CLOSED IDEALS IN THE GROUP ALGEBRA
$L^1(G) \cap L^2(G)$

BY
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0. Introduction. In the following, $G$ will denote a locally compact abelian topological group with character group $\hat{G}$. For $1 \leq p < \infty$, $L^p(G)$ is the Banach space of all complex-valued functions whose $p$th powers are Haar integrable over $G$. ($L^p(G)$ is often written $L^p$ when the group $G$ is obvious from the context.) The linear space $L^1(G) \cap L^2(G)$ (denoted $L^1 \cap L^2$) is normed in such a way that, under convolution as multiplication, it is a commutative Banach algebra ($\S$2). It is also proved in $\S$2 that it is regular, semi-simple and that its regular maximal ideal space is $\hat{G}$. It is shown ($\S$3) that the abstract Šilov theorem [8, p. 86] holds for $L^1 \cap L^2$.

The standard proof of this theorem in $L^1(G)$ seems to depend upon the uniform boundedness of the approximate identity. A novel aspect of the $L^1 \cap L^2$ case is that a similar proof is obtained despite the fact that every approximate identity in $L^1 \cap L^2$ is unbounded.

An important but unsolved problem of harmonic analysis is the classification of the closed ideals in $L^1(G)$. Using the additional structure supplied by $L^1 \cap L^2$ it is to be expected that more precise results can be obtained about the closed ideals in $L^1 \cap L^2$. If $G$ and $\hat{G}$ are both locally compact metric abelian groups, examples of the more precise results that can be obtained are: (a) If $I$ is a closed proper ideal in $L^1 \cap L^2$, then there exists an $x \in I$ such that the hull of $x$ and the hull of $I$ coincide except for a set of measure zero (Theorem 7.2). (b) For every closed invariant proper subspace $N \subseteq L^2(G)$, $N \cap L^1 = k(h(N \cap L^1))$ (Corollary 2 of Theorem 7.4). This permits a new characterization of the kernel of $E$ for a class of perfect sets $E \subset \hat{G}$. (A. Denjoy terms these sets "épais en lui-même" in *Leçons sur le calcul des coefficients d’une série trigonométrique*, Paris, 1941, 2ième Partie, p. 100.) (c) The set $\mathcal{F}$ of all closed proper ideals in $L^1 \cap L^2$ which are

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not dense in $L^2$ is characterized as the set of all closed proper ideals $I$ such that the hull of $I$ has positive measure.

In §6 it is shown (still under the assumption that $G$ is locally compact metric abelian) that if $E \subseteq \hat{G}$ is a closed set, then $E$ is the hull of a principal ideal in $L^1 \cap L^2$ iff $E$ is a $G_\delta$. The theorem holds under rather more general circumstances; in particular, it holds in $L^1(G)$. It follows (from 6.2 and 6.3) that if $G$ and $\hat{G}$ are both locally compact metric abelian, then a hull $E$ for which spectral synthesis holds (if $I$ is any closed ideal having hull $E$, then $I$ is the kernel of $E$), must be a closed $G_\delta$ set. Consequently, the only instances of Helson's Theorem(3) [4] are given by principal ideals.

1. Preliminaries and notation. The following two theorems are useful in the sequel:

**Theorem 1.1.** If $G$ is a locally compact group, then $G$ is normal, and the family of compact neighborhoods of the identity is a basis for the neighborhood system of $G$ at the identity (Kelley [7, 5.32 and 5.17]).

**Theorem 1.2.** If $G$ is a locally compact abelian group whose character group is $\hat{G}$, then the following are equivalent:

(a) $G$ is metrizable;

(b) The neighborhood system for the identity $e \in G$ has a countable basis;

(c) $\hat{G}$ is $\sigma$-compact.

**Proof.** That (a) and (b) are equivalent is proved in Kelley [7, p. 186]. That (b) and (c) are equivalent is proved in Hewitt and Ross [5, p. 397] (actually a more general result is proved).

The notations and definitions in this work are, in general, those of Loomis [8]. In particular, $L(E)$ will denote the set of all continuous functions having compact support in $E$. Also, if $A$ is any set of functions, then $A^+$ will denote the set of non-negative functions in $A$.

2. The Banach algebra $L^1 \cap L^2$. Let $L^1 \cap L^2$ denote the linear space $L^1(G) \cap L^2(G)$ and observe that the function defined by the equation $\|x\| = \|x\|_1 + \|x\|_2$ for each $x \in L^1 \cap L^2$ is a norm. If multiplication is defined by convolution, it follows that $L^1 \cap L^2$ is a commutative Banach algebra. The conjugate space of $L^1 \cap L^2$ is also obtained in this section. The ideal, $S$, of $L^1 \cap L^2$-functions whose Fourier transforms have compact support is shown to be dense in $L^1 \cap L^2$ and the regular maximal ideal space of $L^1 \cap L^2$ is found to be $\hat{G}$.

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(2) What is required is that $A$ should be a commutative regular semi-simple Banach algebra whose regular maximal ideal space is $\sigma$-compact.

(3) If $I$ is a closed proper ideal in $L^1(G)$ such that the boundary of the hull of $I$ contains no nonempty perfect subset, then $I = k(h(I))$. 

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Definitions 2.1. Let μ denote Haar measure on G. The set $E \subset G$ is locally null if for every compact set $C \subset G$, $μ(E \cap C) = 0$. If $x, y$ are $μ$-measurable functions defined on $G$ such that the set $\{s \in G \mid x(s) \neq y(s)\}$ is locally null, then $x = y$ i.a.e. (locally almost everywhere). Let $LL^\infty$ denote the equivalence classes of functions which are i.a.e. equal.

Theorem 2.2. The linear space $L^1 \cap L^2$ is a Banach space.

Proof. We have only to show that $L^1 \cap L^2$ is complete. Suppose $\{x_n\} \subset L^1 \cap L^2$ is an $L^1 \cap L^2$-Cauchy sequence, so that there exist $x \in L^1$ and $y \in L^2$ for which $\| x_n - x \|_1 \to 0$ and $\| x_n - y \|_2 \to 0$. Hence there exists a subsequence $\{y_n\} \subset \{x_n\}$ such that $y_n \to x$ a.e., and a subsequence of it, $\{z_n\}$, such that $z_n \to y$ a.e. Thus $z_n \to x$ a.e., so that $x = y$ a.e.

Lemma 2.3. Let $T$ be a linear functional on $L^1 \cap L^2$ defined by the equation $T(x) = \int x(t) \cdot y(t) + z(t) \, dt$ for each $x \in L^1 \cap L^2$, where $y \in L^2$ and $z \in L^\infty$. Then $T$ is bounded, and $\| T \| \leq \max(\| y \|_2, \| z \|_\infty)$.

Proof. Let $M = \max(\| y \|_2, \| z \|_\infty)$. Then, by Hölder’s Inequality, $\| T(x) \| \leq \| x \|_2 \| y \|_2 + \| x \|_1 \| z \|_\infty \leq \| x \| M$. Hence $\| T \| = \sup_{\| x \| = 1} | T(x) | \leq M$.

Theorem 2.4. The conjugate space of $L^1 \cap L^2$ is $(LL^\infty \times L^2)/Z$, where $Z = \{(g, h) \in LL^\infty \times L^2 \mid g + h = 0 \text{ i.a.e.}\}$.

