BRAID GROUPS OF COMPACT 2-MANIFOLDS WITH ELEMENTS OF FINITE ORDER(1)

BY
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I. **Introduction.** Braid groups were introduced in 1925 by E. Artin [1]. His 1947 paper [2] rigorized this somewhat intuitive first treatment of braids of the plane; and within a year papers on the subject by F. Bohnenblust [4], W.-L. Chow [5] and yet another by Artin [3] appeared. Artin's presentation of the n-string braid group of E^2 as a group on the $n-1$ generators $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ subject to the relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$

$$\sigma_j\sigma_k = \sigma_k\sigma_j \qquad (j-k \geqq 2)$$

is now classical, with proofs of the completeness of these relations by Bohnenblust [4], Chow [5], R. H. Fox and L. Neuwirth [10] and E. Fadell and J. Van Buskirk [9].

The last two papers mentioned use a recent definition of braid group by Fox [10] which reinterprets Artin's definition of braids of the plane and extends it to define braid groups of arbitrary topological spaces. As noted by Neuwirth, if the topological space is a manifold, then the situation gives rise to associated fiber spaces which yield information on the homotopy groups of certain configuration spaces which then furnish information on the braid groups themselves.

Basic results on braid groups of arbitrary manifolds are obtained by Fadell and Neuwirth [8] as an application of their theory of configuration spaces of manifolds. One such result, due originally to Neuwirth, states that neither the plane nor any compact 2-manifold, with the possible exceptions of the 2-sphere and the projective plane, has a braid group with finite order elements. The 2-sphere case is settled in [9], where an element of finite order in the n-string braid group of the 2-sphere is exhibited for each n.

The purpose of this paper is to find a presentation of the n-string braid group of the projective plane and then, in order to obtain the following result, to exhibit an element of finite order for each n.

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(1) The results in this paper are portions of the author's doctoral dissertation which was prepared under the patient guidance of Professor Edward R. Fadell.

THEOREM. *Let M be a compact 2-manifold or the plane. A necessary and sufficient condition that the n-string braid group on M have an element of finite order is that M be the 2-sphere or the projective plane.*

The treatment of the algebraic braid group draws heavily on the methods of Chow [5] while the treatment of the geometric braid group and the isomorphism between the algebraic and geometric braid groups is essentially that of [9].

II. **Topological preliminaries.** 1. If M is a manifold of dimension at least 2 and $Q_m = \{q_1, \cdots, q_m\}$ a fixed set of m distinct points of M, then the *configuration space* $F_{m,n}(M)$ is defined to be the set of n-tuples of points of M which are distinct from the points of Q_m, as well as from one another. That is,

$$F_{m,n}(M) = \{(x_1, x_2, \cdots, x_n) : x_1 \in M - Q_m,\ x_i \neq x_j \quad (i \neq j\}.$$

The topology on $F_{m,n}(M)$ is that induced naturally by the topology of M.

The following basic theorem is found in [8].

THEOREM. *The map $\pi: F_{m,n}(M) \to F_{m,n-r}(M)$ given by $\pi(x_1, \cdots, x_n) = (x_{r+1}, \cdots, x_n)$ is a locally trivial fiber map with fiber $F_{m+n-r,r}(M)$.*

In the above theorem, if $(p_{r+1}, \cdots, p_n)$ is a fixed base point for $F_{m,n-r}(M)$, then the set

$$Q_{m+n-r} = \{q_1, \cdots, q_m, p_{r+1}, \cdots, p_n\}$$

is used in forming $F_{m+n-r,r}(M)$.

2. The following results of Fadell [7] are crucial in the computation of the braid groups of P^2. Let $\mathscr{A}_n(S^2)$ be the set of n-tuples of points of S^2 which are distinct and nonantipodal.

THEOREM. *The map $\lambda: \mathscr{A}_n(S^2) \to \mathscr{A}_{n-1}(S^2)$, $n \geqq 2$, defined by $\lambda(x_1, x_2, \cdots, x_n) = (x_2, \cdots, x_n)$ is a locally trivial fiber map with fiber $F^*_{2(n-1),1}(S^2)$, where $F^*_{2(n-1),1}(S^2)$ is the 2-sphere less $n-1$ distinct antipodal pairs of points.*

LEMMA. $\pi_i(\mathscr{A}_n(S^2)) = \pi_i(F_{0,n}(P^2))$, $i \geqq 2$.

THEOREM. *$(\mathscr{A}_2(S^2), \lambda, S^2)$ is fiber homotopy equivalent to $(V_{3,2}, g, S^2)$, where the Stiefel manifold $V_{3,2}$ is the space of orthogonal 2-frames in 3-space and $g: V_{3,2} \to S^2$ is the fiber map defined by $g(v_1, v_2) = v_2$ with fiber S^1.*

COROLLARY. $\pi_2(F_{0,n}(P^2)) = 0$, $n \geqq 2$.

Proof. The triviality of $\pi_2(V_{3,2})$ implies that $\pi_2(\mathscr{A}_2(S^2))$, and hence $\pi_2(F_{0,2}(P^2))$, is trivial. Now consider the homotopy sequence of the fibration f of $F_{0,n}(P^2)$ over $F_{0,n-1}(P^2)$ with fiber $F_{n-1,1}(P^2)$ and apply induction.

III. **The algebraic braid groups.** The algebraic braid group of the projective plane on n strings, $B_n(P^2)$, is defined to be the group on the $2n-1$ generators $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$, $\rho_1, \rho_2, \cdots, \rho_n$ subject to the following relations:

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ ($|i-j| \geqq 2$)

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

(iii) $\sigma_i\rho_j = \rho_j\sigma_i$ ($j \neq i, i+1$)

(iv) $\rho_i = \sigma_i\rho_{i+1}\sigma_i$

(v) $\rho_{i+1}^{-1}\rho_i^{-1}\rho_{i+1}\rho_i = \sigma_i^2$

(vi) $\rho_1^2 = \sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2\sigma_1$.

The map taking σ_i onto the transposition $(i, i+1)$ and each ρ_j onto the identity is seen to be a homomorphism, say α, of $B_n(P^2)$ onto Σ^n the symmetric group of degree n, with kernel, say, $K_n(P^2)$. Let $D_n(P^2)$ denote the subgroup of all elements of $B_n(P^2)$ which map, under α, onto permutations of Σ^n leaving 1 fixed. Since the permutations of Σ^n take any of $1, 2, \cdots, n$ into 1, $D_n(P^2)$ is of index n in $B_n(P^2)$. The elements $M_i = \sigma_i\sigma_{i-1}\cdots\sigma_1$ ($i = 1, 2, \cdots, n-1$), $M_0 = 1$ serve as right coset representatives of $D_n(P^2)$, since the permutation $\alpha(M_i)$ takes $i+1$ into 1, as does the image under α of any element of the coset $M_iD_n(P^2)$. Note that the M_i's form a Schreier system of coset representatives. That is, each "segment" (starting from the right) of an M_i is also an M_i.

The following identities, derived from the relations (i) to (vi) will be used to determine a presentation of $D_n(P^2)$.

