ON COMPLEX QUADRATIC FIELDS WITH CLASS NUMBER EQUAL TO ONE\(^{(1)}\)

BY

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Let \( R(\sqrt{-p}) \) be a quadratic extension of the rationals, where \( p \) is a positive square free integer. For nine values of \( p \), namely 1, 2, 3, 7, 11, 19, 43, 67, 163, the integers of \( R(\sqrt{-p}) \) form a unique factorization domain. Heilbronn and Linfoot \([1]\) have shown that there is at most one more such value of \( p \), and Lehmer \([2]\) has shown that \( p \) must be a prime greater than \( 5 \cdot 10^9 \). In the present paper we verify and extend the lower bound of \( 5 \cdot 10^9 \) for \( p \). The result is

**Theorem 1.** If the ring of integers of \( R(\sqrt{-p}) \) (\( p \) square free) forms a unique factorization domain, and \( p > 10^4 \), then \( p > \exp(2.2 \cdot 10^9) \).

It will be assumed throughout that \( p \) is an integer satisfying the hypothesis of Theorem 1. We start with a formula equivalent to that given by Lemma 2 of \([1]\).

\[
(1) \quad \zeta(s)L(s) - \zeta(2s) = 2^{2s-1}p^{(1/2)-s}p(2s - 1)\frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} + h(s),
\]

valid for \( \sigma > \frac{1}{2} \), where \( \zeta(s) \) is the Riemann zeta function, \( L(s) \) is the Dirichlet \( L \)-series formed with the quadratic character \((\mod p)\), and

\[
h(s) = \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \left( x - \left[ x \right] - \frac{1}{2} \right) \frac{d}{dx} \left[ \left( \frac{x + j}{2} \right)^2 + \frac{pj^2}{4} \right]^{-s} \, dx.
\]

Let \( x + (j/2) = u(j/\sqrt{p})/2 \) give a change of variable from \( x \) to \( u \) and integrate by parts \( 2m - 1 \) times; we get, for \( m = 1, 2, \cdots \),

\[
(2) \quad h(s) = -\sum_{j=1}^{\infty} \left( \frac{2}{j\sqrt{p}} \right)^{2m+2s-1} \int_{-\infty}^{\infty} B_{2m}(x - [x]) \frac{d^{(2m)}}{du^{(2m)}} \left\{ (u^2 + 1)^{-s} \right\} \, du,
\]

where \( B_k(x) \) is the \( k \)th Bernoulli polynomial \((B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{2} , \cdots)\) and the series converges for \( \text{Re}s > 1 - m \).

It is well known (\([3, p. 245]\), Jordan's \( \varphi_n(x) = B_n(x)/n! \)) that for \( 0 \leq x \leq 1, k > 1, \)

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so that to estimate \( h(s) \), we need only estimate \( \frac{d^{(2m)}}{du^{(2m)}}((u^2 + 1)^{-s}) \). Using the fact that

\[
\frac{d^2}{du^2} ((u^2 + 1)^{-s}) = 2s(2s + 1)(u^2 + 1)^{-s-1} - 2s(2s + 2)(u^2 + 1)^{-s-2},
\]

we see inductively that we can write

\[
\frac{d^{(2m)}}{du^{(2m)}} ((u^2 + 1)^{-s}) = \sum_{k=0}^{m} c_{mk}(s)(u^2 + 1)^{-s-m-k}.
\]

Again, by induction we have

\[
|c_{mk}(s)| \leq \binom{m}{k} \prod_{j=0}^{2m-1} (2 |s| + 2j).
\]

Thus

\[
\left| \frac{d^{(2m)}}{du^{2m}} ((u^2 + 1)^{-s}) \right| \leq \sum_{k=0}^{m} \binom{m}{k} (u^2 + 1)^{-s-m-k} \prod_{j=0}^{2m-1} (2 |s| + 2j)
\]

\[
\leq 2^m (u^2 + 1)^{-s-m} \prod_{j=0}^{2m-1} (2 |s| + 2j)(2 |s| + 2(2m - 1 - j))
\]

\[
\leq 2^m (u^2 + 1)^{-s-m}(2 |s| + 2m - 1)^2m.
\]

In view of (2), (3) and (6), we have for \( \sigma \geq \frac{1}{2} \) and \( m \geq 1 \):

\[
|h(s)| \leq \sum_{y=1}^{\infty} \left( \frac{2}{y \sqrt{p}} \right)^{2m+2\sigma-1} \cdot \frac{2}{(2\pi)^{2m}} \zeta(2m) \cdot 2^m (2 |s| + 2m - 1)^2 \int_{-\infty}^{\infty} (u^2 + 1)^{-1} du
\]

\[
\leq 2\pi \zeta(2m) \left( \frac{4 |s| + 4m - 2}{\pi \sqrt{2p}} \right)^{2m}
\]

\[
\leq 2\pi \left( \frac{2m}{2m - 1} \right)^2 \left( \frac{4 |s| + 4m - 2}{\pi \sqrt{2p}} \right)^{2m}.
\]

