A NECESSARY CONDITION THAT A CELLULAR UPPER SEMICONTINUOUS DECOMPOSITION OF E^n YIELD E^n

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This result was originally announced by the author in Notices American Mathematical Society **10** (1963), 661 for decompositions of E^3. The proof has since been simplified and found to hold in E^n for $n \geqq 5$ as well. As is consistent with the present seeming lack of knowledge of E^4, Theorem 2.2 is not known to be true in E^4. Its validity in E^1 and E^2 follows from already well known results.

Let upper semicontinuous be defined as in [2]. A decomposition of E^n, G, is called monotone if each element of G is connected. It is called cellular if each element is the intersection of a decreasing sequence of n-cells. The letter G will be used to denote both the collection of subsets of E^n and the resulting decomposition space. The letter H will be used to denote the collection of nondegenerate elements of G, and H^* will denote the union of the elements of H.

A space X is called 1-connected at infinity if for each compact subset C of X there is a compact subset B of X such that $C \subseteq B$ and $X - B$ is connected and simply connected. In [5] Edwards defined a triangulated 3-manifold, M, to be 1-connected at infinity if for each compact subset A of M there exists a compact polyhedral subset P of M such that $A \subseteq P$ and $M - P$ is 1-connected. It is easy to show, using the techniques described by D. R. McMillan, Jr. in [6] that if B is a compact subset of a triangulated 3-manifold, M, and if $M - B$ is 1-connected, then there exists a compact polyhedral subset P of M such that $B \subseteq P$ and $M - P$ is 1-connected. Hence the two definitions are equivalent in 3-manifolds.

The definitions of piecewise linear concepts used are as given in [7].

If B is an n-cell, $\dot{B}$ is used to denote the boundary of B.

1. **A sufficient condition that a decomposition of E^n yield E^n as the decomposition space.** Theorem 1.4 states a condition which is sufficient to insure that the decomposition space is again E^n. The condition is also shown to be necessary for countable decompositions of E^3. Unfortunately, the sufficient condition given in

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§1, Theorem 1.4 is considerably stronger than the necessary condition given in §2, Corollary 2.3. It is unknown whether this necessary condition is also sufficient. The sufficient condition is clearly not necessary. Briefly the necessary condition says that each element of the decomposition must have arbitrarily small neighborhoods which are the union of elements of the decompositions and are open n-cells. The sufficient condition requires, in addition, that the closures of these neighborhoods be n-cells whose boundaries miss the nondegenerate elements of the decomposition.

We shall need the following lemmas in the proof of Theorem 1.4.

LEMMA 1.1. *Let G be a monotone upper semicontinuous decomposition of the unit n-ball, $B = \{x / x \in E^n, \|x\| \leqq 1\}$, where n is a positive integer. We also assume that $H^* \cap \dot{B} = \varnothing$. Then given $\varepsilon > 0$ there exists a homeomorphism f of B onto B such that*

(1) *the diameter of $f(h) < \varepsilon$ for every $h \in H$, and*

(2) *$f/\dot{B} =$ identity.*

Proof. Let $B(r) = \{x / x \in E^n, \|x\| \leqq r\}$. There exists an r_1, $0 < r_1 < 1$, such that if $h \in H$ and diameter $h \geqq \varepsilon/2$ then $h \subseteq B(r_1)$. There exists an $r_2 \geqq \frac{1}{2}$, $r_1 < r_2 < 1$ such that if $h \in H$ and $h \cap B(r_1) \neq \varnothing$, then $h \subseteq B(r_2)$. In general, pick $r_i \geqq (i-1)/i$ so that $r_{i-1} < r_i < 1$ and so that if $h \in H$ and $h \cap B(r_{i-1}) \neq \varnothing$, then $h \subseteq B(r_i)$. The homeomorphism f is defined on each interval 0 to x where 0 is the origin and $x \in \dot{B}$. f maps each such interval onto itself. Consider each such interval as being parametrized by distance from the origin.

(1) f maps the interval $[0, r_1]$ linearly onto the interval $[0, a_1]$ where $a_1 =$ minimum $(\varepsilon/4, r_1)$.

