0. Introduction. In the following, $G$ will denote a locally compact abelian topological group with character group $\hat{G}$. For $1 \leq p < \infty$, $L^p(G)$ is the Banach space of all complex-valued functions whose $p$th powers are Haar integrable over $G$. ($L^p(G)$ is often written $L^p$ when the group $G$ is obvious from the context.) The linear space $L^1(G) \cap L^2(G)$ (denoted $L^1 \cap L^2$) is normed in such a way that, under convolution as multiplication, it is a commutative Banach algebra ($\S 2$). It is also proved in $\S 2$ that it is regular, semi-simple and that its regular maximal ideal space is $\hat{G}$. It is shown ($\S 3$) that the abstract Šilov theorem [8, p. 86] holds for $L^1 \cap L^2$. The standard proof of this theorem in $L^1(G)$ seems to depend upon the uniform boundedness of the approximate identity. A novel aspect of the $L^1 \cap L^2$ case is that a similar proof is obtained despite the fact that every approximate identity in $L^1 \cap L^2$ is unbounded.

An important but unsolved problem of harmonic analysis is the classification of the closed ideals in $L^1(G)$. Using the additional structure supplied by $L^1 \cap L^2$ it is to be expected that more precise results can be obtained about the closed ideals in $L^1 \cap L^2$. If $G$ and $\hat{G}$ are both locally compact metric abelian groups, examples of the more precise results that can be obtained are: (a) If $I$ is a closed proper ideal in $L^1 \cap L^2$, then there exists an $x \in I$ such that the hull of $x$ and the hull of $I$ coincide except for a set of measure zero (Theorem 7.2). (b) For every closed invariant proper subspace $N \subseteq L^2(G)$, $N \cap L^1 = k(h(N \cap L^1))$ (Corollary 2 of Theorem 7.4). This permits a new characterization of the kernel of $E$ for a class of perfect sets $E \subseteq \hat{G}$. (A. Denjoy terms these sets "épais en lui-même" in *Leçons sur le calcul des coefficients d'une série trigonométrique*, Paris, 1941, 2ième Partie, p. 100.) (c) The set $\mathcal{I}$ of all closed proper ideals in $L^1 \cap L^2$ which are

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not dense in $L^2$ is characterized as the set of all closed proper ideals $I$ such that the hull of $J$ has positive measure.

In §6 it is shown (still under the assumption that $G$ is locally compact metric abelian) that if $E \subset \hat{G}$ is a closed set, then $E$ is the hull of a principal ideal in $L^1 \cap L^2$ iff $E$ is a $G_\delta$. The theorem holds under rather more general circumstances (2); in particular, it holds in $L^1(G)$. It follows (from 6.2 and 6.3) that if $G$ and $\hat{G}$ are both locally compact metric abelian, then a hull $E$ for which spectral synthesis holds (if $I$ is any closed ideal having hull $E$, then $I$ is the kernel of $E$), must be a closed $G_\delta$ set. Consequently, the only instances of Helson's Theorem (3) (4) are given by principal ideals.

1. Preliminaries and notation. The following two theorems are useful in the sequel:

**Theorem 1.1.** If $G$ is a locally compact group, then $G$ is normal, and the family of compact neighborhoods of the identity is a basis for the neighborhood system of $G$ at the identity (Kelley [7, 5Y, 5.32 and 5.17]).

**Theorem 1.2.** If $G$ is a locally compact abelian group whose character group is $\hat{G}$, then the following are equivalent;

(a) $G$ is metrizable;

(b) the neighborhood system for the identity $e \in G$ has a countable basis;

(c) $\hat{G}$ is $\sigma$-compact.

**Proof.** That (a) and (b) are equivalent is proved in Kelley [7, p. 186]. That (b) and (c) are equivalent is proved in Hewitt and Ross [5, p. 397] (actually a more general result is proved).

The notations and definitions in this work are, in general, those of Loomis [8]. In particular, $L(E)$ will denote the set of all continuous functions having compact support in $E$. Also, if $A$ is any set of functions, then $A^+$ will denote the set of non-negative functions in $A$.

2. The Banach algebra $L^1 \cap L^2$. Let $L^1 \cap L^2$ denote the linear space $L^1(G) \cap L^2(G)$ and observe that the function defined by the equation $\| x \| = \| x \|_1 + \| x \|_2$ for each $x \in L^1 \cap L^2$ is a norm. If multiplication is defined by convolution, it follows that $L^1 \cap L^2$ is a commutative Banach algebra. The conjugate space of $L^1 \cap L^2$ is also obtained in this section. The ideal, $S$, of $L^1 \cap L^2$-functions whose Fourier transforms have compact support is shown to be dense in $L^1 \cap L^2$ and the regular maximal ideal space of $L^1 \cap L^2$ is found to be $\hat{G}$.

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(2) What is required is that $A$ should be a commutative regular semi-simple Banach algebra whose regular maximal ideal space is $\sigma$-compact.

(3) If $I$ is a closed proper ideal in $L^1(G)$ such that the boundary of the hull of $I$ contains no nonempty perfect subset, then $I = k(h(I))$. 

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Definitions 2.1. Let $\mu$ denote Haar measure on $G$. The set $E \subset G$ is locally null if for every compact set $C \subset G$, $\mu(E \cap C) = 0$. If $x, y$ are $\mu$-measurable functions defined on $G$ such that the set $\{s \in G \mid x(s) \neq y(s)\}$ is locally null, then $x = y$ l.a.e. (locally almost everywhere). Let $LL_\infty$ denote the equivalence classes of functions which are l.a.e. equal.

Theorem 2.2. The linear space $L^1 \cap L^2$ is a Banach space.

Proof. We have only to show that $L^1 \cap L^2$ is complete. Suppose $\{x_n\} \subset L^1 \cap L^2$ is an $L^1 \cap L^2$-Cauchy sequence, so that there exist $x \in L^1$ and $y \in L^2$ for which $\|x_n - x\|_1 \to 0$ and $\|x_n - y\|_2 \to 0$. Hence there exists a subsequence $\{y_n\} \subset \{x_n\}$ such that $y_n \to x$ a.e., and a subsequence of it, $\{z_n\}$, such that $z_n \to y$ a.e. Thus $z_n \to x$ a.e., so that $x = y$ a.e.

Lemma 2.3. Let $T$ be a linear functional on $L^1 \cap L^2$ defined by the equation $T(x) = \int x(t) \cdot y(t) + z(t) \, dt$ for each $x \in L^1 \cap L^2$, where $y \in L^2$ and $z \in L^\infty$. Then $T$ is bounded, and $\|T\| \leq \max(\|y\|_2, \|z\|_\infty)$.

Proof. Let $M = \max(\|y\|_2, \|z\|_\infty)$. Then, by Hölder’s Inequality, $\|T(x)\| \leq \|x\|_1 \|y\|_2 + \|z\|_\infty \leq \|x\|_1 M$. Hence $\|T\| = \sup_{\|x\|_1 = 1} |T(x)| \leq M$.

Theorem 2.4. The conjugate space of $L^1 \cap L^2$ is $(LL_\infty \times L^2)/Z$, where $Z = \{(g, h) \in LL_\infty \times L^2 \mid g + h = 0$ l.a.e.$\}$.

