_k at stage $\bar{r}$. Secondly, if $z \leqq \varepsilon_k^{\bar{r}}(e,n)$ and $z \in L_{kf}$ for some $f < e$, then $z \in C_k^{s-1}$ or $z \in D_f$, so that again z does not enter C_k at stage $\bar{r}$. Lastly, if $z \leqq \varepsilon_k^{\bar{r}}(e,n)$ and $z \in \bar{A}_0 \cap \bar{L}_{kf}$ for all $f < e$, then one of the following cases holds: (i) $z \in C_k^{s-1}$, (ii) $z \in D_{e-1}$, (iii) z is restrained from C_k by Θ_e through n at stage s, (iv) z is re-

strained from C_k by Θ_e through some element $m < n$ of P_e^s at stage s. If either (i) or (ii) holds, then it is clear that z does not enter C_k at stage $\bar{r}$. If (iii) holds, then z is restrained from C_k by Θ_e through n at stage $\bar{r}$, since $\varepsilon_k^r(e,n)$ does not change as r varies from s to $\bar{r}$, and lastly, if (iv) holds, then z is restrained from C_k by Θ_e by some element $m < n$ of P_e^s at stage $\bar{r}$, since our induction hypothesis implies that $\varepsilon_k^r(e,m)$ does not change as r varies from s to $\bar{r}$. Therefore, no number $z < \varepsilon_k^{\bar{r}}(e,n)$ enters C_k at stage $\bar{r}$, and so $\varepsilon_k^{\bar{r}+1}(e,n) = \varepsilon_k^{\bar{r}}(e,n)$. This concludes the proof of our claim and hence of the lemma.

The following lemma embodies the combinatorial principle that we mentioned earlier. We say that a number is *permanently restrained* from C_k by Θ_e through n if there is a stage after which it is always restrained from C_k by Θ_e through n; clearly, such a number is an element of D_e.

LEMMA 10. *For each e, if $z \in D_e$, then either there is a number n such that z is permanently restrained from C_k by Θ_e through n or (in the case $e > 0$) $z \in D_{e-1}$.*

Proof. Suppose, for the sake of a *reductio ad absurdum*, that the lemma is false. Then there is a number $\bar{z}$ which belongs to D_e but does not belong to D_{e-1} and is not permanently restrained from C_k by Θ_e through any number. It follows that, at infinitely many stages, $\bar{z}$ begins to be restrained from C_k by Θ_e through some number (not necessarily always the same number). There are two cases to consider:

Case 1. $\bar{z}$ is restrained from C_k by Θ_e through only finitely many numbers. In this case there is at least one number through which $\bar{z}$ is restrained from C_k by Θ_e infinitely often; let $\bar{n}$ be the greatest such number. Then there is a stage $\bar{s}$ such that at no later stage is $\bar{z}$ restrained from C_k by Θ_e through any number $n > \bar{n}$. On the other hand, there are infinitely many stages at which $\bar{z}$ is neither restrained from C_k by Θ_j, for any $j < e$, nor restrained from C_k by Θ_e through any number $n \leqq \bar{n}$, since there are infinitely many stages at which z begins to be restrained from C_k by Θ_e through $\bar{n}$. It follows that there are infinitely many stages succeeding stage $\bar{s}$ at which $\bar{z}$ is not restrained from C_k by Θ_j, for any $j \leqq e$, and so $\bar{z}$ does not belong to D_e. We conclude that Case 1 cannot occur.

Case 2. $\bar{z}$ is restrained from C_k by Θ_e through infinitely many different numbers. It follows that there are numbers $n_0, n_1, \cdots$, and stages $s_0, s_1, \cdots$, such that, for each i, $\bar{z}$ begins to be restrained from C_k by Θ_e through n_i at stage s_i but at no previous stage was restrained from C_k by Θ_e through any number exceeding n_i. Then, in particular, $\bar{z}$ is not restrained from C_k by Θ_j for any $j \leqq e$ at stage $s_i - 1$, for each i. We conclude that Case 2 also cannot occur. Since either Case 1 or Case 2 must occur if the lemma is false, this contradiction proves the lemma.