(2) f maps $[r_1, r_2]$ linearly onto the interval $[a_1, a_2]$ where $a_2 > a_1$, and $a_2 =$ minimum $(a_1 + \varepsilon/4, r_2)$.

(3) f maps $[r_2, r_3]$ linearly onto $[a_2, a_3]$ where $a_3 > a_2$ and $a_3 =$ minimum $(a_2 + \varepsilon/4, r_3)$.

(4) *etc.*

Obviously, f leaves $\dot{B}$ fixed. This clearly defines a homeomorphism of B onto B which is the identity on $\dot{B}$. Note that $\|f(x)\| \leqq \|x\|$ for all $x \in B$. We need only check that diameter $f(h) < \varepsilon$ for every $h \in H$. If diameter $h \geqq \varepsilon/2$ then $h \subseteq B(r_1)$ and hence $f(h) \subseteq B(a_1) \subseteq B(\varepsilon/4)$. If diameter $h < \varepsilon/2$ then let x and y be in h. There exists i so that $h \subseteq B(r_{i+1}) - B(r_{i-1})$. Hence we see that

$$\big| \|f(x)\| - \|f(y)\| \big| < \varepsilon/2.$$

Thus we get

$$\|f(x) - f(y)\| \leqq \big| \|f(x)\| - \|f(y)\| \big| + \|x - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

LEMMA 1.2. *Let C be an n-cell, n a positive integer, and G be a monotone*

upper semicontinuous decomposition of C. Furthermore, assume that for each $h \in H$ there exists a sequence of n-cells $\{B_i^h\}_{i=1}^{\infty}$ such that for each positive integer:

(1) $B_{i+1}^h \subseteq$ *interior* B_i^h,

(2) $\bigcap_{i=1}^{\infty} B_i^h = h$, *and*

(3) *if* $S_i^h = \dot{B}_i^h$, *then* $S_i^h \cap H^* = \emptyset$.

Finally, let A be a closed subset of C such that $A \cap H^* = \emptyset$. Then given $\varepsilon > 0$ there exists a homeomorphism f of C onto C such that

(1) *f is the identity on A, and*

(2) *for each $h \in H$ the diameter of $f(h) < \varepsilon$.*

Proof. Let $G(\varepsilon)$ denote the subcollection of G consisting of those elements of G with diameter $\geqq \varepsilon$. Then $G(\varepsilon)^*$ is compact. Cover $G(\varepsilon)^*$ with a finite collection $\{D_1, D_2, \cdots, D_k\}$ of n-cells which miss A, and whose boundaries miss H^*. It follows from Lemma 1.1, by using uniform continuity and a homeomorphism between D_1 and the unit ball in E^n, that there is a homeomorphism f_1 of D_1 onto itself which is the identity on its boundary. Furthermore, f_1 shrinks each element of G that lies in D_1 to have diameter $< \varepsilon$. Extend f_1 to be the identity outside D_1. It follows from using Lemma 1.1, again using uniform continuity and a homeomorphism between $f_1(D_2)$ and the unit ball in E^n, that there exists a homeomorphism f_2 of $f_1(D_2)$ onto itself which is the identity on its boundary. Furthermore, f_2 shrinks each set $f_1(g)$, where $g \in G$ and $g \subseteq D_2$ to have diameter $< \varepsilon$. Extend f_2 to be the identity outside of $f_1(D_2)$. Then $f_2 \circ f_1$ is a homeomorphism of C onto itself which shrinks each element of G that lies in $D_1 \cup D_2$ to have diameter $< \varepsilon$ and $f_2 \circ f_1$ is the identity except on $D_1 \cup D_2$. By induction we can obtain a homeomorphism $f = f_k \circ \cdots \circ f_2 \circ f_1$ of C onto itself which shrinks each element of G that lies in $D_1 \cup D_2 \cup \cdots \cup D_k$ to have diameter $< \varepsilon$ and f is the identity except on $D_1 \cup \cdots \cup D_k$. The homeomorphism f_i is obtained by using Lemma 1.1, as it was used above, on $f_{i-1} \circ \cdots \circ f_2 \circ f_1(D_i)$. The homeomorphism f satisfies the conclusion of the lemma.

