SEMICONTINUITY OF INTEGRALS(0)

BY
A. W. J. STODDART

1. **Introduction.** Semicontinuity plays a key part in the direct method of the calculus of variations for the existence of a minimum for an integral: a lower semicontinuous functional on a compact space has a minimum. Here we study sufficient conditions for semicontinuity of integrals of a very general form

$$(*) \qquad (A)\int f(\rho(w),\sigma(w))\,d\mu,$$

where A is a space of any dimension, μ a general measure function on a σ-algebra of subsets of A, where $\rho(w)\colon A\to E$, $\sigma(w)\colon A\to F$ are any two maps, and f any scalar function defined on $E\times F$. The semicontinuity is with respect to an appropriate convergence on (ρ,σ,μ).

The second map $\sigma\colon A\to F$ may be interpreted as a very general substitute for the usual normal vector. Both parametric and nonparametric integrals can be interpreted as particular cases of (*).

A general theorem of this nature was proposed to the writer by Professor L. Cesari. He suggested it as a natural development of his treatment of integrals over a general variety [3], [4], abstracting and unifying the work of Weierstrass, Tonelli, and others on curve integrals, and of Cesari himself on surface integrals. In particular, Cesari has shown [4] that integrals of this nature can be expressed quite generally as measure integrals with which we are concerned in this paper.

The actual form of our general semicontinuity theorem was influenced particularly by the special cases treated in [5], [6], [9], [10], and [12]. In later sections, we apply our general theorem to special cases: parametric curve integrals [8], [10], parametric curve integrals involving higher derivatives [5], [6], parametric surface integrals [1], [12], nonparametric curve integrals [8], [9], [10], and nonparametric integrals [8]. In each case, we verify general conditions assumed in our semicontinuity theorem to obtain semicontinuity theorems in the corresponding section of the calculus of variations. Known semicontinuity theorems of Tonelli, Cesari, Cinquini, and Turner, each involving a particular topology, are so obtained as particular cases of only one general statement concerning integral (*).

Received by the editors May 15, 1965.

(0) This paper is based on part of a doctoral thesis written at The University of Michigan under the direction of Professor Lamberto Cesari. The thesis work was partially supported by NSF Grant GP-57.

Quite generally, we consider a set Ω, and a class $\boldsymbol{T}$ of triplets $T=(\rho,\sigma,\mu)$, where, in each T,

(1) μ is a measure on a subset A of Ω;

(2) σ is a μ-integrable mapping from A to E_m; and

(3) ρ is a μ-measurable mapping from A to E_n.

Denote the class of μ-measurable sets in A by $\boldsymbol{B}$.

Let $f(r,s)$ be a real function on $D=\bigcup(\rho\times\sigma)(A)$ in $E_n\times E_m$, such that

(4) $f(r,s)\geqq 0$, and

(5) for each $T=(\rho,\sigma,\mu)\in\boldsymbol{T}$, $f(\rho(w),\sigma(w))$ has an integral

$$I(T)=(A)\int f(\rho(w),\sigma(w))d\mu,$$

finite or $+\infty$. In particular, under conditions (2), (3), and (4), condition (5) is satisfied if f is continuous.

We shall denote the inner product of two vectors x,x' by $x\cdot x'$, and the length of a vector x by $|x|$.

2. **The semicontinuity theorem.** For a particular triplet $T=(\rho,\sigma,\mu)$, consider the following conditions.

(6) The measure μ is finite.

At μ-almost every $v\in A$,

(7) for any $\varepsilon>0$, there exist $\delta(\varepsilon,v)>0$, $\beta(\varepsilon,v)\in E_1$, and $b(\varepsilon,v)\in E_m$, such that, for $|r-\rho(v)|<\delta(\varepsilon,v)$,

(a) $f(r,s)\geqq\beta+b\cdot s$,

(b) $f(r,s)\leqq\beta+b\cdot s+\varepsilon$ for $|s-\sigma(v)|<\delta(\varepsilon,v)$.

There exists a topology $\boldsymbol{G}$ on A such that $\boldsymbol{G}\subseteq\boldsymbol{B}$ and μ is regular with respect to $\boldsymbol{G}$ in the sense that

(8) for any set $B\in\boldsymbol{B}$ and any $\varepsilon>0$, there exists a $\boldsymbol{G}$-closed set $F\subseteq B$ such that $\mu(B-F)<\varepsilon$; and

(9) for any $\varepsilon>0$, there exists a $\boldsymbol{G}$-compact set $K\subseteq A$ such that $K\in\boldsymbol{B}$ and $\mu(A-K)<\varepsilon$. From now on, all topological references on A will be to this topology $\boldsymbol{G}$.

THEOREM 1. *Let T be a triplet satisfying conditions* (6), (7), (8), *and* (9). *Consider any sequence of triplets $T^n=(\rho^n,\sigma^n,\mu^n)$, such that, as $n\to\infty$,*

(10) (a) $\sup\{|\rho^n(w)-\rho(w)|: w\in A^n\cap K\}\to 0$ *for each compact set $K\subseteq A$; and*

(b) $\inf\{|\mu^n(M)-\mu(B)|+|[(M)\int\sigma^n d\mu^n-(B)\int\sigma d\mu]\cdot b|: M\in\boldsymbol{B}^n,\ M\subseteq B\}\to 0$ *for each $B\in\boldsymbol{B}$ and each $b\in E_m$.*

Then $I(T^n)$ is lower semicontinuous at T, that is, $I(T)\leqq\liminf I(T^n)$.

Proof for $I(T)<\infty$. Consider any $\varepsilon>0$. By absolute continuity of the finite integral, there exists $\gamma=\gamma(\varepsilon)>0$, such that $(E)\int f(\rho(w),\sigma(w))d\mu<\varepsilon$ for every μ-measurable set $E\subseteq A$ with $\mu(E)<\gamma$.

Under conditions (2), (3), (6), and (8), we can apply Lusin's theorem to the set

$\{v: v \text{ as in condition (7)}\}$ to obtain a set K with $\mu(A-K)<\gamma$, ρ and σ continuous on K, and condition (7) holding for every $v \in K$. By conditions (8) and (9), we can take K compact.

By continuity of ρ and σ on K, for each $v \in K$, there exists an open neighbourhood $H(v)$ of v such that, for $w \in H(v) \cap K$, $|\rho(w)-\rho(v)|<\delta(\varepsilon,v)/2$ and $|\sigma(w)-\sigma(v)| < \delta(\varepsilon,v)$.

