ON FUNCTION SPACES WHICH ARE LINDELÖF SPACES

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1. **Introduction.** The purpose of this paper is to study the following question. Let X be a topological space, and let L be a locally convex linear topological space. Under what assumptions (on X or L) is the space of all continuous functions from X into L a Lindelöf space in the topology of pointwise convergence or in the compact open topology? Our main interest is in the case where X is a metric space.

We have posed a very general question, and it is not to be expected that it has a simple complete answer. The results and examples we present here give, however, an answer to this question in many standard situations which arise in functional analysis. Our study of the question above arose from attempts to extend the selection theory of Michael to the case in which the range is nonmetrizable. It turned out that Michael's theorems can be generalized practically only in those situations where suitable function spaces are Lindelöf spaces (cf. [3] for details).

In §2 we prove some theorems which exhibit many function spaces which are Lindelöf spaces. A typical example is the following result: the space of all continuous functions from a separable metric space X to a (possibly nonseparable) Hilbert space with the weak (w)-topology is a Lindelöf space, if we endow it with the topology of pointwise convergence. The proof of this result is rather involved even for compact X (only in the case in which X is countable can the proof be simplified considerably). The difficult part of the proof of this result (as well as the proofs of the other results of §2) is the proof of Lemma 2.1. This lemma is stated in the beginning of §2, but its proof is given only in §4.

§3 is devoted to examples which show that in many respects the results of §2 are the best possible ones. At the end of §3 we give a table which summarizes our results concerning spaces of continuous functions from a metric space to a Hilbert space H (taking in H three topologies—the norm topology, the w-topology and the topology of pointwise convergence of the coordinates with respect to a fixed orthogonal basis).

§4 is, as mentioned already, devoted to the proof of Lemma 2.1. The methods used in the proof are refinements of those used by one of the authors in [2].

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There are some important linear spaces L for which we are unable to decide whether they are Lindelöf spaces. For such spaces L we cannot, of course, prove any nontrivial result concerning the question whether the space of continuous functions from a metric space X into L is a Lindelöf space. We do not know, for example, whether the space $L_1(\mu)$ is a Lindelöf space in the w-topology for every finite measure μ. This question was left open in [2] as a special case of a more general conjecture.

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NOTATIONS. The linear spaces we consider are taken (unless stated otherwise) over the reals R. This is only a matter of convenience; all our results hold also in the complex case. For the notions and results in topology and the theory of linear spaces which we use without specific reference the reader may consult the standard books [5], [6], and [7].

Let Γ and L be topological spaces and let 0 be a fixed point in L (if L is a linear space we shall always take 0 as the origin). We will say that a function f from Γ to L vanishes at infinity if, given any open set U around 0 in L there is a compact subset F of Γ such that $f(\gamma)$ is in U for every γ in the complement of F. Let $C_0 = C_0(\Gamma, L)$ denote the space of all continuous functions from Γ to L which vanish at infinity, where C_0 is given the compact open topology. If $L = R$ we denote $C_0(\Gamma, L)$ also by $C_0(\Gamma)$.

For a function $f \in C_0$ and a subset $\Gamma_0 \subset \Gamma$ let f/Γ_0 denote the function from Γ to L which agrees with f on Γ_0 and which has value 0 for $\gamma \in \Gamma \sim \Gamma_0$. For a subset A of C_0 let $A/\Gamma_0 = \{f/\Gamma_0 : f \in A\}$.

Suppose now that Γ is discrete and that $\Gamma_0 \subset \Gamma$. The map $f \to f/\Gamma_0$ takes $C_0 = C_0(\Gamma, L)$ into itself and it is called the canonical projection from C_0 onto C_0/Γ_0. A subset A of C_0 is said to be *invariant under projections* if $A/\Gamma_0 \subset A$ for every countable subset Γ_0 of Γ. A is said to be *almost invariant under projections* if there exists a set $\{\Gamma_\beta\}_{\beta \in B}$ of countable subsets of Γ, directed by inclusion, such that $A/\Gamma_\beta \subset A$ for every β, $\bigcup_\beta \Gamma_\beta = \Gamma$ and such that whenever $\Gamma_{\beta_1} \subset \Gamma_{\beta_2} \subset \Gamma_{\beta_3} \cdots$ then also $\bigcup_{i=1}^{\infty} \Gamma_{\beta_i}$ is one of the Γ_β, $\beta \in B$.

Let X and L be topological spaces. We denote by $C(X, L)$ the space of all continuous functions from X to L in the compact open topology. The same space but with the topology of pointwise convergence will be denoted by $C_p(X, L)$. The spaces $C(X, R)$ and $C_p(X, R)$ will be denoted also by $C(X)$ and $C_p(X)$ respectively. Let X and Y be topological spaces and let $f \in C(X, C(Y))$. The value of the function $f(x)$, $x \in X$ at the point $y \in Y$ will be denoted by $f(x)(y)$. Let Γ be a discrete space, let $\gamma \in \Gamma$ and $f \in C(\Gamma, L)$. The point $f(\gamma)$ in L will be called also the γ coordinate of f.

The closure of a set A in a topological space will be denoted by $\bar{A}$.

2. **Spaces of continuous functions which are Lindelöf spaces.** The following is the basic lemma on which all the theorems of this section depend.

LEMMA 2.1. *Let X be a separable metric space and let $\hat{X}$ be its completion. Let Γ be a discrete space and let A be a subset of $C_0(\Gamma, C_p(X))$ which is almost invariant under projections. Assume further that for every $f \in A$ there is a G_δ subset X_f of $\hat{X}$ and an element $F \in C_0(\Gamma, C_p(X_f))$ such that $X \subset X_f \subset \hat{X}$ and $F(\gamma)(x) = f(\gamma)(x)$ for every $\gamma \in \Gamma$ and every $x \in X$. Then A is a Lindelöf subspace of $C_0(\Gamma, C_p(X))$.*

The proof of this lemma is rather long and will be given in §4. The role of the various assumptions appearing in the lemma will be discussed in the next section. The examples given there show that Lemma 2.1 is in many respects the best possible one in this direction. We just remark here that the assumption concerning the existence of X_f and F is always trivially satisfied if X is a G_δ set in $\hat{X}$, i.e., if X has an equivalent metric in which it is complete.

Several results will now be shown to follow from Lemma 2.1.