LEMMA. *In* $B_n(P^2)$

$$\begin{aligned}
\sigma_k^\varepsilon M_i &= M_i\sigma_{k+1}^\varepsilon \ (k < i,\ \varepsilon = \pm 1)\\
\sigma_i M_i &= M_{i-1}a_{i+1}\\
\sigma_i^{-1}M_i &= M_{i-1}\\
\sigma_{i+1}M_i &= M_{i+1}\\
\sigma_{i+1}^{-1}M_i &= M_{i+1}a_{i+2}^{-1}\\
\sigma_k^\varepsilon M_i &= M_i\sigma_k^\varepsilon \ (k > i+1,\ \varepsilon = \pm 1)\\
\rho_j^\varepsilon M_i &= M_i(\rho_{j+1}a_{j+1})^\varepsilon \ (j \leqq i,\ \varepsilon = \pm 1)\\
\rho_{i+1}^\varepsilon M_i &= M_i(a_{i+1}^{-1}a_i^{-1}\cdots a_2^{-1}\rho_1)^\varepsilon \ (\varepsilon = \pm 1)\\
\rho_j^\varepsilon M_i &= M_i\rho_j^\varepsilon \ (j > i+1,\ \varepsilon = \pm 1)\\
\textit{where } a_i &= M_{i-2}^{-1}\sigma_{i-1}^2M_{i-2}.
\end{aligned}$$

Proof. The last identity follows, by (iii), from the definition of M_i and the first six identities were derived by Chow [5], using the relations (i) and (ii) which $B_n(P^2)$ has in common with $B_n(E^2)$. Now $a_2a_3\cdots a_i = \sigma_1\sigma_2\cdots\sigma_{i-2}\sigma_{i-1}^2\sigma_{i-2}\cdots\sigma_2\sigma_1$ implies $\rho_{i+1}M_i = \rho_{i+1}\sigma_iM_{i-1} = \sigma_i^{-1}\rho_iM_{i-1} = M_ia_{i+1}^{-1}a_i^{-1}\cdots a_2^{-1}\rho_1$. Finally if $j \leqq i$, $\rho_jM_i = \sigma_i\cdots\sigma_{j+1}\rho_j\sigma_j^{-1}\sigma_{j-1}\cdots\sigma_1a_{j+1} = \sigma_1\cdots\sigma_{j+1}\sigma_j\sigma_{j-1}\cdots\sigma_1\rho_{j+1}a_{j+1} = M_i\rho_{j+1}a_{j+1}$.

LEMMA. $D_n(P^2)$ *is generated by* $\sigma_2, \sigma_3, \cdots, \sigma_{n-1}$, $\rho_1, \rho_2, \cdots, \rho_n$, $a_2, a_3, \cdots, a_n$.

Proof. Application of the Reidemeister-Schreier method [11, p. 309] gives the following elements as generators of $D_n(P^2)$

$$\beta(\sigma_k M_i)^{-1}\sigma_k M_i \qquad (k=1,2,\cdots,n-1,\ i=0,1,\cdots,n-1),$$
$$\beta(\rho_j M_i)^{-1}\rho_j M_i \qquad (j=1,2,\cdots,n,\ i=0,1,\cdots,n),$$

where β maps each element of $B_n(P^2)$ onto its coset representative. Chow [5] obtained the generators $\sigma_2,\sigma_3,\cdots,\sigma_{n-1}$, $a_2,a_3,\cdots,a_n$ from the $\beta(\sigma_k M_i)^{-1}\sigma_k M_i$. Since $\alpha(\rho_j)$ is the identity permutation, $\beta(\rho_k M_i)^{-1}\rho_k M_i = M_i^{-1}\rho_k M_i$. If $j \leqq i$, $M_i^{-1}\rho_j M_i = M_j^{-1}\rho_j M_j = \rho_{j+1}a_{j+1}$. If $j = i+1$, $M_i^{-1}\rho_{i+1}M_i = a_{i+1}^{-1}a_i^{-1}\cdots a_2^{-1}\rho_1$. And finally, if $j > i+1$, $M_i^{-1}\rho_j M_i = \rho_j$.

Note that any one of the a_i's could be deleted from the above set of generators of $D_n(P^2)$, since $a_2a_3\cdots a_n = \sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2\sigma_1 = \rho_1^2$. However, future computations would be complicated if some one of the a_i's were singled out for exclusion.

LEMMA. *A presentation of $D_n(P^2)$ is given by the relations*

(1) $\sigma_j\sigma_k = \sigma_k\sigma_j\ (|j-k| \geqq 2)$
(2) $\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}$
(3) $\sigma_j a_k \sigma_j^{-1} = a_k\ (k \neq j,\ j+1)$
(4) $\sigma_j a_j \sigma_j^{-1} = a_{j+1}$
(5) $\sigma_j a_{j+1}\sigma_j^{-1} = a_{j+1}^{-1}a_j a_{j+1}$
(6) $\rho_1^2 = a_2a_3\cdots a_n$
(7) $\rho_2 a_2 \rho_2 = \sigma_2\sigma_3\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_3\sigma_2$.
(8) $\sigma_j\rho_{j+1}\sigma_j = \rho_j$.
(9) $\rho_j\sigma_k = \sigma_k\rho_j\ (j \neq k,\ k+1)$
(10) $\rho_j a_k = a_k\rho_j\ (j > k)$
(11) $\rho_j^{-1}a_k\rho_j = a_j a_k a_j^{-1}\ (1 < j < k)$.
(12) $\rho_{j+1}^{-1}\rho_j^{-1}\rho_{j+1}\rho_j = \sigma_j^2\ (j > 1)$
(13) $\rho_j^{-1}\rho_1\rho_j = \rho_1 a_j^{-1}\ (j > 1)$
(14) $\rho_j a_j \rho_j^{-1} = \rho_1^{-1}a_2\cdots a_{j-1}a_j^{-1}a_{j-1}^{-1}\cdots a_2^{-1}\rho_1$.

on the generators $a_2,\cdots,a_n,\rho_1,\rho_2,\cdots,\rho_n,\sigma_2,\sigma_3,\cdots\sigma_{n-1}$.

Proof. Since the M_i's form a Schreier system, a set of defining relators for $D_n(P^2)$ is given by expressing the relators

$M_i^{-1}\sigma_j\sigma_k\sigma_j^{-1}\sigma_k^{-1}M_i\ (|j-k| \geqq 2)$
$M_i^{-1}\sigma_j\sigma_{j+1}\sigma_j\sigma_{j+1}^{-1}\sigma_j^{-1}\sigma_{j+1}^{-1}M_i$
$M_i^{-1}\rho_1^{-2}\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_2\sigma_1 M_i$
$M_i^{-1}\sigma_j\rho_{j+1}\sigma_j\rho_j^{-1}M_i$
$M_i^{-1}\rho_j\sigma_k\rho_j^{-1}\sigma_k^{-1}M_i\ (j \neq k,\ k+1)$
$M_i^{-1}\sigma_j^2\rho_j^{-1}\rho_{j+1}^{-1}\rho_j\rho_{j+1}M_i$

in terms of the generators of $D_n(P^2)$ [11, p. 311].

The first two relators yield relations equivalent to (1) through (5) according to Chow [5]. Using the identities derived in the first lemma of this section, the remaining relators can be expressed in terms of the generators of $D_n(P^2)$.

$$M_i^{-1}\rho_1^{-2}\sigma_1\cdots\sigma_{n-1}^2\cdots\sigma_1 M_i = \begin{cases} \rho_1^{-2}\, a_2 a_3\cdots a_n & (i=0), \\ a_2^{-1}\rho_2^{-1}a_2^{-1}\rho_2^{-1}\sigma_2\cdots\sigma_{n-1}^2\cdots\sigma_2 a_2 & (i>0) \end{cases}$$

yields relations (6) and (7).

$$M_i^{-1}\sigma_j\rho_{j+1}\sigma_j\rho_j^{-1}M_i = \begin{cases} \sigma_j\rho_{j+1}\sigma_j\rho_j^{-1} & (j>i+1), \\ 1 & (j=i,\ i+1), \\ \sigma_{j+1}\rho_{j+2}a_{j+2}\sigma_{j+1}a_{j+1}^{-1}\rho_{j+1} & (j<i) \end{cases}$$

yields the single relation (8).

$$M_i^{-1}\rho_j\sigma_k\rho_j^{-1}\sigma_k^{-1}M_i = \begin{cases} \rho_j\sigma_k\rho_j^{-1}\sigma_k^{-1} & (i+1<j<k \text{ or } i+2<k+1<j), \\ \rho_j a_{k+1}\rho_j^{-1}a_{k+1}^{-1} & (i+2 = k+1<j), \\ 1 & (i+1 = k+1<j \text{ or } j<k=1), \\ \rho_j\sigma_{k+1}\rho_j^{-1}\sigma_{k+1}^{-1} & (k+1 < i+1<j), \\ \rho_1\sigma_k\rho_1^{-1}a_2a_3\cdots a_j\sigma_k^{-1}a_j^{-1}\cdots a_3^{-1}a_2^{-1} & (i+1=j<k), \\ \rho_1\sigma_{k+1}\rho_1^{-1}a_2a_3\cdots a_j\sigma_{k+1}^{-1}a_j^{-1}\cdots a_3^{-1}a_2^{-1} & (k+1<i+1=j), \\ \rho_{j+1}a_{j+1}\sigma_k a_{j+1}^{-1}\rho_{j+1}^{-1}\sigma_k^{-1} & (j<i+1=k), \\ \rho_{j+1}a_{j+1}a_{k+1}a_{j+1}^{-1}\rho_{j+1}^{-1}a_{k+1}^{-1} & (j+1\leqq i+1=k) \end{cases}$$

yields the relations (9), (10), and (11).