It follows immediately from Lemma 10 that if $z \in D_e$ then there exist numbers j, n such that $j \leqq e$ and z is permanently restrained from C_k by Θ_j through n. We shall now prove that, for each e, P_e and D_e are finite, and $\Theta_e^{C_k}$ is not the representing function of B. This follows from the next three lemmas.

LEMMA 11. *P_0 is finite and, for each $e > 0$, if D_{e-1} is finite, then P_e is finite.*

Proof. As we have observed above, P_0 is recursively enumerable in A and, if $e > 0$ and D_{e-1} is finite, P_e is recursively enumerable in A. Therefore, as B has been chosen to be a recursively enumerable set whose complement is retraceable, P_e cannot be an infinite subset of $\bar{B}$, for the reasons given at the beginning of the proof of the present theorem. Consider now a number e for which there is a number n^* such that $n^* \in B \cap P_e$. By Lemma 9 there is a stage s^* such that $n^* \in B^s$, $\theta_k^s(e, n^*) = 1$ and $\varepsilon_k^s(e, n^*) = \varepsilon_k^{s^*}(e, n^*)$ for all $s \geqq s^*$. It follows that if $s \geqq s^*$ then F_{ke}^s contains no numbers exceeding both n^* and s^*. Consequently, P_e again cannot be infinite, which proves the lemma.

There is a simple observation that we shall need in the proof of each of the next two lemmas. This is that if there exist numbers which are permanently restrained from C_k by Θ_e through n then there is a stage s such that $\theta_k^s(e, n) = 1$ and $\varepsilon_k^r(e, n) = \varepsilon_k(e, n)$ for all $r > s$, and so $\Theta_e^{C_k}(n) = 1$.

LEMMA 12. *For each e, if P_e is finite and the set $\{n \mid \Theta_e^{C_k}(n) = 1\}$ is infinite, then there is an element m of B such that $\Theta_e^{C_k}(m) = 1$.*

Proof. We shall prove that if $\{n \mid \Theta_e^{C_k}(n) = 1\}$ is an infinite subset of $\bar{B}$ then $\{n \mid \Theta_e^{C_k}(n) = 1\} \subset P_e$ and so P_e is infinite; the lemma clearly follows from this. Let us suppose then that $\{n \mid \Theta_e^{C_k}(n) = 1\} \subset \bar{B}$. We claim that if $\Theta_e^{C_k}(n) = 1$ then there is a stage such that $n \in P_e^s$ at every later stage s. We shall prove this by induction on n, so suppose that $\bar{n}$ is fixed and that if $\bar{n} > 0$ then our claim is true for $n < \bar{n}$. First, there is a stage s_0 such that $\bar{n} \in F_{ke}^s$ for all $s \geqq s_0$. For, otherwise, there would be a number $m < \bar{n}$ such that $m \in B$, $\varepsilon_k^s(e, m) = \varepsilon_k^{\bar{n}}(e, m)$ and $\theta_k^s(e, m) = \theta_k^{\bar{n}}(e, m) = 1$ for all $s \geqq \bar{n}$, which clearly contradicts our assumption that $\{n \mid \Theta_e^{C_k}(n) = 1\} \subset \bar{B}$. Secondly, there is a stage $s_1 \geqq s_0$ such that $\varepsilon_k^s(e, \bar{n}) = \varepsilon_k^{s_1}(e, \bar{n})$ for all $s \geqq s_1$, as otherwise $\Theta_e^{C_k}(\bar{n})$ would not be defined. Now, we wish to show that there is a stage $s_2 \geqq s_1$ such that if $z \leqq \lim_s \varepsilon_k^s(e, \bar{n})$ and $s \geqq s_2$ then the following three statements are true:

(1) $z \in A_0 \rightarrow z \in A_0^s$,

(2) $(j)_{j<e}\,(z \in L_{kj} \rightarrow (z \in C_k^s \vee z \in D_j))$,

(3) $(j)_{j<e}\,(z \in \bar{A}_0 \cap \bar{L}_{kj}) \rightarrow (z \in C_k^s \vee z \in D_{e-1} \vee z$ is restrained from C_k by Θ_e through $\bar{n}$ or some element $n < \bar{n}$ of P_e^{s+1}, at stage $s + 1)$.