The collection $\{H(v): v \in K\}$ covers K, which is compact. Hence we can take a finite subcover $H(v_i)$: $i=1,2,\cdots h$. We shall write $H(v_i)$ as H_i, $\delta(\varepsilon,v_i)$ as δ_i, $\beta(\varepsilon,v_i)$ as β_i, and $b(\varepsilon,v_i)$ as b_i.

Put $B_i=(H_i-\bigcup_{j<i}H_j)\cap K$. The sets B_i are disjoint and $K=\bigcup B_i$. Also $B_i \subseteq H_i \cap K$, so if $w \in B_i$, then $|\rho(w)-\rho(v_i)|<\delta_i/2$ and $|\sigma(w)-\sigma(v_i)|<\delta_i$; hence, by condition (7b), $f(\rho(w),\sigma(w)) \leqq \beta_i + b_i \cdot \sigma(w) + \varepsilon$. Consequently

$$\begin{aligned} I(T) &< (\textstyle\bigcup B_i)\displaystyle\int f(\rho(w),\sigma(w))\,d\mu + \varepsilon \\ &\leqq \sum (B_i)\int (\beta_i + b_i\cdot\sigma(w)+\varepsilon)\,d\mu + \varepsilon \\ &\leqq \sum \beta_i\mu(B_i) + \sum b_i\cdot(B_i)\int \sigma(w)\,d\mu + \varepsilon\mu(A) + \varepsilon. \end{aligned}$$

By condition (10), there exists an integer $N(\varepsilon)$ such that, for $n>N(\varepsilon)$,

$$\sup\{|\rho^n(w)-\rho(w)|: w \in A^n \cap K\} < \min\{\delta_i/2: i=1,2,\cdots h\},$$

and, for each $i=1,2,\cdots h$,

$$\inf\left\{|\mu^n(M)-\mu(B_i)| + \left|\left[(M)\int \sigma^n d\mu^n - (B_i)\int \sigma d\mu\right]\cdot b_i\right| : M \in \boldsymbol{B}^n, M \subseteq B_i\right\}$$
$$< \min\{\varepsilon/h|\beta_i|, \varepsilon/h\}.$$

We can take $M_i \subseteq B_i$, $M_i \in \boldsymbol{B}^n$, such that

$$|\mu^n(M_i)-\mu(B_i)| < \varepsilon/h|\beta_i|,$$

and

$$\left|\left[(M_i)\int \sigma^n d\mu^n - (B_i)\int \sigma d\mu\right]\cdot b_i\right| < \varepsilon/h.$$

If $w \in M_i$, then $w \in A^n \cap K$; hence $|\rho^n(w)-\rho(w)|<\delta_i/2$, $|\rho^n(w)-\rho(v_i)|<\delta_i$, and thus, by condition (7a), $f(\rho^n(w), \sigma^n(w)) \geqq \beta_i + b_i\cdot\sigma^n(w)$. Consequently

$$\begin{aligned} I(T^n) &\geqq (\textstyle\bigcup M_i)\displaystyle\int f(\rho^n(w),\sigma^n(w))d\mu^n \\ &\geqq \sum (M_i)\int (\beta_i + b_i\cdot\sigma^n(w))d\mu^n \\ &> \sum \beta_i\mu(B_i) - \sum|\beta_i|\varepsilon/h|\beta_i| + \sum b_i\cdot(B_i)\int \sigma(w)\,d\mu - \sum \varepsilon/h \\ &> I(T) - \varepsilon\mu(A) - 3\varepsilon. \end{aligned}$$

Thus $I(T^n)$ is lower semicontinuous at T.

We require a substitute for absolute continuity of $\int f(\rho(w),\sigma(w))\,d\mu$ when $I(T)=\infty$.

LEMMA. *Suppose that g is a nonnegative, μ-measurable function on A, with $(A)\int g(w)\,d\mu=\infty$. Then, for any real number k, there exists $\gamma=\gamma(k)>0$ such that $(E)\int g(w)\,d\mu>k$ for all μ-measurable sets E with $\mu(A-E)<\gamma$.*

Proof of Lemma. Define $g_r(w)=\min\{r,g(w)\}$. Then $g_r(w)\uparrow g(w)$ and $(A)\int g_r(w)\,d\mu\to\infty$ as $r\to\infty$; thus $(A)\int g_r(w)\,d\mu>2k$ for r greater than some $R(k)$. Then $(E)\int g\,d\mu\geqq(E)\int g_r\,d\mu=(A)\int g_r\,d\mu-(A-E)\int g_r\,d\mu>2k-r\mu(A-E)$ for $r>R(k)$. Thus we have the required result for $\mu(A-E)<k/(R(k)+1)$.

Proof of Theorem 1 for $I(T)=\infty$. By the lemma, for any k, there exists $\gamma>0$ such that $(E)\int f(\rho(w),\sigma(w))\,d\mu>2k$ for $\mu(A-E)<\gamma$.

As in the case $I(T)<\infty$, apply Lusin's theorem to get K. Then get $H(v)$, H_i, δ_i, β_i, b_i, and B_i as in the case $I(T)<\infty$, but with $\varepsilon=k/(2+\mu(A))$. Then

$$\begin{aligned}2k &< (\textstyle\bigcup B_i)\displaystyle\int f(\rho(w),\sigma(w))\,d\mu\\ &\leqq \sum\beta_i\mu(B_i)+\sum b_i\cdot(B_i)\int\sigma(w)\,d\mu+\varepsilon\mu(A).\end{aligned}$$

Consider T^n as in the case $I(T)<\infty$, but with $\varepsilon=k/(2+\mu(A))$. Then

$$I(T^n)>2k-\varepsilon\mu(A)-2\varepsilon=k.$$

REMARK. For A and μ fixed in $\boldsymbol{T}$, the convergence condition (10b) is satisfied by weak convergence of σ^n to σ in $L_1(A,\mu)$ (and so also by weak convergence in $L_p(A,\mu)$, $p>1$).

3. **The convergence condition (10b).** In condition (10b), the regularity of μ allows us to replace $\boldsymbol{B}$ by $\boldsymbol{F}$, the class of closed sets. Under suitable extra conditions, we can replace $\boldsymbol{B}$ by $\boldsymbol{C}$, any subclass of $\boldsymbol{B}$.