$$M_i^{-1}\sigma_j^2\rho_j^{-1}\rho_{j+1}^{-1}\rho_j\rho_{j+1}M_i = \begin{cases} \sigma_j^2\rho_j^{-1}\rho_{j+1}^{-1}\rho_j\rho_{j+1} & (j>i+1), \\ a_{j+1}\rho_1^{-1}\rho_{j+1}^{-1}\rho_1\rho_{j+1} & (j=i+1), \\ \rho_{j+1}\rho_i\rho_{j+1}^{-1}\rho_1 a_2\cdots a_j a_{j+1}a_j^{-1}\cdots a_2^{-1} & (j=i), \\ \sigma_{j+1}^2 a_{j+1}^{-1}\rho_{j+1}^{-1}a_{j+2}^{-1}\rho_{j+2}^{-1}\rho_{j+1}a_{j+1}\rho_{j+2}a_{j+2} & (j<i) \end{cases}$$

yields (12), (13), and

$$\rho_j\rho_1\rho_j^{-1} = a_2\cdots a_{j-1}a_j^{-1}a_{j-1}^{-1}\cdots a_2^{-1}\rho_1.$$

The addition of (14) to the relations (1) to (13) is equivalent to the addition of this remaining relation.

Let the subgroup of $D_n(P^2)$ generated by $\rho_1, a_2, a_3, \cdots, a_n$ be denoted by $A_n(P^2)$.

Since $A_n(P^2)$ is a finitely generated subgroup of the finitely generated group $D_n(P^2)$, by direct computation one can prove the following

LEMMA. *$A_n(P^2)$ is a normal subgroup of $D_n(P^2)$.*

Since the generators $\rho_1, a_2, a_3, \cdots, a_n$ of $A_n(P^2)$ satisfy $\rho_1^2 = a_2a_3\cdots a_n$, $A_n(P^2)$ is generated by ρ_1 and by any $n-2$ of the a_i's and is thus a homomorph of a free

group of rank $n-1$. It will now be shown that $A_n(P^2)$ is in fact, for $n>2$, a free group of rank $n-1$. For this, it suffices to exhibit a representation of $D_n(P^2)$ in which the image of $A_n(P^2)$ is free of rank $n-1$.

Let $U_n=\{v,u_2,u_3,\cdots,u_n\colon v^2=u_2u_3\cdots u_n\}$, a free group of rank $n-1$. The representation of $D_n(P^2)$ to be exhibited is given by taking for the generators $\sigma_2,\sigma_3,\cdots,\sigma_{n-1}$, $a_2,a_3,\cdots,a_n$, $\rho_1,\rho_2,\cdots,\rho_n$ the following automorphisms of U_n.

$$\sigma_i\colon\ u_j\to u_j\quad (j\neq i,\ i+1),\quad u_i\to u_{i+1},\quad u_{i+1}^{-1}\to u_{i+1}^{-1}u_iu_{i+1},\quad v\to v$$

$$a_i\colon\ u_j\to u_iu_ju_i^{-1},\quad v\to u_ivu_j^{-1}$$

$$\rho_1\colon\ u_j\to vu_jv^{-1},\ v\to v$$

and if $i>1$,

$$\rho_i\colon\ u_j\to w^{-1}u_jw\qquad (j>i)\text{ where } w=v^{-1}u_2\cdots u_{i-1}u_iu_{i-1}^{-1}\cdots u_2^{-1}v$$

$$u_j\to u_j\quad (j<i),\quad u_i\to v^{-1}u_2\cdots u_{i-1}u_i^{-1}u_{i-1}^{-1}\cdots u_2^{-1}v$$

$$v\to u_2\cdots u_{i-1}u_i^{-1}\cdots u_2^{-1}v.$$

These automorphisms, which were suggested by the proof of the above lemma, can now be shown to satisfy relations (1) to (14) of $D_n(P^2)$. Note that the inner automorphism group of U_n is generated by the automorphisms ρ_1, $a_2,a_3,\cdots,a_n$ and that these generators satisfy relation (6), $\rho_1^2=a_2a_3\cdots a_n$. Since $U_n=\{v,u_2,\cdots,u_n\colon v^2=u_2\cdots u_n\}$, a free group of rank $n-1$, is centerless for $n>2$, it is isormorphic with its inner automorphism group under the correspondence $v\leftrightarrow\rho_1$, $u_i\leftrightarrow a_i$. This is sufficient to prove the following

LEMMA. *If $n>2$, $A_n(P^2)$ is a free group of rank $n-1$ with the presentation $\{\rho_1,a_2,a_3,\cdots,a_n:\rho_1^2=a_2a_3\cdots a_n\}$.*

COROLLARY. *$B_n(P^2)$ is an infinite group for $n>2$.*

Since $A_n(P^2)=\{\rho_1,a_2,a_3,\cdots,a_n\colon\rho_1^2=a_2a_3\cdots a_n\}$ is normal in $D_n(P^2)$, a presentation of the factor group $D_n(P^2)/A_n(P^2)$ is obtained from the presentation of $D_n(P^2)$ given above by adding the relations $\rho_1=1$, $a_2=1$, $a_3=1,\cdots,a_n=1$. The addition of these relations annihililates relations (3), (4), (5), (6), (10), (11), (13) and (14) while altering (7) to $\rho_2^2=\sigma_2\sigma_3\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_3\sigma_2$. Hence $D_n(P^2)/A_n(P^2)$ is generated by $\sigma_2,\sigma_3,\cdots,\sigma_{n-1},\rho_2,\cdots,\rho_n$ subject to the relations:

(1) $\sigma_j\sigma_k=\sigma_k\sigma_j\ (|j-k|\geqq 2)$
(2) $\sigma_j\sigma_{j+1}\sigma_j=\sigma_{j+1}\sigma_j\sigma_{j+1}$
(9) $\rho_j\sigma_k=\sigma_k\rho_j\ (j\neq k,\ k+1)$
(8) $\sigma_j\rho_{j+1}\sigma_j=\rho_j$
(12) $\rho_{j+1}^{-1}\rho_j^{-1}\rho_{j+1}\rho_j=\sigma_j^2\ (j>1)$
(7) $\rho_2^2=\sigma_2\sigma_3\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_3\sigma_2$.

The factor group $D_n(P^2)/A_n(P^2)$ is now seen to be the braid group $B_{n-1}(P^2)$ on $\sigma_2, \sigma_3, \cdots, \sigma_{n-1}, \rho_2, \rho_3, \cdots, \rho_n$. The natural homomorphism of $D_n(P^2)$ onto $B_{n-1}(P^2)$ will be denoted by j and called the *Chow homomorphism*. The above result is restated in the following

LEMMA. *The sequence*

$$1 \to A_n(P^2) \xrightarrow{i} D_n(P^2) \xrightarrow{j} B_{n-1}(P^2) \to 1$$

is exact.

The proof of the following result is found in [9, p. 245].

LEMMA. *The Chow homomorphism induces an exact sequence*

$$1 \to A_n(P^2) \xrightarrow{i} K_n(P^2) \xrightarrow{j} K_{n-1}(P^2) \to 1$$

called the fundamental exact sequence for $B_n(P^2)$.

LEMMA. $B_2(P^2)$ *is a dicyclic group of order* 16.