THEOREM 2.2. *Let X be a topological space which is a continuous image of a complete separable metric space. Let Γ be a discrete space and let L be a subset of $C_0(\Gamma)$ which is almost invariant under projections. Then $C_p(X, L)$ is a Lindelöf space.*

Proof. Let ψ be a continuous function from a complete separable metric space Y onto X. We define a map Ψ from $C_p(X, L)$ into $C_0(\Gamma, C_p(Y))$ by

$$\Psi f(\gamma)(y) = f(\psi y)(\gamma), \quad \gamma \in \Gamma, \quad y \in Y. \tag{2.1}$$

For every $y \in Y$ and $\varepsilon > 0$ the number of the $\gamma \in \Gamma$ for which $|\Psi f(\gamma)(y)| > \varepsilon$ is finite, since $f(\psi y) \in L \subset C_0(\Gamma)$, and hence $\Psi f \in C_0(\Gamma, C_p(Y))$. Ψ is a one-to-one map since ψ is onto. Ψ is also a homeomorphism (into) since a net $\{f_\alpha\}$ in $C_p(X, L)$ converges to f iff $\{f_\alpha(x)(\gamma)\}$ converges to $f(x)(\gamma)$ for every $x \in X$ and $\gamma \in \Gamma$, i.e., iff $\{f_\alpha(\psi(y))(\gamma)\}$ converges to $f(\psi(y))(\gamma)$ for every $y \in Y$ and $\gamma \in \Gamma$. The range of Ψ is almost invariant under projections. Indeed if $\Gamma_0 \subset \Gamma$ is such that $L/\Gamma_0 \subset L$ and if $f \in C_p(X, L)$, then $(\Psi f)/\Gamma_0 = \Psi(P_0 f)$ where P_0 is the canonical projection from $C_0(\Gamma)$ onto $C_0(\Gamma)/\Gamma_0$. Since $P_0 f \in C_p(X, L)$, it follows that

$$\Psi(C_p(X, L))/\Gamma_0 \subset \Psi(C_p(X, L)).$$

The theorem follows now by applying Lemma 2.1 to the subset $\Psi(C_p(X, L))$ of $C_0(\Gamma, C_p(Y))$.

We point out that any countable topological space X satisfies the assumption of Theorem 2.2, since it is a continuous image of the integers.

Our next result is similar to Theorem 2.2. The only difference is that now we require that L is compact (though not necessarily almost invariant under projections) and allow more general X.

THEOREM 2.3. *Let X be a topological space which is the continuous image of a separable metric space. Let Γ be a discrete space and L be a compact subset of $C_0(\Gamma)$. Then $C_p(X,L)$ is a Lindelöf space.*

Proof. Let ψ be a continuous map from a separable metric space Y onto X. Define $\Psi: C_p(X,L) \to C_0(\Gamma, C(Y))$ by the equation (2.1). As above Ψ is a homeomorphism into (but in general $\Psi(C_p(X,L))$ is not almost invariant under projections). Let A be the subset of $C_0(\Gamma, C_p(Y))$ consisting of all the functions f which satisfy

(i) There is a $g \in C_p(X, C_0(\Gamma))$ such that $f = \Psi g$ (Ψ is defined in (2.1), the definition clearly makes sense for the whole of $C_p(X, C_0(\Gamma))$ and not only for $C_p(X,L)$.

(ii) There is a G_δ subset Y_f of the completion $\hat{Y}$ of Y with $Y_f \supset Y$ and an $F \in C_0(\Gamma, C_p(Y_f))$ such that $F(\gamma)(y) = f(\gamma)(y)$ for every $\gamma \in \Gamma$ and $y \in Y$.

We shall prove that

(a) $\Psi(C_p(X,L))$ is a closed subset of A,

and

(b) A is invariant under projections.

This will conclude the proof of the theorem since, by Lemma 2.1, (b) implies that A is a Lindelöf space and hence by (a) and the fact that Ψ is a homeomorphism we get that also $C_p(X,L)$ is a Lindelöf space.

We pass to the proof of (a). Let $g \in C_p(X,L)$, we have first to show that $\Psi g \in A$ i.e. that Ψg satisfies condition (ii). Let $\{x_i\}_{i=1}^{\infty}$ be a dense sequence of points in X. Since $L \subset C_0(\Gamma)$ there is a countable subset Γ_0 of Γ such that $g(x_i)(\gamma) = 0$ if $\gamma \in \Gamma \sim \Gamma_0$ and $i = 1, 2, \cdots$. It follows, by the continuity of g, that

$$g(X) \subset L \cap (C_0(\Gamma)/\Gamma_0).$$

The set $L \cap (C_0(\Gamma)/\Gamma_0)$ is compact and metrizable.

We need now the following known fact (cf. [7]). Let h be a continuous function from a subset M of a metric space $\hat{M}$ into a complete metric space Z. Then h can be extended to a continuous function H from M_0 to Z for some G_δ subset M_0 of $\hat{M}$ containing M.

We apply this result to the situation we have by taking $h(y) = g(\psi(y))$, $M = Y$, $\hat{M} = \hat{Y}$, and $Z = L \cap (C_0(\Gamma)/\Gamma_0)$. We get that there is a G_δ subset Y_0 of $\hat{Y}$ with $Y \subset Y_0 \subset \hat{Y}$ and an element $H \in C_p(Y_0, L)$ such that $H(y) = g(\psi(y))$ for $y \in Y$. Let $F \in C_0(\Gamma, C_p(Y_0))$ be defined by $F(\gamma)(y) = H(y)(\gamma)$, $y \in Y_0$, $\gamma \in \Gamma$. For $y \in Y$, we have

$$F(\gamma)(y) = H(y)(\gamma) = g(\psi(y))(\gamma) = \Psi g(\gamma)(y),$$

and hence $f = \Psi g$ satisfies (ii). This shows that $\Psi(C_p(X,L))$ is a subset of A. The fact that it is a closed subset of A is a consequence of our assumption that L is compact and thus closed in $C_0(\Gamma)$.

We turn to the proof of assertion (b). Let $f = \Psi g$ for some $g \in C_p(X, C_0(\Gamma))$ and let Γ_0 be a countable subset of Γ. Then $f/\Gamma_0 = \Psi(P_0 g)$, where P_0 is the canonical projection from $C_0(\Gamma)$ onto $C_0(\Gamma)/\Gamma_0$, and hence also f/Γ_0 satisfies (i). Assume now that f satisfies (ii) and let Y_f and F be a suitable G_δ subset of Y and function respectively. Then $F/\Gamma_0 \in C_0(\Gamma, C_p(Y_f))$ and $F/\Gamma_0(\gamma)(y) = f/\Gamma_0(\gamma)(y)$ for every $\gamma \in \Gamma$ and $y \in Y$, and hence f/Γ_0 also satisfies (ii). This concludes the proof of assertion (b) and thus of Theorem 2.3.

We turn now to some examples of spaces L which satisfy the assumptions in Theorem 2.2 or Theorem 2.3. Let $0 < p < \infty$ and let Γ be a set. The space $L = l_p(\Gamma)$ of all real-valued functions on Γ for which $\sum_{\gamma \in \Gamma} |f(\gamma)|^p < \infty$ clearly satisfies the assumption of Theorem 2.2 if we take in it the topology of pointwise convergence (i.e. the topology induced on it by $C_0(\Gamma)$).