Since there is a stage after which (1) and (2) always hold, the only nontrivial case is (3). Suppose then that $z \in \bar{C}_k$, $z \in \bar{A}_0$ and $z \in \bar{L}_{kj}$ for all $j < e$. If z is restrained from C_k by Θ_e through $\bar{n}$ at some stage after stage s_1, then z is permanently restrained from C_k by Θ_e through $\bar{n}$. Let us suppose then that z is not restrained from C_k by Θ_e through n at any stage after stage s_1. It follows that at each stage after stage s_1 there is either a number $j < e$ such that z is restrained from C_k by Θ_j or there is a number $n < \bar{n}$ such that z is restrained from C_k by Θ_e through n. If $z \notin D_{e-1}$, then there are infinitely many stages at which z is restrained from C_k

by Θ_e through some number $n < \bar{n}$. We deduce, by essentially the argument under Case 1 in Lemma 10, that there is a number $n < \bar{n}$ such that z is permanently restrained from C_k by Θ_e through n. But then $\Theta_e^{C_k}(n) = 1$, by the remark which precedes this lemma, and therefore, by our induction hypothesis, there is a stage such that $n \in P_e^s$ at every later stage s. We may now conclude that there is a stage such that $\bar{n} \in P_e$ at every later stage s, and this completes the proof of the lemma.

LEMMA 13. *For each e, if P_e is finite, then D_e is finite.*

Proof. By Lemma 10 it is sufficient to prove that only finitely many numbers are permanently restrained from C_k by Θ_e. Certainly, only finitely many numbers are permanently restrained from C_k by Θ_e through any particular number n, since there must then be a stage after which $\varepsilon_k^s(e, n)$ does not change as s increases. It remains to prove that Θ_e permanently restrains numbers from C_k through only finitely many n. Suppose, for *reductio ad absurdum*, that Θ_e permanently restrains numbers from C_k through infinitely many n. Then $\{n \mid \Theta_e^{C_k}(n) = 1\}$ is infinite, by the remark that precedes Lemma 12, and so by Lemma 12 itself there is an element n^* of B such that $\Theta_e^{C_k}(n^*) = 1$. It follows as in Lemma 11 that there is a stage s^* such that if $s \geqq s^*$ then F_{ke}^s contains no number exceeding both n^* and s^*. But then Θ_e only restrains numbers from C_k through finitely many n, so that our supposition is absurd. This proves the lemma.

It follows from Lemmas 11 and 13 that P_e and D_e are finite for all e, and then from Lemma 12 that B is not recursive in C_k. This concludes the discussion of C_k for $k \in \bar{S}$.

Now let k be a fixed element of S. In this case there is a least number e such that L_{ke} is of degree $\boldsymbol{b}$ and L_{kj} is recursive for all $j < e$. It follows, by the reasoning contained in Lemmas 11, 12 and 13, that P_j and D_j are finite for all $j \leqq e$. Hence, $L_{ke} \cap C_k$ differs from L_{ke} by a finite set and so is of degree $\boldsymbol{b}$. But $L_{ke} \cap C_k$ is recursive in C_k, as the sequence $\{L_{kj}\}$ is disjoint, and so C_k is of degree $\geqq \boldsymbol{b}$. In order to complete the proof of the theorem we need only prove the following lemma.

LEMMA 14. *For all k, C_k is recursive in B.*

Proof. We need only prove that $\bar{C}_k$ is recursively enumerable in B. Let k be fixed but arbitrary. If a number is neither of the form $2^e \cdot 3^u \cdot 5^v$ nor of the form 7^x then it belongs to $\bar{C}_k$; the set of all these numbers is recursive. If a number is of the form 7^{x+1}, then it belongs to $\bar{C}_k$ if and only if $x \in A$; the set of all such numbers is recursive in A and so certainly recursive in B. Therefore, in order to prove the lemma we have to show that the set $\{2^e \cdot 3^u \cdot 5^v \mid 2^e \cdot 3^u \cdot 5^v \in \bar{C}_k\}$ is recursively enumerable in B. It is clear from the construction and from the definition of $\{L_{kj}\}$ in Lemma 4 that for all e, u and v:

$$2^e \cdot 3^u \cdot 5^v \in \bar{C}_k \equiv (2^e \cdot 3^u \cdot 5^v \in \bar{L}_{ke} \vee 2^e \cdot 3^u \cdot 5^v \in D_e).$$