THEOREM 2. *Suppose that, as $n\to\infty$,*

(10b′) $\inf\{|\mu^n(M)-\mu(C)|+|[(M)\int\sigma^n d\mu^n-(C)\int\sigma d\mu]\cdot b|: M\in\boldsymbol{B}^n,\ M\subseteq C\}\to 0$ *for each $C\in\boldsymbol{C}$ and each $b\in E_m$. In addition to the assumptions other than* (10b) *in Theorem* 1, *let us assume the following.*

(11) *The mapping ρ is continuous on A.*

(12) *For any $\varepsilon>0$ and any $G\in\boldsymbol{G}$, there exists $C\in\boldsymbol{C}$ with its closure $C^c\subseteq G$, C^c compact, and $\mu(G-C)<\varepsilon$.*

Then $I(T^n)$ is lower semicontinuous at T.

Proof. We proceed at first as in Theorem 1, except that, in the construction of K, we use $\gamma/2$ instead of γ. Also, since ρ is continuous on A, $|\rho(w)-\rho(v)|<\delta(\varepsilon,v)/2$ for $w\in H(v)$, not just $H(v)\cap K$. Having reached the construction of H_i,δ_i,β_i, and b_i, we put $\alpha=\max\{|\beta_i|,\ |b_i|: i=1,2,\cdots h\}$.

For any $\varepsilon' > 0$, we have, by absolute continuity of the integral, $(E)\int|\sigma|\,d\mu < \varepsilon'$ for $\mu(E)$ less than some $\lambda(\varepsilon') > 0$. By the regularity condition (8), we can take $G \in \mathbf{G}$ with $K \subseteq G$ and $\mu(G-K) < \min\{\lambda(\varepsilon/\alpha), \varepsilon/\alpha, \gamma\}$. Put $G_i = G \cap H_i$. Use condition (12) to take $C_1 \in \mathbf{C}$ with compact closure $C_1^c \subseteq G_1$ and $\mu(G_1 - C_1) < \gamma/2h$. Then $G_1 - C_1^c \in \mathbf{G}$. Inductively, take $C_i \in \mathbf{C}$ with compact closure $C_i^c \subseteq G_i - \bigcup_{j<i} C_j^c$ and

$$\mu\Big(G_i - \bigcup_{j<i} C_j^c - C_i\Big) < \gamma/2h$$

for $i = 2, 3, \cdots h$. The sets C_i are disjoint and contained in G, while $C_i \subseteq H_i$. Note that they are not the same as B_i in Theorem 1; in particular, $C_i \in \mathbf{C}$ and C_i need not be contained in K.

Then

$$\begin{aligned}
\mu(A - \textstyle\bigcup C_i) &< \mu(K) + \gamma/2 - \textstyle\sum \mu(C_i) \\
&\leqq \mu(\textstyle\bigcup G_i) - \textstyle\sum \mu(C_i) + \gamma/2 \\
&= \textstyle\sum \mu\Big(G_i - \bigcup_{j<i} G_j\Big) - \textstyle\sum \mu(C_i) + \gamma/2 \\
&\leqq \textstyle\sum \mu\Big(G_i - \bigcup_{j<i} C_j^c - C_i\Big) + \gamma/2 \\
&< \gamma.
\end{aligned}$$

Also

$$\begin{aligned}
\Big|\textstyle\sum (C_i - K)\int (\beta_i + b_i \cdot \sigma)\,d\mu\Big| &\leqq \textstyle\sum (C_i - K)\int \alpha(1 + |\sigma|)\,d\mu \\
&\leqq \alpha(G - K)\int (1 + |\sigma|)\,d\mu \\
&< 2\varepsilon.
\end{aligned}$$

Hence

$$\begin{aligned}
I(T) &< (\textstyle\bigcup C_i)\int f(\rho,\sigma)\,d\mu + \varepsilon \\
&< (\textstyle\bigcup C_i \cap K)\int f(\rho,\sigma)\,d\mu + 2\varepsilon \\
&\leqq \textstyle\sum (C_i \cap K)\int (\beta_i + b_i \cdot \sigma + \varepsilon)\,d\mu + 2\varepsilon \\
&\leqq \textstyle\sum (C_i)\int (\beta_i + b_i \cdot \sigma)\,d\mu - \sum (C_i - K)\int (\beta_i + b_i \cdot \sigma)\,d\mu + \varepsilon\mu(A) + 2\varepsilon \\
&< \textstyle\sum \beta_i \mu(C_i) + \sum b_i \cdot (C_i)\int \sigma\,d\mu + \varepsilon\mu(A) + 4\varepsilon.
\end{aligned}$$

By conditions (10a) and (10b'), there exists an integer $N(\varepsilon)$ such that, for $n > N(\varepsilon)$ and each $i = 1, 2, \cdots h$,

$$\sup\{|\rho^n(w)-\rho(w)|: w\in A^n\cap C_i^c\}<\delta_i/2,$$

and

$$\inf\left\{|\mu^n(M)-\mu(C_i)|+\left|\left[(M)\int\sigma^n d\mu^n-(C_i)\int\sigma d\mu\right]\cdot b_i\right|: M\in \boldsymbol{B}^n, M\subseteq C_i\right\}$$
$$<\min\{\varepsilon/h|\beta_i|,\varepsilon/h\}.$$

We now proceed as in Theorem 1, noting that in using inequality (7a), it does not matter that C_i is not contained in K, because of condition (11).

Note that the requirement that C^c should be compact can be omitted from condition (12) if condition (10a) is strengthened to

$$\sup\{|\rho^n(w)-\rho(w)|: w\in A^n\cap A\}\to 0.$$

4. **The convexity condition (7).** Condition (7) could be termed "convexity of $f(r,s)$ in s uniformly in some neighborhood of $\rho(v)$" in view of the following result.

THEOREM 3. *Let $f(s)$ be a real function on a convex set $S\subseteq E_m$. Then f is continuous and convex if and only if, for any $s_0\in S$ and any $\varepsilon>0$, there exist $\delta>0$, $\beta\in E_1$, and $b\in E_m$ such that*

(a) *$f(s)\geqq\beta+b\cdot s$ for all $s\in S$, and*

(b) *$f(s)\leqq\beta+b\cdot s+\varepsilon$ for $|s-s_0|<\delta$, $s\in S$.*

Proof. Continuity follows immediately from these inequalities. If f is not convex, then, for some $s_0, s_1, s_2\in S$ and some α, $0<\alpha<1$, we have

$$s_0=\alpha s_1+(1-\alpha)s_2,\qquad f(s_0)>\alpha f(s_1)+(1-\alpha)f(s_2).$$

Take $2\varepsilon=f(s_0)-\alpha f(s_1)-(1-\alpha)f(s_2)$. Condition (a) gives $f(s_1)\geqq\beta+b\cdot s_1$, $f(s_2)\geqq\beta+b\cdot s_2$, so $f(s_0)>\beta+b\cdot s_0+\varepsilon$, contradicting condition (b).