Another example. Let G be a compact Abelian topological group and let Γ be the (discrete) dual group. The Fourier transform T maps $L_1(\mu_G)$ (μ_G denotes the Haar measure on G, we use here the complex L_1 space) into $C_0(\Gamma)$. The image of $L_1(\mu_G)$ is almost invariant under projections. In fact, let Γ_0 be a countable subgroup of Γ, let G_0 be the subgroup of G orthogonal to Γ_0 and let $f \in L_1(\mu_G)$. The function $f_0 \in L_1(\mu_G)$ defined by

$$f_0(g) = \int_{G_0} f(gg_0)d\mu_{G_0},$$

(μ_{G_0} is the Haar measure on G_0) satisfies

$$\int_G f_0(g)\gamma(g)d\mu_G = \begin{cases} \int_G f(g)\gamma(g)d\mu_G & \text{if } \gamma \in \Gamma_0, \\ 0 & \text{if } \gamma \in \Gamma \sim \Gamma_0. \end{cases}$$

Since the set of countable subgroups of Γ is directed by inclusion and closed under the operation of taking the union of an increasing sequence we have thus verified the assertion made on $L_1(\mu_G)$. (A similar assertion holds also for non-commutative G, cf. [2].) Observe also that if the function f above belongs to $L_p(\mu_G)$ for some $1 \leqq p \leqq \infty$ then the same is true for f_0, and hence for every $1 \leqq p \leqq \infty$ the subset $TL_p(\mu_G)$ of $C_0(\Gamma)$ is almost invariant under projections. It follows now from Theorem 2.2 that $C_p(X, L_p(\mu_G))$ *is a Lindelöf space if X is a complete separable metric space and if the topology in $L_p(\mu_G)$ is the topology of pointwise convergence on the characters* (i.e. $f_\alpha \to f$ iff

$$\int_G f_\alpha(g)\gamma(g)d\mu_G \to \int_G f(g)\gamma(g)d\mu_G$$

for every $\gamma \in \Gamma$). This topology is weaker than the w-topology of $L_p(\mu_G)$ (for $p = \infty$ it is also weaker than the w^*-topology since the characters belong to $L_1(\mu_G)$).

Let now μ be a measure on a general measure space. By a theorem of Maharam [8] there is a set $\{G_\alpha\}_{\alpha\in\Omega}$ of compact Abelian groups (with Haar measure μ_α) so that $L_p(\mu)$ is isometric to $(\sum_\alpha \oplus L_p(\mu_\alpha))_p$ for every $1 \leqq p \leqq \infty$ (the direct sum is defined to be the set of all functions u defined on Ω such that $u(\alpha) \in L_p(\mu_\alpha)$ and $\|u\| = (\sum_\alpha \|u(\alpha)\|^p)^{1/p} < \infty$). If μ is σ-finite, the set Ω will be countable. It follows that, for every measure μ and every $1 \leqq p < \infty$, there is a locally convex Hausdorff topology on $L_p(\mu)$ which is weaker than the w-topology and is such that $L_p(\mu)$ with this topology is linearly homeomorphic with a linear subspace of $C_0(\Gamma)$ (Γ a discrete space) which is almost invariant under projections. In the proof we gave Γ is the disjoint union of the sets $\{\Gamma_\alpha\}_{\alpha\in\Omega}$ where Γ_α is the dual group of G_α. If μ is σ-finite, there is also a locally convex Hausdorff topology on $L_\infty(\mu)$, which is weaker than the w^*-topology such that $L_\infty(\mu)$ with this topology is linearly homeomorphic with a linear subspace of $C_0(\Gamma)$ which is almost invariant under projections.

If L is a w-compact subset of $L_p(\mu)$ for some $1 \leqq p < \infty$ then the topology on $L_p(\mu)$ which was described in the preceding paragraph coincides on L with the w-topology. Hence we may apply Theorem 2.3 to a w-compact subset L of $L_p(\mu)$, $1 \leqq p < \infty$. The same is true for a w-compact and even w^*-compact subset of $L_\infty(\mu)$ if μ is σ-finite.

Theorem 2.2 shows that if X is a complete separable metric space then it is possible to introduce in many of the common linear spaces L a locally convex topology which is rather weak so that $C_p(X, L)$ is a Lindelöf space. The following question arises naturally: Let L be a locally convex space; is $C_p(X, L)$ a Lindelöf space if we take in L the given topology? Let us take for example a nonseparable Hilbert space H. If we take in H the norm topology (we denote H with the norm topology by H^n) then clearly $C_p(X, H^n)$ is not Lindelöf even if X consists of a single point. If we take in H the topology p of pointwise convergence of the coordinates (with respect to a fixed orthonormal basis), then we can apply Theorem 2.2 and we get that $C_p(X, H^p)$ is Lindelöf if X is separable complete metric. An example, due to Michael, (cf. (2) in §3) shows that $C_p(X, H^p)$ may fail to be Lindelöf if X is separable metric but not complete. What is the situation if we consider H^w—the Hilbert space with the w-topology? We cannot apply Theorem 2.2, since H^w is not homeomorphic to a subset of $C_0(\Gamma)$ with discrete Γ even if H is separable infinite-dimensional. Indeed, if H is separable then H^w is also separable and every separable subset of $C_0(\Gamma)$ is metrizable.

However, by using Theorem 2.3 and the fact that H^w is σ-compact, we are able to give an answer to the question concerning H^w. $C_p(X, H^w)$ is Lindelöf for every separable metric space X. The proof of this result actually works in more general situations (e.g. for general $L_p(\mu)$ spaces in the w-topology if $1 < p < \infty$). Some of the ideas in this proof were suggested by E. Michael who used them, together with Theorem 2.3, to prove a special case of Theorem 2.4.

THEOREM 2.4. *Let X be a topological space which is a continuous image of a separable metric space, and let H^w be (a not necessarily separable) Hilbert space in the weak topology. Then $C_p(X, H^w)$ is a Lindelöf space.*

Proof. It is well known that every compact metric space is a continuous image of the Cantor set and that every separable metric space is homeomorphic to a subset of a compact metric space (e.g. the Hilbert cube). Hence there is a totally disconnected subset Y of R and a continuous function ψ from Y onto X. Let $\Psi: C_p(X, H^w) \to C_p(Y, H^w)$ be defined by $\Psi f(y) = f(\psi(y))$. Ψ is a homeomorphism into. Let S^w denote the unit cell of H (in the w-topology) and let N be the integers (in the discrete topology). We define $\phi: N \times S^w \to H^w$ by $\phi(n, z) = nz$, and $\Phi: C_p(Y, N \times S^w) \to C_p(Y, H^w)$ by $\Phi f(y) = \phi(f(y))$. Φ is clearly a continuous map. We show now that Φ is onto. Let $g \in C_p(Y, H^w)$. Since g is locally bounded, there is an open covering $\mathscr{U}$ of Y such that g maps every element of the covering to a bounded subset of the Hilbert space. Since Y is a totally disconnected subset of R the covering $\mathscr{U}$ has a refinement $\mathscr{V} = \{V_i\}_{i=1}^{\infty}$ consisting of mutually disjoint open and closed subsets V_i of Y. Let n_i be an integer which is larger than $\sup_{y \in V_i} \| g(y) \|$ $(i = 1, 2, \cdots)$ and define $f: Y \to N \times S^w$ by $f(y) = (n_i, g(y)/n_i)$ if $y \in V_i$. Then $f \in C_p(Y, N \times S^w)$ and $g = \Phi f$.