Proof. Let $\rho$ be defined on $L^1 \times L^2$ by the equation $\rho(x, y) = \| x \|_1 + \| y \|_2$, and $r$ be defined on $LL^\infty \times L^2$ by the equation $r(g, h) = \max(\| g \|_\infty, \| h \|_2)$. If $z \in L^p(G)$ and $w \in L^q(G)$, where $1 \leq p \leq \infty$, and $1/p + 1/q = 1$, let $\langle z, w \rangle = \int z(s) \cdot w(s) \, ds$. If $L^1 \times L^2$ is equipped with the norm $\rho$, it becomes a Banach space whose conjugate is the Banach space $LL^\infty \times L^2$ equipped with the norm $r$ (Schatten [13]). Identify $L^1 \cap L^2$ with $\Delta = \{(x, x) \mid x \in L^1 \cap L^2\}$, which is a closed linear subspace of $L^1 \times L^2$. Let $f \in (L^1 \cap L^2)^*$ (= Banach space conjugate of $L^1 \cap L^2$), and let $\phi$ be defined on $\Delta$ by the equation $\phi(x, x) = f(x)$. By the Hahn-Banach theorem $\phi$ may be extended without change of norm from the closed linear subspace $\Delta$ to all of $L^1 \times L^2$, i.e. $\phi$ may be extended to a bounded linear functional $F \in LL^\infty \times L^2$. Let $F = (F_1, F_2)$, and observe that, if $(x, y) \in L^1 \times L^2$, $F(x, y) = \langle x, F_1 \rangle + \langle y, F_2 \rangle$. If $(x, x) \in \Delta$ it follows that $F(x, x) = \langle x, F_1 \rangle + \langle x, F_2 \rangle = \langle x, F_1 + F_2 \rangle$. It is clear that if $F_1 + F_2 = 0$ i.a.e., $F \equiv 0$ on $\Delta$, hence it follows that we must identify elements of $\Delta^*$ (= Banach space conjugate of $\Delta$) differing by an element of $Z$, where $Z = \{(g, h) \in LL^\infty \times L^2 \mid g + h = 0 \text{ i.a.e.}\}$. Let $\{(g_n, h_n)\} \subset Z$, and suppose that $(g_n, h_n) \to (g, h)$. Then $\| g_n - g \|_\infty \to 0$ and $\| h_n - h \|_2 \to 0$, and $g_n + h_n = 0$ i.a.e. for $n = 1, 2, 3, \cdots$; but, since $\| h_n - h \|_2 \to 0$, there exists a subsequence $\{(p_n, k_n)\} \subset \{(g_n, h_n)\}$ such that $\| k_n - h \|_\infty \to 0$, and for each $n = 1, 2, 3, \cdots$, $p_n + k_n = 0$ i.a.e. Therefore,

$$\| g + h \|_\infty = \| p_n + k_n - g - h \|_\infty \leq \| p_n - g \|_\infty + \| k_n - h \|_\infty \to 0.$$
Hence $g + h = 0$ I.a.e.; i.e., $(g, h) \in Z$, so that $Z$ is closed. Therefore $(L^\infty \times L^2) / Z$ is a Banach space, and $\phi \in (L^\infty \times L^2) / Z$; i.e., $(L^1 \cap L^2)^* \subset (L^\infty \times L^2) / Z$. Now, let $H \in (L^\infty \times L^2) / Z$. We can find $F \in L^\infty \times L^2$ such that $F$ belongs to the coset $H$ of $L^\infty \times L^2$, and for this $F$, let $f(x) = F(x, x)$. Then $f(x) = \langle x, F_1 \rangle + \langle x, F_2 \rangle$, so that

$$|f(x)| \leq \|x\|_1 \|F_1\|_\infty + \|x\|_2 \|F_2\|_2 \leq \|x\| \|F\|;$$

i.e., $f \in (L^1 \cap L^2)^*$. Therefore $(L^\infty \times L^2) / Z = (L^1 \cap L^2)^*$.

**Lemma 2.5.** Let $x, y \in L^1 \cap L^2$. Then $\|x \ast y\| \leq \min(\|x\|_1 \|y\|_1, \|x\| \|y\|_1)$.

**Corollary 1.** If $x \neq 0$ and $y \neq 0$, $\|x \ast y\| < \|x\| \|y\|_1$.

**Corollary 2.** $L^1 \cap L^2$ is a Commutative Banach Algebra.

**Definition 2.6.** Let $A$ be a Banach algebra and let $P$ be a directed set. Then the net $\{v_p \in A \mid p \in P\}$ is an approximate identity for $A$ if $\lim_{p} v_p x = x$ for each $x \in A$.

Note that it is not required that $\{\|v_p\|\}$ should be bounded. In fact this cannot be required, in general: $L^1 \cap L'$ with norm $\|\cdot\|_1 + \|\cdot\|_r$ is a Banach algebra if its multiplication is convolution. If $G$ is neither compact nor discrete (of course $G$ is still assumed to be locally compact abelian), $L^1 \cap L'$ has an approximate identity in the above sense, but it can be shown(*) that any approximate identity in the above sense must be unbounded if $1 < r \leq 2$.

Let $\mathcal{V}$ denote the family of all precompact (closure is compact) neighborhoods of $e$, the identity of the group $G$. Partially order $\mathcal{V}$ by inclusion and designate it by $\{V_p\}$. Then $\{V_p\}$ is a directed set, and we may define a net $\{v_p\}$ of functions on it, by choosing, for each $V_p$, $v_p \in L^+ (V_p)$ such that $\int v_p(s) ds = 1$.

**Theorem 2.7.** The net $\{v_p\}$ defined above is an approximate identity for $L^1 \cap L^2$ (cf. Loomis [8, p. 124]).

**Theorem 2.8.** Let $\Sigma$ be a closed subset of $L^1 \cap L^2$. Then $\Sigma$ is an ideal iff it is a translation-invariant subspace of $L^1 \cap L^2$ (cf. Loomis [8, p. 125]).

**Lemma 2.9.** Let $v \in L^1 +$, $\int v(t) dt = 1$ and $\varepsilon > 0$ be given. Then there exists $q \in (L^1 \cap L^2)^+$ such that $q \in L(G)$, $\int q(t) dt = 1$, and $\|q - v\|_1 < \varepsilon$ (Edwards [2, pp. 165-166]).

**Theorem 2.10.** The ideal $S = \{x \in L^1 \cap L^2 \mid x \in L(G)\}$ is dense in $L^1 \cap L^2$.

**Proof.** Let $x \in L^1 \cap L^2$ and $\varepsilon > 0$ be given. Assume that $x \neq 0$, since $0 \in S$. Choose $v$ from an approximate identity so that $v \in L^1 +$, and $\|x \ast v - x\| < \varepsilon / 2$. Then by Lemma 2.9, choose $q \in S$ so that $\|q - v\|_1 < \varepsilon / 2\|x\|$. Hence

(*) By application of the Hausdorff-Young Inequality.
\[ \| q * x - x \| \leq \| x * (q - v) \| + \| x * v - x \| < \varepsilon. \] Thus \( \| q * x - x \| \leq \varepsilon \) and \( (q * x)^\ast = \hat{\delta} \hat{x} \in L(\hat{G}). \)

**Theorem 2.11.** Let \( K \subset \hat{G} \) be any compact set containing \( \hat{e} \), and let \( U \) be an open neighborhood of \( K \). Then there exists a function \( x \in L^1 \cap L^2 \) such that \( \hat{x} \equiv 1 \) on \( K \), \( \hat{x} \equiv 0 \) off \( U \), and \( 0 \leq \hat{x} \leq 1 \).

**Proof.** Let \( V \) be a symmetric compact neighborhood of \( \hat{e} \) sufficiently small that \( V V K \subset U \). Set \( \Sigma = V K \). Then \( \Sigma \) is compact. Let \( y \), \( z \) be the characteristic functions of \( \Sigma \), \( V \) respectively. Since \( V \), \( \Sigma \) are both compact, each has finite measure, so that \( y \), \( z \in (L^1 \cap L^2)(\hat{G}) \) and therefore \( y * z \in (L^1 \cap L^2)(\hat{G}) \).

Let \( \bar{y} \), \( \bar{z} \), \( (y * z)^\ast \) be the inverse Fourier transforms of \( y \), \( z \) and \( y * z \) respectively. Then \( (y * z)^\ast = \bar{y} \cdot \bar{z} \). Let \( u = (y * z)^\ast = \bar{y} \cdot \bar{z} \). Then \( u \in L^2(G) \), since \( y * z \in L^2(\hat{G}) \).

Also \( u \in L^1(G) \) since \( \bar{y}, \bar{z} \in L^2(G) \). Thus \( u \in L^1 \cap L^2(G) \), and \( \hat{u}(\alpha) = (y * z)(\alpha) \) a.e., and since each of \( \hat{u} \) and \( y * z \) is continuous, \( \hat{u} = y * z \).