Conversely, if f is convex, then the set

$$\{(s,y): y\geqq f(s), s\in S\}$$

in E_{m+1} is convex. By continuity of f, $(s_0,f(s_0)-\frac{1}{2}\varepsilon)$ is at positive distance from that set. Thus there exists a separating plane; that is, there exist $\beta\in E_1$, $b\in E_m$, and $\gamma\in E_1$ such that

$$(b,\gamma)\cdot(s,y)+\beta<0\quad\text{for } y\geqq f(s), s\in S,$$

and

$$(b,\gamma)\cdot(s_0,f(s_0)-\tfrac{1}{2}\varepsilon)+\beta>0.$$

If $\gamma\geqq 0$, the inequalities would be contradictory for $s=s_0$, $y=f(s_0)$. Thus $\gamma<0$, and we can take $\gamma=-1$. Then

$$b\cdot s-f(s)+\beta<0\quad\text{for all } s\in S,$$

and

$$b\cdot s_0-f(s_0)+\tfrac{1}{2}\varepsilon+\beta>0.$$

Condition (b) follows by continuity.

COROLLARY. *If $f(r_0,s)$ is convex and continuous in s, and $f(r,s)$ on D is continuous in r at r_0 uniformly in s, then condition* (7) *is satisfied at* $(r_0,s_0)\in D$.

Proof. For any $\varepsilon>0$, Theorem 3 gives $\delta>0$, β, b such that $f(r_0,s)\geqq\beta+b\cdot s$ for all s, and, for $|s-s_0|<\delta$, $f(r_0,s)\leqq\beta+b\cdot s+\varepsilon/3$. Also, $|f(r,s)-f(r_0,s)|<\varepsilon/3$ for $|r-r_0|<\delta'$ independent of s. Hence, for $|r-r_0|<\min\{\delta,\delta'\}$,

$$f(r,s)\geqq\beta+b\cdot s-\varepsilon/3=\beta'+b\cdot s$$

where $\beta'=\beta-\varepsilon/3$; and for $|s-s_0|<\min\{\delta,\delta'\}$ in addition,

$$f(r,s)\leqq\beta+b\cdot s+2\varepsilon/3=\beta'+b\cdot s+\varepsilon.$$

Thus we have the required result.

In particular, the continuity conditions here are satisfied if, for some $\Delta>0$, $f(r,s)$ is continuous on some compact set containing $D\cap\{(r,s):|r-r_0|<\Delta\}$. However, in many important cases, D need not be bounded. The example

$$f(r,s) \;=\; rs \text{ on } E_1\times E_1 \text{ with } r_0=0$$

$$(\text{or } f(r,s) \;=\; (rs+1)^+ \text{ if } f(r,s)\geqq 0 \text{ is required})$$

shows that convexity and continuity in such circumstances is not sufficient for condition (7). In these cases, the following theorem of Turner [10], involving a strengthened convexity condition, is of use:

Let R be a closed set in E_n and S a closed convex set in E_m such that $R\times S$ contains D. Suppose that $f(r,s)$ is a continuous function on $R\times S$, such that, as a function of s, $f(r,s)$ is convex and its graph contains no whole straight lines. Then f satisfies condition (7).

However, if $(A^n)\int|\sigma^n|d\mu^n$ is bounded, then condition (7) in Theorem 1 can be replaced by the condition that $f(r,s)$ is convex in s and continuous.

THEOREM 4. *Suppose that $(A^n)\int|\sigma^n|d\mu^n$ is bounded. Let the conditions of Theorem* 1 *hold, except that* (7) *is replaced by*

(7′) *$f(r,s)$ is convex in s and continuous on a set $R\times S$ as above.*

Then $I(T^n)$ is lower semicontinuous at T.

Proof. Let M be an upper bound for $(A^n)\int|\sigma^n|d\mu^n$. For any $\varepsilon>0$, put $f_\varepsilon(r,s)=f(r,s)+\varepsilon|s|/2M$. Then f_ε satisfies Turner's strengthened convexity condition. Consequently $(A^n)\int f_\varepsilon(\rho^n,\sigma^n)d\mu^n$ is lower semicontinuous at T. Thus

$$I(T^n)+\varepsilon(A^n)\int|\sigma^n|d\mu^n/2M>I(T)+\varepsilon(A)\int|\sigma|d\mu/2M-\varepsilon/2$$

for all n sufficiently large. This gives $I(T^n)>I(T)-\varepsilon$.

5. **The homogeneous case.** Suppose that $f(r,s)$ is positively homogeneous of degree one in s. Then condition (7a) gives

$$tf(r,s) \geqq \beta + tb \cdot s$$

for all $t > 0$. Hence $0 \geqq \beta$ and $f(r,s) \geqq b \cdot s$. With $\beta \leqq 0$, $f(r,s) \leqq b \cdot s + \varepsilon$ in (7b). Thus β can be taken zero in (7).

Now the term $|\mu^n(M) - \mu(B)|$ in condition (10b) is used in Theorem 1 only in relation to the β terms. Consequently, if $f(r,s)$ is positively homogeneous of degree one in s, then condition (10b) in Theorem 1 can be weakened by omitting the term $|\mu^n(M) - \mu(B)|$.

6. **The condition (4)** $f(r,s) \geqq 0$. In certain cases, condition (4) can be weakened. Suppose that $f(r,s) \geqq -m$ for some fixed positive number m. Then $f_m(r,s) = f(r,s) + m$ satisfies condition (4). The other conditions on f in Theorem 1 carry over to f_m. Thus (A^n) $\int f_m(\rho^n, \sigma^n) d\mu^n = I(T^n) + m\mu^n(A^n)$ is lower semicontinuous at T. Consequently, if, in addition, $\mu^n(A^n)$ is upper semicontinuous at T, then $I(T^n)$ is lower semicontinuous at T.

This is only one of several possible relaxations of condition (4) in Theorem 1. More generally, suppose that

$$f(r,s) \geqq -g(r,s)$$

where f and g satisfy the conditions on f in Theorem 1 other than condition (4), and (A^n) $\int g(\rho^n, \sigma^n) d\mu^n$ is upper semicontinuous at T. Then $f + g$ satisfies all the conditions of Theorem 1. Consequently, (A^n) $\int (f+g) d\mu^n$, and so (A^n) $\int f d\mu^n$, are lower semicontinuous at T.