Proof. Recall that

$$B_2(P^2) = \{\rho_1, \rho_2, \sigma_1 : \rho_2^{-1}\rho_1^{-1}\rho_2\rho_1 = \sigma_1^2, \sigma_1\rho_2\sigma_1 = \rho_1, \sigma_1^2 = \rho_1^2\}.$$

Using the relation $\rho_2 = \sigma_1^{-1}\rho_1\sigma_1^{-1}$ as the definition of ρ_2,

$$B_2(P^2) = \{\rho_1, \sigma_1 : \sigma_1\rho_1^{-1}\sigma_1\rho_1^{-1}\sigma_1^{-1}\rho_1\sigma_1^{-1}\rho_1 = \sigma_1^2, \sigma_1^2 = \rho_1^2\}.$$

The relation $\sigma_1^2 = \rho_1^2$ implies that $\sigma_1\rho_1^{-1} = \sigma_1^{-1}\rho_1$ and hence that $(\sigma_1\rho_1^{-1})^4 = \sigma_1\rho_1^{-1}\sigma_1\rho_1^{-1}\sigma_1^{-1}\rho_1\sigma_1^{-1}\rho_1 = \sigma_1^2$. Conversely the relations $(\sigma_1\rho_1^{-1})^4 = \sigma_1^2$ and $\sigma_1^2 = \rho_1^2$ imply that $\sigma_1\rho_1^{-1}\sigma_1\rho_1^{-1}\sigma_1^{-1}\rho_1\sigma_1^{-1}\rho_1 = \sigma_1^2$. Then

$$B_2(P^2) = \{\rho_1, \sigma_1 : (\sigma_1\rho_1^{-1})^4 = \rho_1^2 = \sigma_1^2\},$$

which is, by [6], a presentation of the dicyclic group of order 16.

LEMMA. $K_2(P^2)$ *is the quaternion group.*

Proof. The map $\alpha : B_2(P^2) \to \Sigma^2$ takes σ_1 onto the transposition $(1, 2)$ and ρ_1, ρ_2 onto the identity permutation. Since $\sigma_1^2 = \rho_1^2$ in $B_2(P^2)$, adding the relations $\rho_1 = 1$ and $\rho_2 = 1$ to $B_2(P^2)$ gives a presentation of Σ^2, namely $\{\sigma_1 : \sigma_1^2 = 1\}$. The kernel $K_2(P^2)$ of α is therefore the consequence of these two additional relations, that is the smallest normal subgroup of $B_2(P^2)$ containing ρ_1 and ρ_2. $K_2(P^2)$ is thus the subgroup of $B_2(P^2)$ generated by ρ_1 and ρ_2, provided it is normal in $B_2(P^2)$ (this fact will soon be evident since it is of index 2 in $B_2(P^2)$). First note, since $(\sigma_1\rho_1^{-1})^4 = \sigma_1^2$, that $(\rho_1^{-1}\sigma_1)^4 = \sigma_1^2$ and then, since $\rho_1^{-1}\sigma_1 = \rho_1\sigma_1^{-1}$ that $\sigma_1^2 = (\rho_1^{-1}\sigma_1)^4 = (\rho_1\sigma_1^{-1})^4 = (\sigma_1\rho_1^{-1})^4 = \sigma_1^{-2}$. But then $\sigma_1^4 = 1$ which in turn implies $\rho_1^4 = 1$. Next note, since $\sigma_1^2 = \rho_1^2$, that $\rho_2^2 = (\sigma_1^{-1}\rho_1\sigma_1^{-1})^2 = \sigma_1^{-1}\rho_1\sigma_1^{-2}\rho_1\sigma_1^{-1} = \sigma_1^{-1}\rho_1\rho_1^{-2}\rho_1\sigma_1^{-1} = \sigma_1^{-2}$ and thus $\rho_2^2 = \sigma_1^2 = \rho_1^{-2} = \rho_1^2$. Since $\rho_2^2 = \rho_1^{-2}$

implies $\rho_2\rho_1 = \rho_2^{-1}\rho_1^{-1}$, the relation $\rho_2^{-1}\rho_1^{-1}\rho_2\rho_1 = \sigma_1^2$ yields $(\rho_2\rho_1)^2 = \sigma_1^2 = \rho_1^2$. It is now clear that ρ_1 and ρ_2 satisfy $(\rho_2\rho_1)^2 = \rho_1^2 = \rho_2^2$. Therefore if $\{\rho_1, \rho_2 : \rho_2^2 = \rho_1^2 = (\rho_2\rho_1)^2\}$ is not a presentation of $K_2(P^2)$, then it presents a group which is too large and relations, not a consequence of these, must be added to it in order to get the desired presentation. However $K_2(P^2)$, being a group of index 2 in a group of order 16, has order 8 and $\{\rho_1, \rho_2 : \rho_2^2 = \rho_1^2 = (\rho_2\rho_1)^2\}$ is, by [6], a presentation of the quaternion group, a group of order 8.

COROLLARY. *$(\rho_1\rho_2)^2 = \sigma_1^2$ has order 2 in $B_2(P^2)$.*

Proof. First note that $(\rho_1\rho_2)^2 = \sigma_1^2$, since $(\rho_2\rho_1)^2 = \rho_1^2 = \sigma_1^2$, and recall that $\sigma_1^4 = 1$. It suffices to show that $\sigma_1^2 = (\rho_1\rho_2)^2 \neq 1$. The assumption that $(\rho_1\rho_2)^2 = 1$ implies that $K_2(P^2)$ is Klein's four-group, contrary to the above lemma.

LEMMA. *$A_2(P^2)$ is cyclic of order 4.*

Proof. Consider the exact sequence $1 \to A_2(P^2) \to K_2(P^2) \to K_1(P^2) \to 1$. Since Σ^1 is the identity, $K_1(P^2) = B_1(P^2) = \{\rho_2 : \rho_2^2 = 1\}$. $A_2(P^2)$ is thus of index 2 in $K_2(P^2)$ and hence of order 4. But then, since the generators ρ_1 and $a_2 = \sigma_1^2$ satisfy $a_2 = \rho_1^2$, $\rho_1^4 = 1$ in $B_2(P^2)$, $A_2(P^2) = \{\rho_1 : \rho_1^4 = 1\}$.

LEMMA. *$(\sigma_1\sigma_2\cdots\sigma_{n-1})^n \neq 1$ in $B_n(P^2)$.*

Proof. It is shown in [9, p. 247] that the relation $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n = \sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_2\sigma_1(\sigma_2\sigma_3\cdots\sigma_{n-1})^{n-1}$ is a consequence of relations (i) and (ii) which are common to $B_n(P^2)$ and $B_n(S^2)$. Thus $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ equals $\rho_1^2(\sigma_2\sigma_3\cdots\sigma_{n-1})^{n-1}$ in $B_n(P^2)$ and is contained in $K_n(P^2)$. Consider the fundamental exact sequence of $B_n(P^2)$

$$1 \to A_n(P^2) \xrightarrow{i} K_n(P^2) \xrightarrow{j_n} K_{n-1}(P^2) \to 1$$

and note that j_n maps $\rho_1^2(\sigma_2\sigma_3\cdots\sigma_{n-1})^{n-1}$ onto $(\sigma_2\sigma_3\cdots\sigma_{n-1})^{n-1}$. But then $j_3j_4\cdots j_n$ maps $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ onto $\sigma_{n-1}^2 \in K_2(P^2)$ and thus the assumption, for purpose of contradiction, that $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n = 1$ in $B_n(P^2)$ implies $\sigma_{n-1}^2 = 1$ in $B_2(P^2)$, contrary to the above corollary.

IV. **The geometric braid groups.** 1. If M is a manifold of dimension $\geqq 2$ then the symmetric group Σ^n of degree n acts freely on $F_{0,n}(M)$ by permuting coordinates. Let $B_{0,n}(M) = F_{0,n}(M)/\Sigma^n$ and $p : F_{0,n} \to B_{0,n}$ be the associated covering space with fiber Σ^n. The following definition is a reinterpretation and extension by R. H. Fox [10] of a definition of E. Artin [2].

DEFINITION. $\pi_1(B_{0,n}(M))$ is called the *geometric braid group* of M on n strings and will be denoted by $G_n(M)$.

2. $G_n(P^2)$. Fix n distinct points $x_1, x_2, \cdots, x_n$ in P^2 to serve, in the given order, as the base point of $F_{0,n}(P^2)$.

Let s_i, $1 \leqq i \leqq n-1$, and r_i, $1 \leqq i \leqq n$, denote the elements of $G_n(P^2)$ indicated

by the paths shown in Figure 1, where the projective plane is represented as a disc with antipodal points of the boundary identified.