Let \( x = u / m(V) \). This \( x \) is the desired function.

**Remark.** By translation, it follows that if \( K \) is a compact subset of \( \hat{G} \) and if \( U \) is any open neighborhood of \( K \), there exists a function \( x \in L^1 \cap L^2 \) such that \( \hat{x} \equiv 1 \) on \( K \), \( \hat{x} \equiv 0 \) off \( U \), and \( 0 \leq \hat{x} \leq 1 \).

**Notation 2.12.** Let \( A \) be a commutative Banach algebra, and let \( \Delta(A) \) denote the set of all continuous homomorphisms of \( A \) onto the complex numbers. If \( A^* \) is the conjugate space of \( A \), then \( \Delta(A) \) is a subset of \( A^* \), and \( \Delta(A) \) is locally compact in the weak*-topology of \( A^* \).

Let \( \mathcal{M} \) be the set of all regular maximal ideals of \( A \). Then \( \Delta(A) \) may be identified with \( \mathcal{M} \) by associating with \( \phi \in \Delta(A) \) the corresponding regular maximal ideal \( M_\phi \in \mathcal{M} \): \( M_\phi \equiv \phi^{-1}(0) \). If \( I \) is an ideal of \( A \), the hull \( h(I) \) of \( I \) is the set of all regular maximal ideals containing \( I \). If \( S \subset \mathcal{M} \), the kernel \( k(S) \) of \( S \) is the ideal which is the intersection of all the regular maximal ideals \( M \in S \). If \( S \subset \mathcal{M} \) is defined as \( h(k(S)) \), the hull-kernel topology is induced on \( \mathcal{M} \). If this topology coincides with the weak*-topology on \( \mathcal{M} \), \( A \) is said to be regular.

Let \( A \) be the algebra \( L^1 \cap L^2 \) and let \( M \in \mathcal{M} \). Then \( M \) is \( \phi^{-1}(0) \) for some homomorphism \( \phi \in \Delta(L^1 \cap L^2) \). Denote \( \phi(x) \) by \( x'(M) \) for each \( x \in L^1 \cap L^2 \).

Thus \( x' \) is a function defined on \( \mathcal{M} \). Fix \( M \in \mathcal{M} \), and let \( \alpha_M(s) = x'_s(M) / x'(M) \), where \( x \in L^1 \cap L^2 \) is so chosen that \( x'(M) \neq 0 \) (i.e. \( x \notin M \)).

**Theorem 2.13.**

(i) \( \alpha_M \) is a character of \( G \),

(ii) \( \alpha_M \) is continuous on \( G \times \mathcal{M} \),

(iii) If \( u \) runs through an approximate identity, \( u'_s(M) \) converges uniformly to \( \alpha_M(s) \).

The proof is the same as that in Loomis [8, pp. 135–136].

**Theorem 2.14.** The mapping \( M \to \alpha_M \) is a one-to-one mapping of \( \mathcal{M} \) onto the set of all characters of \( G \), and \( x'_s(M) = \int x(s) \cdot \alpha_M(s) \, ds \) (cf. Loomis [8, p. 136]).
Theorem 2.15. The topology of $\hat{G}$ (the weak*-topology of $(L^1 \cap L^2)^*$ induced on $\hat{G}$) is the usual topology of $\hat{G}$ as the dual of $G$.

Proof. Let $F$ be the set of Fourier transforms of all functions in $L^1 \cap L^2$. Then $F \subset C_0(\hat{G})$, the set of continuous functions vanishing at infinity on $\hat{G}$. Suppose $\alpha, \beta \in \hat{G}$ and $\alpha \neq \beta$. Since $\hat{G}$ is normal, there exist open disjoint neighborhoods $U(\alpha), V(\beta)$, and disjoint compact neighborhoods $K(\alpha) \subset U$ and $C(\beta) \subset V$.

By Theorem 2.11 there exist $x, y \in L^1 \cap L^2$ such that $x = 1$ on $U$, $x = 0$ on $C$ and $y = 1$ on $C$, $y = 0$ on $K$. Therefore, for each $\alpha \in \hat{G}$ there exists $\hat{x} \in F$ such that $\hat{x}(\alpha) = 1$, and also, $F$ separates the points of $\hat{G}$. By a theorem (5G) of Loomis [8], it follows that the weak topology induced on $\hat{G}$ by $F$ is precisely the one in which the functions of $F$ are continuous; i.e., it is the usual topology for $\hat{G}$ as the dual of $G$.

Theorem 2.16. $L^1 \cap L^2$ is semi-simple and regular.

Proof. We have established that if $x \in L^1 \cap L^2$, $x' = \hat{x}$. It follows that if $x' \equiv 0$, $x \equiv 0$ a.e.; i.e., $L^1 \cap L^2$ is semi-simple. By a theorem in Loomis [8, p. 57] to prove that $L^1 \cap L^2$ is regular we have only to prove that if $F \subset \mathfrak{M}$ is closed in the hull-kernel topology, and $x \notin F$, then there exists $x \in L^1 \cap L^2$ such that $\hat{x} \equiv 0$ on $F$ and $\hat{x}(x) \neq 0$. Let $U = \hat{G} - F$ so that $U$ is open and $x \in U$. Choose a compact neighborhood $K$ of $x$ such that $K \subset U$. Apply Theorem 2.11 to obtain $x \in L^1 \cap L^2$ such that $\hat{x} \equiv 0$ off $U$, $\hat{x} \equiv 1$ on $K$ so that $x \equiv 0$ on $F$, and $\hat{x}(x) = 1 \neq 0$.

3. Šilov’s Theorem for $L^1 \cap L^2$. Let $A$ be a commutative Banach algebra. Then $A$ satisfies the condition $D$ if, given $x \in M \in \mathfrak{M}$ there exists a sequence $\{x_n\} \subset A$ such that $\hat{x}_n \equiv 0$ in a neighborhood $V_n$ of $M$ for $n = 1, 2, 3, \ldots$, and $\|xx_n - x\| \to 0$. If $\mathfrak{M}$ is not compact the condition must also be satisfied for the point at infinity; i.e., for each point $x \in A$, there exists a sequence $\{x_n\} \subset A$ such that $\{\hat{x}_n\} \subset L(\mathfrak{M})$, and $\lim_{n \to \infty} \|xx_n - x\| = 0$.

ŠiLOV’S THEOREM 3.1 (LOOMIS [8, p. 86]). Let $A$ be a regular semisimple commutative Banach algebra satisfying condition $D$, and let $I$ be a closed ideal in $A$. Then $I$ contains every element $x \in k(h(I))$ such that $[bd h(x)] \cap h(I)$ includes no nonempty perfect set; i.e., is scattered (a closed scattered set is one which contains no nonempty perfect subset).

Since we have already established that $L^1 \cap L^2$ is regular, commutative, and semi-simple, we have only to show that $L^1 \cap L^2$ satisfies the condition $D$. We shall first prove that $L^1 \cap L^2$ satisfies the condition $D$ at infinity. After that, the remaining part of this section will be devoted to showing that $L^1 \cap L^2$ satisfies the condition $D$ at finite points. The proof given in Loomis [8, p. 151] that $L^1(G)$ satisfies Ditkin’s Condition at finite points appears to depend upon the uniform boundedness of the approximate identity. Since this boundedness is never available.
in the $L^1 \cap L^2$ case (cf. 2.6), it is somewhat surprising that in spite of this lack a proof similar to the $L^1(G)$ case can be constructed.

