ON COMPLEX QUADRATIC FIELDS WITH CLASS NUMBER EQUAL TO ONE\(^{1)}\)

BY

HAROLD STARK

Let \(R(\sqrt{-p})\) be a quadratic extension of the rationals, where \(p\) is a positive square free integer. For nine values of \(p\), namely 1, 2, 3, 7, 11, 19, 43, 67, 163, the integers of \(R(\sqrt{-p})\) form a unique factorization domain. Heilbronn and Linfoot [1] have shown that there is at most one more such value of \(p\), and Lehmer [2] has shown that \(p\) must be a prime greater than \(5 \cdot 10^9\). In the present paper we verify and extend the lower bound of \(5 \cdot 10^9\) for \(p\). The result is

**Theorem 1.** If the ring of integers of \(R(\sqrt{-p})\) (\(p\) square free) forms a unique factorization domain, and \(p > 10^4\), then \(p > \exp(2.2 \cdot 10^7)\).

It will be assumed throughout that \(p\) is an integer satisfying the hypothesis of Theorem 1. We start with a formula equivalent to that given by Lemma 2 of [1].

\[
\zeta(s)L(s) - \zeta(2s) = 2^{2s-1}p^{(1/2)-s}\zeta(2s-1)/\pi \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} + h(s),
\]

valid for \(\sigma > \frac{1}{2}\), where \(\zeta(s)\) is the Riemann zeta function, \(L(s)\) is the Dirichlet \(L\)-series formed with the quadratic character (mod \(p\)), and

\[
h(s) = \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \left( x - \lfloor x \rfloor - \frac{1}{2} \right) \frac{d}{dx} \left[ \left( x + \frac{j}{2} \right)^2 + \frac{p^2}{4} \right]^{-s} \, dx.
\]

Let \(x + (j/2) = u(j/\sqrt{p})/2\) give a change of variable from \(x\) to \(u\) and integrate by parts \(2m - 1\) times; we get, for \(m = 1, 2, \ldots\),

\[
(2) \quad h(s) = - \sum_{j=1}^{\infty} \left( \frac{2}{j/\sqrt{p}} \right)^{2m+2s-1} \int_{-\infty}^{\infty} \frac{B_{2m}(x - \lfloor x \rfloor)}{(2m)!} \frac{d^{(2m)}}{du^{(2m)}} \{(u^2 + 1)^{-s}\} \, du,
\]

where \(B_k(x)\) is the \(k\)th Bernoulli polynomial (\(B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{2}, \ldots\)) and the series converges for \(\text{Re}s > 1 - m\).

It is well known ([3, p. 245], Jordan's \(\varphi_n(x) = B_n(x)/n!\)) that for \(0 \leq x \leq 1, k > 1,\)

\footnote{Received by the editors May 6, 1965.}

\footnote{A portion of my Ph. D. dissertation was written under the supervision of Professor D. H. Lehmer.}
so that to estimate \( h(s) \), we need only estimate \( d^{(2m)}/du^{(2m)}((u^2 + 1)^{-s}) \). Using the fact that

\[
\frac{d^2}{du^2} ((u^2 + 1)^{-s}) = 2s(2s + 1)(u^2 + 1)^{-s-1} - 2s(2s + 2)(u^2 + 1)^{-s-2},
\]

we see inductively that we can write

\[
\frac{d^{(2m)}}{du^{(2m)}} ((u^2 + 1)^{-s}) = \sum_{k=0}^{m} c_{mk}(s)(u^2 + 1)^{-s-m-k}.
\]

Again, by induction we have

\[
|c_{mk}(s)| \leq \binom{m}{k} \prod_{j=0}^{2m-1} (2|s| + 2j).
\]

Thus

\[
\left| \frac{d^{(2m)}}{du^{2m}} ((u^2 + 1)^{-s}) \right| \leq \sum_{k=0}^{m} \binom{m}{k} (u^2 + 1)^{-s-m-k} \prod_{j=0}^{2m-1} (2|s| + 2j)
\]

\[
\leq 2^m(u^2 + 1)^{-s-m} \prod_{j=0}^{m-1} (2|s| + 2j)(2|s| + 2(2m - 1 - j))
\]

\[
\leq 2^m(u^2 + 1)^{-\frac{s}{2}}(2|s| + 2m - 1)^{2m}.
\]

In view of (2), (3) and (6), we have for \( \sigma \geq \frac{1}{2} \) and \( m \geq 1 \):

\[
|h(s)|
\leq \sum_{y=1}^{\infty} \left( \frac{2}{y \sqrt{p}} \right)^{2m+2s-1} \cdot \frac{2}{(2\pi)^{2m}} \zeta(2m) \cdot 2^m(2|s| + 2m - 1)^{2m} \int_{-\infty}^{\infty} (u^2 + 1)^{-1}du
\]

\[
\leq 2\pi \zeta(2m) \left( \frac{4|s| + 4m - 2}{\pi \sqrt{2p}} \right)^{2m}
\]

\[
< 2\pi \left( \frac{2m}{2m - 1} \right)^2 \left( \frac{4|s| + 4m - 2}{\pi \sqrt{2p}} \right)^{2m}.
\]