Proof. Let $\rho$ be defined on $L^1 \times L^2$ by the equation $\rho(x, y) = \|x\|_1 + \|y\|_2$, and $r$ be defined on $LL_\infty \times L^2$ by the equation $r(g, h) = \max(\|g\|_\infty, \|h\|_2)$. If $z \in L^p(G)$ and $w \in L^q(G)$, where $1 \leq p \leq \infty$, and $1/p + 1/q = 1$, let $\langle z, w \rangle = \int z(s) \cdot w(s) \, ds$. If $L^1 \times L^2$ is equipped with the norm $\rho$, it becomes a Banach space whose conjugate is the Banach space $LL_\infty \times L^2$ equipped with the norm $r$ (Schatten [13]). Identify $L^1 \cap L^2$ with $\Delta = \{\langle x, x \rangle \mid x \in L^1 \cap L^2\}$, which is a closed linear subspace of $L^1 \times L^2$. Let $f \in (L^1 \cap L^2)^*$ ( = Banach space conjugate of $L^1 \cap L^2$), and let $\phi$ be defined on $\Delta$ by the equation $\phi(x, x) = f(x)$. By the Hahn-Banach theorem $\phi$ may be extended without change of norm from the closed linear subspace $\Delta$ to all of $L^1 \times L^2$, i.e. $\phi$ may be extended to a bounded linear functional $F \in LL_\infty \times L^2$. Let $F = (F_1, F_2)$, and observe that, if $(x, y) \in L^1 \times L^2$, $F(x, y) = \langle x, F_1 \rangle + \langle y, F_2 \rangle$. If $(x, x) \in \Delta$ it follows that $F(x, x) = \langle x, F_1 \rangle + \langle x, F_2 \rangle = \langle x, F_1 + F_2 \rangle$. It is clear that if $F_1 + F_2 = 0$ l.a.e., $F \equiv 0$ on $\Delta$, hence it follows that we must identify elements of $\Delta^*$ ( = Banach space conjugate of $\Delta$) differing by an element of $Z$, where $Z = \{(g, h) \in LL_\infty \times L^2 \mid g + h = 0$ l.a.e.$\}$. Let $\{(g_n, h_n)\} \subset Z$, and suppose that $(g_n, h_n) \to (g, h)$. Then $\|g_n - g\|_\infty \to 0$ and $\|h_n - h\|_2 \to 0$, and $g_n + h_n \to 0$ l.a.e. for $n = 1, 2, 3, \ldots$; but, since $\|h_n - h\|_2 \to 0$, there exists a subsequence $\{(p_n, k_n)\} \subset \{(g_n, h_n)\}$ such that $\|k_n - h\|_\infty \to 0$, and for each $n = 1, 2, 3, \ldots$, $p_n + k_n = 0$ l.a.e. Therefore,

$$\|g + h\|_\infty = \|p_n + k_n - g - h\|_\infty \leq \|p_n - g\|_\infty + \|k_n - h\|_\infty \to 0.$$
Hence $g + h = 0$ I.a.e.; i.e., $(g, h) \in Z$, so that $Z$ is closed. Therefore $(L^{\infty}L^2)/Z$ is a Banach space, and $\phi \in (L^{\infty}L^2)/Z$; i.e., $(L^1 \cap L^2)^* \subset (L^{\infty}L^2)/Z$. Now, let $H \in (L^{\infty}L^2)/Z$. We can find $F \in L^{\infty}L^2$ such that $F$ belongs to the coset $H$ of $L^{\infty}L^2$, and for this $F$, let $f(x) = F(x, x)$. Then $f(x) = \langle x, F_1 \rangle + \langle x, F_2 \rangle$, so that

$$|f(x)| \leq \|x\|_1 \|F_1\|_{L^\infty} + \|x\|_2 \|F_2\|_2 \leq \|x\| \|F\|,$$

i.e., $f \in (L^1 \cap L^2)^*$. Therefore $(L^{\infty}L^2)/Z = (L^1 \cap L^2)^*$.

**Lemma 2.5.** Let $x, y \in L^1 \cap L^2$. Then $\|x \ast y\| \leq \min \{\|x\|_1 \|y\|, \|x\|_1 \|y\|_1\}$.

**Corollary 1.** $\|x \ast y\| < \|x\| \|y\|$.

**Corollary 2.** If $x \neq 0$ and $y \neq 0$, $\|x \ast y\| < \|x\| \|y\|$.

**Corollary 3.** $L^1 \cap L^2$ is a Commutative Banach Algebra.

**Definition 2.6.** Let $A$ be a Banach algebra and let $P$ be a directed set. Then the net $\{v_p \in A : p \in P\}$ is an approximate identity for $A$ if $\lim_p v_p x = \lim_p x v_p = x$ for each $x \in A$.

Note that it is not required that $\{\|v_p\|\}$ should be bounded. In fact this cannot be required, in general: $L^1 \cap L'$ with norm $\|x\|_1 + \|x\|_r$ is a Banach algebra if its multiplication is convolution. If $G$ is neither compact nor discrete (of course $G$ is still assumed to be locally compact abelian), $L^1 \cap L'$ has an approximate identity in the above sense, but it can be shown that any approximate identity in the above sense must be unbounded if $1 < r \leq 2$.

Let $\mathcal{V}$ denote the family of all precompact (closure is compact) neighborhoods of $e$, the identity of the group $G$. Partially order $\mathcal{V}$ by inclusion and designate it by $\{V_p\}$. Then $\{V_p\}$ is a directed set, and we may define a net $\{v_p\}$ of functions on it, by choosing, for each $V_p$, $v_p \in L^+(V_p)$ such that $\int v_p(s) \, ds = 1$.

**Theorem 2.7.** The net $\{v_p\}$ defined above is an approximate identity for $L^1 \cap L^2$ (cf. Loomis [8, p. 124]).

**Theorem 2.8.** Let $\Sigma$ be a closed subset of $L^1 \cap L^2$. Then $\Sigma$ is an ideal iff it is a translation-invariant subspace of $L^1 \cap L^2$ (cf. Loomis [8, p. 125]).

**Lemma 2.9.** Let $v \in L^1$, $\int v(t) \, dt = 1$ and $\varepsilon > 0$ be given. Then there exists $q \in (L^1 \cap L^2)^+$ such that $\hat{q} \in L_0(G)$, $\int q(t) \, dt = 1$, and $\|q - v\|_1 < \varepsilon$ (Edwards [2, pp. 165–166]).

**Theorem 2.10.** The ideal $S = \{x \in L^1 \cap L^2 : x \in L(G)\}$ is dense in $L^1 \cap L^2$.

**Proof.** Let $x \in L^1 \cap L^2$ and $\varepsilon > 0$ be given. Assume that $x \neq 0$, since $0 \in S$. Choose $v$ from an approximate identity so that $v \in L^1$, and $\|v \ast v - x\| < \varepsilon/2$. Then by Lemma 2.9, choose $q \in S$ so that $\|q - v\|_1 < \varepsilon/2 \|x\|$. Hence

**By application of the Hausdorff-Young Inequality.**
\[ \| q * x - x \| \leq \| x * (q - v) \| + \| x * v - x \| < \varepsilon. \] Thus \( \| q * x - x \| \leq \varepsilon \) and 
\( (q * x) = \hat{\delta} \chi \in L(\hat{G}). \)

**Theorem 2.11.** Let \( K \subset \hat{G} \) be any compact set containing \( \hat{\varepsilon} \), and let \( U \) be an open neighborhood of \( K \). Then there exists a function \( x \in L^1 \cap L^2 \) such that 
\( \hat{x} = 1 \) on \( K \), \( \hat{x} = 0 \) off \( U \), and \( 0 \leq \hat{x} \leq 1. \)

**Proof.** Let \( V \) be a symmetric compact neighborhood of \( \hat{\varepsilon} \) sufficiently small that \( V V K \subset U \). Set \( \Sigma = V K \). Then \( \Sigma \) is compact. Let \( y, z \) be the characteristic functions of \( \Sigma, V \) respectively. Since \( V, \Sigma \) are both compact, each has finite measure, so that \( y, z \in (L^1 \cap L^2)(\hat{G}) \) and therefore \( y * z \in (L^1 \cap L^2)(\hat{G}). \)

Let \( \tilde{y}, \tilde{z}, (y * z)^\sim \) be the inverse Fourier transforms of \( y, z \) and \( y * z \) respectively. Then \( (y * z)^\sim = \tilde{y} \cdot \tilde{z}. \) Let \( u = (y * z)^\sim = \tilde{y} \cdot \tilde{z}. \) Then \( u \in L^2(G) \), since \( y * z \in L^2(\hat{G}). \)

Also \( u \in L^1(G) \) since \( \tilde{y}, \tilde{z} \in L^2(G). \) Thus \( u \in L^1 \cap L^2(G) \), and \( \hat{u}(\alpha) = (y * z)(\alpha) \) a.e., and since each of \( \hat{u} \) and \( y * z \) is continuous, \( \hat{u} = y * z \). Let \( x = u / m(V) \). This \( x \) is the desired function.