Letting \( m = 30 \), we see that if \( |s| \leq 22 \) and \( \sigma \geq \frac{1}{2} \) (and of course \( p > 10,000 \)), then

\[
|h(s)| < 10^{-19}.
\]

Let \( \theta \) denote a number, complex or real, not necessarily the same each time it occurs, which satisfies \( |\theta| \leq 1 \). We find that for \( |s| < 22 \) and \( \sigma \geq \frac{1}{2} \), (1) becomes
\( \zeta(s)L(s) - \zeta(2s) = \zeta(2-2s) \frac{\Gamma(1-s)}{\Gamma(s)} \left( \frac{\sqrt{p}}{2\pi} \right)^{1-2s} + 10^{-19}\theta, \)

where the functional equation for \( \zeta(s) \) was used to obtain the first term on the right.

Let
\( s_n = \frac{1}{2} + i\gamma_n \)

denote the \( n \)th zero of \( \zeta(s) \) above the real axis. It is known that \( \gamma_1 \approx 14 \) and \( \gamma_2 \approx 21 \) (see Appendix); in particular, \( |s_1| < |s_2| < 22 \) and thus
\( \zeta(2s_n) = -\zeta(2-2s_n) \frac{\Gamma(1-s_n)}{\Gamma(s_n)} \left( \frac{p}{4\pi^2} \right)^{-i\gamma_n} + 10^{-19}\theta, \quad (n = 1, 2). \)

Multiplying both sides by \( (p/4\pi^2)^{i\gamma_n}(1/\zeta(2s_n)) \) and using the fact that \( |\zeta(2s_n)| > \frac{1}{2} \) for \( n = 1, 2 \) (see Appendix), we get
\[
\left( \frac{p}{4\pi^2} \right)^{i\gamma_n} = -\frac{\zeta(2-2s_n)}{\zeta(2s_n)} \frac{\Gamma(1-s_n)}{\Gamma(s_n)} + 2 \cdot 10^{-19}\theta
\]
\( (11) \]
\[
\frac{\zeta(1-2i\gamma_n)\Gamma\left(\frac{1}{2} - i\gamma_n\right)}{\zeta(1+2i\gamma_n)\Gamma\left(\frac{1}{2} + i\gamma_n\right)} (1 + 2 \cdot 10^{-19}\theta), \quad (n = 1, 2).
\]

Taking arguments of both sides of (11) gives
\( (12) \]
\[
\gamma_n \log \left( \frac{p}{4\pi^2} \right) = a_n + 2\pi x_n + 3 \cdot 10^{-19}\theta, \quad (n = 1, 2)
\]

where \( x_n \) is an integer and
\( (13) \]
\[
a_n \equiv \pi - 2 \arg \zeta(2s_n) - 2 \arg \Gamma(s_n) \pmod{2\pi},
\]
\[
0 \leq a_n < 2\pi.
\]

Eliminating \( \log (p/4\pi^2) \) from the equations (12), and solving for \( x_2 \), we obtain
\( (14) \]
\[
x_2 = \frac{\gamma_2}{\gamma_1} x_1 + a_0 + 10^{-18}\theta,
\]

where
\( (15) \]
\[
a_0 = \frac{1}{2\pi} \left( \frac{\gamma_2}{\gamma_1} a_1 - a_2 \right).
\]

From the Appendix,
\( (16) \]
\[
\frac{\gamma_2}{\gamma_1} = 1.487 \; 262 \; 003 \; 892 \; 890 \; 048 + 10^{-18}\theta,
\]
\[
a_0 = a + 4 \cdot 10^{-9}\theta \quad \text{where} \quad a = -0.461 \; 786 \; 352.
\]
We can rewrite (14) as

\[(17) \quad x_2 = \frac{\gamma_2}{\gamma_1} x_1 + a + \frac{1}{2} \cdot 10^{-9} \theta.\]

Note that

\[(18) \quad 3.999999660 = \frac{\gamma_2}{\gamma_1} \cdot 3 + a + \frac{1}{2} \cdot 10^{-9} \theta.\]

It is not accidental that $3(\gamma_2/\gamma_1) + a$ should be close to an integer; $x_1 = 3$ corresponds to $p = 163$ (see introduction). In fact

\[\gamma_1 \log \left( \frac{163}{4\pi^2} \right) = 20.042984673072\ldots,\]

\[a_1 + 2\pi \cdot 3 = 20.042984673470\ldots.\]

(Compare this with (12), where these numbers would agree to at least 19 decimal places if $p > 10,000$.) From (12), we now see that $p > 10^4$ implies that $x_1 > 3$.