Two further useful choices for g are as follows.

(i) $g = m_1 + m_2 |s|^\alpha$ with m_1, m_2 positive constants, $\alpha \geqq 1$, and $\mu^n(A^n)$, (A^n) $\int |\sigma^n|^\alpha d\mu^n$ continuous at T.

(ii) $g = p + q \cdot s$ with p, q constant, and $\mu^n(A^n)$, (A^n) $\int \sigma^n d\mu^n$ continuous at T.

7. **Upper semicontinuity and continuity.** A functional $I(T^n)$ is upper semicontinuous at T if and only if $-I(T^n)$ is lower semicontinuous at T. Hence conditions sufficient for upper semicontinuity can be obtained from Theorem 1 by reversing the inequalities in conditions (4) and (7); thus, in an appropriate sense, f would be assumed concave in s.

Continuity is equivalent to lower semicontinuity and upper semicontinuity combined. Consequently, we can obtain sufficient conditions for continuity from Theorem 1 by adding conditions (4) and (7) with reversed inequalities. Thus f would be assumed linear in s. But in order that conditions (4) and (4) reversed should not make f trivial, we must use some relaxation of condition (4), as discussed in §6.

Thus, in Theorem 1, if condition (7) is strengthened to

(7*) $f(r,s)$ is linear in s and continuous, and condition (4) is replaced by any one of the following (i), (ii), or (iii), then I is continuous at T:

(i) f is bounded with $\mu^n(A^n)$ continuous at T;

(ii) $|f(r,s)| \leqq m_1 + m_2|s|^\alpha$ with $\alpha \geqq 1$ and $\mu^n(A^n)$, $(A^n)\int|\sigma^n|^\alpha d\mu^n$ continuous at T;

(iii) $p + q \cdot s \leqq f(r,s) \leqq p' + q' \cdot s$ with $\mu^n(A^n)$, $(A^n)\int \sigma^n d\mu^n$ continuous at T.

8. **Parametric curve integrals.** Let the sets A be the same finite closed interval $\{w: a \leqq w \leqq b\}$ in E_1, with $\boldsymbol{G}$ the Euclidean topology. Consider any class of continuous mappings $\rho: A \to E_n$ of bounded variation, that is, continuous curves of finite length in E_n. Each measure μ (arc length on the curve) and a corresponding signed measure ν are constructed by the process described in [4] from the interval function

$$\phi[u,v] = \rho(v) - \rho(u).$$

Let $\sigma = d\nu/d\mu$; then σ is μ-integrable and $\int \sigma d\mu = \nu$. The triplets $T = (\rho, \sigma, \mu)$ are now determined by the curves ρ.

Since each mapping ρ is of bounded variation, $\mu(A)$ is finite. Each measure μ is $\boldsymbol{G}$-regular by the general theory of [4]. Condition (9) is trivial here. Condition (12) with $\boldsymbol{C} = \boldsymbol{G}$ follows from (H_4) of [4].

The function $f(r,s)$ will be assumed to be positively homogeneous of degree one in s, so we shall consider the appropriate weakening of condition (10b) in §5. We shall prove that uniform convergence in ρ is sufficient for that condition in Theorem 2 with $\boldsymbol{C} = \boldsymbol{G}$.

THEOREM 5. *For each* $G \in \boldsymbol{G}$, *if* $\sup\{|\rho^n(w) - \rho(w)|: w \in A\} \to 0$, *then* $\inf\{|\nu^n(M) - \nu(G)|: M \in \boldsymbol{B}^n, M \subseteq G\} \to 0$.

Proof. In the setting of [4], we have

$$\sum_{I \in D_G} \phi(I) \to \nu(G) \quad \text{as } \delta(D_G) \to 0$$

[4: (1i),(3iv)]. Hence, for any $\varepsilon > 0$, $|\nu(G) - \Sigma\phi(I)| < \varepsilon$ for some finite number (m, say) of nonoverlapping intervals $I \subseteq G$. Here $\phi[u,v] = \rho(v) - \rho(u) = \nu(u,v)$. Consider n such that $\sup\{|\rho^n(w) - \rho(w)|: w \in A\} < \varepsilon/m$. Then $|\nu^n(I^0) - \phi(I)| < 2\varepsilon/m$. Hence $|\nu^n(\bigcup I^0) - \nu(G)| < 3\varepsilon$.

We can now deduce immediately from Theorem 2 the Tonelli-Turner theorem:

Let $f(r,s)$ be nonnegative, positively homogeneous of degree one in s, convex in s in the sense (7), and such that $f(\rho, d\nu/d\mu)$ is measurable. Then the integral $(A)\int f(\rho, d\nu/d\mu)\, d\mu$ is lower semicontinuous with respect to uniform convergence in ρ.

9 **Nonparametric curve integrals.** Let each set A be a compact interval $[a,b]$ in E_1 with the relative Euclidean topology $\boldsymbol{G}$. Let μ be Lebesgue measure on A. Consider absolutely continuous mappings $X: A \to E_m$. Put $\rho(w) = (w, X(w))$;

thus $n = m+1$, and our mappings ρ are essentially nonparametric curves in E_{m+1}. Take $\sigma(w) = X'(w)$; then σ is integrable and $[\alpha,\beta] \int \sigma\, d\mu = X(\beta) - X(\alpha)$ for intervals $[\alpha,\beta] \subseteq [a,b]$.

Conditions (6), (8), and (9) are obviously satisfied. An elementary construction shows that condition (12) is satisfied with $\boldsymbol{C} = \boldsymbol{G}$.

We consider a convergence on X sufficient for condition (10b′) with $\boldsymbol{C} = \boldsymbol{G}$.

THEOREM 6. *If $a^n \to a$, $b^n \to b$, and*

$$\sup \{|X_n(w) - X(w)| : w \in A^n \cap A\} \to 0,$$

then, for each $G \in \boldsymbol{G}$,

$$\inf \left\{ |\mu(M) - \mu(G)| + \left| (M) \int X_n' d\mu - (G) \int X' d\mu \right| : M \in \boldsymbol{B}^n,\ M \subseteq G \right\} \to 0.$$

Proof. By absolute continuity, for any $\varepsilon > 0$, there exists $\gamma(\varepsilon) > 0$ such that $(E) \int |X'|\, d\mu < \varepsilon$ for $\mu(E) < \gamma(\varepsilon)$. Now $G = G^+ \cap A$ for some set G^+ open in E_1. Take a compact set $F^+ \subseteq G^+$ with $\mu(G^+ - F^+) < \min\{\varepsilon, \gamma(\varepsilon)/2\}$. The open set G^+ is the union of a countable number of open intervals. The compact set F^+ is covered by a finite number of these intervals. Contract them to closed intervals still covering F^+, and decompose these into closed nonoverlapping intervals J_i. Put $S^+ = \bigcup J_i$, $S = S^+ \cap A$.