**Remark.** By translation, it follows that if \( K \) is a compact subset of \( \hat{G} \) and if \( U \) is any open neighborhood of \( K \), there exists a function \( x \in L^1 \cap L^2 \) such that 
\( \hat{x} = 1 \) on \( K \), \( \hat{x} = 0 \) off \( U \), and \( 0 \leq \hat{x} \leq 1. \)

**Notation 2.12.** Let \( A \) be a commutative Banach algebra, and let \( \Delta(A) \) denote the set of all continuous homomorphisms of \( A \) onto the complex numbers. If \( A^* \)

**Theorem 2.13.**

(i) \( \alpha_M \) is a character of \( \hat{G} \),

(ii) \( \alpha_M \) is continuous on \( G \times M \),

(iii) If \( u \) runs through an approximate identity, \( u'(M) \) converges uniformly to \( \alpha_M(s) \).

The proof is the same as that in Loomis [8, pp. 135–136].

**Theorem 2.14.** The mapping \( M \to \alpha_M \) is a one-to-one mapping of \( M \) onto the set of all characters of \( G \), and \( x'(M) = \int x(s) \cdot \alpha_M(s) \) ds (cf. Loomis [8, p. 136]).
Theorem 2.15. The topology of \( \hat{G} \) (the weak*-topology of \((L^1 \cap L^2)^*\) induced on \( \hat{G} \)) is the usual topology of \( \hat{G} \) as the dual of \( G \).

Proof. Let \( F \) be the set of Fourier transforms of all functions in \( L^1 \cap L^2 \). Then \( F \subset C_0(\hat{G}) \), the set of continuous functions vanishing at infinity on \( \hat{G} \).

Suppose \( \alpha, \beta \in \hat{G} \) and \( \alpha \neq \beta \). Since \( \hat{G} \) is normal, there exist open disjoint neighborhoods \( U(\alpha), V(\beta) \), and disjoint compact neighborhoods \( K(\alpha) \subset U \) and \( C(\beta) \subset V \). By Theorem 2.11 there exist \( x, y \in L^1 \cap L^2 \) such that \( \hat{x} \equiv 1 \) on \( K \), \( \hat{x} \equiv 0 \) on \( C \) and \( \hat{y} \equiv 1 \) on \( C \), \( \hat{y} \equiv 0 \) on \( K \). Therefore, for each \( \alpha \in \hat{G} \) there exists \( \hat{x} \in F \) such that \( \hat{x}(\alpha) = 1 \), and also, \( F \) separates the points of \( \hat{G} \). By a theorem (5G) of Loomis [8], it follows that the weak topology induced on \( \hat{G} \) by \( F \) is precisely the one in which the functions of \( F \) are continuous; i.e., it is the the usual topology for \( \hat{G} \) as the dual of \( G \).

Theorem 2.16. \( L^1 \cap L^2 \) is semi-simple and regular.

Proof. We have established that if \( x \in L^1 \cap L^2 \), \( x' = \hat{x} \). It follows that if \( x' \equiv 0 \), \( x \equiv 0 \) a.e.; i.e., \( L^1 \cap L^2 \) is semi-simple. By a theorem in Loomis [8, p. 57] to prove that \( L^1 \cap L^2 \) is regular we have only to prove that if \( F \subset M \) is closed in the hull-kernel topology, and \( \alpha \notin F \), then there exists \( x \in L^1 \cap L^2 \) such that \( \hat{x} \equiv 0 \) on \( F \) and \( \hat{x}(\alpha) \neq 0 \). Let \( U = \hat{G} \setminus F \) so that \( U \) is open and \( \alpha \in U \). Choose a compact neighborhood \( K \) of \( \alpha \) such that \( K \subset U \). Apply Theorem 2.11 to obtain \( x \in L^1 \cap L^2 \) such that \( \hat{x} \equiv 0 \) off \( U \), \( \hat{x} \equiv 1 \) on \( K \) so that \( x \equiv 0 \) on \( F \), and \( \hat{x}(\alpha) = 1 / 0 \).

3. Šilov's Theorem for \( L^1 \cap L^2 \). Let \( A \) be a commutative Banach algebra. Then \( A \) satisfies the condition \( D \) if, given \( x \in M \in M \) there exists a sequence \( \{ x_n \} \subset A \) such that \( \hat{x}_n \equiv 0 \) in a neighborhood \( V_n \) of \( M \) for \( n = 1,2,3, \ldots \), and \( \| xx_n - x \| \to 0 \).

If \( M \) is not compact the condition must also be satisfied for the point at infinity; i.e., for each point \( x \in A \), there exists a sequence \( \{ x_n \} \subset A \) such that \( \{ \hat{x}_n \} \subset L(M) \), and \( \lim_{n \to \infty} \| xx_n - x \| = 0 \)

Šilov’s Theorem 3.1 (Loomis [8, p. 86]). Let \( A \) be a regular semi-simple commutative Banach algebra satisfying condition \( D \), and let \( I \) be a closed ideal in \( A \). Then \( I \) contains every element \( x \in k(h(I)) \) such that \( [\text{bd } h(x)] \cap h(I) \) includes no nonempty perfect set; i.e., is scattered (a closed scattered set is one which contains no nonempty perfect subset).

Since we have already established that \( L^1 \cap L^2 \) is regular, commutative, and semi-simple, we have only to show that \( L^1 \cap L^2 \) satisfies the condition \( D \). We shall first prove that \( L^1 \cap L^2 \) satisfies the condition \( D \) at infinity. After that, the remaining part of this section will be devoted to showing that \( L^1 \cap L^2 \) satisfies the condition \( D \) at finite points. The proof given in Loomis [8, p. 151] that \( L^1(G) \) satisfies Ditkin’s Condition at finite points appears to depend upon the uniform boundedness of the approximate identity. Since this boundedness is never available
in the $L^1 \cap L^2$ case (cf. 2.6), it is somewhat surprising that in spite of this lack a proof similar to the $L^1(G)$ case can be constructed.

**Lemma 3.2.** $L^1 \cap L^2$ satisfies the condition $D$ at infinity.

*Proof.* Assume that $\hat{G}$ is not compact. Let $x \in L^1 \cap L^2$, and $\varepsilon > 0$ be given. Use the construction of Theorem 2.10 to obtain $q \in S$ (so that $\hat{q} \in L(\hat{G})$) such that $\| q * x - x \| < \varepsilon$. Clearly then, there exists a sequence $\{x_n\} \subset S$ for which $\lim_n \| x_n - x \| = 0$. Since this can be done for every $x \in L^1 \cap L^2$, $L^1 \cap L^2$ satisfies the condition $D$ at infinity.

Let $\mathcal{U} = \{ U_\lambda \}_{\lambda \in \Lambda}$ denote the family of all symmetric Baire neighborhoods of $\hat{e}$ of measure less than or equal to one. Then $\mathcal{U}$ is a directed system under inclusion. Let $\{ V_\lambda \}_{\lambda \in \Lambda}$ denote any net of symmetric Baire neighborhoods of $\hat{e}$ defined on $\mathcal{U}$, and satisfying the following conditions:

(i) If $V \in \{ V_\lambda \}$, $V$ is compact;

(ii) Given $U_\lambda \in \mathcal{U}$, $V_\lambda \subset U_\lambda$ and $m(U_\lambda) < 4m(V_\lambda)$ (where $m$ is Haar measure on $\hat{G}$);

(iii) Given $U_\lambda \in \mathcal{U}$ and $V_\lambda$, there exists a neighborhood $W(\hat{e})$ depending on $U_\lambda$ and $V_\lambda$, such that $V_\lambda \subset W(\hat{e})$.

**Lemma 3.3.** There exists a net $\{ z_\lambda \} \subset L^1 \cap L^2$ defined on $\mathcal{U}$ such that for every $\lambda \in \Lambda$:

(i) $\| z_\lambda \| < 3$,

(ii) $z_\lambda \equiv 1$ on some neighborhood of $\hat{e}$.