Subtracting (18) from (17) gives:

\[(19) \quad x_2 - 4 = \frac{\gamma_2}{\gamma_1} (x_1 - 3) - b + 10^{-9} \theta, \quad \text{where } b = .000\ 000\ 340.\]

Now let

\[(20) \quad p_1 = 83,532,765, \quad p_2 = 12,832,922,\]

\[q_1 = 56,165,467, \quad q_2 = 8,628,555.\]

Then $p_1q_2 - q_1p_2 = 1$, so that $p_1$ and $q_1$ are relatively prime.

Also,

\[(21) \quad \left| \left( \frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1} \right) \right| < 2.3 \cdot 10^{-16}.\]

Let

\[(22) \quad Q + \frac{R}{q_1} = \frac{p_1}{q_1} (x_1 - 3),\]

where $0 \leq R < q_1$ and $Q$ and $R$ are integers. Subtracting (22) from (19) gives

\[(23) \quad x_2 - Q - 4 = \left( \frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1} \right) (x_1 - 3) + \left( \frac{R}{q_1} - b \right) + 10^{-9} \theta.\]

If $x_1 \leq 5.1 \times 10^7$, then

\[(24) \quad \left| \left( \frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1} \right) (x_1 - 3) \right| < 12 \cdot 10^{-9},\]

and thus $x_1 \leq 5.1 \cdot 10^7$ implies
On the other hand, since
\[
\frac{18}{q_1} < 321 \cdot 10^{-9} < b = 340 \cdot 10^{-9} < 356 \cdot 10^{-9} < \frac{20}{q_1},
\]
we find that \( x_1 \leq 5.1 \cdot 10^7 \) and \( R \neq 19 \) implies
\[
|x_2 - Q - 4| \leq |\frac{R}{q_1} - b| - \left| \left( \frac{y_2}{\gamma_1} - \frac{p_1}{q_1} \right) (x_1 - 3) \right| - 10^{-9}
\]
\[
> 16 \cdot 10^{-9} - 12 \cdot 10^{-9} - 10^{-9} > 0.
\]
Inequalities (25) and (26) are contradictory, and therefore \( x_1 \leq 5.1 \cdot 10^7 \) implies \( R = 19 \).

But if \( R = 19 \), then we see from (22) that
\[
19 \equiv p_1(x_1 - 3) \pmod{q_1},
\]
and this implies
\[
x_1 - 3 \equiv 51,611,611 \pmod{q_1}.
\]
Thus under all circumstances, \( x_1 > 5.1 \cdot 10^7 \). Hence by (12),
\[
\log \left( \frac{p}{4\pi^2} \right) = \frac{a_1 + 2\pi x_1 + 3 \cdot 10^{-19} \theta}{\gamma_1}
\]
\[
> \frac{2\pi(5.1 \cdot 10^7) - 3 \cdot 10^{-19}}{14.2}
\]
\[
> 2.2 \cdot 10^7,
\]
and Theorem 1 follows.

**APPENDIX**

I wish to express my thanks to M.D. Bigg who furnished values of \( \gamma_1 \) and \( \gamma_2 \) to fifty decimal places and to R.S. Lehman who furnished values of \( \arg \zeta(2s_n) \) and \( |\zeta(2s_n)| \) for \( n = 1 \) and \( 2 \), with a proved accuracy of \( \pm 10^{-10} \). The values of \( \gamma_1 \) and \( \gamma_2 \) were confirmed independently by Robert Spira to fifteen decimal places. Their values are:
\[
\gamma_1 = 14.134 \ 725 \ 141 \ 734 \ 693 \ 790 \ 457 + 10^{-21} \theta,
\]
\[
\gamma_2 = 21.022 \ 039 \ 638 \ 771 \ 554 \ 992 \ 628 + 10^{-21} \theta,
\]
\[
\frac{1}{\pi} \arg \zeta(2s_1) = - .108 \ 452 \ 737 \ 083 \ 095 + 10^{-10} \theta \pmod{2},
\]
\[
\frac{1}{\pi} \arg \zeta(2s_2) = .067 \ 103 \ 865 \ 503 \ 910 + 10^{-10} \theta \pmod{2},
\]
\[
|\zeta(2s_1)| = 1.948 \ 757 \ 313 \ 817 \ 40 + 10^{-10} \theta,
\]
\[
|\zeta(2s_2)| = .830 \ 962 \ 021 \ 546 \ 955 + 10^{-10} \theta.
\]
All other numbers were calculated on a Monroe desk calculator to fifteen places.