It follows from the preceding arguments that in order to prove the theorem it is enough to show that $A = \Phi^{-1}\Psi(C_p(X, H^w))$ is a Lindelöf subspace of $C_p(Y, N \times S^w)$. Let $\{e_\gamma\}_{\gamma \in \Gamma^*}$ be an orthogonal basis of H and let $\Gamma = \{1\} \cup \Gamma^*$. We embed $N \times S^w$ in $C_0(\Gamma)$ by letting (n, z) correspond to the point whose first coordinate is n and whose γ coordinate $(\gamma \in \Gamma^*)$ is the inner product (z, e_γ). We can thus consider $C_p(Y, N \times S^w)$ (and therefore A) as a subspace of $C_p(Y, C_0(\Gamma)) = C_0(\Gamma, C_p(Y))$. It is easy to check that A is a subset of $C_0(\Gamma, C_p(Y))$ which is invariant under projections. Also for every $f \in C_p(Y, N \times S^w)$ there is a G_δ subset Y_f of R (or, what amounts to the same thing, of $\tilde{Y}$), with $Y_f \supset Y$, such that f can be extended to a continuous function F from Y_f to $N \times S^w$ (cf. the proof of Theorem 2.3). The fact that A is a Lindelöf subspace of $C_0(\Gamma, C_p(Y))$ follows now from Lemma 2.1. This concludes the proof of the theorem.

In Lemma 2.1 we consider functions from a discrete space to a space of the form $C_p(X)$. Our final result in this section shows that, if X is countable then Lemma 2.1 is still true if the discrete space Γ is replaced by an arbitrarily locally compact metric space Z.

THEOREM 2.5. *Let Z be a locally compact metric space and let L be a separable metric space. Then $C_0(Z, L)$ is a Lindelöf space.*

Proof. Let $\{Z_\gamma\}_{\gamma \in \Gamma}$ be pairwise disjoint open σ-compact subsets of Z such that $Z = \bigcup_{\gamma \in \Gamma} Z_\gamma$. (The existence of such Z_γ is an easy consequence of the fact that Z is paracompact.) Let Y be the disjoint union of a countable number of copies of the Cantor set. Since every compact metric space is a continuous image of the

Cantor set it follows that every Z_γ is a continuous image of Y. Moreover it is easy to see that for every γ there is a continuous function ψ_γ from Y onto Z_γ such that $\psi_\gamma^{-1}(K)$ is a compact subset of Y for every compact subset K of Z_γ. Define now $\Psi: C_0(Z,L) \to C_0(\Gamma \times Y, L)$ by putting $\Psi f(\gamma, y) = f(\psi_\gamma(y))$, $f \in C_0(Z,L)$. That $\Psi f \in C_0(\Gamma \times Y, L)$ follows from our assumption on the ψ_γ. Since L, like any separable metric space, is homeomorphic to a subset of $C_p(N)$, we may consider $C_0(\Gamma \times Y, L)$ as a subset of $C_0(\Gamma \times Y, C_p(N))$ (we assume, as we may, that the homeomorphism from L into $C_p(N)$ takes the point "0" in L to the zero function on the integers). There is a natural injection (homeomorphism into) Φ from $C_0(\Gamma \times Y, C_p(N))$ into $C_0(\Gamma, C_p(N, C_0(Y)))$ defined by $\Phi f(\gamma)(n)(y) = f(\gamma, y)(n)$. The space $C_0(Y)$ is separable metric and hence it is homeomorphic to a subset of $C_p(N)$. Thus we can identify $C_0(\Gamma, C_p(N, C_0(Y)))$ with a subset of

$$C_0(\Gamma, C_p(N, C_p(N))) = C_0(\Gamma, C_p(N \times N)).$$

It is easy to check that, with these identifications, the map $\Phi\Psi$ maps $C_0(Z,L)$ homeomorphically onto a subset of $C_0(\Gamma, C_p(N \times N))$ which is invariant under projections. An application of Lemma 2.1 gives the desired result.

Note. Theorem 1 of [2] states that $C_0(Z)$ is Lindelöf under the w-topology derived from giving $C_0(Z)$ the uniform norm. Let S be the unit cell of the Banach space $C_0(Z)$. Then $C_0(Z)$ is w-Lindelöf iff S is w-Lindelöf. Using the Riesz representation theorem for $C_0(Z)^*$ it is easy to prove that the w-topology on S is weaker than the compact open topology. Consequently Theorem 2.5 with $L = R$ implies the result of [2].

3. **Examples.** We begin with comments and examples concerning the various assumptions in the statement of Lemma 2.1.

(1) It is easily seen that Lemma 2.1 no longer holds if we drop the assumption that A is almost invariant under projections. In fact, even $C_0(\Gamma)$ has subspaces which are not Lindelöf if Γ is uncountable. Let y_γ, $\gamma \in \Gamma$, be the element of $C_0(\Gamma)$ whose γ coordinate is 1 and all the rest are 0, and let $A = \{y_\gamma\}_{\gamma \in \Gamma}$. A is a discrete uncountable space and hence not Lindelöf. A is in a sense "nearly" invariant under projections—if we add to A the zero function we obtain a subset of $C_0(\Gamma)$ which is invariant under projections (this set is, of course, also homeomorphic to the one point compactification of A).

(2) We cannot drop the assumption concerning the existence of X_f and F in Lemma 2.1. The example which shows this is due to E. Michael. It is known [7] that there is an uncountable separable metric space X such that every countable subset of X is a G_δ set. In fact X can be taken to be a suitable subset of R. Let us recall the construction. To every ordinal α, less than the first uncountable ordinal Ω, we assign a G_δ subset X_α of R with Lebesgue measure 0 so that for every α $X_\alpha \supset \bigcup_{\beta<\alpha} X_\beta$ with proper inclusion. This can be done since every subset of R

with measure 0 is contained in a G_δ set with measure 0. For every α pick a point $x_\alpha \in X_\alpha \sim \bigcup_{\beta<\alpha} X_\beta$. The set $X = \{x_\alpha : \alpha < \Omega\}$ has the desired properties.