*Proof.* Given $U_\lambda$, let $V_\lambda$ be the corresponding set in the net of sets defined above. Let $\hat{u}_\lambda$, $\hat{v}_\lambda$ be the characteristic functions of $U_\lambda$, $V_\lambda$ respectively, and let $z_\lambda = \hat{u}_\lambda * \hat{v}_\lambda / m(V_\lambda)$. Since $\hat{u}_\lambda$, $\hat{v}_\lambda$, and $z_\lambda$ all belong to $L^1 \cap L^2(\hat{G})$, the inverse Fourier-Plancherel transform of each exists. These may be designated as $u_\lambda$, $v_\lambda$ and $z_\lambda$, respectively.

**Proof of (i).** $\| z_\lambda \|_2 = \| \hat{z}_\lambda \|_2 = [1 / m(V_\lambda)] \cdot \| \hat{u}_\lambda * \hat{v}_\lambda \|_2 \leq [1 / m(V_\lambda)] \cdot \| \hat{u}_\lambda \|_2 \cdot \| \hat{v}_\lambda \|_2 \leq 1$. Thus $\| z_\lambda \|_2 \leq 1$. Similarly $\| z_\lambda \|_1 = [1 / m(V_\lambda)] \cdot \| u_\lambda v_\lambda \|_1 < 2$. Hence $\| z_\lambda \| < 3$.

**Proof of (ii).** Corresponding to $U_\lambda$ and $V_\lambda$ there exists a neighborhood $W = W(\hat{e})$ such that $V_\lambda W \subset U_\lambda$. Let $\beta \in W$. Then

$$
\hat{z}_\lambda(\beta) = [1 / m(V_\lambda)] \cdot (\hat{u}_\lambda * \hat{v}_\lambda)(\beta) = [1 / m(V_\lambda)] \int_{V_\lambda} \hat{u}_\lambda(\alpha \beta) \, d\alpha = 1.
$$

Hence $z_\lambda \equiv 1$ on $W(\hat{e})$.

Let $C \subset G$ be a compact subset, and let $\varepsilon > 0$ be given. Then $U(C, \varepsilon / 5, \hat{e})$ is open in $\hat{G}$, where $U(C, \varepsilon / 5, \hat{e}) = \{ \alpha \in \hat{G} \mid 1 - (s, \alpha) < \varepsilon / 5, \text{all } s \in C \}$. Since $\mathcal{U}$ contains a basis for the topology of $\hat{G}$ at $\hat{e}$, there exists a $\lambda_0$ such that if $\lambda < \lambda_0$, $U^2_\lambda \subset U(C, \varepsilon / 5, \hat{e})$. For convenience of notation, let $S_\lambda$ denote the net $\{ z_\lambda \}$ constructed in Lemma 3.3, and let $S_0 = \{ z_\lambda \mid \lambda > \lambda_0 \}$.
Lemma 3.4. Given $\varepsilon > 0$, there exists $\lambda_0$ such that if $z \in S_{\lambda_0}$, then

$$\| z - z_s \| < \varepsilon$$

for every $s \in C$.

Proof. Choose an appropriate $\lambda_0$ as above so that if $\lambda > \lambda_0$, $U_\lambda^2 \subset U(C, \varepsilon/5, \delta)$. Let $z \in S_{\lambda_0}$ and suppose that $z = uv/m(V)$. Let $s \in C$, and note that $\delta = 0$ off $UV$. Then

$$\| z - z_s \|_2^2 = \| \varepsilon - \varepsilon_s \|_2^2 = \int_U \varepsilon(x)^2 \cdot |1 - (s, x)|^2 \, dx < (\varepsilon/5)^2.$$

Hence $\| z - z_s \|_2 < \varepsilon/5$. Similarly $\| u - u_s \|_2 < [m(U)]^{1/2}(\varepsilon/5)$, and

$$\| v - v_s \|_2 < [m(V)]^{1/2}(\varepsilon/5).$$

We observe that

$$\| z - z_s \|_1 \leq \left[ 1/m(V) \right] \left[ \| u(v - v_s) \|_1 + \| v_s(u - u_s) \|_1 \right]$$

$$< 2(\varepsilon/5) \left[ \frac{m(U)}{m(V)} \right]^{1/2} < 4\varepsilon/5,$$

so that $\| z - z_s \| < \varepsilon$.

Corollary. If $x \in L^1 \cap L^2$, and $x(\mathbf{e}) = 0$, then $\lim_{\lambda} \| x * z_\lambda \| = 0$.

Proof. Let $\delta > 0$ be given, and choose $C \subset G$ to be compact, symmetric and such that $\int_{G-C} |x(s)| \, ds < \delta/12$. Set $\varepsilon = \delta/2 \| x \|_1$, and choose $\lambda_0$ as before so that if $z \in S_{\lambda_0}$, then $\| z - z_s \| < \varepsilon$ for every $s \in C$. Hence

$$(x * z)(t) = \int x(s) \left[ z(ts^{-1}) - z(t) \right] \, ds.$$

We observe that $\| z * x \|_1 = \sup_{\| h \|_\infty = 1} |\langle z * x, h \rangle|$, and that

$$\| z * x \|_2 = \sup_{\| p \|_2 = 1} |\langle z * x, p \rangle|.$$

Thus, by a straightforward computation,

$$\| x * z \| \leq \int_C |x(s)| \| z_{s-1} - z \|_1 \, ds + \int_{G-C} |x(s)| \| z_{s-1} - z \|_2 \, ds.$$

If $z \in S_{\lambda_0}$, then $s \in C$ implies that $\| z_{s-1} - z \| < \varepsilon$. In this case the inequality $(\ast)$ becomes

$$\| x * z \| < \varepsilon \| x \|_1 + 2 \| z \| \delta/12 < \delta,$$

so that $\lim_{\lambda} \| x * z_\lambda \| = 0$.

Theorem 3.5. There exists a net $\{ v_q \} \subset L^1 \cap L^2$ such that each $\delta_q = 0$ in a neighborhood of $\mathbf{e}$ (depending on $v_q$) and such that if $x \in L^1 \cap L^2$, and $\hat{x}(\mathbf{e}) = 0$, then $\lim_{q} \| x * v_q - x \| = 0$.

Proof. Let $\{ u_p \}$ be the approximate identity defined in Theorem 2.7 and let $\{ z_\lambda \}$ denote the net defined in Lemma 3.3. Let $v(p, \lambda) = u_p - z_\lambda u_p$. Clearly $v(p, \lambda) \in L^1 \cap L^2$. The set of all ordered pairs $(p, \lambda)$ may be directed by: $(p_1, \lambda_1)$.
> (p_2, \lambda_2) \text{ iff } p_1 > p_2 \text{ and } \lambda_1 > \lambda_2. \text{ If we allow } q \text{ to run through this directed set, }
\{v_q\} \text{ is a net, and we note that } \delta(p, \lambda) = \hat{u}_p\{
(1 - \hat{z}_\lambda) \equiv 0 \text{ in the neighborhood of } \hat{e} \text{ where } \hat{z}_\lambda \equiv 1. \text{ Finally } \left\| v(p, \lambda) \ast x - x \right\| \leq 2 \left\| u_p \ast x - x \right\| + \left\| u_p \right\|_1 \left\| x \ast z_\lambda \right\|. \text{ Hence } \lim_{q} \left\| v_q \ast x - x \right\| \leq 2 \lim_{p} \left\| u_p \ast x - x \right\| + \lim \left\| u_p \right\|_1 \left\| x \ast z_\lambda \right\| = 0.

COROLLARY 1. \( L^1 \cap L^2 \) satisfies condition D.

Proof. In the above theorem we have just established that \( L^1 \cap L^2 \) satisfies condition D at \( \hat{e} \). The condition D follows for all other finite points upon translation. It was established for the point at infinity in Lemma 3.2.

COROLLARY 2. Šilov's theorem is valid for \( L^1 \cap L^2 \).

4. Translation-invariant subspaces of \( L^2(G) \). Let the notation \([m]\) following an assertion denote that the assertion is valid except for sets of zero \( m \)-measure (on \( \hat{G} \)).

Example. \( E \subset F \) \([m]\) means that \( m(E - F) = 0 \).