Letting \( m = 30 \), we see that if \( |s| \leq 22 \) and \( \sigma \geq \frac{1}{2} \) (and of course \( p > 10,000 \)), then

\[
|h(s)| < 10^{-19}.
\]

Let \( \theta \) denote a number, complex or real, not necessarily the same each time it occurs, which satisfies \( |\theta| \leq 1 \). We find that for \( |s| < 22 \) and \( \sigma \geq \frac{1}{2} \), (1) becomes
\[ \zeta(s)L(s) - \zeta(2s) = \zeta(2 - 2s) \frac{\Gamma(1 - s)}{\Gamma(s)} \left( \frac{\sqrt{p}}{2\pi} \right)^{1-2s} + 10^{-19}\theta, \]

where the functional equation for \( \zeta(s) \) was used to obtain the first term on the right.

Let
\[ s_n = \frac{1}{2} + iy_n \]
denote the \( n \)th zero of \( \zeta(s) \) above the real axis. It is known that \( \gamma_1 \approx 14 \) and \( \gamma_2 \approx 21 \) (see Appendix); in particular, \( |s_1| < |s_2| < 22 \) and thus
\[ \zeta(2s_n) = -\zeta(2 - 2s_n) \frac{\Gamma(1 - s_n)}{\Gamma(s_n)} \left( \frac{p}{4\pi^2} \right)^{-iy_n} + 10^{-19}\theta, \quad (n = 1, 2). \]

Multiplying both sides by \( (p/4\pi^2)^{iy_n} (1/\zeta(2s_n)) \) and using the fact that \( |\zeta(2s_n)| > \frac{1}{2} \) for \( n = 1, 2 \) (see Appendix), we get
\[ \frac{p}{4\pi^2} = -\frac{\zeta(2 - 2s_n)}{\zeta(2s_n)} \frac{\Gamma(1 - s_n)}{\Gamma(s_n)} + 2 \cdot 10^{-19}\theta \]
\[ = -\frac{\zeta(1 - 2iy_n)\Gamma\left(\frac{1}{2} - iy_n\right)}{\zeta(1 + 2iy_n)\Gamma\left(\frac{1}{2} + iy_n\right)} \left(1 + 2 \cdot 10^{-19}\theta\right), \quad (n = 1, 2). \]

Taking arguments of both sides of (11) gives
\[ \gamma_n \log \left( \frac{p}{4\pi^2} \right) = a_n + 2\pi x_n + 3 \cdot 10^{-19}\theta, \quad (n = 1, 2) \]
where \( x_n \) is an integer and
\[ a_n \equiv \pi - 2 \arg \zeta(2s_n) - 2 \arg \Gamma(s_n) \pmod{2\pi}, \quad 0 \leq a_n < 2\pi. \]

Eliminating \( \log (p/4\pi^2) \) from the equations (12), and solving for \( x_2 \), we obtain
\[ x_2 = \frac{\gamma_2}{\gamma_1} x_1 + a_0 + 10^{-18}\theta, \]
where
\[ a_0 = \frac{1}{2\pi} \left( \frac{\gamma_2}{\gamma_1} a_1 - a_2 \right). \]

From the Appendix,
\[ \frac{\gamma_2}{\gamma_1} = 1.487262003892890048 + 10^{-18}\theta, \]
\[ a_0 = a + 4 \cdot 10^{-9}\theta \quad \text{where} \quad a = -0.461786352. \]
We can rewrite (14) as
\[ x_2 = \frac{\gamma_2}{\gamma_1} x_1 + a + \frac{1}{2} \cdot 10^{-9} \theta. \]  

Note that
\[ 3.999 999 660 = \frac{\gamma_2}{\gamma_1} \cdot 3 + a + \frac{1}{2} \cdot 10^{-9} \theta. \]

It is not accidental that \( 3(\gamma_2/\gamma_1) + a \) should be close to an integer; \( x_1 = 3 \) corresponds to \( p = 163 \) (see introduction). In fact
\[
\gamma_1 \log \left( \frac{163}{4\pi^2} \right) = 20.042 \, 984 \, 673 \, 072\ldots,
\]
\[
a_1 + 2\pi \cdot 3 = 20.042 \, 984 \, 673 \, 470\ldots.
\]
(Compare this with (12), where these numbers would agree to at least 19 decimal places if \( p > 10,000 \).) From (12), we now see that \( p > 10^4 \) implies that \( x_1 > 3 \).