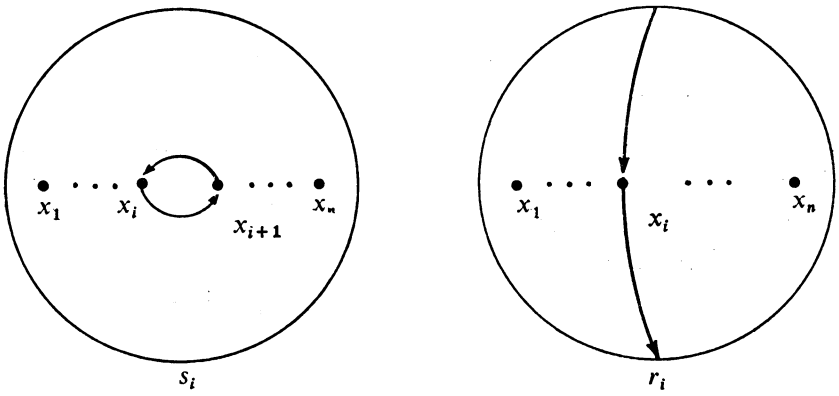


FIGURE 1

LEMMA. $r_1^2 = s_1 s_2 \cdots s_{n-2} s_{n-1}^2 s_{n-2} \cdots s_2 s_1$.

Proof. Let x_i have polar coordinates $((i-1)/2n, 0)$, $i = 1, 2, \cdots, n$. The homotopy $h: I \times I \to B_{0,n}(P^2)$ defined by $h(s,t) = (x_1(s,t), x_2, \cdots, x_n)$ where

$$x_1(s,t) = \begin{cases} \left(\frac{1}{2}\cos\frac{8st-1}{2}\pi, \frac{8st-1}{2}\pi\right), & 0 \leqq t \leqq \frac{1}{4}, \\ \left(\frac{1}{2}\cos\frac{2s-1}{2}\pi + (4t-1)\left(1 - \frac{1}{2}\cos\frac{2s-1}{2}\pi\right), \frac{2s-1}{2}\pi\right), & \frac{1}{4} \leqq t \leqq \frac{1}{2}, \\ \left(2t-2, \frac{2s-1}{2}\pi\right), & \frac{1}{2} \leqq t \leqq 1 \end{cases}$$

shows that $h(0,t)$, a representative of r_1, is homotopic to $h(1,t)$, a representative of $s_1 s_2 \cdots s_{n-1} s_{n-1} \cdots s_2 s_1 r_1^{-1}$, relative to $(x_1, x_2, \cdots, x_n)$, as illustrated in Figure 2.

One can similarly prove the next lemma. Relations (iii) and (iv) are held in common with $G_n(E^2)$, while relations (i) and (ii) are illustrated in Figures 3 and 4 respectively.

LEMMA. *The relations*

(i) $r_i = s_i r_{i+1} s_i$,

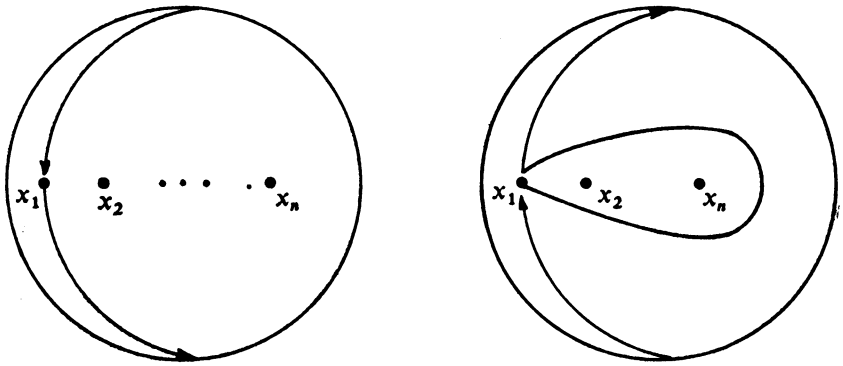
(ii) $r_{i+1}^{-1} r_i^{-1} r_{i+1} r_i = s_i^2$,

(iii) $s_i s_j = s_j s_i$ *if* $i+1 < j$ *or* $i > j+1$,

(iv) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$,

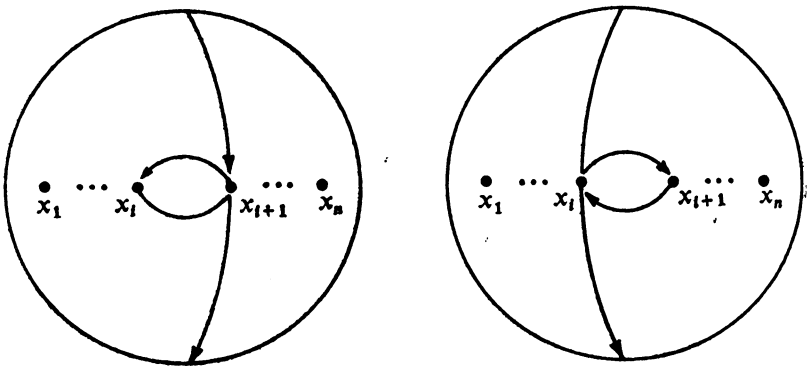
(v) $s_i s_j = s_j s_i$ *if* $i+1 < j$ *or* $i > j$,

hold in $G_n(P^2)$.



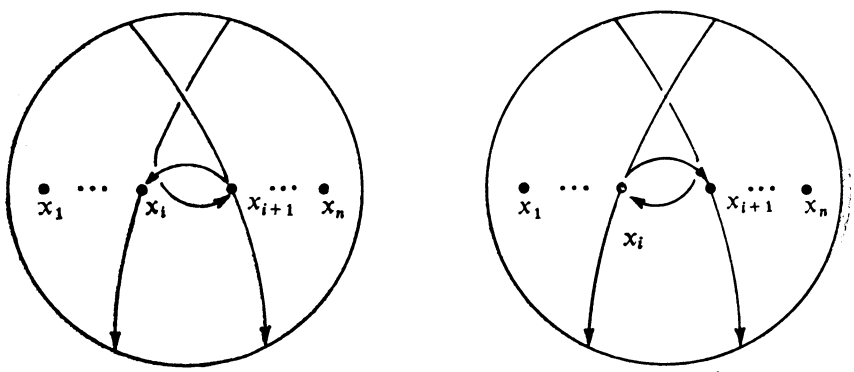
r_1 $\qquad$ $s_1 \cdots s_{n-2} s_{n-1}^2 s_{n-2} \cdots s_1 r_1^{-1}$

FIGURE 2



$s_i r_{i+1}$ $\qquad$ $r_i s_i^{-1}$

FIGURE 3



$r_i r_{i+1} s_i$ $\qquad$ $r_{i+1} r_i s_i^{-1}$

FIGURE 4

LEMMA. $s_1^{-1}\cdots s_{i-2}^{-1}s_{i-1}^2 s_{i-2}\cdots s_1$, *which will be denoted by* b_i, *is represented by the path indicated in Figure* 5.

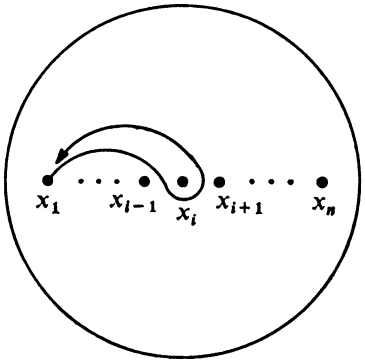


FIGURE 5

Let $\gamma: G_n(P^2) \to \Sigma^n$ denote the homomorphism induced by the covering space $p: F_{0,n}(P^2) \to B_{0,n}(P^2)$ and observe that the kernel of γ is $\pi_1(F_{0,n}(P^2))$ and that γ maps s_i onto the transposition $(i, i+1)$ and r_1 onto the identity permutation of Σ^n. Consider the fiber map $f: F_{0,n}(P^2) \to F_{0,n-1}(P^2)$ where $f(p_1, p_2, \cdots, p_n) = (p_2, \cdots, p_n)$. The fiber $F_{n-1,1}(P^2)$ over $(x_2, \cdots, x_n)$ consists of those points of P^2 which are distinct from $x_2, \cdots, x_n$. Let x_1 be the base point in $F_{n-1,1}(P^2)$ and $\eta: F_{n-1,1}(P^2) \to F_{0,n}(P^2)$ the inclusion map given by $\eta(p) = (p, x_2, \cdots, x_n)$. Recall that, when $n \geqq 2$, $\pi_2(F_{0,n}(P^2)) = 0$. This gives, for $n \geqq 3$, the homotopy sequence of the fibering $f: F_{0,n}(P^2) \to F_{0,n-1}(P^2)$

$$1 \to \pi_1(F_{n-1,1}(P^2)) \xrightarrow{\eta_*} \pi_1(F_{0,n}(P^2)) \xrightarrow{f_*} \pi_1(F_{0,n-1}(P^2)) \to 1$$

which will be called the *fundamental exact sequence* for $G_n(P^2)$.