If \( x \in L^2(G) \), \( N(x) \) will denote the set of all finite linear combinations of translates of \( x \). \( H(x) \) will denote the \( L^2 \)-closure of \( N(x) \). The spaces \( L^1(G) \), \( L^2(G) \) will be written as \( L^1, L^2 \) respectively unless possible ambiguity prevents this. If \( N \) is any subspace of \( L^2 \), \( N \) is invariant if, for every \( s \in G \), \( x \in N \) implies that \( x_s \in N \). If \( x \in L^2 \), define \( h(x) = \{ x \in \hat{G} \mid \hat{x}(x) = 0 \} \) \([m]\).

The result (4.1) of this section is taken from S. Bochner and K. Chandrasekharan [1, pp. 148-149], where it is established for the case of \( G = R \). Their proof carries over without change to the general case, so there is no need to reproduce it here.

THEOREM 4.1. Let \( x, y \in L^2 \). Then \( x \in H(y) \) if and only if \( h(y) \subset h(x) \) \([m]\).

In this section, let \( N \) denote an arbitrary closed proper \( (\{0\} \neq N \text{ and } N \neq L^2) \) subspace of \( L^2 \), invariant under translation. If \( E \subset L^2 \), \( cl(E) \) will denote the \( L^2 \)-closure of the set \( E \).

LEMMA 4.2. If \( x \in N \), \( m(h(x)) > 0 \).

LEMMA 4.3. Let \( x, y \in N \). Then there exists \( z \in N \) for which

\[ h(z) = h(x) \cap h(y)[m]. \]

THEOREM 4.4. Let \( \{x_n\} \subset N \). Then there exists \( x_0 \in N \) such that

\[ h(x_0) = \bigcap_{n=1}^{\infty} h(x_n)[m]. \]

Proof. Without loss of generality, assume that \( \left\| x_n \right\|_2 > 0 \) for \( n = 1, 2, 3, \ldots \), and let \( c_k = \left\{ 2^k \cdot \left\| x_k \right\|_2 \right\}^{-1} \) for each \( k = 1, 2, 3, \ldots \). Let \( \hat{p}_n = \sum_{k=1}^{\infty} c_k \hat{x}_k \), and let \( p_n \) be the inverse Fourier-Plancherel transform of \( \hat{p}_n \). Then it is clear that \( \{p_n\} \subset N \), and \( h(p_n) = \bigcap_{k=1}^{\infty} h(x_k)[m] \) by Lemma 4.3. There exists \( x_0 \in N \) for which \( \lim_{n} \left\| p_n - x_0 \right\|_2 = 0 \). Hence \( \bigcap_{n=1}^{\infty} h(x_n) = \bigcap_{n=1}^{\infty} h(p_n) \subset h(x_0)[m] \), and
consequently, we have only to prove that for each $n$, $h(x_0) \subseteq h(p_n)$ \([m]\). These remarks lead to a straightforward proof by contradiction.

**Theorem 4.5.** Let $E$ be a measurable subset of $\hat{G}$, and suppose that for some $x' \in N$, $m(E \cap h(x'))$ is finite. Then there exists a $z \in N$ such that for every $x \in N$, $E \cap h(z) \subseteq E \cap h(x)$ \([m]\).

**Proof.** Let $c = \inf_{x \in N} m(E \cap h(x)) < \infty$. Choose a sequence $\{x_n\} \subseteq N$ such that $m(E \cap h(x_n)) < c + 1/n + 1$ \((n = 1, 2, 3, \ldots)\). Then, by Theorem 4.4, there exists a $z \in N$ such that $h(z) = \bigcap_{n=1}^{\infty} h(x_n)$ \([m]\). Hence $m(E \cap h(z)) = c$. Let $x \in N$. Then, by Lemma 4.3 there exists $y \in N$ for which $h(y) = h(x) \cap h(z)$ \([m]\). Thus $m(E \cap h(x) \cap h(z)) = c$. Observe that $h(x) \cap h(z)$ and $h(z) - h(x)$ are disjoint and that $h(z) = [h(x) \cap h(z)] \cup [h(z) - h(x)]$. Therefore $c = m(E \cap h(z)) = c + m(E \cap [h(z) - h(x)])$. Thus $m(E \cap h(z)) = 0$.

**Theorem 4.6.** Let $G$ be a measurable subset of $\hat{G}$. Then there exists a $z \in N$ such that $H(z) = N$.

**Proof.** Since $\hat{G}$ is $\sigma$-compact, we may set $\hat{G} = \bigcup_{n=1}^{\infty} K_n$, where each $K_n$ is compact; thus $0 \leq m(K_n) < \infty$, for $n = 1, 2, 3, \ldots$. Hence if $x \in N$, $m(K_n \cap h(x))$ is finite, and we apply Theorem 4.5 to obtain a sequence $\{z_n\} \subseteq N$ such that for every $x \in N$, $K_n \cap h(z_n) \subseteq K_n \cap h(x)$ \([m]\). By Theorem 4.4, there exists a $z \in N$ such that $h(z) = \bigcap_{n=1}^{\infty} h(z_n)$ \([m]\). Suppose $x \in N$; then except for a null set, $h(z) = \bigcup_{n=1}^{\infty} [K_n \cap h(z)] \subseteq \bigcup_{n=1}^{\infty} [K_n \cap h(z_n)] \subseteq \bigcup_{n=1}^{\infty} [K_n \cap h(x)] = h(x)$.

By Theorem 4.1, $x \in H(z)$; i.e., $N \subseteq H(z)$. Since $z \in N$, $H(z) \subseteq N$, so that $N = H(z)$.

5. **The closed ideals** $I$ and $I^\perp$ in $L^1 \cap L^2$. If $I$ is an ideal in $L^1 \cap L^2$, $\text{cl} I$ will be denoted by $J$. The class of all closed proper ideals $I$ in $L^1 \cap L^2$ for which $J \neq L^2$ will be denoted by $\mathcal{J}$. $I$ is symmetric if $x \in I$ implies that $x^* \in I$, where $x^*(s) = x(s^{-1})$ \((s \in G)\). If $N$ is a closed subspace of $L^2(G)$, we shall denote the orthogonal complement of $N$ by $N^\perp$. Let $I \in \mathcal{J}$, and let $\langle x, y \rangle = \int x(s)y(s) \, ds$. Then, define $I^\perp = \{x \in L^1 \cap L^2 | \langle x, y \rangle = 0 \text{ for all } y \in I\}$.

**Remark 5.1.** If $M$ is a regular maximal ideal of $L^1 \cap L^2$, then $M$ is symmetric. Thus, if $I$ is a closed ideal in $L^1 \cap L^2$ such that $I = k(h(I))$, then $I$ is symmetric.

**Lemma 5.2.** Let $N$ be a closed proper invariant subspace of $L^2$ such that $N \cap L^1 \neq \{0\}$. Then $N \cap L^2 \in \mathcal{J}$ and $N \cap L^1$ is symmetric.

**Theorem 5.3.** Let $I \in \mathcal{J}$. Then $I^\perp = J^\perp \cap L^1$.

**Proof.** Let $y \in J$. Then there exists a sequence $\{y_n\} \subseteq I$ such that $\|y_n - y\|_2 \to 0$. If $x \in I^\perp$, $\langle x, y_n \rangle = 0$ \(n = 1, 2, 3, \ldots\), and since strong convergence implies weak convergence, it follows that $\langle x, y \rangle = 0$. Thus if $x \in I^\perp$, $\langle x, y \rangle = 0$ for every $y \in J$.,
so that \( x \in J^\perp \cap L^1 \); i.e., \( I^\perp \subset J^\perp \cap L^1 \). Now let \( x \in J^\perp \cap L^1 \). Then if \( y \in I \), \( (x, y) = 0 \) since \( y \in J \). Thus \( x \in I^\perp \).

**Lemma 5.4.** Let \( I \in \mathcal{I} \). Then \( J \) and \( J^\perp \) are closed proper invariant subspaces of \( L^2 \).

**Corollary.** If \( I \in \mathcal{I} \), \( I^\perp = \{0\} \), or \( I^\perp \in \mathcal{I} \). In either event, \( I^\perp \) is symmetric.