We claim that for this space X the spaces $C_p(X, C_0(\Gamma))$ and $C_p(X, H^p)$ are not Lindelöf spaces if Γ is discrete uncountable and H^p is a nonseparable Hilbert space with the topology of coordinatewise convergence (with respect to a fixed orthonormal basis). The proof is actually valid for every separable space X_1 which has an uncountable set X_2 such that every countable subset of X_2 is a G_δ subset of X_1. For the proof it is clearly enough to consider the case where the cardinality of Γ (or of the basis of the Hilbert space) is $\aleph_1$, and hence let $\Gamma = \{\alpha : \alpha < \Omega\}$. Let U_α, $\alpha \in \Gamma$, be the open subset of $C_p(X, C_0(\Gamma))$ defined by

$$U_\alpha = \{f : |f(x_\alpha)(\alpha)| < 1\}.$$

Since X is separable there is for every $f \in C_p(X, C_0(\Gamma))$ a countable subset Γ_f of Γ such that $f(\alpha) \equiv 0$ (and hence $f(x_\alpha)(\alpha) = 0$) if $\alpha \in \Gamma \sim \Gamma_f$. Consequently the $\{U_\alpha\}_{\alpha \in \Gamma}$ form an open cover of $C_p(X, C_0(\Gamma))$. Let Γ_0 be a countable subset of Γ. The set $X_0 = \{x_\alpha : \alpha \in \Gamma_0\}$ is a G_δ subset of X, and hence there is a decreasing sequence $\{V_n\}_{n=1}^\infty$ of open subsets of X such that $X_0 = \bigcap_{n=1}^\infty V_n$. Let $\{\alpha_n\}_{n=1}^\infty$ be an enumeration of the elements of Γ_0, and let $f_n \in C_p(X, [0,1])$ be such that $f_n(x_{\alpha_n}) = 1$, $f_n(x_{\alpha_i}) = 0$ if $i < n$ and $f_n(x) = 0$ if $x \in X \sim V_n$. Then for every $x \in X$ there is only a finite number of indices n such that $f_n(x) \neq 0$. Let $f \in C_p(X, C_0(\Gamma))$ be defined by

$$f(\alpha)(x) = \begin{cases} f_n(x) & \text{if } \alpha = \alpha_n,\ n = 1, 2, \cdots, \\ 0 & \text{if } \alpha \notin \Gamma_0. \end{cases}$$

Note that f maps X into the subspace of $C_0(\Gamma)$ consisting of all the elements which vanish off a finite subset of Γ and in particular into H^p (in its canonical embedding in $C_0(\Gamma)$). For every $\alpha \in \Gamma_0$ we have that $f(\alpha)(x_\alpha) = 1$ and hence $\bigcup_{\alpha \in \Gamma_0} U_\alpha$ does not contain $C_p(X, C_0(\Gamma))$ and it does not cover even the entire space $C_p(X, H^p)$. This concludes the proof of the fact that neither $C(X, C_0(\Gamma))$ nor $C_p(X, H^p)$ are Lindelöf. This example shows also that the completeness assumption cannot be discarded in the statement of Theorem 2.2.

(3) Lemma 2.1, as well as Theorems 2.2, 2.3, and 2.4, no longer holds if we do not require that X be separable. Let X be the disjoint union of an uncountable number of copies of $I = [0,1]$. Then X is locally compact but $C_p(X, I)$ is not a Lindelöf space. In fact, $C_p(X, I)$ is homeomorphic to the product of an uncountable number of copies of $C_p(I, I)$ and, since $C_p(I, I)$ is not countably compact, $C_p(X, I)$ is not even normal (cf. A.H. Stone [10]). If Γ is a discrete space, then $C_p(\Gamma, I)$ is compact and hence Lindelöf, but if Γ is uncountable, $C_p(\Gamma, R)$ is not even normal ($C_p(\Gamma, R)$ is, of course, the product of an uncountable number of copies of R). Consequently we cannot replace $C_0(\Gamma, C_p(X))$ in Lemma 2.1 by $C_p(\Gamma, C_p(X))$ or replace $C_0(\Gamma)$ by $C_p(\Gamma)$ in Theorems 2.2 and 2.3.

(4) Theorem 2.5 shows that if X is a countable space then Lemma 2.1 is still valid if we replace the discrete space Γ by a locally compact metric space Z. We now present an example which shows that this may fail to hold for uncountable separable (in fact, compact) metric spaces X. We shall show that *the spaces* $C(I, C_p(I))$ *and* $C_p(I, C_p(I))$ *are not normal spaces and therefore not Lindelöf.* We identify $C(I, C_p(I))$ with the space $C^s(I \times I)$ consisting of all separately continuous real-valued functions on $I \times I$ with the topology of uniform convergence on sets of the form $\{x\} \times I$, $x \in I$. Similarly, $C_p(I, C_p(I))$ can be identified with $C_p^s(I \times I)$ which consists of the same functions as $C^s(I \times I)$ but its topology is that of pointwise convergence on $I \times I$. The polynomials in two variables and rational coefficients are dense in both $C^s(I \times I)$ and $C_p^s(I \times I)$. We shall show that $C_p^s(I \times I)$ has a subset A of cardinality $\aleph$ (the cardinality of I) which is discrete and closed. This set will, of course, be also discrete and closed in $C^s(I \times I)$. By Tietze's extension theorem the existence of such a set A implies that $C^s(I \times I)$ and $C_p^s(I \times I)$ are not normal. Indeed, the cardinality of $C(A, I)$ is $2^{\aleph}$ while for every separable B (and hence for $C^s(I \times I)$ and $C_p^s(I \times I)$)the cardinality $C(B, I)$ is $\leqq \aleph$.

Let $t \in I$ and let $f_t \in C^s(I \times I)$ be a function which satisfies

$$(*) \qquad \begin{aligned} &f_t(x, y) = 1 \text{ if } x \leqq t \leqq y \text{ or if } y \leqq t \leqq x, \\ &f_t(x, x) = 0 \text{ if } t \neq x. \end{aligned}$$

Such an f_t is easy to construct. Take $A = \{f_t : t \in I\}$. This set has the required properties. Indeed, assume that there is a net $\{f_{t_\alpha}\}$ in A which is not ultimately constant and which converges pointwise on $I \times I$ to some $g \in C^s(I \times I)$. By passing to a subnet, if necessary, we may assume that $t_\alpha \to t$ for some $t \in I$. We have

$$g(t, t) = \lim_\alpha f_{t_\alpha}(t, t) = 0.$$

Since g is separately continuous there is an $h > 0$ such that $|g(t + h, t)| < 1/2$ and $|g(t - h, t)| < 1/2$ (if $t = 1$ we omit the requirement concerning $t + h$, similarly if $t = 0$). Since the f_{t_α} converge pointwise to g there is an α_0 such that $\alpha > \alpha_0$ implies

$$(**) \qquad |f_{t_\alpha}(t + h, t)| < 1 \text{ and } |f_{t_\alpha}(t - h, t)| < 1.$$

By (*) and (**) $t_\alpha \notin (t - h, t + h)$ for $\alpha > \alpha_0$ and this contradicts the assumption that $t_\alpha \to t$.

This concludes our discussion of the conditions appearing in Lemma 2.1 and the other results of §2 which deal with the topology of pointwise convergence in the function spaces. We now turn our attention to the compact open topology.

Let X be a locally compact metric space, and let L be a linear space. If L is

metrizable and separable (e.g. $L = H^n$ or $L = H^p$ where H is a separable Hilbert space) then, by Theorem 2.5, $C_0(X, L)$ is a Lindelöf space. This is true also for some nonmetrizable L; H^w is an example if H is separable. Indeed, for every $f \in C_0(X, H^w)$, $f(X)$ is a bounded subset of the Hilbert space and hence $C_0(X, H^w) = \bigcup_{n=1}^{\infty} C_0(X, nS^w)$ where S is the unit cell of H. Since S^w is metrizable Theorem 2.5 implies also that $C_0(X, H^w)$ is a Lindelöf space.