V. **The isomorphism.** First note that the correspondence $\sigma_i \to s_i$, $\rho_i \to r_i$ defines a homomorphism $\phi_n: B_n(P^2) \to G_n(P^2)$, and that the following diagram commutes.

$$\begin{tikzcd} B_n(P^2) \arrow[rr, "\phi_n"] \arrow[dr, "\alpha"'] & & G_n(P^2) \arrow[dl, "\gamma"] \\ & \Sigma_n & \end{tikzcd}$$

The following result is proved as for E^2 [9].

LEMMA. ϕ_n *is an isomorphism if and only if the restriction* ψ_n *of* ϕ_n *to* $K_n(P^2)$ *is an isomorphism.*

Now compare the fundamental exact sequences of the algebraic and geometric braid groups of the projective plane. If $n \geqq 3$, one obtains the diagram

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & A_n(P^2) \arrow[r, hook, "i"] \arrow[d, "\theta_n"] & K_n(P^2) \arrow[r, "j"] \arrow[d, "\psi_n"] & K_{n-1}(P^2) \arrow[r] \arrow[d, "\psi_{n-1}"] & 1 \arrow[d] \\
1 \arrow[r] & \pi_1(F_{n-1,1}(P^2)) \arrow[r, "\eta_*"] & \pi_1(F_{0,n}(P^2)) \arrow[r, "f_*"] & \pi_1(F_{0,n-1}(P^2)) \arrow[r] & 1
\end{tikzcd}$$

where the generators of $B_{n-1}(P^2)$ are taken to be $\sigma_2, \sigma_3, \cdots, \sigma_{n-1}, \rho_2, \rho_3, \cdots, \rho_n$ and the base point for $\pi_1(F_{0,n-1}(P^2))$ is $(x_2, x_3, \cdots, x_n)$ if the base point for $\pi_1(F_{0,n}(P^2))$ is $(x_1, x_2, \cdots, x_n)$. The map θ_n will now be defined.

Let x_1 be the base point for $F_{n-1,1}(P^2)$, the set of points of P^2 distinct from $x_2, x_3, \cdots, x_n$. If $F_{n-1,1}(P^2)$ is considered to be the set of points of the open Möbius strip $P^2 - x_n$ which are distinct from $x_2, x_3, \cdots, x_{n-1}$ then its fundamental group is readily seen to be the free group generated by r_1, $b_2, b_3, \cdots, b_{n-1}$. However, in order to effect the isomorphism θ_n, the generator b_n and its annihilating relation $r_1^2 = b_1 b_2 \cdots b_n$ are included in the presentation

$$\pi_1(F_{n-1,1}(P^2)) = \{r_1, b_2, b_3, \cdots, b_n : \ r_1^2 = b_2 b_3 \cdots b_n\}.$$

Recall that

$$A_n(P^2) = \{\rho_1, a_2, a_3, \cdots, a_n : \ \rho_1^2 = a_2 a_3 \cdots a_n\}.$$

The map θ_n, $n \geqq 3$, is defined to be the isomorphism between these two free groups of rank $n-1$ given by $\theta_n : \rho_1 \to r_1$, $a_i \to b_i$. With this definition of θ_n, the above diagram is commutative. On applying the "Five Lemma" to this diagram the following lemma follows inductively.

LEMMA. *ψ_n is an isomorphism, $n \geqq 2$, provided ψ_2 is an isomorphism.*

To show that ψ_2 is an isomorphism, the homotopy sequence of the fibering $f : F_{0,2}(P^2) \to F_{0,1}(P^2)$ and the fundamental exact sequence of $B_2(P^2)$ will now be brought into play.

$$\begin{tikzcd}
 & 1 \arrow[r] & A_2(P^2) \arrow[r] & K_2(P^2) \arrow[r] \arrow[d, "\psi_2"] & K_1(P^2) \arrow[r] \arrow[d, "\psi_1"] & 1 \arrow[d] \\
1 \arrow[r] & \pi_2(F_{0,1}(P^2)) \arrow[r, "\Delta"] & \pi_1(F_{1,1}(P^2)) \arrow[r] & \pi_1(F_{0,2}(P^2)) \arrow[r] & \pi_1(F_{0,1}(P^2)) \arrow[r] & 1
\end{tikzcd}$$

First note that $\psi_1 : \rho_2 \to r_2$ is an isomorphism since $\pi_1(F_{0,1}(P^2)) = \pi_1(P^2) = \{r_2 : r_2^2 = 1\}$ while the triviality of Σ^1 implies that $K_1(P^2) = B_1(P^2) = \{\rho_2 : \rho_2^2 = 1\}$. Since (x_1, x_2) is a base point of $\pi_1(F_{0,2}(P^2))$, it follows that $F_{1,1}(P^2)$ is P^2 with the point x_2 deleted, that is an open Möbius strip, and that $\pi_1(F_{1,1}(P^2) = \{r_i :\}$. $\pi_2(F_{0,1}(P^2)) = \pi_2(P^2)$ is likewise infinite cyclic. To show ψ_2 is an isomorphism it will suffice to show that a generator of the infinite cyclic group $\pi_2(F_{0,1}(P^2))$ is

mapped by Δ onto r_1^4 in $\pi_1(F_{1,1}(P^2))$. To see this, recall that $A_2(P^2)=\{\rho_1 : \rho_1^4=1\}$ and note that if θ_2 is defined by $\theta_2 : \rho_1 \to r_1$, the resulting diagram

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & A_2(P^2) \arrow[r] \arrow[d, "\theta_2"] & K_2(P^2) \arrow[r] \arrow[d, "\psi_2"] & K_1(P^2) \arrow[r] \arrow[d, "\psi_1"] & 1 \arrow[d] \\
1 \arrow[r] & \{r_1 : r_1^4 = 1\} \arrow[r] & \pi_1(F_{0,2}(P^2)) \arrow[r] & \pi_1(F_{0,1}(P^2)) \arrow[r] & 1
\end{tikzcd}$$

is commutative and, since θ_2 and ψ_1 are isomorphisms, ψ_2 is, by the "Five Lemma," an isomorphism. To show that Δ has this property, the homotopy sequence of the fibering $\lambda : \mathscr{A}_2(S^2) \to \mathscr{A}_1(S^2)$ will be employed. The fiber here is the set of points of S^2 which avoid an antipodal pair of points, that is an open annulus, which will be denoted by $F^*_{2,1}(S^2)$. Since $\pi_2(\mathscr{A}_2(S^2))=0$ and $\pi_1(\mathscr{A}_1(S^2))=\pi_1(S^2)=1$, one obtains the exact sequence

$$1 \to \pi_2(\mathscr{A}_1(S^2)) \xrightarrow{d} \pi_1(F^*_{2,1}(S^2)) \longrightarrow \pi_1(\mathscr{A}_2(S^2)) \to 1.$$