The Euler-Maclaurin sum formula for \( \log \Gamma\left(\frac{1}{2} + iy\right) \) can be expressed ([4, p. 132]. It should be noted that the exponent \((m + 1)\) in Nörlund’s remainder term should be replaced by \(m\). See also [5, p. 131]):

\[
\log \Gamma\left(\frac{1}{2} + iy\right) = iy \log(iy) - iy + \frac{1}{2} \log(2\pi)
\]

\[
+ \sum_{j=2}^{11} (-1)^j \frac{B_j\left(\frac{1}{2}\right)}{j(j-1)} \cdot \frac{1}{(i\pi)^{j-1}}
\]

\[
- \int_0^\infty B_{11} \left( x - \frac{1}{2} - \left[ x - \frac{1}{2} \right] \right) \frac{dx}{12(x + iy)^{11}}
\]

From this we see that

\[
\arg \Gamma\left(\frac{1}{2} + iy\right) = \gamma(\log \gamma - 1) + \sum_{k=1}^5 \frac{(-1)^kB_{2k}\left(\frac{1}{2}\right)}{2k(2k-1)\gamma^{2k-1}}
\]

\[
- \text{Im} \int_0^\infty B_{11} \left( x - \frac{1}{2} - \left[ x - \frac{1}{2} \right] \right) \frac{dx}{12(x + iy)^{11}}
\]

From the formula [5, p. 123],

\[
B_k\left(\frac{1}{2}\right) = -(1 - 2^{1-k})B_k,
\]

we get the following:

\[
\begin{align*}
B_2\left(\frac{1}{2}\right) & = -0.041666666667 + 10^{-15}\theta, \\
B_4\left(\frac{1}{2}\right) & = 0.002430555556 + 10^{-15}\theta, \\
B_6\left(\frac{1}{2}\right) & = -0.000768849206349 + 10^{-15}\theta, \\
B_8\left(\frac{1}{2}\right) & = 0.000590587797619 + 10^{-15}\theta, \\
B_{10}\left(\frac{1}{2}\right) & = -0.000840106797138 + 10^{-15}\theta.
\end{align*}
\]
From [6], we get
\[ \log \gamma_1 = 2.648 \ 634 \ 545 \ 730 \ 790 + 10^{-13} \theta, \]
\[ \log \gamma_2 = 3.045 \ 571 \ 393 \ 984 \ 561 + 10^{-13} \theta. \]

Finally, for \( t > 14 \), we get from (3),
\[ \int_0^\infty B_{11} \left( x - \frac{1}{2} - \left[ x - \frac{1}{2} \right] \right) \frac{dx}{11(x + it)^{11}} \leq \frac{(2\pi)^{2.11!}}{11^{11}} \cdot \frac{11}{10} \int_0^\infty \frac{dx}{x^2 + t^2} \]
\[ = \frac{11!\pi}{10(2\pi)^{11} t^{10}} < 10^{-13}. \]

Putting these numbers into (A2), we obtain
\[ \frac{1}{\pi} \arg \Gamma(s_1) = 7.418 \ 512 \ 651 \ 985 \ 173 + 2 \cdot 10^{-13} \theta, \]
\[ \frac{1}{\pi} \arg \Gamma(s_2) = 13.688 \ 619 \ 111 \ 000 \ 235 + 2 \cdot 10^{-13} \theta. \]

We may rewrite (12) thusly:
\[ \frac{a_n}{2\pi} \equiv \frac{1}{2} - \frac{1}{\pi} \arg \zeta(2s_n) - \frac{1}{\pi} \arg \Gamma(s_n) \quad (\text{mod} \ 1), \]
\[ 0 \leq \frac{a_n}{2\pi} < 1. \]

Using (A1) and (A3), we find that
\[ \frac{a_1}{2\pi} \equiv .189 \ 940 \ 085 \ 097 \ 922 + 1.002 \cdot 10^{-10}\theta \quad (\text{mod} \ 1), \]
\[ \frac{a_2}{2\pi} \equiv .744 \ 277 \ 023 \ 495 \ 855 + 1.002 \cdot 10^{-10}\theta \quad (\text{mod} \ 1). \]

Since these numbers are between 0 and 1, the above is actually an equality.

Finally,
\[ \frac{\gamma_2}{\gamma_1} = 1.487 \ 262 \ 003 \ 892 \ 890 \ 048 + 10^{-18}\theta \]
and
\[ a_0 = \frac{\gamma_2}{\gamma_1} \left( \frac{a_1}{2\pi} \right) - \frac{a_2}{2\pi} \]
\[ = -.461 \ 786 \ 351 \ 913 \ 533 + 3 \cdot 10^{-10}\theta. \]
References