Consider n such that $|a - a|$ and $|b^n - b| < \min\{\varepsilon, \gamma(\varepsilon)/4\}$, and

$$\sup \{|X_n(w) - X(w)| : w \in A^n \cap A\} < \varepsilon/N,$$

where N is the number of intervals J_i. Let $M = A^n \cap S$. Then $M \subseteq G$ and $G - M \subseteq (G^+ - S^+) \cup (A - A^n)$, so

$$|\mu(M) - \mu(G)| = \mu(G - M) \leqq \mu(G^+ - S^+) + \mu(A - A^n) < 3\varepsilon \text{ and } \gamma(\varepsilon).$$

Consequently

$$\begin{aligned} \left|(M) \int X_n' d\mu - (G) \int X' d\mu\right| &< \left|(M) \int (X_n' - X') d\mu\right| + \varepsilon \\ &\leqq \Sigma |X_n(w_i) - X(w_i)| + \varepsilon \\ &< 2N\varepsilon/N + \varepsilon = 3\varepsilon \end{aligned}$$

since there are not more than $2N$ endpoints w_i of the closed nonoverlapping intervals making up M.

We can now deduce from Theorem 2 and §6 the standard semicontinuity theorem:

Let $f(w,p,s): E_1 \times E_m \times E_m \to E_1$ be bounded below, convex in s in the sense (7), and such that $f(w, X(w), X'(w))$ is measurable on A. Then the integra

$$(A)\int f(w,X,X')\,d\mu$$

is lower semicontinuous with respect to the convergence

$$a^n \to a,\quad b^n \to b,\quad \sup\{|X_n(w)-X(w)|: w\in A^n\cap A\}\to 0.$$

10. **Parametric curve integrals involving higher derivatives.** Cinquini [5], [6] deals with variational problems for invariant curve integrals of functions involving derivatives up to the third order. We shall relate our theorems to Cinquini's results for semicontinuity in these cases.

Second order problems. Corresponding to the second order problems of [5], we have the following system. The sets A are the same compact interval in E_1, with $\boldsymbol{G}$ the Euclidean topology. Let X be any absolutely continuous mapping from A to E_3 such that, when parametrized by its arc length l, $X'=dX/dl$ is also absolutely continuous. Put $\rho=(X,X')$ and $\sigma=X'\wedge X''$ where $\wedge$ denotes the vector product in E_3; thus $n=6$, $m=3$. Each measure μ corresponds to the arc length l. The total length $L=\mu(A)$ is finite by absolute continuity of X.

We consider Cinquini's convergence of such X. A sequence X_n converges to X with $L>0$ if

$$\sup\{|l_n'(w)-1|+|X_n(w)-X(w)|+|X_n'(w)-X'(w)|\}\to 0.$$

Here $l_n'=dl_n/dl$ with l_n treated as a function of l, but $X_n'=dX_n/dl_n$; the supremum is taken over $w\in A$. If $L=0$, that is, X is constant, then we require

$$\sup|X_n(w)-X|+\sup|X_n(w_1)-X_n'(w_2)|\to 0$$

where the first supremum is over A, the second over pairs of points in A.

If we restrict considerations to a class on which $(A)\int|X''|\,d\mu$ is bounded, then this convergence satisfies the condition (10b′) in Theorem 2 with $\boldsymbol{C}=\boldsymbol{G}$. Indeed, essentially the same method as Cinquini's in [5, p. 33] proves

THEOREM 7. *If a sequence X_n converges to X in Cinquini's sense, and $(A)\int|X_n''|\,d\mu^n$ is bounded, then, for each $G\in\boldsymbol{G}$,*

$$\inf\left\{|\mu^n(M)-\mu(G)|+\left|(M)\int X_n'\wedge X_n''\,d\mu^n-(G)\int X'\wedge X''\,d\mu\right|: M\in\boldsymbol{B}^n, M\subseteq G\right\}\to 0.$$

We can now deduce from Theorem 2 the semicontinuity theorem used by Cinquini:

Let $f(p,q,s)$: $E_3\times E_3\times E_3\to E_1$ be nonnegative, convex in s in the sense (7), and such that $f(X,X',X'\wedge X'')$ is μ-measurable. Then the integral

$$(A)\int f(X,X',X'\wedge X'')\,d\mu$$

is lower semicontinuous with respect to Cinquini's convergence in any class in which $(A)\int|X''|\,d\mu$ is bounded.

Third order problems. Take the sets A as the same compact interval in E_1, with $\boldsymbol{G}$ the Euclidean topology. Let X be any absolutely continuous mapping from A to E_3 such that, when parametrized by its arc length l, $X' = dX/dl$ and $X'' = d^2X/dl^2$ are also absolutely continuous. We put $\rho = (X, X', X' \wedge X'')$ and $\sigma = X' \wedge X'''$; thus $n = 9$ and $m = 3$. Each measure μ corresponds to the arc length l.

Consider Cinquini's convergence for such X: A sequence X_n converges to X with $L > 0$ if

$$\sup\{|l_n'(w) - 1| + |X_n(w) - X(w)| + |X_n'(w) - X'(w)| + |X_n''(w) - X''(w)|\} \to 0.$$

If $L = 0$, that is, X is constant, then we require that

$$\sup |X_n(w) - X| + \sup \{|X_n'(w_1) - X_n'(w_2)| + |X_n''(w_1) - X_n''(w_2)|\} \to 0.$$

This convergence satisfies condition (10b′) of Theorem 2 with $\boldsymbol{C} = \boldsymbol{G}$. Specifically, essentially the same method as Cinquini's [6, p. 54] proves

THEOREM 8. *If a sequence X_n converges to X in Cinquini's sense for third order problems, then, for each $G \in \boldsymbol{G}$,*

$$\inf\left\{|\mu^n(M) - \mu(G)| + \left|(M)\int X_n' \wedge X_n''' d\mu^n - (G)\int X' \wedge X''' d\mu\right| : M \in \boldsymbol{B}^n,\ M \subseteq G\right\} \to 0.$$

We can now deduce from Theorem 2 Cinquini's semicontinuity theorem for third order problems;

Let $f(p, q, t, s): E_3 \times E_3 \times E_3 \times E_3 \to E_1$ be nonnegative, convex in s in the sense (7), and such that $f(X, X', X' \wedge X'', X' \wedge X''')$ is μ-measurable. Then the integral $(A)\ \int f(X, X', X' \wedge X'', X' \wedge X''') d\mu$ is lower semicontinuous with respect to Cinquini's convergence above.