Let us now assume that H is a nonseparable Hilbert space. Clearly $C_0(X, H^n)$ is not Lindelöf even if X consists of a single point. The spaces $C_0(X, H^p)$ and $C_0(X, H^w)$ are not Lindelöf unless the locally compact metric space X is discrete and countable. If X is countable and discrete then $C_0(X, H^p)$ and $C_0(X, H^w)$ are Lindelöf spaces by the results of §2. In order to show that for all other locally compact metric X, $C_0(X, H^w)$ and $C_0(X, H^p)$ are not Lindelöf it is enough to show that for these X, $C_0(X, S^w)$ is not Lindelöf. If X is discrete and uncountable then it is easy to see that $C_0(X, S^w)$ has a closed subset which is homeomorphic to a product of an uncountable number of copies of $C_0(N, I)$ ($N =$ the integers, $I = [0,1]$) and therefore $C_0(X, S^w)$ is not even normal. Assume now that X contains a subset X_0 homeomorphic to the one point compactification of N, and let $\{e_\gamma\}_{\gamma \in \Gamma}$ be an orthonormal basis of H. For every $\gamma \in \Gamma$ let U_γ be the open subset of $C_0(X, S^w)$ defined by

$$U_\gamma = \{f : f \varepsilon C_0(X, S^w), \left|(f(x), e_\gamma)\right| < 1 \text{ for all } x \varepsilon X_0\}.$$

Since $f(X_0)$ is separable for every $f \in C_0(X, S^w)$ there is always a $\gamma \in \Gamma$ such that $(f(x), e_\gamma) = 0$ for every $x \in X_0$. Hence $\bigcup_{\gamma \in \Gamma} U_\gamma \supset C_0(X, S^w)$. Let $X_0 = \{x_1, x_2, \cdots, \tilde{x}\}$ with $x_i \to \tilde{x}$ and let $\{\gamma_i\}_{i=1}^{\infty} \in \Gamma$. There is an $\tilde{f} \in C_0(X, S^w)$ such that $\tilde{f}(x_i) = e_{\gamma_i}$, $i = 1, 2, \cdots$ (and $\tilde{f}(\tilde{x}) = 0$). This can be easily shown directly and it follows also from the extension theorem of Dugundji [4]. We have that $\tilde{f} \notin \bigcup_{i=1}^{\infty} U_{\gamma_i}$ and hence $C_0(X, S^w)$ is not Lindelöf.

Take now, in particular, a compact infinite metric space X. $C(X)$ is a complete separable metric space. $C(X, S^w)$ can be identified with the closed subspace of $\prod = \prod_{\gamma \in \Gamma} C_\gamma$ (where $C_\gamma = C(X)$ for every γ) consisting of the $f \in \prod$ for which $\sum_\gamma f^2(\gamma)(x) \leqq 1$, $x \in X$. This subspace of $\prod$ is a closed subspace of the Σ product (cf. [1]) of the C_γ, i.e., the set of all elements of $\prod$ whose coordinates vanish for the γ outside a countable subset of Γ. These remarks, together with the observation made above that $C(X, S^w)$ is not Lindelöf, show that $C(X, S^w)$ can replace the space F_0 used in [1] for giving a counterexample to a conjecture of Kelley (we refer to [1] for details). This may be of interest since $C(X, S^w)$ arises by standard operations while F_0 is constructed purely for its unusual properties.

Let X be a separable metric space and let H be a separable Hilbert space. Then $C(X, H^n)$, $C(X, H^p)$ and $C(X, H^w)$ are Lindelöf spaces. For $C(X, H^n)$ and $C(X, H^p)$ this fact is an immediate consequence of Lemma 4.1 (H^n and H^p are separable metric spaces), while for $C(X, H^w)$ we have to use, besides Lemma 4.1, the observation that every element in $C(X, H^w)$ is locally bounded. Michael [9]

has proved a stronger result; namely that for these X and H, $C(X,H^n)$, $C(X,H^p)$ and $C(X,H^w)$ are $\aleph_0$ spaces, a property which is stronger than the Lindelöf property.

Let us sum up the results concerning mappings from metric spaces into Hilbert spaces. In the squares of the first row of the table below we have written down *all locally compact metric spaces* X for which $C_0(X,Y)$ is a Lindelöf space. In the other two rows we have written down *all the metric spaces* X for which the corresponding function space is Lindelöf. All the results contained in the table were either stated before or follow easily from the preceding discussions.

	H separable			H nonseparable		
	$Y=H^n$	$Y=H^w$	$Y=H^p$	$Y=H^n$	$Y=H^w$	$Y=H^p$
$C_0(X, Y)$	all X	all X	all X	no X	iff X is discrete countable	iff X is discrete countable
$C_p(X, Y)$	iff X is separable	iff X is separable	iff X is separable	no X	iff X is separable	if X is a continuous image of a complete separable metric space. Not for all separable X. For no nonseparable X.
$C(X, Y)$	iff X is separable	iff X is separable	iff X is separable	no X	iff X is discrete countable	iff X is discrete countable

4. **The proof of Lemma 2.1.** This section is devoted to the proof of Lemma 2.1. The statement of Lemma 2.1 becomes much simpler if we consider the special case where X is complete metric and $A = C_0(\Gamma, C_p(X))$. However our proof of this special case is not simpler than the proof of the more general result.

We state now three simple results which we shall need in the proof of Lemma 2.1. Lemma 4.1 is due, essentially, to Michael (cf. [9]).

LEMMA 4.1. *Let X be a topological space with a countable base, and let Γ_0 be a countable discrete space. Then for any subset $A \subset C_p(\Gamma_0, C_p(X))$ there is a countable collection of subsets $\mathcal{S}$ of A such that, given any $f \in A$ and any open set U containing f, $f \in S \subset U$ for some S in $\mathcal{S}$.*

Proof. $C_p(\Gamma_0, C_p(X))$ is homeomorphic to $C_p(Y)$ where Y is the disjoint union

of countably many copies of X. Consequently, we need only to prove the lemma for subsets A of $C_p(Y)$. Note that Y has a countable base $\mathscr{B}$ since X does. Let $\mathscr{D}$ be a countable base for the reals.

For each $B \in \mathscr{B}$ and each $D \in \mathscr{D}$ let $(B, D) = \{f \in A : f(B) \subset D\}$. The collection of all such (B, D) has the required properties.

LEMMA 4.2. *Let Y be an uncountable separable metric space. Then Y has a countable subset K such that no point of K is isolated in K.*

Proof. Take as K a countable dense set in the perfect kernel of Y.