Now $\pi_2(\mathscr{A}_1(S^2))=\pi_2(S^2)$ is infinite cyclic, as is $\pi_1(F^*_{2,1}(S^2))$, since $F^*_{2,1}(S^2)$ is of the same homotopy type as S^1. But then, since $\pi_1(\mathscr{A}_2(S^2))=\pi_1(V_{3,2})$ is cyclic of order 2, d takes the generator of $\pi_2(\mathscr{A}_1(S^2))$ onto the square of a generator of $\pi_1(F^*_{2,1}(S^2))$. Recall that $\xi : S^2 \to P^2$, the map which identifies antipodal points, induces the covering map $\xi_n : \mathscr{A}_n(S^2) \to F_{0,n}(P^2)$ with a discrete fiber of 2^n points, and consider the diagram

$$\begin{tikzcd}
1 & \pi_2(F_{0,1}(P^2)) \arrow[l] & \pi_1(F_{1,1}(P^2)) \arrow[l, "\Delta"'] & \pi_1(F_{0,2}(P^2)) \arrow[l] & \pi_1(F_{0,1}(P^2)) \arrow[l] & 1 \arrow[l] \\
1 \arrow[u] & \pi_2(A_1(S^2)) \arrow[u, "\xi_{1*}"'] & \pi_1(F^*_{2,1}(S^2)) \arrow[l, "d"'] \arrow[u, "\xi'_*"'] & \pi_1(A_2(S^2)) \arrow[l] \arrow[u, "\xi_{2*}"'] & 1 \arrow[l] \arrow[u, "\xi_{1*}"'] &
\end{tikzcd}$$

where $\xi_{1*} : \pi_2(\mathscr{A}_1(S^2)) \to \pi_2(F_{0,1}(P^2))$ is an isomorphism, while ξ_{2*} and $\xi_{1*} : 1 \to \pi_1(F_{0,1}(P^2))$ are monomorphisms. Also recall that $F_{1,1}(P^2)$ is P^2 with the point x_2 deleted and consider $F^*_{2,1}(S^2)$ to be S^2 with the antipodal pair $\pm x_2$ deleted. The map $\xi' : F^*_{2,1}(S^2) \to F_{1,1}(P^2)$ induced by ξ maps a path between a point of $F^*_{2,1}(P^2)$ and its antipodal point, along a great circle, onto the generator r_1 of $\pi_1(F_{1,1}(P^2))$ and hence maps a generator of the infinite cyclic group $\pi_1(F^*_{2,1}(P^2))$, a path along a great circle beginning and terminating with x_1, onto r_1^2 in $\pi_1(F_{1,1}(P^2))$. But then, since ξ_{1*}^{-1} is an isomorphism while d maps the generator of $\pi_2(A_1(S^2)J)$ onto the square of the generator $\pi_1(F^*_{2,1}(S^2))$ and ξ'_* maps the generator of $\pi_1(F^*_{2,1}(S^2))$ onto r_1^2 in $\pi_1(F_{1,1}(P^2))$, it follows that $\xi_* d \xi_{1*}^{-1}$ maps the generator of $\pi_2(F_{0,1}(P^2)$ onto r_1^4 in $\pi_1(F_{1,1}(P^2))$ and hence, by the commutativity of the above diagram, so does Δ. As noted above, this gives the following lemma.

LEMMA. *ψ_2 is an isomorphism.*

On observing that $B_1(P^2) = \{\rho_1 : \rho_1^2 = 1\}$ and $G_1(P^2) = \pi_1(P^2) = \{r_1 : r_1^2 = 1\}$, one now obtains the following theorem.

THEOREM. *The algebraic and geometric braid groups, $B_n(P^2)$ and $G_n(P^2)$, of the projective plane are isomorphic under ϕ: $\rho_i \to r_i$, $\sigma_i \to s_i$.*

VI. **Main results and conclusion.** 1. Properties of the braid groups of P^2.

THEOREM. *$B_1(P^2)$ is cyclic of order 2, $B_2(P^2)$ is a dicyclic group of order 16, while $B_n(P^2)$ is infinite for $n \geqq 3$.*

Proof. $B_1(P^2) = \pi_1(F_{0,1}(P^2)) = \pi_1(P^2)$ is cyclic of order 2, $B_2(P^2)$ is identified in §III, and $B_n(P^2)$ contains the free group $A_n(P^2)$ if $n \geqq 3$.

Though it has already been shown algebraically that $(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n \neq 1$ in $B_n(P^2)$, a geometric proof of this fact, essentially that used by Fadell [7] to solve the Dirac String Problem, is now given.

LEMMA. $(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n \neq 1$ *in* $B_n(P^2)$.

Proof. The isomorphism $\psi_n : K_n(P^2) \to \pi_1(F_{0,n}(P^2))$ maps $(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$ onto $[(s_1 s_2 \cdots s_{n-1})]^n$, which is in turn mapped, by the inverse of the monomorphism ϕ_{n*} induced by the covering map $\phi_n : A_n(S^2) \to F_{0,n}(P^2)$ onto an element of $\pi_1(A_n(S^2))$ represented by the loop in $A_n(S^2)$, say ζ_n, described by a complete revolution of the points $p_k = (\cos(\pi k/2n), 0, \sin(\pi k/2n))$, $k = 1, 2, \cdots, n$, of the unit sphere about the x_3-axis. Consider the fundamental group of the rotation group R_3 of E^3 which is cyclic of order 2 with generator

$$\gamma(t) = \begin{bmatrix} \cos 2t & -\sin 2t & 0 \\ \sin 2t & \cos 2t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad 0 \leqq t \leqq 1$$

corresponding to a complete rotation of E^3 about the x_3-axis and note that $\zeta_n(t) = (\gamma(t)p_1, \cdots, \gamma(t)p_n)$, $0 \leqq t \leqq 1$, Let λ_* denote the homomorphism: $\pi_1(\mathscr{A}_n(S^2)) \to \pi_1(\mathscr{A}_2(S^2))$, induced by the composition $\lambda_3\lambda_4 \cdots \lambda_n$ of locally trivial fiber maps λ_i: $A_i(S^2) \to A_{i-1}(S^2)$, which picks off the last two coordinates. Now $\lambda_*[(\zeta_n(t))] = [(\gamma(t)p_{n-1}, \gamma(t)p_n)]$ corresponds, under the isomorphism p of $\pi_1(A_2(S^2))$ induced by change of the base point of $A_2(S^2)$, to $[\zeta_2(t)] = [(\gamma(t)(1,0,0), \gamma(t)(0,0,1))]$. The monomorphism induced by the inverse of the inclusion map β: $V_{3,2} \to A_2(S^2)$ now takes $[\zeta^2(t)]$ onto the generator

$$[(\cos(-2\pi t), \sin(-2\pi t), 0)], \qquad 0 \leqq t \leqq 1$$

of $\pi_1(V_{3,2})$ which in turn corresponds to the generator $[\gamma]$ of $\pi_1(R_3)$ under the identification of $V_{3,2}$ with R_3. But then $(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n \in B_n(P^2)$ is mapped by the homomorphism $\beta_*^{-1}\rho\lambda_*\phi_{n*}^{-1}\psi_n$ onto the (nontrivial) generator of $\pi_1(R_3)$.

Since $\alpha: B_n(E^2) \to \Sigma^n$ maps $(\sigma_1\sigma_2\cdots\sigma_{n-1})$ onto a permutation of order n and $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n \neq 1$, it suffices to show $(\sigma_1\sigma_2\cdots\sigma_{n-1})^{2n} = 1$ to prove the following

THEOREM. *$(\sigma_1\sigma_2\cdots\sigma_{n-1})$ has order $2n$ in $B_n(P^2)$.*

Proof (M. H. A. Newman [12]). Since $[\gamma^2] = 1$ in $\pi_1(R_3)$, there is a homotopy $h: I \times I \to R_3$, relative to the identity rotation I_3 such that

$$\begin{aligned} h(0,t) &= \gamma^2(t) & 0 \leqq t \leqq 1\,, \\ h(1,t) &= I_3 & 0 \leqq t \leqq 1\,, \\ h(s,0) &= h(s,1) = I_3 & 0 \leqq t \leqq 1. \end{aligned}$$

But then

$$h'(s,t) = (h(s,t)p_1,\ h(s,t)p_2, \cdots, h(s,t)p_n)$$

is a homotopy: $I \times I \to \mathscr{A}_n(S^2)$ showing that $[\zeta_n^2] = 1$ in $\pi_1(\mathscr{A}_n(S^2))$. Finally note that the monomorphism $\phi_{n*}^{-1}\psi_n$ takes $(\sigma_1\sigma_2\cdots\sigma_{n-1})^{2n}$ onto $[\zeta_n^2]$.