**Lemma 3.2.** $L^1 \cap L^2$ satisfies the condition $D$ at infinity.

**Proof.** Assume that $\hat{G}$ is not compact. Let $x \in L^1 \cap L^2$, and $\varepsilon > 0$ be given. Use the construction of Theorem 2.10 to obtain $q \in S$ (so that $\hat{q} \in L(\hat{G})$) such that $\| q \ast x - x \| < \varepsilon$. Clearly then, there exists a sequence $\{x_n\} \subset S$ for which $\lim_{n} x_n = x$ in $L^1 \cap L^2$. Since this can be done for every $x \in L^1 \cap L^2$, $L^1 \cap L^2$ satisfies the condition $D$ at infinity. Let $\mathcal{U} = \{U_x\}_{x \in \Lambda}$ denote the family of all symmetric Baire neighborhoods of $\hat{e}$ of measure less than or equal to one. Then $\mathcal{U}$ is a directed system under inclusion. Let $\{V_x\}_{x \in \Lambda}$ denote any net of symmetric Baire neighborhoods of $\hat{e}$ defined on $\mathcal{U}$, and satisfying the following conditions:

(i) If $V \in \{V_x\}$, $V$ is compact;
(ii) Given $U \in \mathcal{U}$, $V \subset U$ and $m(U) < 4m(V)$ (where $m$ is Haar measure on $G$);
(iii) Given $U \in \mathcal{U}$ and $V$, there exists a neighborhood $W(\hat{e})$ depending on $U$ and $V$, such that $V \subset W \subset U$.

**Lemma 3.3.** There exists a net $\{z_x\} \subset L^1 \cap L^2$ defined on $\mathcal{U}$ such that for every $\lambda \in \Lambda$:

(i) $\| z_x \| < 3$, and
(ii) $z_x \equiv 1$ on some neighborhood of $\hat{e}$.

**Proof.** Given $U$, let $V$ be the corresponding set in the net of sets defined above. Let $\hat{u}_x$, $\hat{v}_x$ be the characteristic functions of $U$, $V$ respectively, and let $z_x = \hat{u}_x \ast \hat{v}_x / m(V)$. Since $\hat{u}_x$, $\hat{v}_x$ and $z_x$ all belong to $L^1 \cap L^2(\hat{G})$, the inverse Fourier-Plancherel transform of each exists. These may be designated as $u_x$, $v_x$ and $z_x$, respectively.

**Proof of (i).** $\| z_x \|_2 = \| z_x \|_2 = \| [1/m(V)] \cdot \hat{u}_x \ast \hat{v}_x \|_2 \leq \| [1/m(V)] \cdot \hat{u}_x \|_2 \cdot \| \hat{v}_x \|_1 \leq 1$. Thus $\| z_x \|_2 \leq 1$. Similarly $\| z_x \|_1 = \| [1/m(V)] \cdot u_x v_x \|_1 < 2$. Hence $\| z_x \| < 3$.

**Proof of (ii).** Corresponding to $U$ and $V$ there exists a neighborhood $W = W(\hat{e})$ such that $V \subset W \subset U$. Let $\beta \in W$. Then

$$z_x(\beta) = [1/m(V)] \cdot (\hat{u}_x \ast \hat{v}_x)(\beta) = [1/m(V)] \int_{V} \hat{v}_x(\alpha \beta) \, d\alpha = 1.$$ 

Hence $z_x \equiv 1$ on $W(\hat{e})$.

Let $C \subset G$ be a compact subset, and let $\varepsilon > 0$ be given. Then $U(C, \varepsilon / 5, \hat{e})$ is open in $\hat{G}$, where $U(C, \varepsilon / 5, \hat{e}) = \{\alpha \in \hat{G} \mid 1 - (s, \alpha) < \varepsilon / 5, \text{ all } s \in C\}$. Since $\mathcal{U}$ contains a basis for the topology of $\hat{G}$ at $\hat{e}$, there exists a $\lambda_0$ such that if $\lambda < \lambda_0$, $U^2 \subset U(C, \varepsilon / 5, \hat{e})$. For convenience of notation, let $S_{\lambda}$ denote the net $\{z_x\}$ constructed in Lemma 3.3, and let $S_{\lambda_0} = \{z_x \mid \lambda > \lambda_0\}$.
**Lemma 3.4.** Given \( \varepsilon > 0 \), there exists \( \lambda_0 \) such that if \( z \in S_{\lambda_0} \), then

\[
\| z - z_s \| < \varepsilon
\]

for every \( s \in C \).

**Proof.** Choose an appropriate \( \lambda_0 \) as above so that if \( \lambda > \lambda_0 \), \( U^2_\lambda \subset U(C, \varepsilon/5, \varepsilon) \). Let \( z \in S_{\lambda_0} \) and suppose that \( z = uv / m(V) \). Let \( s \in C \), and note that \( \varepsilon \equiv 0 \) off \( UV \).

Then \( \| z - z_s \|_2^2 = \| \varepsilon - \varepsilon_s \|_2^2 = \int_U |\varepsilon(x)|^2 \cdot |1 - (s, x)|^2 \, dx < (\varepsilon/5)^2 \). Hence \( \| z - z_s \|_2 < \varepsilon/5 \). Similarly \( \| u - u_s \|_2 < \| m(U) \|^{1/2} (\varepsilon/5) \), and

\[
\| v - v_s \|_2 < \| m(V) \|^{1/2} (\varepsilon/5) .
\]

We observe that

\[
\| z - z_s \| \leq \frac{1}{m(V)} \left[ \| u(v - v_s) \|_1 + \| v_s(u - u_s) \|_1 \right] < 2(\varepsilon/5) \left[ \frac{m(U)}{m(V)} \right]^{1/2} < 4\varepsilon/5,
\]

so that \( \| z - z_s \| < \varepsilon \).

**Corollary.** If \( x \in L^1 \cap L^2 \), and \( \hat{x}(\varepsilon) = 0 \), then \( \lim_{\varepsilon} \| x * z_x \| = 0 \).

**Proof.** Let \( \delta > 0 \) be given, and choose \( C \subset G \) to be compact, symmetric and such that \( \int_{G-C} |x(s)| \, ds < \delta/12 \). Set \( \varepsilon = \delta/2 \| x \|_1 \), and choose \( \lambda_0 \) as before so that if \( z \in S_{\lambda_0} \), then \( \| z - z_s \| < \varepsilon \) for every \( s \in C \). Hence

\[
(x * z)(t) = \int x(s) [z(ts^{-1}) - z(t)] \, ds.
\]

We observe that \( \| z * x \|_1 = \sup_{\| h \|_\infty = 1} |\langle z * x, h \rangle| \), and that

\[
\| z * x \|_2 = \sup_{\| p \|_1 = 1} |\langle z * x, p \rangle| .
\]

Thus, by a straightforward computation,

\[
\| x * z \| \leq \int_C |x(s)| \left\| z_{s-1} - z \right\| \, ds + \int_{G-C} |x(s)| \left\| z_{s-1} - z \right\| \, ds.
\]

If \( z \in S_{\lambda_0} \), then \( s \in C \) implies that \( \| z_{s-1} - z \| < \varepsilon \). In this case the inequality \((*)\) becomes \( \| x * z \| < \varepsilon \| x \|_1 + 2 \| z \| \delta/12 < \delta \), so that \( \lim_{\varepsilon} \| x * z \| = 0 \).

**Theorem 3.5.** There exists a net \( \{ v_q \} \subset L^1 \cap L^2 \) such that each \( \delta_q = 0 \) in a neighborhood of \( \varepsilon \) (depending on \( v_q \)) and such that if \( x \in L^1 \cap L^2 \), and \( \hat{x}(\varepsilon) = 0 \), then \( \lim_q \| x * v_q - x \| = 0 \).

**Proof.** Let \( \{ p_\lambda \} \) be the approximate identity defined in Theorem 2.7 and let \( \{ z_\lambda \} \) denote the net defined in Lemma 3.3. Let \( v(p, \lambda) = u_p - z_\lambda u_p \). Clearly \( v(p, \lambda) \in L^1 \cap L^2 \). The set of all ordered pairs \( (p, \lambda) \) may be directed by: \( (p_1, \lambda_1) \)
> \langle p_2, \lambda_2 \rangle \text{ iff } p_1 > p_2 \text{ and } \lambda_1 > \lambda_2. \text{ If we allow } q \text{ to run through this directed set, } 
\{v_q\} \text{ is a net, and we note that } \tilde{v}(p, \lambda) = \tilde{u}_p(1 - \tilde{z}_\lambda) = 0 \text{ in the neighborhood of } \tilde{e} \text{ where } \tilde{z}_\lambda = 1. \text{ Finally } \|v(p, \lambda) * x - x\| \leq \lim_p\|u_p * x - x\| + \lim\|u_p\|_1 \|x * z_\lambda\|. \text{ Hence } \lim_q\|v_q * x - x\| \leq \lim_p\|u_p * x - x\| + \lim\|u_p\|_1 \|x * z_\lambda\| = 0.