11. **Parametric surface integrals.** Let the sets A be the same admissible set in E_2 [2, p. 27]. The dimensions m and n are both 3. Consider any class of mappings $\rho: A \to E_3$, continuous in the Euclidean topology $\boldsymbol{U}$ and of bounded variation; that is, continuous surfaces of finite area. Each ρ determines a topology $\boldsymbol{G}$ on A, namely the class of $\boldsymbol{U}$-open ρ-whole sets in A [2, §10.2]. Each measure μ (Lebesgue area on the surface) and a corresponding signed measure ν are constructed by the general process described in [4] from an interval function defined from ρ [3, p. 106]. Let $\sigma = d\nu/d\mu$; then σ is μ-integrable and $\int \sigma\, d\mu = \nu$. The triplets (ρ, σ, μ) are now determined by the surfaces ρ.

Since each mapping ρ is of bounded variation, $\mu(A)$ is finite. Each measure μ is $\boldsymbol{G}$-regular by the general theory of [4]. Condition (9) is satisfied [12, p. 196]. Each ρ is $\boldsymbol{G}$-continuous. Condition (12) for $\boldsymbol{C} = \boldsymbol{G}$ follows from (H_4) of [4].

The function $f(r,s)$ will be assumed positively homogeneous of degree one in s, so we shall consider the weakening of condition (10b) discussed in §5. We shall extract part of Theorem 1 of [12] to show that uniform convergence of ρ is sufficient for that condition in Theorem 2 with $\boldsymbol{C} = \boldsymbol{G}$.

THEOREM 9. *If* $\sup\{|\rho^n(w) - \rho(w)| : w \in A\} \to 0$, *then, for each* $G \in \boldsymbol{G}$ *and each (unit) vector* $b \in E_3$,

$$\inf\{|(v^n(M) - v(G)) \cdot b| : M \in \boldsymbol{B}^n, M \subseteq G\} \to 0.$$

Proof. Let P be a rotation taking b to the z-axis. Then

$$v(G) \cdot b = Pv(G) \cdot Pb = v_3(G, P\rho) = v(G, \tau P\rho)$$

where we have considered v explicitly as a function of ρ, v_3 is the z component, and τ is projection on the (x, y) plane. The second equality follows from a rotational property of v [11, Theorem 3].

According to Lemma 3 of [12], simplified for our purpose, for any $\varepsilon > 0$ and any plane mapping T_0 on G with bounded variation, there exists $\delta > 0$ such that, for any other plane mapping T on G with $\sup\{|T(w) - T_0(w)| : w \in G\} < \delta$, there exists $M \subseteq G$ such that $|v(M, T) - v(G, T_0)| < \varepsilon$. Now, if we take n such that $\sup\{|\rho^n(w) - \rho(w)| : w \in A\} < \delta$, then

$$|\tau P\rho^n(w) - \tau P\rho(w)| \leqq |P\rho^n(w) - P\rho(w)| = |\rho^n(w) - \rho(w)| < \delta$$

on A and so certainly on G. Hence

$$|(v^n(M) - v(G)) \cdot b| = |v(M, \tau P\rho^n) - v(G, \tau P\rho)| < \varepsilon.$$

We can now deduce immediately from Theorem 2 Turner's theorem [12, Theorem 1]:

Let $f(r,s)$ be nonnegative, positively homogeneous of degree one in s, convex in s in the sense (7), and such that $f(\rho, dv/d\mu)$ is measurable. Then the integral $(A)\int f(\rho, dv/d\mu)\,d\mu$ is lower semicontinuous with respect to uniform convergence in ρ.

12. **Nonparametric integrals.** Let the sets A be the same open set in E_k with finite Lebesgue k-measure μ. Take $\boldsymbol{G}$ as the Euclidean topology on A.

Consider mappings $X: A \to E_l$ which are continuous and absolutely continuous in the sense of Tonelli. Thus, for each coordinate w_i of $w \in A$, $\partial X/\partial w_i$ exists μ-almost everywhere in A, is μ-integrable, and

$$\int_\alpha^\beta \partial X/\partial w_i \; dw_i = X_{w_i=\beta} - X_{w_i=\alpha}$$

for each segment $\{\alpha \leqq w_i \leqq \beta\}$ in A on μ^*-almost every line parallel to the w_i axis. Here μ^* is Lebesgue $(k-1)$-measure; if $k = 1$, then we take μ^* as enumeration.

Put $\rho(w) = (w, X(w))$. Thus $n = k + l$. In the case $k = 1$, our mappings ρ are ρ essentially nonparametric curves on A in E_{l+1}. In the case $l = 1$, our mappings are essentially nonparametric hypersurfaces on A in E_{k+1}.

Put $\sigma = X' = [\partial X / \partial w_i]$. This is a matrix of order $k \times l$, but here we treat it as a vector of dimension kl.

The Lebesgue measure μ is $\boldsymbol{G}$-regular. Indeed, the closed set F in the regularity condition can be taken compact, since its compact intersections F_n with the spheres $\{w: |w| \leqq n\}$ have $\mu(F_n) \to \mu(F)$; thus condition (9) is satisfied. We use this compact regularity of μ to show that condition (12) is satisfied with $\boldsymbol{C} = \boldsymbol{G}$. For any $G \in \boldsymbol{G}$ and $\varepsilon > 0$, we take compact $F \subseteq G$ with $\mu(G - F) < \varepsilon$. The open set G is the union of a countable number of closed intervals, and also the union of the corresponding open intervals. A finite number of these open intervals I covers F. Take $C = \bigcup I$. Its closure $C^c = \bigcup I^c \subseteq G$ and is compact. Of course, C is open and $\mu(G - C) \leqq \mu(G - F) < \varepsilon$.

We now show that convergence of X uniformly on each compact subset of A is sufficient for condition (10b′) of Theorem 2 with $\boldsymbol{C} = \boldsymbol{G}$.