Since $\{L_{kj}\}$ is uniformly of degree $\leqq \boldsymbol{b}$ by Lemma 4, it remains to show that the sequence $\{D_e\}$ is uniformly recursively enumerable in B. By Lemma 10 we know that for all e and z:

$$z \in D_e \equiv (\exists f)_{f \leqq e}(\exists n) \quad (z \in H_{fn}),$$

where H_{fn} is the set of all numbers which are permanently restrained from C_k by Θ_f through n. Obviously, it is sufficient to show that the sequence $\{H_{fn}\}$ is uniformly recursively enumerable in B. Now, if $z \in H_{fn}$, then there is a stage p such that z is restrained from C_k by Θ_f through n at every stage $q < p$. This can only be so if $\varepsilon_k^q(f,n) = \varepsilon_k^p(f,n)$ for all $q < p$, and hence there is a stage $s > p$ such that $y \in C_k \equiv y \in C_k^s$ for all $y \leqq \varepsilon_k^s(f,n)$. In fact, it follows from the construction and the proof of Lemma 9 that we may write:

$$\begin{aligned} z \in H_{fn} \equiv (\exists s)\ [&z \text{ is restrained from } C_k \text{ by } \Theta_f \text{ through } n \text{ at stage } s \\ &\& \ (y)_{y \leqq \varepsilon_k^s(f,n)}((y \in A_0 \to y \in A_0^s) \& (j)_{j<f}(y \in L_{kj} \to \\ &(y \in C_k^s \vee (\exists i)_{i \leqq j}(\exists m)(y \in H_{im}))) \& (j)_{j \geqq f}(y \in L_{kj} \to \\ &(y \in C_k^s \vee y \text{ is restrained from } C_k \text{ by } \Theta_f \text{ through } n \text{ at stage} \\ &s \vee (\exists m)_{m<n}(y \in H_{fm}) \vee (\exists i)_{i<f}(\exists m)(y \in H_{im}))))]. \end{aligned}$$

Clearly, H_{00} is recursively enumerable in B and, since the sequence $\{L_{kj}\}$ is uniformly recursive in B, it may be seen that in fact there is a recursive function ψ such that for all f, n and z:

$$z \in H_{fn} \equiv (\exists y) T_1^B(\psi(f,n), z, y);$$

the existence of computations of the values of ψ may be verified by induction over the recursive well-ordering $\prec$ (of all ordered pairs) defined by

$$(f_1, n_1) \prec (f_2, n_2) \equiv f_1 < f_2 \vee (f_1 = f_2 \& \ m_1 < m_2),$$

observing that every descending sequence in this ordering is finite. It follows that $\{H_{fn}\}$ is uniformly recursively enumerable in B and so $\bar{C}_k$ is recursively enumerable in B. This completes the proof of the lemma.

We have now proved that if $k \in S$ then C_k is of degree $\boldsymbol{b}$. In other words, we have finally proved that S is one-one reducible to $G(\boldsymbol{b})$, which completes the proof of the theorem.

THEOREM 4. *If $\boldsymbol{b}$ is any recursively enumerable degree, then the degree of $G(\boldsymbol{b})$ is $\boldsymbol{b}^{(3)}$. Also $G(\boldsymbol{b}) \in \Sigma_3(\boldsymbol{b}) - \Pi_3(\boldsymbol{b})$.*

Proof. It follows from Lemma 2 that the degree of $G(\boldsymbol{b})$ is $\leqq \boldsymbol{b}^{(3)}$ and it follows Theorem 3 that the degree of $G(\boldsymbol{b})$ is $\geqq \boldsymbol{b}^{(3)}$. The rest again follows from remarks in §1.