Corollary 1. \( L^1 \cap L^2 \) satisfies condition D.

Proof. In the above theorem we have just established that \( L^1 \cap L^2 \) satisfies condition D at \( \tilde{e} \). The condition D follows for all other finite points upon translation. It was established for the point at infinity in Lemma 3.2.

Corollary 2. Šilov’s theorem is valid for \( L^1 \cap L^2 \).

4. Translation-invariant subspaces of \( L^2(G) \). Let the notation \([m]\) following an assertion denote that the assertion is valid except for sets of zero \( m \)-measure (on \( \tilde{G} \)).

Example. \( E \subset F [m] \) means that \( m(E - F) = 0 \).

If \( x \in L^2(G) \), \( N(x) \) will denote the set of all finite linear combinations of translates of \( x \). \( H(x) \) will denote the \( L^2 \)-closure of \( N(x) \). The spaces \( L^1(G), L^2(G) \) will be written as \( L^1, L^2 \) respectively unless possible ambiguity prevents this. If \( N \) is any subspace of \( L^2 \), \( N \) is invariant if, for every \( s \in G \), \( x \in N \) implies that \( x_s \in N \). If \( x \in L^2 \), define \( h(x) = \{x \in \tilde{G} \mid \tilde{x}(x) = 0\} [m] \).

The result (4.1) of this section is taken from S. Bochner and K. Chandrasekharan [1, pp. 148–149], where it is established for the case of \( G = R \). Their proof carries over without change to the general case, so there is no need to reproduce it here.

Theorem 4.1. Let \( x, y \in L^2 \). Then \( x \in H(y) \) iff \( h(y) \subset h(x) [m] \).

In this section, let \( N \) denote an arbitrary closed proper \((\{0\} \neq N \text{ and } N \neq L^2)\) subspace of \( L^2 \), invariant under translation. If \( E \subset L^2 \), \( cl(E) \) will denote the \( L^2 \)-closure of the set \( E \).

Lemma 4.2. If \( x \in N \), \( m(h(x)) > 0 \).

Lemma 4.3. Let \( x, y \in N \). Then there exists \( z \in N \) for which

\[ h(z) = h(x) \cap h(y) [m] \].

Theorem 4.4. Let \( \{x_n\} \subset N \). Then there exists \( x_0 \in N \) such that

\[ h(x_0) = \bigcap_{n=1}^{\infty} h(x_n) [m] \].

Proof. Without loss of generality, assume that \( \|x_n\|_2 > 0 \) for \( n = 1, 2, 3, \ldots \), and let \( c_k = \{2^k \|x_k\|_2\}^{-1} \) for each \( k = 1, 2, 3, \ldots \). Let \( \tilde{p}_n = \sum_{k=1}^{\infty} c_k |\hat{x}_k| \), and let \( p_n \) be the inverse Fourier-Plancherel transform of \( \tilde{p}_n \). Then it is clear that \( \{p_n\} \subset N \), and \( h(p_n) = \bigcap_{k=1}^{\infty} h(x_k) [m] \) by Lemma 4.3. There exists \( x_0 \in N \) for which \( \lim_n\|p_n - x_0\|_2 = 0 \). Hence \( \bigcap_{n=1}^{\infty} h(x_n) = \bigcap_{n=1}^{\infty} h(p_n) \subset h(x_0) [m] \), and
consequently, we have only to prove that for each \( n \), \( h(x_0) \subset h(p_n) \). These remarks lead to a straightforward proof by contradiction.

**Theorem 4.5.** Let \( E \) be a measurable subset of \( \hat{G} \), and suppose that for some \( x' \in N \), \( m(E \cap h(x')) \) is finite. Then there exists a \( z \in N \) such that for every \( x \in E \cap h(z) \subset E \cap h(x) \).

**Proof.** Let \( c = \inf_{x \in E} m(E \cap h(x)) \). Choose a sequence \( \{x_n\} \subset N \) such that \( m(E \cap h(x_n)) < c + 1/n + 1 \equiv 0 \). Then, by Theorem 4.4, there exists a \( z \in N \) such that \( h(z) = \bigcap_{n=1}^\infty h(x_n) \). Hence \( m(E \cap h(z)) = c \). Let \( x \in E \). Then, by Lemma 4.3 there exists \( y \in N \) for which \( h(y) = h(x) \cap h(z) \). Thus \( m(E \cap h(x) \cap h(z)) = c \). Observe that \( h(x) \cap h(z) \) and \( h(z) - h(x) \) are disjoint and that \( h(z) = [h(x) \cap h(z)] \cup [h(z) - h(x)] \). Therefore \( c = m(E \cap h(z)) = c + m(E \cap [h(z) - h(x)]) \). Thus \( m([E \cap h(z)] - [E \cap h(x)]) = 0 \).

**Theorem 4.6.** Let \( G \) be metric, and let \( N \) be a closed proper invariant subspace of \( L^2(G) \). Then there exists a \( z \in N \) such that \( H(z) = N \).

**Proof.** Since \( \hat{G} \) is \( \sigma \)-compact, we may set \( \hat{G} = \bigcup_{n=1}^\infty K_n \), where each \( K_n \) is compact; thus \( 0 \leq m(K_n) < \infty \), for \( n = 1, 2, 3, \ldots \). Hence if \( x \in E \), \( m(K_n \cap h(x)) \) is finite, and we apply Theorem 4.5 to obtain a sequence \( \{z_n\} \subset N \) such that for every \( x \in N \), \( K_n \cap h(z_n) \subset K_n \cap h(x) \). By Theorem 4.4, there exists a \( z \in N \) such that \( h(z) = \bigcap_{n=1}^\infty h(z_n) \). Suppose \( z \in N \); then except for a null set,

\[
0 \leq m(E \cap h(z)) \subset m(E \cap h(z_n)) \subset m(K_n \cap h(z_n)) = m(E \cap h(x)).
\]

By Theorem 4.1, \( x \in H(z) \); i.e., \( N \subset H(z) \). Since \( z \in N \), \( H(z) \subset N \), so that \( N = H(z) \).

5. **The closed ideals** \( I \) **and** \( I^\perp \) **in** \( L^1 \cap L^2 \). If \( I \) is an ideal in \( L^1 \cap L^2 \), \( \operatorname{cl} I \) will be denoted by \( J \). The class of all closed proper ideals \( I \) in \( L^1 \cap L^2 \) for which \( J \neq L^2 \) will be denoted by \( \mathcal{J} \). \( I \) is symmetric if \( x \in I \) implies that \( x^\ast \in I \), where \( x^\ast(s) = x(s^{-1}) \) (\( s \in G \)). If \( N \) is a closed subspace of \( L^2(G) \), we shall denote the orthogonal complement of \( N \) by \( N^\perp \). Let \( I \in \mathcal{J} \), and let \( (x, y) = \int x(s) \overline{y(s)} \, ds \). Then, define \( I^\perp = \{x \in L^1 \cap L^2 \mid (x, y) = 0 \text{ for all } y \in I \} \).

**Remark 5.1.** If \( M \) is a regular maximal ideal of \( L^1 \cap L^2 \), then \( M \) is symmetric. Thus, if \( I \) is a closed ideal in \( L^1 \cap L^2 \) such that \( I = k(h(I)) \), then \( I \) is symmetric.

**Lemma 5.2.** Let \( N \) be a closed proper invariant subspace of \( L^2 \) such that \( N \cap L^1 \neq \{0\} \). Then \( N \cap L^2 \in \mathcal{J} \) and \( N \cap L^1 \) is symmetric.