LEMMA 4.3. *Let Y be a complete separable metric space and let K be a countable dense subset of Y. Let θ be a function from K to the open sets in Y such that $k \in \theta(k)$ for each $k \in K$. If no point of K is isolated in Y, then there is some $y \in Y$ such that $y \in \theta(k)$ for infinitely many k.*

Proof. Let $\theta(k, n)$, $k \in K$, $n = 1, 2, \cdots$, be an open subset of $\theta(k)$ which contains k and has a diameter less than $1/n$. Let A_n be the complement of $\bigcup_{k \in K} \theta(k, n)$. A_n is closed and nowhere dense in Y. If each point of Y were in $\theta(k)$ for only finitely many k, then every point of Y is either in K or some A_n. This contradicts Baire's theorem.

The main step in the proof of Lemma 2.1 is the proof of the next lemma. This lemma is, in a sense, a local version of Lemma 2.1. Before stating this lemma, we introduce some notations.

A net $\{f_\alpha\}$ in $C_0 = C_0(\Gamma, C_p(X))$ converges to an element $f \in C_0$ if and only if $f_\alpha(\gamma)(x) \to f(\gamma)(x)$ for every $\gamma \in \Gamma$ and $x \in X$. It follows that C_0 has a base composed of sets constructed as follows: Pick a finite subset P of $\Gamma \times X$ and a function ψ from P to the open subsets of the real numbers. Let $[P, \psi]$ denote the set of all $f \in C_0$ such that $f(\gamma)(x) \in \psi(\gamma, x)$ for each $(\gamma, x) \in P$. The $[P, \psi]$ are sets to be used as a base for C_0. For every set $P \subset \Gamma \times X$, let

$$P' = \{\gamma : (\gamma, x) \in P \text{ for some } x \in X\}$$

and

$$P'' = \{x : (\gamma, x) \in P \text{ for some } \gamma \in \Gamma\}.$$

LEMMA 4.4. *Assume that X is a separable metric space and that Γ is discrete. Let A be a subset of $C_0 = C_0(\Gamma, C_p(X))$ such that, for every $f \in A$, there is G_δ subset X_f of $\hat{X}$ (the completion of X) containing X, and an $F \in C_0(\Gamma, C_p(X_f))$ such that $F(\gamma)(x) = f(\gamma)(x)$ for every $\gamma \in \Gamma$ and $x \in X$. Let n be an integer and let $\mathscr{D}$ be a finite collection of open sets around 0 in R such that $R \in \mathscr{D}$. Let $\mathscr{U}$ be a collection of subsets of C_0 consisting of sets of the form $A \cap [P, \psi] = U$ where P is a subset of $\Gamma \times X$ with at most n elements, and ψ is a function from P to $\mathscr{D}$ (P and ψ depend on U in $\mathscr{U}$ but $\mathscr{D}$ is fixed). Then there is a countable subset $\mathscr{V}$ of $\mathscr{U}$ such that $\bigcup \mathscr{U} = \bigcup \mathscr{V}$.*

Proof. The proof is by induction on n. The lemma is obvious for $n = 0$, so suppose that it is true for $n - 1$. Moreover, suppose that it is not valid for the collection $\mathscr{U}$. We will argue for a contradiction.

Suppose that there is a countable subset Γ_0 of Γ such that, for each

$$[P, \psi] \cap A \in \mathscr{U},$$

some $\gamma \in P'$ is also in Γ_0. Let $\mathscr{S}$ be a countable collection of subsets of A/Γ_0 having the properties of Lemma 4.1. For each $S \in \mathscr{S}$, let $\mathscr{U}(S) = \{U \in \mathscr{U}: U/\Gamma_0 \supset S\}$, let $S^* = \{f \in C_0: f/\Gamma_0 \in S\}$, and let $\mathscr{U}(S^*) = \{U \cap S^*: U \in \mathscr{U}(S)\}$. It is easy to see that each element of $\mathscr{U}(S^*)$ is of the form $[F, \psi] \cap A \cap S^*$, where F has fewer than n elements. By the inductive assumption, there is a countable subcover $\mathscr{V}(S^*)$ for $\mathscr{U}(S^*)$. Let $\mathscr{V}(S)$ be a countable subset of $\mathscr{U}(S)$ such that each $V^* \in \mathscr{V}(S^*)$ is of the form $V \cap S^*$ for some $V \in \mathscr{V}(S)$. Then

$$\mathscr{V} = \bigcup\{\mathscr{V}(S): S \in \mathscr{S}\}$$

is a countable subcover for $\mathscr{U}$.

Since we have assumed that the lemma is not valid for $\mathscr{U}$, it must be true that, for each countable $\Gamma_0 \subset \Gamma$, there is a $[P, \psi] \cap A \in \mathscr{U}$ such that $P' \cap \Gamma_0 = \varnothing$. For each ordinal α less than the first uncountable one Ω, let us pick a $U_\alpha = [P_\alpha, \psi_\alpha] \cap A$ such that $P'_\alpha \cap P'_\beta = \varnothing$ if $\alpha \neq \beta$. This is possible because $\bigcup\{P'_\alpha: \alpha < \beta\}$ is countable and there is consequently some $[P, \psi]$ such that P' does not meet this set.

We will now assume that each P_α has exactly n elements. This is possible, since we may just add some more elements to P_α if necessary and define ψ_α on these new elements to be R.

Let X^n denote the product of n copies of X. For each $\alpha < \Omega$ we pick a point $x^\alpha = (x_1^\alpha, x_2^\alpha, \cdots, x_n^\alpha)$ in X^n such that each $z \in P''_\alpha$ is equal to x_i^α for exactly as many i as there are (γ, x) in P_α with $x = z$.

First suppose that $\{x^\alpha: \alpha < \Omega\}$ is a countable set. Then some fixed $y \in X^n$ equals x^α for uncountably many α. Let K be a countable (infinite) set consisting of such α and let $D = \cap \mathscr{D}$. Since the lemma is not valid for $\mathscr{U}$ there is a function f in $\bigcup \mathscr{U}$, and hence in C_0, such that f is not in U_α for any $\alpha \in K$. This means that, whenever $\alpha \in K, f(\gamma_\alpha)(y_j) \notin \psi_\alpha(\gamma_\alpha, y_j)$ for some $\gamma_\alpha \in P_\alpha$ and some j. Since y has only n coordinates, there is some fixed j_0 which has the latter property for infinitely many $\alpha \in K$. Consequently, for all such α, $f(\gamma_\alpha)(y_{j_0})$ is not in D. That contradicts the fact that f vanishes at infinity on Γ.

The previous argument implies that $\{x^\alpha: \alpha < \Omega\}$ is an uncountable subset of X^n. By Lemma 4.2, there is a countable set K of ordinals less than Ω such that $\{x^\alpha: \alpha \in K\}$ has no isolated point. Since the lemma is not valid for $\mathscr{U}$, there is some $f \in A$ which is not in U_α for any $\alpha \in K$. Let X_f be a G_δ subset of $\hat{X}$ containing X such that there is an $F \in C_0(\Gamma, C_p(X_f))$ for which $f(\gamma)(x) = F(\gamma)(x)$, $\gamma \in \Gamma$, $x \in X$.