THEOREM 10. *If* $\sup\{|X_n(w) - X(w)|: w \in K\} \to 0$ *for each compact subset* K *of* A, *then, for each open set* $G \subseteq A$,

$$\inf\left\{|\mu(M) - \mu(G)| + \left|(M)\int X_n' d\mu - (G)\int X' d\mu\right| : M \in \boldsymbol{B},\ M \subseteq G\right\} \to 0.$$

Proof. By absolute continuity of the integral, for any $\varepsilon > 0$, there exists $\gamma(\varepsilon) > 0$ such that $(E)\int |X'|\, d\mu < \varepsilon$ for $\mu(E) < \gamma(\varepsilon)$. Let F be a compact subset of G with $\mu(G - F) < \min\{\varepsilon, \gamma(\varepsilon)\}$. The open set G is the union of a countable number of open intervals. A finite number of those intervals covers F. These can be contracted to closed intervals still covering F, and then decomposed into closed nonoverlapping intervals I. Then $\mu(G - \bigcup I) < \varepsilon$ and $\gamma(\varepsilon)$. Hence

$$\left|(G - \bigcup I)\int X' d\mu\right| < \varepsilon.$$

Consider n such that

$$\sup\{|X_n(w) - X(w)|: w \in \bigcup I^*\} < \varepsilon / \sum \mu^*(I^*)$$

where I^* is the boundary of I. Now

$$\left|(I)\int (X_n' - X')d\mu\right| \leqq \sum \left|(I)\int \partial(X_n - X)/\partial w_i\, d\mu\right| \leqq (I^*)\int |X_n - X|\, d\mu^*,$$

so $|(\bigcup I)\int (X_n' - X')d\mu| < \varepsilon$. Consequently

$$|\mu(\bigcup I) - \mu(G)| + \left|(\bigcup I)\int X_n' d\mu - (G)\int X' d\mu\right| < 3\varepsilon.$$

We can now deduce from Theorem 2 and §6 the semicontinuity theorem:

Let $f(w,p,s): A \times E_l \times E_{kl} \to E_1$ be bounded below, convex in s in the sense (7), and such that $f(w,X(w),X'(w))$ is measurable on A. Then the integral $(A) \int f(w,X,X')\,d\mu$ is lower semicontinuous with respect to convergence of X uniformly on compact subsets of A.

In [13], Kazimirov has obtained a semicontinuity theorem for integrals of the form (in our notation)

$$(A) \int f(\rho(w),\sigma(w))\,d\mu$$

where A is a fixed set in E_k and μ is Lebesgue measure. The semicontinuity is relative to strong convergence of ρ in $L_p(A,\mu)$ and weak convergence of σ in $L_p(A,\mu)$. (See also Morrey's work, for example in [7], where σ is a generalized derivative of ρ.) We can obtain Kazimirov's result from Theorem 1 by a standard procedure.

THEOREM 11. *Let A be a set in E_k with finite Lebesgue measure μ. Consider mappings $\rho: A \to E_n$ and $\sigma: A \to E_m$ with components in $L_p(A,\mu)$, $p \geqq 1$. Let $f(r,s): E_n \times E_m \to E_1$ be continuous, nonnegative, and convex in s in the sense* (7). *Then the integral $(A) \int f(\rho,\sigma)\,d\mu$ is lower semicontinuous in the convergence*

$$(A) \int |\rho_n - \rho|^p d\mu \to 0, (A) \int \sigma_n \phi\, d\mu \to (A) \int \sigma\phi\, d\mu \text{ for each } \phi \in L_p^*(A,\mu).$$

Proof. There exists a subsequence $n(k)$ such that

$$(A) \int f(\rho_{n(k)},\sigma_{n(k)})d\mu \to \liminf (A) \int f(\rho_n,\sigma_n)\,d\mu \text{ and } \rho_{n(k)}(w) \to \rho(w)$$

almost everywhere in A. For any $\eta < (A) \int f(\rho,\sigma)\,d\mu$, there exists a set $B \subseteq A$ such that $\eta < (B) \int f(\rho,\sigma)\,d\mu$ and $\rho_{n(k)}(w) \to \rho(w)$ uniformly on B. By Theorem 1,

$$\begin{aligned}(B)\int f(\rho,\sigma)\,d\mu &\leqq \liminf (B) \int f(\rho_{n(k)},\sigma_{n(k)})\,d\mu \\ &\leqq \lim (A) \int f(\rho_{n(k)},\sigma_{n(k)})\,d\mu \\ &= \liminf (A) \int f(\rho_n,\sigma_n)\,d\mu.\end{aligned}$$

REFERENCES

1. L. Cesari, *Condizioni sufficienti per la semicontinuità degli integrali sopra una superficie in forma parametrica*, Ann. Scuola Norm. Sup. Pisa (2) **14** (1945), 47–79.

2. ———, *Surface area*, Princeton Univ. Press, Princeton, N. J., 1956.

3. ———, *Quasi additive set functions and the concept of integral over a variety*, Trans. Amer. Math. Soc. **102** (1962), 94–113.

4. ———, *Extension problem for quasi additive set functions and Radon-Nikodym derivatives*, Trans. Amer. Math. Soc. **102** (1962), 114–146.

5. S. Cinquini, *Sopra i problemi variazionali in forma parametrica dependenti dalle derivate di ordine superiore*, Ann. Scuola Norm. Sup. Pisa (2) **13** (1944), 19–49.

6. ———, *Sopra l'esistenza dell'estremo per una classe di integrali curvilinei in forma parametrica*, Ann. Mat. Pura Appl. (4) **49** (1960), 25–71.

7. C. B. Morrey, *Multiple integral problems in the calculus of variations and related topics*, Ann. Scuola Norm. Sup. Pisa (3) **14** (1960), 1–61.

8. L. Tonelli, *Fondamenti di calcolo delle variazioni*. Vols. I, II, Zanichelli, Bologna, 1921, 1923.

9. ———, *Sugli integrali del calcolo delle variazioni, in forma ordinaria*, Ann. Scuola Norm. Sup. Pisa (2) **3** (1934), 401–450.

10. L. H. Turner, *The direct method in the calculus of variations*, Thesis, Purdue Univ., Lafayette, Ind., 1957.

11. ———, *An invariant property of Cesari's surface integral*, Proc. Amer. Math. Soc. **9** (1958), 920–925.

12. ———, *Sufficient conditions for semicontinuous surface integrals*, Michigan Math. J. **10** (1963), 193–206.

13. V. I. Kazimirov, *On the semicontinuity of integrals of the calculus of variations*, Uspehi Mat. Nauk **11** (1956), 125–129.

University of Michigan,
Ann Arbor, Michigan,
University of Otago,
Dunedin, New Zealand