In fact, we have proved that $G(\boldsymbol{b})$ is of the highest isomorphism-type possible for elements of $\Sigma_3(\boldsymbol{b})$. In particular, since $\Sigma_4 = \Sigma_3(\mathbf{0}^{(1)})$ we have proved Rogers' conjecture [9] that $G(\mathbf{0}^{(1)})$ is of the highest isomorphism-type possible for elements of Σ_4.

THEOREM 5. *If $\boldsymbol{a}$ and $\boldsymbol{b}$ are recursively enumerable degrees such that $\boldsymbol{a} < \boldsymbol{b}$, then $G(\boldsymbol{a})$ is one-one reducible to $G(\boldsymbol{b})$.*

Proof. Since $\boldsymbol{a} < \boldsymbol{b}$, it follows from Lemma 2 that $G(\boldsymbol{a}) \in \Sigma_3(\boldsymbol{b})$ and so by Theorem 3 is one-one reducible to $G(\boldsymbol{b})$.

The next theorem provides a sample of the results which follow from Theorem 3 and Sacks' work on the jump operator [10].

THEOREM 6. (i) *If $\boldsymbol{c} \geqq \mathbf{0}^{(3)}$ and $\boldsymbol{c}$ is recursively enumerable in $\mathbf{0}^{(3)}$, then there is a recursively enumerable degree $\boldsymbol{b}$ such that $\boldsymbol{c}$ is the degree of $G(\boldsymbol{b})$.* (ii) *There is a recursively enumerable degree $\boldsymbol{b}$ such that $\mathbf{0} < \boldsymbol{b} < \mathbf{0}^{(1)}$ and $G(\boldsymbol{b})$ is of the highest isomorphism-type possible for elements of Σ_3.* (iii) *There is a recursively enumerable degree $\boldsymbol{b}$ such that $\mathbf{0} < \boldsymbol{b} < \mathbf{0}^{(1)}$ and $G(\boldsymbol{b})$ is of the highest isomorphism-type possible for elements of Σ_4.*

The most interesting corollary of Theorem 3, however, is that the recursively enumerable degrees are dense:

THEOREM 7. *If $\boldsymbol{a}$ and $\boldsymbol{b}$ are recursively enumerable degrees such that $\boldsymbol{a} < \boldsymbol{b}$, then there is a recursively enumerable degree $\boldsymbol{c}$ such that $\boldsymbol{a} < \boldsymbol{c} < \boldsymbol{b}$.* (*Sacks* [11].)

Proof. $G(\boldsymbol{a})$ is one-one reducible to $G(\boldsymbol{b})$ by Theorem 5, and so there is a recursive function γ such that for all e:

$$e \in G(\boldsymbol{a}) \equiv \gamma(e) \in G(\boldsymbol{b}).$$

Moreover, the proof of Theorem 3 shows that γ can be arranged so that $R_{\gamma(e)}$ is of degree between $\boldsymbol{a}$ and $\boldsymbol{b}$ for all e. By the fixed-point theorem there is a number e^* such that $R_{e^*} = R_{\gamma(e^*)}$. $G(\boldsymbol{a})$ and $G(\boldsymbol{b})$ are disjoint since $\boldsymbol{a} < \boldsymbol{b}$, so it follows that the degree $\boldsymbol{c}$ of R_{e^*} lies strictly between $\boldsymbol{a}$ and $\boldsymbol{b}$.

We note that the proof of Theorem 7 is fully effective, since the fixed-point theorem enables us to actually compute e^* from indices of A and B (fixed recursively enumerable sets of degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, as in the proof of Theorem 3). In this connection it is worth mentioning that a different and more effective proof of another theorem of Sacks can be obtained using this indirect method, namely the theorem that if $\mathbf{0} < \boldsymbol{a} < \mathbf{0}^{(1)}$ then there is a recursively enumerable degree $\boldsymbol{c}$ such that $\boldsymbol{a} \mid \boldsymbol{c}$. Sacks' original proof of the latter result (which may be found for example in [10]) suffers from the minor defect that two recursively enumerable sets are produced one of which has the desired property but we do not (in a sense) know which one it is; the alternative proof (various extensions of