THEOREM [9]. *$B_2(S^2)$ is cyclic of order 2, $B_3(S^2)$ is a ZS-metacyclic group of order 12, while $B_n(S^2)$ is infinite for $n \geqq 4$.*

The groups $B_1(P^2)$, $B_2(P^2)$, $B_2(S^2)$, $B_3(S^2)$ are thus finite groups; while $\sigma_1\sigma_2\cdots\sigma_{n-1}$ has order $2n$ in $B_n(P^2)$ for $n \geqq 3$ and in $B_n(S^2)$ for $n \geqq 4$ as shown in §III and in [9] respectively. Combining these results with the theorem of Fadell and Neuwirth on the existence of elements of finite order, one obtains the following

THEOREM. *If M^2 is a compact 2-manifold or E^2, then the n-string braid group of M^2 has elements of finite order if and only if M^2 is S^2 or P^2.*

2. The Word Problem for the braid groups of P^2. Chow's solution of the Word Problem for $B_n(E^2)$ [5] makes use of the fact that the exact sequence

$$1 \to A_n(E^2) \xrightarrow{i} D_n(E^2) \xrightarrow{j} B_{n-1}(E^2) \to 1$$

gives $D_n(E^2)$ as a direct product of $A_n(E^2)$ by $B_{n-1}(E^2)$. An attempt to carry this proof through for $B_n(S^2)$ fails however, since $\sigma_1\sigma_2\cdots\sigma_{n-2} = \sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-2}^{-1}$ holds in $B_{n-1}(S^2)$ but not in $B_n(S^2)$, so that $B_{n-1}(S^2)$ is not naturally imbedded in $B_n(S^2)$ if $n \geqq 3$.

In solving the Word Problem for $B_n(S^2)$, Fadell [9] considered the isomorphic fundamental exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_n(S^2) & \longrightarrow & K_n(S^2) & \longrightarrow & K_{n-1}(S^2) & \longrightarrow & 1 \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ 1 & \to & \pi_1(F_{n-1}(S^2)) & \to & \pi_1(F_{0,n}(S^2)) & \to & \pi_1(F_{0,n-1}(S^2)) & \longrightarrow & 1 \end{array}$$

of $B_n(S^2)$ and $G_n(S^2)$, and used the fact that the locally trivial fiber space $(F_{0,n}(S^2), \pi, F_{0,n-1}(S^2))$ admits cross sections for $n \geqq 4$.

Recall that the fiber map $\lambda: \mathscr{A}_{n+1}(S^2) \to \mathscr{A}_n(S^2)$ with fiber $F^*_{2n,1}(S^2)$ consisting of n distinct pairs of antipodal points of S^2, and consider the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(F^*_{2n,1}(S^2)) \arrow[r] \arrow[d, "\phi_*"] & \pi_1(\mathscr{A}_{n+1}(S^2)) \arrow[r, "\lambda_*"] \arrow[d, "\phi_{n+1*}"] & \pi_1(\mathscr{A}_n(S^2)) \arrow[r] \arrow[d, "\phi_{n*}"] \arrow[l, bend left, "\sigma_*"] & 1 \\
1 \arrow[r] & \pi_1(F_{n,1}(P^2)) \arrow[r] & \pi_1(F_{0,n+1}(P^2)) \arrow[r, "\pi_*"] & \pi_1(F_{0,n}(P^2)) \arrow[r] & 1
\end{tikzcd}$$

arising from the homomorphism from the exact homotopy sequence of $(\mathscr{A}_{n+1}(S^2), \lambda, \mathscr{A}_n(S^2))$ to the fundamental exact sequence of $G_n(P^2)$ where $\phi_n: \mathscr{A}_n(S^2) \to F_{0,n}(P^2)$ is the map induced by the identification ϕ of antipodal points of S^2.

Fadell has shown [7] that the fiber space $(\mathscr{A}_{n+1}(S^2), \lambda, \mathscr{A}_n(S^2))$ admits cross sections by exhibiting a cross section

$$\sigma(x_1, x_2, \cdots, x_n) = (x_0(x_1, x_2, \cdots, x_n), x_1, x_2, \cdots, x_n),$$

where $x_0(x_1, x_2, \cdots, x_n)$ lies on the geodesic from x_2 to x_1, sufficiently close to x_1 to avoid $\pm x_1, \pm x_2, \cdots, \pm x_n$. The fact that the locally trivial fiber space $(\mathscr{A}_{n+1}(S^2), \lambda, \mathscr{A}_n(S^2))$ admits cross sections implies that $\pi_1(\mathscr{A}_{n+1}(S^2))$ is a semidirect product of $\pi_1(F^*_{2n,1}(S^2))$ by $\pi_1(\mathscr{A}_n(S^2))$.

One hopes that the cross section σ will induce a cross section $(F_{0,n+1}(P^2), \pi, F_{0,n}(P^2))$ which will in turn give $\pi_1(F_{0,n+1}(P^2))$ as a semidirect product of $\pi_1(F_{n,1}(P^2))$ by $\pi_1(F_{0,n}(P^2))$ so that the solution of the Word Problem for $B_n(P^2)$ will go through as it did for $B_n(S^2)$. But this is not the case as will be shown next.

In order that the correspondence $\mu: F_{0,n+1}(P^2) \to F_{0,n}(P^2)$ induced by σ be continuous, it is necessary that

$$x_0(\varepsilon_1 x_1, \varepsilon_2 x_2, \cdots, \varepsilon_n x_n) = \pm x_0(x_1, x_2, \cdots, x_n)$$

for any assignment of 1's and -1's to the ε_i's. That is, if any one or more of the arguments $x_1, x_2, \cdots, x_n$ of $x_0(x_1, x_2, \cdots, x_n)$ is replaced by its antipode, then the map $x_0: \mathscr{A}_n(S^2) \to S^2$ must yield $x_0(x_1, x_2, \cdots, x_n)$ or its antipode. But if x_1 and x_2 are orthogonal, then $x_0(x_1, x_2, x_3, \cdots, x_n)$ and $x_0(x_1, -x_2, x_3, \cdots, x_n)$ are distinct and nonantipodal, since they lie on opposite sides of x_1 in the interior of the half of the great circle containing x_1 and $\pm x_2$.

LEMMA. *The fiber space $(F_{0,3}(P^2), \pi, F_{0,2}(P^2))$ admits cross sections.*

Proof. Let P^2 be represented as the unit sphere in E^3 with antipodes identified. A cross section is given by $\mu(x_1, x_2) = (x_1 \times x_2, x_1, x_2)$.

COROLLARY. *$K_3(P^2)$ is a semidirect product of $A_3(P^2)$ by $K_2(P^2)$.*

Proof. Since the locally trivial fiber space $(F_{0,3}(P^2), \pi, F_{0,2}(P^2))$ with fiber $F_{2,1}(P^2)$ admits cross sections, $\pi_1(F_{0,3}(P^2))$ is a semidirect product of $\pi_1(F_{2,1}(P^2))$ by $\pi_1(F_{0,2}(P^2))$. The isomorphism theorem now completes the proof.

The proof of the following corollary is that given in [9] as a solution of the Word Problem for $B_n(S^2)$.

COROLLARY (WORD PROBLEM FOR $B_3(P^2)$). *Each element of the infinite group $B_3(P^2)$ has a unique representation of the form $x = M(x)x_3x_2$, where $M(x)$ depends only on the permutation $\alpha(x)$, x_3 belongs to the free group $A_3(P^2)$ and x_2 belongs to the finite group $K_2(P^2)$.*

The following question is of interest since

$$(F_{0,n+1}(P^2), \pi, F_{0,n}(P^2))$$

admits cross sections for $n = 2$, but not (by the fixed point property of P^2) for $n = 1$. For what n does $(F_{0,n+1}(P^2), \pi, F_{0,n}(P^2))$ admit cross sections?

Added in proof. A solution to the Word Problem depends on *effectively* expressing a braid in canonical form. Joint work with R. M. Gillette gives an algorithm for $B_n(S^2)$ based on Artin's "combing" [2, p. 464] in $B_n(E^2)$.

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