**Theorem 5.3.** Let \( I \in \mathcal{J} \). Then \( I^\perp = J^\perp \cap L^1 \).

**Proof.** Let \( y \in J \). Then there exists a sequence \( \{y_n\} \subset I \) such that \( \|y_n - y\|_2 \to 0 \). If \( x \in I^\perp \), \( (x, y_n) = 0 \) \( n = 1, 2, 3, \ldots \), and since strong convergence implies weak convergence, it follows that \( (x, y) = 0 \). Thus if \( x \in I^\perp \), \( (x, y) = 0 \) for every \( y \in J \),
so that \( x \in J^\perp \cap L^1 \); i.e., \( I^\perp \subset J^\perp \cap L^1 \). Now let \( x \in J^\perp \cap L^1 \). Then if \( y \in I \), 
\((x, y) = 0 \) since \( y \in J \). Thus \( x \in I^\perp \).

**Lemma 5.4.** Let \( I \in \mathcal{I} \). Then \( J \) and \( J^\perp \) are closed proper invariant subspaces of \( L^2 \).

**Corollary.** If \( I \in \mathcal{I} \), \( I^\perp = \{0\} \), or \( I^\perp \in \mathcal{I} \). In either event, \( I^\perp \) is symmetric.

Let the ideal \( I \oplus I^\perp \) be defined as the direct sum of the two ideals \( I \) and \( I^\perp \). The notation \( I \oplus I^\perp \) will denote the \( L^1 \cap L^2 \)-closure of \( I \oplus I^\perp \).

**Theorem 5.5.** Let \( I_1 \) and \( I_2 \) be closed proper ideals of \( L^1 \cap L^2 \) such that \( I_1 \cap I_2 = \{0\} \). Then \( h(I_1) \cup h(I_2) = \hat{G} \).

**Proof.** Let \( E = h(I_1) \cup h(I_2) \), and suppose that \( E \neq \hat{G} \). Then there exists \( \alpha \in \hat{G} \) such that \( \alpha \notin E \), and there exist open disjoint neighborhoods \( V_1(\alpha) \) and \( V_2(E) \). The compact neighborhoods of \( \alpha \) are a basis for the topology of \( \hat{G} \) at \( \alpha \), so there exists an open neighborhood \( U_1(\alpha) \) such that \( U_1(\alpha) \subset \hat{G} - V_1(\alpha) \subset h(y) \) so that \( E \subset h_0(y) \) (= interior of \( h(y) \)). Thus \( h(I_1) \subset h(y)^0 \) and \( h(I_2) \subset h(y)^0 \), so that \( y \in I_1 \) and \( y \in I_2 \); i.e., \( y \in I_1 \cap I_2 \), i.e., \( y = 0 \). This is a contradiction.

**Corollary.** Let \( I \in \mathcal{I} \). Then \( h(I) \cup h(I^\perp) = \hat{G} \).

**Theorem 5.6.** If \( I \in \mathcal{I} \), then \( I^\perp \in \mathcal{I} \) iff \( h(I)^0 \neq \emptyset \).

**Proof.** Part 1. (Necessity) Claim: If \( h(I)^0 = \emptyset \), then \( I^\perp = \{0\} \). In fact, it follows from the above corollary that \( h(I) \cup h(I^\perp) = \hat{G} \), and \( \hat{G} - h(I^\perp) \subset h(I)^0 = \emptyset \); so that \( \hat{G} - h(I^\perp) = \emptyset \), and \( h(I^\perp) = \hat{G} \); i.e., \( I^\perp = \{0\} \).

Part 2. (Sufficiency) Claim: If \( h(I)^0 \neq \emptyset \), then \( I^\perp \neq \{0\} \). In fact, let \( \alpha \in h(I)^0 \), and let \( C = C(\alpha) \) be a compact neighborhood of \( \alpha \) such that \( C \subset h(I)^0 \). By Theorem 2.11 there exists \( 0 \neq y \in L^1 \cap L^2 \) such that \( \hat{y} \equiv 0 \) on \( C \), \( \hat{y} \equiv 0 \) off \( h(I)^0 \), and \( 0 \leq \hat{y} \leq 1 \). Thus if \( x \in I \), and \( \beta \in h(I) \), \( \hat{x}(\beta) \equiv 0 \); and if \( \beta \notin h(I), \beta \notin h(I)^0 \), so that \( \hat{y}(\beta) = 0 \). Therefore \( \hat{x}(y) \equiv 0 \), and therefore \( \int \hat{x}(\beta) \cdot \hat{y}(\beta) \, d\beta = 0 \). Thus \( (x, y) = (\hat{x}, \hat{y}) = 0 \) for every \( x \in I \). Hence \( 0 \neq y \in I^\perp \); i.e., \( I^\perp \neq \{0\} \).

**Theorem 5.7.** Let \( I_1 \) and \( I_2 \) be closed proper ideals in \( L^1 \cap L^2 \) such that \( I_1 \cap I_2 = \{0\} \). Then \( h(I_1) \cap h(I_2) = h(I_1 \oplus I_2) \).

**Corollary** \( h(I_1 \oplus I_2) = h(I_1) \cap h(I_2) \).

6. **Closed ideals in \( L^1 \cap L^2 \) when \( G \) is metric.** The topological group \( G \) has been assumed to be a locally compact abelian group. It will be assumed from this point on that, in addition to this, \( G \) is metric. It follows by Theorem 1.2 that \( \hat{G} \) is \( \sigma \)-compact. Hence every \( F_\sigma \) set in \( \hat{G} \) is also \( \sigma \)-compact.
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Definition 6.1. If $x \in L^1 \cap L^2$, the ideal $I(x)$ denotes the closed ideal generated by $x$ together with its translates. The linear subspace $N(x)$ denotes the set of all finite linear combinations of translates of $x$. Thus the $L^1 \cap L^2$-closure of $N(x)$ is $I(x)$.

Lemma 6.2. If $x \in L^1 \cap L^2$, then $h[I(x)] = h(x)$.

Proof. Observe that if $y \in N(x)$, $h(x) \subset y$. Let $z \in I(x)$. Then, since $N(x)$ is $L^1 \cap L^2$-dense in $I(x)$, there exists a sequence $\{z_n\} \subset N(x)$ such that $\|z - z_n\| \to 0$. If $z \in I(x)$, then $\hat{z}_n(a) = 0$ for $n = 1, 2, 3, \ldots$. Hence

$$|\hat{z}(a)| \leq \|\hat{z} - \hat{z}_n\| \leq \|z - z_n\|,$$

and since $\|z - z_n\| \to 0$, $\hat{z}(a) = 0$. Thus $h(x) \subset h(z)$.

Proof. $h[I(x)] = \bigcap \{h(z) \mid z \in I(x)\}$, it follows that $h(x) \subset h[I(x)]$. But $x \in I(x)$, so that $h[I(x)] \subset h(x)$. Consequently, $h(x) = h[I(x)]$.

Theorem 6.3. Let $I$ be a closed ideal of $L^1 \cap L^2$. Then there exists an $x \in I$ such that $h(x) = h(I)$ iff $h(I)$ is a $G_\delta$ set.

Proof. If $h(I) = h(x)$ for some $x \in I$, then since $\hat{x}$ is continuous, $h(I)$ is a $G_\delta$ set. Now suppose $h(I)$ is a $G_\delta$ set, and let $U = \hat{G} - h(I)$ so that $U$ is an open $F_\sigma$-set and $U = \bigcup_{n=1}^\infty K_n$, with $K_n \subset U$ and $K_n$ compact. Since $K_n \cap h(I) = \emptyset$, there exist disjoint open neighborhoods $U_n(K_n)$ and $V_n(h(I))$. By Theorem 2.11, there exists $w_n \in L^1 \cap L^2$ such that $\hat{w}_n \equiv 1$ on $K_n$, $\hat{w}_n \equiv 0$ off $U_n$, and $0 \leq \hat{w}_n \leq 1$. Observe that $V_n$ (open) $\subset h(w_n)$, so that $V_n \subset h(w_n)^0$. Therefore $h(I) \subset h(w_n)^0$, since $h(I) \subset V_n$. Hence $\{w_n\} \subset I$.