Let the ideal \( I \oplus I^\perp \) be defined as the direct sum of the two ideals \( I \) and \( I^\perp \). The notation \( I \oplus I^\perp \) will denote the \( L^1 \cap L^2 \)-closure of \( I \oplus I^\perp \).

**Theorem 5.5.** Let \( I_1 \) and \( I_2 \) be closed proper ideals of \( L^1 \cap L^2 \) such that \( I_1 \cap I_2 = \{0\} \). Then \( h(I_1) \cup h(I_2) = \hat{G} \).

**Proof.** Let \( E = h(I_1) \cup h(I_2) \), and suppose that \( E \neq \hat{G} \). Then there exists \( \alpha \in \hat{G} \) such that \( \alpha \notin E \), and there exist open disjoint neighborhoods \( V_1(\alpha) \) and \( V_2(E) \). The compact neighborhoods of \( \alpha \) are a basis for the topology of \( \hat{G} \) at \( \alpha \), so there exists an open neighborhood \( U_1(\alpha) \) such that \( U_1 \) is a compact subset of \( V_1 \). By Theorem 2.11 there exists \( 0 \neq y \in L^1 \cap L^2 \) such that \( \hat{y} \equiv 0 \) on \( U_1 \), \( \hat{y} \equiv 0 \) off \( V_1 \), and \( 0 \leq \hat{y} \leq 1 \). Observe that \( E \subset V_2(\text{open}) \subset \hat{G} - V_1 \subset h(y) \) so that \( E \subset h(y)^0 \) (= interior of \( h(y) \)). Thus \( h(I_1) \subset h(y)^0 \) and \( h(I_2) \subset h(y)^0 \), so that \( y \in I_1 \) and \( y \in I_2 \); i.e., \( y \in I_1 \cap I_2 \); i.e., \( y = 0 \). This is a contradiction.

**Corollary.** Let \( I \in \mathcal{I} \). Then \( h(I) \cup h(I^\perp) = \hat{G} \).

**Theorem 5.6.** If \( I \in \mathcal{I} \), then \( I^\perp \in \mathcal{I} \) iff \( h(I)^0 \neq \emptyset \).

**Proof.** Part 1. (Necessity) Claim: If \( h(I)^0 = \emptyset \), then \( I^\perp = \{0\} \). In fact, it follows from the above corollary that \( h(I) \cup h(I^\perp) = \hat{G} \), and \( \hat{G} - h(I^\perp) \subset h(I)^0 = \emptyset \); so that \( \hat{G} - h(I^\perp) = \emptyset \), and \( h(I^\perp) = \hat{G} \); i.e., \( I^\perp = \{0\} \).

Part 2. (Sufficiency) Claim: If \( h(I)^0 \neq \emptyset \), then \( I^\perp \neq \{0\} \). In fact, let \( \alpha \in h(I)^0 \), and let \( C = C(\alpha) \) be a compact neighborhood of \( \alpha \) such that \( C \subset h(I)^0 \). By Theorem 2.11 there exists \( 0 \neq y \in L^1 \cap L^2 \) such that \( \hat{y} \equiv 1 \) on \( C \), \( \hat{y} \equiv 0 \) off \( h(I)^0 \), and \( 0 \leq \hat{y} \leq 1 \). Thus if \( x \in I \), and \( \beta \in h(I) \), \( \hat{x}(\beta) = 0 \); and if \( \beta \notin h(I) \), \( \hat{\beta} \notin h(I)^0 \), so that \( \hat{y}(\beta) = 0 \). Therefore \( \hat{x} \hat{y} = \hat{x} \hat{y} \equiv 0 \), and therefore \( \int \hat{x}(\beta) \cdot \hat{y}(\beta) \, d\beta = 0 \). Thus \( (x, y) = (\hat{x}, \hat{y}) = 0 \) for every \( x \in I \). Hence \( 0 \neq y \in I^\perp \); i.e., \( I^\perp \neq \{0\} \).

**Theorem 5.7.** Let \( I_1 \) and \( I_2 \) be closed proper ideals in \( L^1 \cap L^2 \) such that \( I_1 \cap I_2 = \{0\} \). Then \( h(I_1) \cap h(I_2) = h(I_1 \oplus I_2) \).

**Corollary** \( h(I_1 \oplus I_2) = h(I_1) \cap h(I_2) \).

6. Closed ideals in \( L^1 \cap L^2 \) when \( G \) is metric. The topological group \( G \) has been assumed to be a locally compact abelian group. It will be assumed from this point on that, in addition to this, \( G \) is metric. It follows by Theorem 1.2 that \( \hat{G} \) is \( \sigma \)-compact. Hence every \( F_\sigma \) set in \( \hat{G} \) is also \( \sigma \)-compact.
Definition 6.1. If $x \in L^1 \cap L^2$, the ideal $I(x)$ denotes the closed ideal generated by $x$ together with its translates. The linear subspace $N(x)$ denotes the set of all finite linear combinations of translates of $x$. Thus the $L^1 \cap L^2$-closure of $N(x)$ is $I(x)$.

Lemma 6.2. If $x \in L^1 \cap L^2$, then $h[I(x)] = h(x)$.

Proof. Observe that if $y \in N(x)$, $h(x) \subset h(y)$. Let $z \in I(x)$. Then, since $N(x)$ is $L^1 \cap L^2$-dense in $I(x)$, there exists a sequence $\{z_n\} \subset N(x)$ such that $\|z - z_n\| \to 0$. If $\alpha \in h(x)$, then $\hat{z}_n(\alpha) = 0$ for $n = 1, 2, 3, \ldots$. Hence

$$|\hat{z}(\alpha)| \leq \|\hat{z} - \hat{z}_n\|_\infty \leq \|z - z_n\|,$$

and since $\|z - z_n\| \to 0$, $\hat{z}(\alpha) = 0$. Thus $h(x) \subset h(z)$. Since $h[I(x)] = \bigcap \{h(z) \mid z \in I(x)\}$, it follows that $h(x) \subset h[I(x)]$. But $x \in I(x)$, so that $h[I(x)] \subset h(x)$. Consequently, $h(x) = h[I(x)]$.

Theorem 6.3. Let $I$ be a closed ideal of $L^1 \cap L^2$. Then there exists an $x \in I$ such that $h(x) = h(I)$ iff $h(I)$ is a $G_\delta$ set.

Proof. If $h(I) = h(x)$ for some $x \in I$, then since $\hat{x}$ is continuous, $h(I)$ is a $G_\delta$ set. Now suppose $h(I)$ is a $G_\delta$ set, and let $U = \hat{G} - h(I)$ so that $U$ is an open $F_\sigma$-set and $U = \bigcup_{n=1}^\infty K_n$, with $K_n \subset U$ and $K_n$ compact. Since $K_n \cap h(I) = \emptyset$, there exist disjoint open neighborhoods $U_n(K_n)$ and $V_n(h(I))$. By Theorem 2.11, there exists $w_n \in L^1 \cap L^2$ such that $\hat{w}_n \equiv 1$ on $K_n$, $\hat{w}_n \equiv 0$ off $U_n$, and $0 \leq \hat{w}_n \leq 1$. Observe that $V_n(\text{open}) \subset h(w_n)$, so that $V_n \subset h(w_n)^0$. Therefore $h(I) \subset h(w_n)^0$, since $h(I) \subset V_n$. Hence $\{w_n\} \subset I$.

Let $x_m = \sum_{k=1}^m \hat{w}_k \cdot \{2^k \|w_k\|^2\}^{-1}$. Then $\{x_m\} \subset I$, and $\{x_m\}$ is $L^1 \cap L^2$-Cauchy so there exists $x \in L^1 \cap L^2$ such that $x_m \to x$. But $I$ is closed, so it follows that $x \in I$. Therefore $h(I) \subset h(x)$. Let $\alpha \notin h(I)$; i.e., $\alpha \in U$. Then there exists some $n$ for which $x \in K_n$, so that $\hat{w}_n(\alpha) = 1$, and hence $\hat{x}(\alpha) \neq 0$. Therefore $\alpha \notin h(x)$; i.e., $h(x) \subset h(I)$.