Subtracting (18) from (17) gives:
\[ x_2^2 - 4 = \left( \frac{\gamma_2}{\gamma_1} \right)^2 \left( x_1 - 3 \right) - b + 10^{-9} \theta, \quad \text{where } b = .000 \, 000 \, 340. \]

Now let
\[ p_1 = 83,532,765, \quad p_2 = 12,832,922, \]
\[ q_1 = 56,165,467, \quad q_2 = 8,628,555. \]
Then \( p_1 q_2 - q_1 p_2 = 1 \), so that \( p_1 \) and \( q_1 \) are relatively prime. Also,
\[ \left| \left( \frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1} \right) \right| < 2.3 \cdot 10^{-16}. \]
Let
\[ Q + \frac{R}{q_1} = \frac{p_1}{q_1} (x_1 - 3), \]
where \( 0 \leq R < q_1 \) and \( Q \) and \( R \) are integers. Subtracting (22) from (19) gives
\[ x_2 - Q - 4 = \left( \frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1} \right) (x_1 - 3) + \left( \frac{R}{q_1} - b \right) + 10^{-9} \theta. \]

If \( x_1 \leq 5.1 \cdot 10^7 \), then
\[ \left| \left( \frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1} \right) (x_1 - 3) \right| < 12 \cdot 10^{-9}, \]
and thus \( x_1 \leq 5.1 \cdot 10^7 \) implies
(25) \[ |x_2 - Q - 4| < 12 \cdot 10^{-9} + (1 - 340 \cdot 10^{-9}) + 10^{-9} < 1. \]

On the other hand, since
\[
\frac{18}{q_1} < 321 \cdot 10^{-9} < b = 340 \cdot 10^{-9} < 356 \cdot 10^{-9} < \frac{20}{q_1},
\]
we find that \( x_1 \leq 5.1 \cdot 10^7 \) and \( R \neq 19 \) implies
\[
| x_2 - Q - 4 | < \frac{R}{q_1} - b = \left| \frac{y_2}{\gamma_1} - \frac{p_1}{q_1} \right| (x_1 - 3) < 10^{-9}
\]
\[
> 16 \cdot 10^{-9} - 12 \cdot 10^{-9} - 10^{-9} > 0.
\]

Inequalities (25) and (26) are contradictory, and therefore \( x_1 \leq 5.1 \cdot 10^7 \) implies \( R = 19 \).

But if \( R = 19 \), then we see from (22) that
\[
19 \equiv p_1(x_1 - 3) \pmod{q_1},
\]
and this implies
\[
x_1 - 3 \equiv 51,611,611 \pmod{q_1}.
\]
Thus under all circumstances, \( x_1 > 5.1 \cdot 10^7 \). Hence by (12),
\[
\log \left( \frac{p}{4 \pi^2} \right) = \frac{a_1 + 2 \pi x_1 + 3 \cdot 10^{-19} \theta}{\gamma_1} > \frac{2 \pi (5.1 \cdot 10^7) - 3 \cdot 10^{-19}}{14.2} > 2.2 \cdot 10^7,
\]
and Theorem 1 follows.

Appendix

I wish to express my thanks to M.D. Bigg who furnished values of \( \gamma_1 \) and \( \gamma_2 \) to fifty decimal places and to R.S. Lehman who furnished values of \( \arg \zeta(2s) \) and \( |\zeta(2s_n)| \) for \( n = 1 \) and 2, with a proved accuracy of \( \pm 10^{-10} \). The values of \( \gamma_1 \) and \( \gamma_2 \) were confirmed independently by Robert Spira to fifteen decimal places. Their values are:

\[
\begin{align*}
\gamma_1 &= 14.134 \ 725 \ 141 \ 734 \ 693 \ 790 \ 457 + 10^{-21} \theta, \\
\gamma_2 &= 21.022 \ 039 \ 638 \ 771 \ 554 \ 992 \ 628 + 10^{-21} \theta, \\
\frac{1}{\pi} \arg \zeta(2s_1) &= -0.108 \ 452 \ 737 \ 083 \ 095 + 10^{-10} \theta \quad (\text{mod } 2), \\
\frac{1}{\pi} \arg \zeta(2s_2) &= 0.067 \ 103 \ 865 \ 503 \ 910 + 10^{-10} \theta \quad (\text{mod } 2), \\
|\zeta(2s_1)| &= 1.948 \ 757 \ 313 \ 817 \ 40 + 10^{-10} \theta, \\
|\zeta(2s_2)| &= .830 \ 962 \ 021 \ 546 \ 955 + 10^{-10} \theta.
\end{align*}
\]
All other numbers were calculated on a Monroe desk calculator to fifteen places.