By the choice of f we see that for every $\alpha \in K$ there is some $\gamma_\alpha \in P'_\alpha$ and some j_α such that $f(\gamma_\alpha)(x^\alpha_{j_\alpha})$ is not in $\psi_\alpha(\gamma_\alpha, x^\alpha_{j_\alpha})$. Consequently,

$$F(\gamma_\alpha)(x^\alpha_{j_\alpha}) = f(\gamma_\alpha)(x^\alpha_{j_\alpha}) \notin D \qquad \text{if} \quad \alpha \in K.$$

Let $\varepsilon > 0$ be chosen so that all real numbers within 2ε of 0 are contained in D. For each $\alpha \in K$ let B_α be an open set in X_f containing $x^\alpha_{j_\alpha}$ such that $F(\gamma_\alpha)$ varies by less than ε over B_α. Then $|F(\gamma_\alpha)(x)| > \varepsilon$ for each x in B_α.

We denote by W_α the open set in X^n_f (the Cartesian product of n copies of X_f) around x^α defined by: $y \in W_\alpha$ iff $y_{j_\alpha} \in B_\alpha$. By Lemma 4.3, there is a point $q \in X^n_f$ such that $q \in W_\alpha$ for each $\alpha \in K_0$ where K_0 is an infinite subset of K. (We apply Lemma 4.3 by taking as Y the closure of $\{x^\alpha : \alpha \in K)$ in X^n_f, and noticing that X_f, and hence X^n_f, can be given a complete metric.) It follows that $q_{j_\alpha} \in B_\alpha$ for each $\alpha \in K_0$. Since q has only a finite number of coordinates there is a j_0 and an infinite subset K_1 of K_0 such that $q_{j_0} \in B_\alpha$ for each $\alpha \in K_1$. Hence $|F(\gamma_\alpha)(q_{j_0})| > \varepsilon$ for each $\alpha \in K_1$. This contradicts the fact that F vanishes at infinity and completes the proof of the lemma.

We pass now to the proof of Lemma 2.1 itself. Let A be a subset of $C_0(\Gamma, C_p(X))$ which satisfies the assumptions in the statement of the lemma. We call a countable subset of Γ admissible if it is one of the subsets whose existence is ensured by the assumption that A is almost invariant under projections.

Let $\mathscr{U}$ be an open cover of A by sets of the form $[P, \psi] \cap A$. Let $\mathscr{B}$ be a fixed countable base of R and let $\mathscr{B}_0 = \{B \in \mathscr{B} : 0 \in B\}$. We may assume that the range of each ψ is contained in $\mathscr{B}$. Let Γ_0 be a countable subset of Γ. By Lemma 4.1 there is a countable collection $\mathscr{S}$ of subsets of A/Γ_0 such that, whenever $f \in U/\Gamma_0$ for some $U \in \mathscr{U}$, there is an $S \in \mathscr{S}$ with $f \in S \subset U/\Gamma_0$. Define for each $S \in \mathscr{S}$ and each finite subset $\mathscr{D}$ of $\mathscr{B}_0$

$$\mathscr{U}(S, \mathscr{D}) = \{U = [P, \psi] \cap A \in \mathscr{U} : S \subset U/\Gamma_0 \text{ and } \psi(\gamma, x) \in \mathscr{D} \text{ if } (\gamma, x) \in P \text{ and } \gamma \notin \Gamma_0\}.$$

Let $S^* = \{f \in A : f/\Gamma_0 \in S\}$. Define $\mathscr{U}^*(S, \mathscr{D}) = \{U \cap S^* : U \in \mathscr{U}(S, \mathscr{D})\}$. Each element of $\mathscr{U}^*(S, \mathscr{D})$ is of the form $S^* \cap A \cap [P, \psi]$ for $[P, \psi]$ such that $\psi(\gamma, x) \in \mathscr{D}$ for every $(\gamma, x) \in P$. Consequently, by Lemma 4.4 there is a countable subcover $\mathscr{V}^*(S, \mathscr{D})$ of $\mathscr{U}^*(S, \mathscr{D})$. Let $\mathscr{V}(S, \mathscr{D})$ be a countable subset of $\mathscr{U}(S, \mathscr{D})$ such that each $V^* \in \mathscr{V}^*(S, D)$ is of the form $V \cap S^*$ for some $V \in \mathscr{V}(S, \mathscr{D})$. Let $\mathscr{V}_0 = \bigcup\{\mathscr{V}(S, \mathscr{D}) : S \in \mathscr{S} \text{ and } \mathscr{D} \subset \mathscr{B}_0\}$. $\mathscr{V}_0$ is countable and $\bigcup \mathscr{V}_0$ contains $\bigcup \mathscr{U}(S, \mathscr{D})$ for all $S \in \mathscr{S}$ and $\mathscr{D} \subset \mathscr{B}_0$.

It follows from the preceding argument that we can construct, inductively, a sequence Γ_i of subsets of Γ and a sequence $\mathscr{V}_i$ of countable subsets of $\mathscr{U}$ so that

(i) Each Γ_i is countable and admissible,

(ii) $\Gamma_i \subset \Gamma_{i+1}$, $i = 1, 2 \cdots$,

(iii) For every i, $\bigcup \mathscr{V}_i$ contains each $[P, \psi] \cap A \in \mathscr{U}$ for which $\psi(\gamma, x) \in \mathscr{B}_0$ whenever $(\gamma, x) \in P$ and $\gamma \notin \Gamma_i$,

(iv) For $i \geqq 2$,

$$\Gamma_i \supset \bigcup\{P' : [P,\psi] \cap A \in \mathscr{V}_{i-1}\}.$$

Let $\Gamma_\infty = \bigcup_{i=1}^{\infty} \Gamma_i$, and let $f \in A$. Since Γ_∞ is admissible, $f/\Gamma_\infty \in A$. Hence there is a $U = [P,\psi] \cap A \in \mathscr{U}$ such that $f/\Gamma_\infty \in U$. Since P is finite, there is a finite i such that $P' \cap \Gamma_\infty = P' \cap \Gamma_i$. Since f/Γ_∞ vanishes outside Γ_∞, $\psi(\gamma, x) \in B_0$ if $(\gamma, x) \in P$ and $\gamma \notin \Gamma_\infty$, and hence (by (iii)) U is contained in $\bigcup \mathscr{V}_i$. Therefore $f/\Gamma_\infty \in \bigcup \mathscr{V}_i$. Since $f/\Gamma_\infty = f$ on Γ_{i+1}, it follows (use (iv)) that $f/\Gamma_\infty \in \bigcup \mathscr{V}_i$ implies that also $f \in \bigcup \mathscr{V}_i$. Therefore any $f \in A$ is in $\bigcup \mathscr{V}$, where $\mathscr{V} = \bigcup\{\mathscr{V}_i : i = 1, 2, \cdots\}$. The subcover $\mathscr{V}$ of $\mathscr{U}$ is countable. This concludes the proof of Lemma 2.1.

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