Corollary. Let $I$ be a closed ideal in $L^1 \cap L^2$. If $h(I)$ is a $G_\delta$ set with a scattered boundary, then there exists an $x \in I$ such that $I = I(x)$.

Proof. This follows from Šilov's Theorem (Theorem 3.5).

Remark. This corollary shows that if $\hat{G}$ is also metric, then the only instances of Helson's Theorem [4] for $L^1 \cap L^2$ (and, similarly for $L^1(G)$) are given by principal ideals. Theorem 6.3 shows that if $E \subset \hat{G}$ is closed, then $E$ is the hull of a closed principal ideal in $L^1 \cap L^2$ iff $E$ is a $G_\delta$ set. Thus, for example, if $G$ is $\sigma$-compact, every closed set in $\hat{G}$ is the hull of a closed principal ideal (since $\hat{G}$ would then be metric), and any nonprincipal closed ideal in $L^1 \cap L^2$ would
therefore provide an example of an ideal for which spectral synthesis fails \([I \neq I(x)]\), but \(h(I(x)) = h(x) = h(I)\).

It should be remarked here that Schwartz' counterexample to spectral synthesis in \(L^1(\mathbb{R}^n) (n \geq 3)\) carries over to \(L^1 \cap L^2(\mathbb{R}^n) (n \geq 3)\) with only minor modifications. The proof of the following theorem, therefore, is omitted.

**Theorem.** There exists \(x \in L^1 \cap L^2(\mathbb{R}^n)\), for \(n \geq 3\), such that \(x \notin I(x \ast x)\) (Reiter \([11, pp. 469-470]\)).

7. **The family \(\mathcal{J}\) of ideals of \(L^1 \cap L^2\).**

**Theorem 7.1.** Let \(I\) be a closed nonzero ideal of \(L^1 \cap L^2\). If \(m(h(I)) > 0\), then \(I \in \mathcal{J}\).

**Proof.** We have only to prove that \(J \neq L^2\). We will accomplish this by assuming that \(J = L^2\) and showing that this leads to a contradiction.

By Theorem 6.3, there exists a function \(u_1 \in L^1 \cap L^2\) such that \(\hat{u}_1 > 0\) on \(\hat{G}\). Let \(y = \text{the characteristic function of } h(I)\), and set \(\hat{u} = \hat{u}_1 y\). Then \(\hat{u} \in L^2(\hat{G})\), so that, by the Plancherel theorem, there exists \(u \in L^2(G)\) such that the Fourier transform of \(u\) is equal to \(\hat{u}\) a.e. Let \(\int_{h(I)} |\hat{u}(x)|^2 dx = p^2 > 0\). Since \(u \in L^2\), there exists a sequence \(\{x_n\} \subset I\) such that \(\lim_n \| u - x_n \|_2 = 0\). Hence \(0 = \lim_n \| \hat{u} - \hat{x}_n \|_2^2 \geq p^2 > 0\). This is the desired contradiction.

**Corollary.** If \(\hat{G}\) has a closed subset \(E\) of positive measure such that \(E^0 = \emptyset\), then there exists a closed proper invariant subspace \(N \subset L^2(G)\) for which \(N \cap L^1 = \{0\}\).

**Proof.** Let \(I = k(E)\), so that \(h(I) = E\) and \(m(h(I)) > 0\). Then \(I \in \mathcal{J}\) by the above theorem and \(h(I)^0 = \emptyset\). Hence \(I^\perp = \{0\}\) by Theorem 5.6. But \(I^\perp = J^\perp \cap L^1\) by Theorem 5.3 and \(J^\perp\) is a closed proper invariant subspace of \(L^2(G)\) by Theorem 5.4. Therefore the desired subspace is \(N = J^\perp\).

**Example.** Let \(G\) be the real line under addition, and let \(E\) be a Cantor set of positive measure. Note that if \(G\) is compact, \(\hat{G}\) is discrete so that no set \(E \neq \emptyset\) can be found such that \(E^0 = \emptyset\). However, in this case, \(N \cap L^1 = N\) for every \(N \subset L^2(G)\).

**Theorem 7.2.** If \(I\) is any closed proper ideal in \(L^1 \cap L^2\), there exists \(x \in I\) such that \(h(x) = h(I) [m]\).

**Proof.** Let \(\{U_n\}\) be a family of open neighborhoods of \(h(I)\) such that \(m(U_n - h(I)) < 1/n (n = 1, 2, 3, \ldots)\). If \(F_n = \hat{G} - U_n\), then \(F_n\) is closed, and therefore \(\sigma\)-compact. Hence, let \(F_n = \bigcup_{k=1}^{\infty} K_{n_k}\), where \(K_{n_k}\) is compact for each

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(5) A proof of the fact that every nondiscrete locally compact group contains a compact nowhere dense subset of positive measure was communicated to the author in the summer of 1963 by K. A. Ross and K. Stromberg.
n, k = 1, 2, 3, … . Observe that \( h(I) \cap K_n = \emptyset \) so that we may choose disjoint open neighborhoods \( U_{nk}, V_{nk} \) of \( h(I) \) and \( K_n \) respectively, and we can always choose \( U_{nk} \subset U_n \). Let \( \hat{x}_{nk} \in L^1 \cap L^2 \) be constructed by Theorem 2.11 so that \( \hat{x}_{nk} \equiv 1 \) on \( K_n \), and \( \hat{x}_{nk} \equiv 0 \) off \( V_{nk} \). Then \( h(I) \subset U_{nk} \subset h(x_{nk}) \), and it follows that \( x_{nk} \in I \). Let \( y_{np} = \sum_{k=1}^{\infty} x_{nk} \cdot \{2^k \|x_{nk}\| \}^{-1} \). Then \( \{y_{np}\} \subset I \) and \( \{y_{np}\} \) is \( L^1 \cap L^2 \)-Cauchy and \( I \) is closed, so that \( \lim_{p} \|y_{np} - x_n\| = 0 \) for some \( x_n \in I \). By proceeding this manner, we obtain a sequence \( \{x_n\} \subset I \). By construction \( \hat{x}_n > 0 \) on each \( K_n \) (for \( k = 1, 2, 3, \ldots \); i.e., \( \hat{x}_n > 0 \) on \( F_n \), and \( \hat{x}_n \equiv 0 \) on each \( U_{nk} \) (for \( k = 1, 2, 3, \ldots \)).

Now let \( E = \bigcap_{n=1}^{\infty} U_n \), so that \( m(E - h(I)) = 0 \), and proceeding as before, let \( x = \sum_{n=1}^{\infty} x_n \cdot \{2^k \|x_n\| \}^{-1} \). Then \( x \in I \), and \( \hat{x} > 0 \) on each \( F_n = G - U_n \) (for \( n = 1, 2, 3, \ldots \)). Hence \( \hat{x} > 0 \) on \( \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} (G - U_n) = G - \bigcap_{n=1}^{\infty} U_n = G - E \). Moreover \( \hat{x} \equiv 0 \) on \( h(I) \), so that \( h(x) = h(I) \) \([m]\).

**Corollary 1.** If \( I \) is a closed proper ideal in \( L^1 \cap L^2 \), then \( J = H(x) \).

**Proof.** Let \( y \in J \). Then there exists a sequence \( \{y_n\} \subset I \) such that \( \|y - y_n\|_2 \to 0 \). Let \( F = h(x) - h(y) \) \([m]\), where \( x \in E \), and \( h(I) = h(x) \) \([m]\) as in the above theorem. Suppose \( F \neq \emptyset \) \([m]\), and let \( \int_F |\hat{y}(x)|^2 \, dx = \delta > 0 \), since \( |\hat{y}(x)|^2 > 0 \) a.e. on \( F \). Then \( \int_F |\hat{y}(x) - \hat{y}_n(x)|^2 \, dx = \int_F |\hat{y}(x)|^2 \, dx = \delta > 0 \), since each \( \hat{y}_n \equiv 0 \) a.e. on \( F \). It follows that \( \lim_n \|y - y_n\|_2 \neq 0 \), which contradicts our initial assumption. Hence \( F = \emptyset \) \([m]\), and if \( y \in J \), \( h(x) \subset h(y) \) \([m]\). By Theorem 4.1, it follows that \( y \in H(x) \). Hence \( J \subset H(x) \), so that \( J = H(x) \).