The Euler-Maclaurin sum formula for \( \log \Gamma \left( \frac{1}{2} + i\gamma \right) \) can be expressed \([4, \text{p. 132}]\).
It should be noted that the exponent \((m+1)\) in Nörlund’s remainder term should be replaced by \(m\). See also \([5, \text{p. 131}]\):

\[
\log \Gamma \left( \frac{1}{2} + i\gamma \right) = i\gamma \log(i\gamma) - i\gamma + \frac{1}{2} \log(2\pi)
\]

\[
+ \sum_{j=2}^{11} \frac{(-1)^j B_j \left( \frac{1}{2} \right)}{j(j-1)} \cdot \frac{1}{(i\gamma)^{j-1}}
\]

\[
- \int_{0}^{\infty} \frac{B_{11} \left( x - \frac{1}{2} - \left[ x - \frac{1}{2} \right] \right)}{11(x + i\gamma)^{11}} \, dx.
\]

From this we see that

\[
\arg \Gamma \left( \frac{1}{2} + i\gamma \right) = \gamma(\log \gamma - 1) + \sum_{k=1}^{5} \frac{(-1)^k B_{2k} \left( \frac{1}{2} \right)}{2k(2k-1)\gamma^{2k-1}}
\]

\[
- \Im \int_{0}^{\infty} \frac{B_{11} \left( x - \frac{1}{2} - \left[ x - \frac{1}{2} \right] \right)}{11(x + i\gamma)^{11}} \, dx.
\]

From the formula \([5, \text{p. 123}]\),

\[
B_k \left( \frac{1}{2} \right) = -(1 - 2^{1-k})B_k,
\]

we get the following:

\[
B_2 \left( \frac{1}{2} \right) = \frac{1}{2 \cdot 1} = -0.0416666666667 + 10^{-15} \theta,
\]

\[
B_4 \left( \frac{1}{2} \right) = \frac{1}{4 \cdot 3} = 0.002430555555556 + 10^{-15} \theta,
\]

\[
B_6 \left( \frac{1}{2} \right) = \frac{1}{6 \cdot 5} = -0.000768849206349 + 10^{-15} \theta,
\]

\[
B_8 \left( \frac{1}{2} \right) = \frac{1}{8 \cdot 7} = 0.000590587797619 + 10^{-15} \theta,
\]

\[
B_{10} \left( \frac{1}{2} \right) = \frac{1}{10 \cdot 9} = -0.000840106797138 + 10^{-15} \theta.
\]
From [6], we get
\[ \log \gamma_1 = 2.648\,634\,545\,730\,790 + 10^{-13}\theta, \]
\[ \log \gamma_2 = 3.045\,571\,393\,984\,561 + 10^{-13}\theta. \]

Finally, for \( t > 14 \), we get from (3),
\[
\left| \int_0^\infty \frac{B_{11} \left( x - \frac{1}{2} - \left\lfloor x - \frac{1}{2} \right\rfloor \right)}{11(x + it)^{11}} \, dx \right| \leq \frac{(2\pi)^{11} \cdot 11}{10} \int_0^\infty \frac{dx}{x^2 + t^2} \\
= \frac{11!\pi}{10(2\pi)^{11}t^{10}} < 10^{-13}. 
\]

Putting these numbers into (A2), we obtain
\[
\frac{1}{\pi} \arg \Gamma(s_1) = 7.418\,512\,651\,985\,173 + 2 \cdot 10^{-13}\theta, \\
\frac{1}{\pi} \arg \Gamma(s_2) = 13.688\,619\,111\,000\,235 + 2 \cdot 10^{-13}\theta. 
\]

We may rewrite (12) thusly:
\[
\frac{a_n}{2\pi} \equiv \frac{1}{2} - \frac{1}{\pi} \arg \zeta(2s_n) - \frac{1}{\pi} \arg \Gamma(s_n) \pmod{1}, \\
0 \leq \frac{a_n}{2\pi} < 1. 
\]

Using (A1) and (A3), we find that
\[
\frac{a_1}{2\pi} \equiv .189\,940\,085\,097\,922 + 1.002 \cdot 10^{-10}\theta \pmod{1}, \\
\frac{a_2}{2\pi} \equiv .744\,277\,023\,495\,855 + 1.002 \cdot 10^{-10}\theta \pmod{1}. 
\]

Since these numbers are between 0 and 1, the above is actually an equality. Finally,
\[
\frac{\gamma_2}{\gamma_1} = 1.487\,262\,003\,892\,890\,048 + 10^{-18}\theta 
\]
and
\[
\frac{a_0}{\gamma_1} = \frac{\gamma_2}{\gamma_1} \left( \frac{a_1}{2\pi} \right) - \frac{a_2}{2\pi} \\
= -.461\,786\,351\,913\,533 + 3 \cdot 10^{-10}\theta. 
\]
REFERENCES


UNIVERSITY OF MICHIGAN

ANN ARBOR, MICHIGAN