Let $x_m = \sum_{k=1}^n w_k \cdot \{2^k \|w_k\|\}^{-1}$. Then $\{x_m\} \subset I$, and $\{x_m\}$ is $L^1 \cap L^2$-Cauchy so there exists $x \in L^1 \cap L^2$ such that $x_m \to x$. But $I$ is closed, so it follows that $x \in I$. Therefore $h(I) \subset h(x)$. Let $x \notin h(I)$; i.e., $x \in U$. Then there exists some $n$ for which $x \in K_n$, so that $\hat{w}_n(x) = 1$, and hence $\hat{x}(a) \neq 0$. Therefore $x \notin h(x)$; i.e., $h(x) \subset h(I)$.

Corollary. Let $I$ be a closed ideal in $L^1 \cap L^2$. If $h(I)$ is a $G_\delta$ set with a scattered boundary, then there exists an $x \in I$ such that $I = I(x)$.

Proof. This follows from Šilov's Theorem (Theorem 3.5).

Remark. This corollary shows that if $\hat{G}$ is also metric, then the only instances of Helson's Theorem [4] for $L^1 \cap L^2$ (and, similarly for $L^1(G)$) are given by principal ideals. Theorem 6.3 shows that if $E \subset \hat{G}$ is closed, then $E$ is the hull of a closed principal ideal in $L^1 \cap L^2$ iff $E$ is a $G_\delta$ set. Thus, for example, if $G$ is $\sigma$-compact, every closed set in $\hat{G}$ is the hull of a closed principal ideal (since $\hat{G}$ would then be metric), and any nonprincipal closed ideal in $L^1 \cap L^2$ would
therefore provide an example of an ideal for which spectral synthesis fails \([I \neq I(x), \text{but } h(I(x)) = h(x) = h(I)]\).

It should be remarked here that Schwartz' counterexample to spectral synthesis in \(L^1(R^n)\) \((n \geq 3)\) carries over to \(L^1 \cap L^2(R^n)\) \((n \geq 3)\) with only minor modifications. The proof of the following theorem, therefore, is omitted.

**THEOREM.** There exists \(x \in L^1 \cap L^2(R^n)\), for \(n \geq 3\), such that \(x \notin I(x\ast x)\) (Reiter [11, pp. 469–470]).

7. The family \(\mathcal{J}\) of ideals of \(L^1 \cap L^2\).

**THEOREM 7.1.** Let \(I\) be a closed nonzero ideal of \(L^1 \cap L^2\). If \(m(h(I)) > 0\), then \(I \in \mathcal{J}\).

**Proof.** We have only to prove that \(J \neq L^2\). We will accomplish this by assuming that \(J = L^2\) and showing that this leads to a contradiction.

By Theorem 6.3, there exists a function \(u_1 \in L^1 \cap L^2\) such that \(\hat{u}_1 > 0\) on \(\hat{G}\). Let \(y\) be the characteristic function of \(h(I)\), and set \(\hat{u} = \hat{u}_1 y\). Then \(\hat{u} \in L^2(\hat{G})\), so that, by the Plancherel theorem, there exists \(u \in L^2(G)\) such that the Fourier transform of \(u\) is equal to \(\hat{u}\) a.e. Let \(\int_{h(I)} |\hat{u}(x)|^2 dx = p^2 > 0\). Since \(u \in L^2\), there exists a sequence \(\{x_n\} \subset I\) such that \(\lim_n \|u - x_n\|_2 = 0\). Hence \(0 = \lim_n \|\hat{u} - \hat{x}_n\|_2^2 \geq p^2 > 0\). This is the desired contradiction.

**COROLLARY (5).** If \(\hat{G}\) has a closed subset \(E\) of positive measure such that \(E^0 = \emptyset\), then there exists a closed proper invariant subspace \(N \subset L^2(G)\) for which \(N \cap L^1 = \{0\}\).

**Proof.** Let \(I = k(E)\), so that \(h(I) = E\) and \(m(h(I)) > 0\). Then \(I \in \mathcal{J}\) by the above theorem and \(h(I)^0 = \emptyset\). Hence \(I^\perp = \{0\}\) by Theorem 5.6. But \(I^\perp = J^\perp \cap L^1\) by Theorem 5.3 and \(J^\perp\) is a closed proper invariant subspace of \(L^2(G)\) by Theorem 5.4. Therefore the desired subspace is \(N = J^\perp\).

**EXAMPLE.** Let \(G\) be the real line under addition, and let \(E\) be a Cantor set of positive measure. Note that if \(G\) is compact, \(\hat{G}\) is discrete so that no set \(E \neq \emptyset\) can be found such that \(E^0 = \emptyset\). However, in this case, \(N \cap L^1 = N\) for every \(N \subset L^2(G)\).

**THEOREM 7.2.** If \(I\) is any closed proper ideal in \(L^1 \cap L^2\), there exists \(x \in I\) such that \(h(x) = h(I) [m]\).

**Proof.** Let \(\{U_n\}\) be a family of open neighborhoods of \(h(I)\) such that \(m(U_n - h(I)) < 1/n\) \((n = 1, 2, 3, \ldots)\). If \(F_n = \hat{G} - U_n\), then \(F_n\) is closed, and therefore \(\sigma\)-compact. Hence, let \(F_n = \bigcup_{k=1}^{\infty} K_{nk}\), where \(K_{nk}\) is compact for each

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(5) A proof of the fact that every nondiscrete locally compact group contains a compact nowhere dense subset of positive measure was communicated to the author in the summer of 1963 by K. A. Ross and K. Stromberg.
n, k = 1, 2, 3, ... . Observe that \( h(I) \cap K_n = \emptyset \) so that we may choose disjoint open neighborhoods \( U_{n,k}, V_{n,k} \) of \( h(I) \) and \( K_n \) respectively, and we can always choose \( U_{n,k} \subset U_n \). Let \( x_{n,k} \in L^1 \cap L^2 \) be constructed by Theorem 2.11 so that \( x_{n,k} \equiv 1 \) on \( K_n \) and \( x_{n,k} \equiv 0 \) off \( V_{n,k} \). Then \( h(I) \subset U_{n,k} \subset h(x_{n,k})^0 \), and it follows that \( x_{n,k} \in I \). Let \( y_{n,p} = \sum_{k=1}^{\infty} x_{n,k} \cdot \{2^k \| x_{n,k} \| \}^{-1} \). Then \( \{y_{n,p}\} \subset I \) and \( \{y_{n,p}\} \) is \( L^1 \cap L^2 \)-Cauchy and \( I \) is closed, so that \( \lim_{n} \| y_{n,p} - x_n \| = 0 \) for some \( x_n \in I \). By proceeding this manner, we obtain a sequence \( \{x_n\} \subset I \). By construction \( x_n > 0 \) on each \( K_n \) (for \( k = 1, 2, 3, ... \)); i.e., \( x_n > 0 \) on \( F_n \), and \( x_n \equiv 0 \) on each \( U_{n,k} \) (\( k = 1, 2, 3, ... \)).

Now let \( E = \bigcap_{n=1}^{\infty} U_n \), so that \( m(E - h(I)) = 0 \), and proceeding as before, let \( x = \sum_{n=1}^{\infty} x_n \cdot \{2^n \| x_n \| \}^{-1} \). Then \( x \in I \), and \( x > 0 \) on each \( F_n = \hat{G} - U_n \) (for \( n = 1, 2, 3, ... \)). Hence \( x > 0 \) on \( \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} (G - U_n) = G - \bigcap_{n=1}^{\infty} U_n = \hat{G} - E \). Moreover \( x \equiv 0 \) on \( E \), so that \( h(x) = h(I) \) \([m]\).