The proofs of the following results are direct applications of Theorem 7.1, Theorem 4.1, and the above Corollary 1.

**Corollary 2.** Let \( I \) be a closed proper ideal in \( L^1 \cap L^2 \). Then \( I \in \mathcal{I} \) iff \( m(h(I)) > 0 \).

**Corollary 3.** Let \( I_1, I_2 \) be closed proper ideals in \( L^1 \cap L^2 \). Then \( \text{cl } I_1 = \text{cl } I_2 \) iff \( h(I_1) = h(I_2) \) \([m]\).

**Theorem 7.3.** The group \( \hat{G} \) is connected iff for every pair \( I, I^\perp \in \mathcal{I}, I \oplus I^\perp \) is a proper ideal in \( L^1 \cap L^2 \).

**Proof.** Suppose that \( \hat{G} \) is connected and that for some pair \( I, I^\perp \in \mathcal{I}, I \oplus I^\perp \) is not proper; i.e., \( I \oplus I^\perp \) is \( L^1 \cap L^2 \). Then by the corollary to Theorem 5.7, \( \emptyset = h(I) \cap h(I^\perp) \) and by the corollary to Theorem 5.5, \( h(I) \cup h(I^\perp) = \hat{G} \). Hence, \( \hat{G} \) is not connected, contrary to our assumption. Conversely, suppose that \( \hat{G} \) is not connected; i.e., suppose \( \hat{G} = P \cup Q \) where \( P \) and \( Q \) are open-closed and disjoint in \( \hat{G} \). Let \( j(P) = \{x \in L^1 \cap L^2 \mid \hat{x} \in \hat{G} \text{ and } P \subset h(x)^0 \} \) and let \( I = L^1 \cap L^2 \)-closure of \( j(P) \). Thus, \( I \in \mathcal{I}, I^\perp \in \mathcal{I} \) and \( h(I) = P \). Let \( I' = L^1 \cap L^2 \)-closure of \( j(Q) \), and observe that since \( Q \) is open-closed, \( x \in I' \) iff \( Q < h(x) \). Note that \( I' \subset I^\perp \) and that \( Q = \hat{G} - h(I) = h(I^\perp) \). Therefore, if \( x \in I^\perp \), \( Q < h(x) \), so that \( x \in I' \); i.e., \( I^\perp = L^1 \cap L^2 \)-closure of \( j(Q) \). Hence
\[ h(I \oplus I^\perp) = h(I) \cap h(I^\perp) = P \cap Q = \emptyset, \]
so that \( I \oplus I^\perp = L^1 \cap L^2. \)

**Theorem 7.4.** If \( I \in \mathcal{J}, \) then \( k(h(I)) \subset J \cap L^1. \)

**Proof.** By Theorem 7.2, there exists \( x \in I \) such that \( h(I) = h(x) [m], \) and by Corollary 1 of the same theorem \( H(x) = J. \) Let \( y \in k(h(I)), \) so that \( h(I) \subset h(y). \) Then \( h(x) \subset h(y) [m], \) and by Theorem 4.1, \( y \in H(x) = J. \) Since \( y \in L^1 \cap L^2, \) it follows that \( y \in J \cap L^1. \) Hence \( k(h(I)) \subset J \cap L^1. \)

**Corollary 1.** If \( I \in \mathcal{J}, \) then \( J \cap L^1 = k(h(J \cap L^1)). \)

**Corollary 2.** If \( N \) is a closed proper invariant subspace of \( L^2(G), \) then \( N \cap L^1 = k(h(N \cap L^1)). \) In particular, if \( I, I^\perp \in \mathcal{J}, \) then \( I^\perp = k(h(I^\perp)). \)

**Proof.** We assume that \( N \cap L^1 \neq 0 \) without loss of generality. The above Corollary 1 implies that \( N \cap L^1 = k(h(L^1 \cap \text{cl}(N \cap L^1))). \) The observation that \( N \cap L^1 = L^1 \cap \text{cl}(N \cap L^1) \) completes the proof.

**Theorem 7.5.** Let \( I, I^\perp \in \mathcal{J}. \) If \( I \oplus I^\perp = k(h(I \oplus I^\perp)), \) then \( I = k(h(I)). \)

**Proof.** By Theorem 5.7, \( h(I \oplus I^\perp) = h(I) \cap h(I^\perp). \) Hence \( x \in k(h(I)) \) implies that \( x \in k(h(I \oplus I^\perp)), \) and \( x \in k(h(I^\perp)) \) implies that \( x \in k(h(I \oplus I^\perp)) \) so that \( k(h(I)) \cup k(h(I^\perp)) = k(h(I \oplus I^\perp)). \) By the above Corollary 2, \( I^\perp = k(h(I^\perp)), \) so that \( k(h(I)) \cup I^\perp = k(h(I \oplus I^\perp)). \) However, by Theorem 7.4, \( k(h(I)) \subset J \cap L^1, \) so it follows that \( k(h(I)) \cap I^\perp = \{0\}, \) and thus \( k(h(I)) \oplus I^\perp = k(h(I \oplus I^\perp)). \) Hence \( I \oplus I^\perp = k(h(I)) \oplus I^\perp \subset I \oplus I^\perp, \) i.e., \( I \oplus I^\perp = k(h(I)) \oplus I^\perp. \) Since both members of the last equation are direct sums, it follows that \( I = k(h(I)). \)

**Theorem 7.6.** If \( I \in \mathcal{J}, \) then \( h(I)^0 = h(J \cap L^1)^0. \)

**Proof.** Since \( I \subset J \cap L^1, \) and \( \text{cl} I = J, \) it follows that \( \text{cl} I = \text{cl} (J \cap L^1), \) so that by Corollary 3 of Theorem 7.2, \( h(I) = h(J \cap L^1) [m]. \) Let \( A = h(I)^0 - h(J \cap L^1), \) so that \( A \) is open, and thus either \( A = \emptyset \) or \( m(A) > 0. \) But \( A \subset h(I) - h(J \cap L^1) \) so that \( m(A) = 0. \) Hence \( A = \emptyset, \) and therefore \( h(I)^0 \subset h(J \cap L^1). \) Thus \( h(I)^0 \subset h(J \cap L^1)^0. \) However \( h(J \cap L^1) = h(I), \) so that \( h(J \cap L^1)^0 \subset h(I)^0, \) and therefore \( h(I)^0 = h(J \cap L^1)^0. \)

**Theorem 7.7.** Let \( I \in \mathcal{J} \) and let \( F \) be a closed subset of \( h(J \cap L^1). \) If \( h(J \cap L^1) = F [m], \) then \( h(J \cap L^1) = F. \)

**Proof.** As before, \( h(I) = h(J \cap L^1) [m]. \) Hence, by assumption, \( F = h(I) [m]. \) Therefore if \( x \in k(F), \) \( h(I) \subset h(x) [m], \) so that, by Theorem 7.2, Corollary 1, \( x \in J \cap L^1; \) i.e., \( k(F) \subset J \cap L^1. \) But if \( F \) is closed, then \( F = h(k(F)) \supset h(J \cap L^1) \supset F. \) Therefore \( F = h(J \cap L^1). \)

**Corollary.** If \( G \) is not discrete, and if \( I \in \mathcal{J}, \) then \( h(J \cap L^1) \) is perfect.
Proof. Since $h(J \cap L^1)$ is closed, it can be written $h(J \cap L^1) = S \cup P$ where $S$ is scattered, $P$ is perfect (Sierpiński [16, Chapter 1]). By a theorem of Rudin [12, Theorem 5, p. 41] it follows that $m(S) = 0$ so that $m(h(J \cap L^1) - P) = m(S) = 0$. Hence $P = h(J \cap L^1)$.

With suitable modifications, many of the results of §§4–7 that depend upon the restrictive assumption that $G$ is metric can be extended to the more general case where the metric hypothesis is removed. A consequence of this is that a significant generalization of a theorem due to I.E. Segal will be presented in a forthcoming paper.


BIBLIOGRAPHY


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