which will appear in [16]) does not suffer from this defect although of course it is more difficult. It in fact seems strange that such a lot of extra work is necessary to overcome a relatively minor defect; whichever approach we use it appears to be unavoidable. One natural question that this raises, since the fully effective proofs of these two theorems of Sacks have so much in common and since there exists a much easier but less effective proof of one of them, is whether there is a much simpler but less effective proof of the other, in other words of Theorem 7. We cannot at present answer this question but an affirmative answer would be interesting. Lastly, we note that with little extra trouble it is possible to prove the following extension of Theorem 7: if $\boldsymbol{a}$ and $\boldsymbol{b}$ are recursively enumerable degrees such that $\boldsymbol{a} < \boldsymbol{b}$ and $\boldsymbol{a} > \mathbf{0}$, then there is a recursively enumerable degree $\boldsymbol{c}$ such that $\boldsymbol{a} \cup \boldsymbol{c} < \boldsymbol{b}$ and $\boldsymbol{a} \mid \boldsymbol{c}$. This may be proved using the methods of [11] or those that we have used above.

Sacks has proved, for example in [12] and §6 of [10], a number of theorems which assert, given a recursively enumerable degree $\boldsymbol{a}$ such that $\mathbf{0} < \boldsymbol{a}$, the existence of a recursively enumerable degree $\boldsymbol{c}$ which has a preassigned property (for example, in [12], $\boldsymbol{c}$ is the degree of a maximal set) and is such that $\boldsymbol{a} \nleqq \boldsymbol{c}$. The indirect approach which we used to derive Theorem 7 provides a uniform method for strengthening most of these theorems to assert that, if $\mathbf{0} < \boldsymbol{a} < \mathbf{0}^{(1)}$, then the corresponding recursively enumerable degree $\boldsymbol{c}$ can be arranged to be such that $\boldsymbol{a} \mid \boldsymbol{c}$. For example, we can prove:

(i) If $\boldsymbol{a} < \mathbf{0}^{(1)}$ and $\boldsymbol{a}_0 < \boldsymbol{a}_1 < \cdots$ is an infinite ascending sequence of simultaneously recursively enumerable degrees each $< \boldsymbol{a}$, then there is a recursively enumerable degree $\boldsymbol{c}$ such that $\boldsymbol{a}_0 < \boldsymbol{a}_1 < \cdots < \boldsymbol{c}$ and $\boldsymbol{a} \mid \boldsymbol{c}$.

(ii) If $\mathbf{0} < \boldsymbol{a} < \mathbf{0}^{(1)}$ and $\boldsymbol{b}$ is a degree which is $\geqq \mathbf{0}^{(1)}$ and recursively enumerable in $\mathbf{0}^{(1)}$, then there is a recursively enumerable degree $\boldsymbol{c}$ such that $\boldsymbol{c}^{(1)} = \boldsymbol{b}$ and $\boldsymbol{a} \mid \boldsymbol{c}$.

(iii) If $\mathbf{0} < \boldsymbol{a} < \mathbf{0}^{(1)}$, then there is a degree $\boldsymbol{c}$ which contains a maximal set and is such that $\boldsymbol{a} \mid \boldsymbol{c}$.

Martin [6] has proved the pleasing theorem that a recursively enumerable degree $\boldsymbol{c}$ is the degree of a maximal set if and only if $\boldsymbol{c}^{(1)} = \mathbf{0}^{(2)}$, so in fact (iii) is a consequence of (ii). Incidentally, it is relatively easy to derive an exact classification of the index-sets corresponding to the classes of all maximal sets and all hyperhypersimple sets (see [14] for definitions): both of these sets are of the highest isomorphism-type possible for elements of Π_4. (We recall that Rogers [9] proved that the index-sets corresponding to the classes of all simple sets and all hypersimple sets are each of the highest isomorphism-type possible for elements of Π_3.) The proofs of these and other results will appear in [16]. Finally, another application of some of the present techniques is contained in [15], where we prove that there are two incomparable recursively enumerable degrees whose greatest lower bound is $\mathbf{0}$.

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