**Corollary 1.** If \( I \) is a closed proper ideal in \( L^1 \cap L^2 \), then \( J = h(x) \).

**Proof.** Let \( y \in J \). Then there exists a sequence \( \{y_n\} \subset I \) such that \( \| y - y_n \|_2 \to 0 \). Let \( F = h(x) - h(y) \) \([m]\), where \( x \in I \), and \( h(I) = h(x) \) \([m]\) as in the above theorem. Suppose \( F \neq \emptyset \) \([m]\), and let \( \int_F |\tilde{y}(x)|^2 \, dx = \delta > 0 \), since \( |\tilde{y}(x)|^2 > 0 \) a.e. on \( F \). Then \( \int_F |\tilde{y}(x) - \tilde{y}_n(x)|^2 \, dx = \int_F |\tilde{y}(x)|^2 \, dx = \delta > 0 \), since each \( \tilde{y}_n \equiv 0 \) a.e. on \( F \). It follows that \( \lim_{n} \| y - y_n \|_2 \neq 0 \), which contradicts our initial assumption. Hence \( F = \emptyset \) \([m]\), and if \( y \in J \), \( h(x) \subset h(y) \) \([m]\). By Theorem 4.1, it follows that \( y \in H(x) \). Hence \( J \subset H(x) \), so that \( J = H(x) \).

The proofs of the following results are direct applications of Theorem 7.1, Theorem 4.1, and the above Corollary 1.

**Corollary 2.** Let \( I \) be a closed proper ideal in \( L^1 \cap L^2 \). Then \( I \in \mathcal{J} \) iff \( m(h(I)) > 0 \).

**Corollary 3.** Let \( I_1, I_2 \) be closed proper ideals in \( L^1 \cap L^2 \). Then \( \text{cl } I_1 = \text{cl } I_2 \) iff \( h(I_1) = h(I_2) \) \([m]\).

**Theorem 7.3.** The group \( \hat{G} \) is connected iff for every pair \( I, I^\perp \in \mathcal{J} \), \( I \oplus I^\perp \) is a proper ideal in \( L^1 \cap L^2 \).

**Proof.** Suppose that \( \hat{G} \) is connected and that for some pair \( I, I^\perp \in \mathcal{J} \), \( I \oplus I^\perp \) is not proper; i.e., \( I \oplus I^\perp = L^1 \cap L^2 \). Then by the corollary to Theorem 5.7, \( \emptyset = h(I) \cap h(I^\perp) \) and by the corollary to Theorem 5.5, \( h(I) \cup h(I^\perp) = \hat{G} \). Hence, \( \hat{G} \) is not connected, contrary to our assumption. Conversely, suppose that \( \hat{G} \) is not connected; i.e., suppose \( \hat{G} = P \cup Q \) where \( P \) and \( Q \) are open-closed and disjoint in \( \hat{G} \). Let \( j(P) = \{x \in L^1 \cap L^2 \mid x \in L(\hat{G}) \text{ and } P \subset h(x)^0 \} \) and let \( I = L^1 \cap L^2 \)-closure of \( j(P) \). Thus, \( I \in \mathcal{J} \), \( I^\perp \in \mathcal{J} \) and \( h(I) = P \). Let \( I^* = L^1 \cap L^2 \)-closure of \( j(Q) \), and observe that since \( Q \) is open-closed, \( x \in I^* \) iff \( Q \subset h(x) \). Note that \( I^* \subset I^\perp \) and that \( Q = \hat{G} - h(I) \subset h(I^\perp) \). Therefore, if \( x \in I^\perp \), \( Q \subset h(x) \), so that \( x \in I^* \); i.e., \( I^\perp = L^1 \cap L^2 \)-closure of \( j(Q) \). Hence
\[ h(I \oplus I^\perp) = h(I) \cap h(I^\perp) = P \cap Q = \emptyset, \]
so that \( I \oplus I^\perp = L^1 \cap L^2. \)

**Theorem 7.4.** If \( I \in \mathcal{I} \), then \( k(h(I)) \subset J \cap L^1. \)

**Proof.** By Theorem 7.2, there exists \( x \in I \) such that \( h(I) = h(x) [m] \), and by Corollary 1 of the same theorem \( H(x) = J \). Let \( y \in k(h(I)) \), so that \( h(I) \subset h(y) \). Then \( h(x) \subset h(y) [m] \), and by Theorem 4.1, \( y \in H(x) = J \). Since \( y \in L^1 \cap L^2 \), it follows that \( y \in J \cap L^1. \) Hence \( k(h(I)) \subset J \cap L^1. \)

**Corollary 1.** If \( I \in \mathcal{I} \), then \( J \cap L^1 = k(h(J \cap L^1)). \)

**Corollary 2.** If \( N \) is a closed proper invariant subspace of \( L^2(G) \), then \( N \cap L^1 = k(h(N \cap L^1)). \) In particular, if \( I, I^\perp \in \mathcal{I} \), then \( I^\perp = k(h(I^\perp)). \)

**Proof.** We assume that \( N \cap L^1 \neq 0 \) without loss of generality. The above Corollary 1 implies that \( N \cap L^1 = k(h(L^1 \cap \text{cl}(N \cap L^1))). \) The observation that \( N \cap L^1 = L^1 \cap \text{cl}(N \cap L^1) \) completes the proof.

**Theorem 7.5.** Let \( I, I^\perp \in \mathcal{I} \). If \( I \oplus I^\perp = k(h(I \oplus I^\perp)) \), then \( I = k(h(I)). \)

**Proof.** By Theorem 5.7, \( h(I \oplus I^\perp) = h(I) \cap h(I^\perp) \). Hence \( x \in k(h(I)) \) implies that \( x \in k(h(I \oplus I^\perp)) \), and \( x \in k(h(I^\perp)) \) implies that \( x \in k(h(I \oplus I^\perp)) \) so that \( k(h(I)) \cup k(h(I^\perp)) \subset k(h(I \oplus I^\perp)) \). By the above Corollary 2, \( I^\perp = k(h(I^\perp)) \), so that \( k(h(I)) \cup I^\perp \subset k(h(I \oplus I^\perp)) \). However, by Theorem 7.4, \( k(h(I)) \subset J \cap L^1 \), so it follows that \( k(h(I)) \cap I^\perp = \{0\} \), and hence \( k(h(I)) \oplus I^\perp \subset k(h(I \oplus I^\perp)) \).

**Theorem 7.6.** If \( I \in \mathcal{I} \), then \( h(I)^0 = h(J \cap L^1)^0 \).

**Proof.** Since \( I \subset J \cap L^1 \), and \( \text{cl} I = J \), it follows that \( \text{cl} I = \text{cl} (J \cap L^1) \), so that by Corollary 3 of Theorem 7.2, \( h(I) = h(J \cap L^1) [m] \). Let \( A = h(I)^0 - h(J \cap L^1) \), so that \( A \) is open, and thus either \( A = \emptyset \) or \( m(A) > 0 \). But \( A \subset h(I) - h(J \cap L^1) \) so that \( m(A) = 0 \). Hence \( A = \emptyset \), and therefore \( h(I)^0 \subset h(J \cap L^1) \).

**Theorem 7.7.** Let \( I \in \mathcal{I} \) and let \( F \) be a closed subset of \( h(J \cap L^1) \). If \( h(J \cap L^1) = F [m] \), then \( h(J \cap L^1) = F \).

**Proof.** As before, \( h(I) = h(J \cap L^1) [m] \). Hence, by assumption, \( F = h(I) [m] \). Therefore if \( x \in k(F) \), \( h(I) \subset h(x) [m] \), so that, by Theorem 7.2, Corollary 1, \( x \in J \cap L^1 \); i.e., \( k(F) \subset J \cap L^1 \). But if \( F \) is closed, then \( F = h(k(F)) \supset h(J \cap L^1) \supset F \).

**Corollary.** If \( \mathcal{G} \) is not discrete, and if \( I \in \mathcal{I} \), then \( h(J \cap L^1) \) is perfect.
Proof. Since $h(\mathcal{J} \cap L^1)$ is closed, it can be written $h(\mathcal{J} \cap L^1) = S \cup P$ where $S$ is scattered, $P$ is perfect (Sierpiński [16, Chapter 1]). By a theorem of Rudin [12, Theorem 5, p. 41] it follows that $m(S) = 0$ so that $m(h(\mathcal{J} \cap L^1) - P) = m(S) = 0$. Hence $P = h(\mathcal{J} \cap L^1)$.

With suitable modifications, many of the results of §§4-7 that depend upon the restrictive assumption that $G$ is metric can be extended to the more general case where the metric hypothesis is removed. A consequence of this is that a significant generalization of a theorem due to I.E. Segal will be presented in a forthcoming paper.


Bibliography


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