LEMMA 1.3. *Let n be a positive integer. Let G be an upper semicontinuous decomposition of E^n, such that for each $h \in H$ there exists a sequence of n-cells $\{B_i^h\}_{i=1}^{\infty}$ with the following properties. For each positive integer i,*

(1) $B_{i+1}^h \subseteq$ *interior* B_i^h,

(2) $\bigcap_{i=1}^{\infty} B_i^h = h$, *and*

(3) if $S_i^h = \dot{B}_i^k$ *then* $S_i^h \cap H^* = \emptyset$.

Let $0 = \{C_1, C_2, \cdots\}$ be a locally finite collection (possibly finite) of n-cells such that $\dot{C}_j \cap H^* = \emptyset$ for each j. Finally, let ε be a positive real number. Then there exists a homeomorphism f of E^n onto itself such that

(1) $f \big| E^n - \bigcup_{j=1}^{\infty} C_j =$ *identity,*

(2) $f \big| \dot{C}_j =$ *identity for each j, and*

(3) *diameter $f(h) < \varepsilon$ for each $h \in H$ such that $h \subseteq$ some C_j.*

Proof. We use Lemma 1.2 to get a homeomorphism $f_1\colon C_1 \to C_1$ such that
(i) $f_1 \big| \bigcup_{j=1}^{\infty} \dot{C}_j =$ identity, and
(ii) diameter $f_1(h) < \varepsilon$ for each $h \in H$ such that $h \subseteq C_1$.
Extend f_1 to take E^n onto E^n by letting it be the identity outside C_1. We shall continue to call it f_1. Now use Lemma 1.2 again to get a homeomorphism f_2 which takes $f_1(C_2) = C_2$ onto itself and such that
(i) $f_2 \big| \bigcup_{j=1}^{\infty} \dot{C}_j =$ identity, and
(ii) diameter $f_2 \circ f_1(h) < \varepsilon$ for each $h \in H$ such that $h \subseteq C_2$.
Extend f_2 to all of E^n by letting it be the identity outside $f_1(C_2) = C_2$. We still call it f_2 though. We continue this procedure, using Lemma 1.2 to get a homeomorphism f_k which takes $f_{k-1} \circ \cdots \circ f_2 \circ f_1(C_k) = C_k$ onto itself and such that
(i) $f_k \big| \bigcup_{j=1}^{\infty} \dot{C}_j =$ identity, and
(ii) diameter $f_k \circ f_{k-1} \circ \cdots \circ f_2 \circ f_1(h) < \varepsilon$ for each $h \in H$ such that $h \subseteq C_k$.
Define $f = \operatorname{limit}_{k\to\infty} f_k \circ f_{k-1} \circ \cdots \circ f_2 \circ f_1$. This limit exists and is a homeomorphism because each f_k was a homeomorphism and the collection $\{C_1, C_2, \cdots\}$ is a locally finite collection. It has the desired properties because the f_k's have the desired properties.

THEOREM 1.4. *Let G be an upper semicontinuous decomposition of E^n where n is a positive integer. Suppose, furthermore, that for each $h \in H$ there exists a sequence $\{B_i^h\}_{i=1}^{\infty}$ of n-cells such that for each positive integer i*
(1) $B_{i+1}^h \subseteq$ *interior* B_i^h,
(2) $\bigcap_{i=1}^{\infty} B_i^h = h$, *and*
(3) if $S_i^h = \dot{B}_i^h$, *then* $S_i^h \cap H^* = \varnothing$.
Then the decomposition space is homeomorphic to E^n. Furthermore, if $n = 3$ and if H is a countable collection, then the existence of such a sequence of 3-cells for each $h \in H$ is necessary for the decomposition space to be homeomorphic to E^3.

Proof of necessity in E^3. Since H is countable, we assume that H, regarding it as a subset of the decomposition space, which is E^3, is contained in the rational points of E^3, i.e., those points all of whose coordinates are rational. Now for each point h of H we choose a sequence $\{C_i^h\}_{i=1}^{\infty}$ of cubes with center h and irrational "radii" which tend to zero. Clearly the boundaries of these cubes will contain no rational points at all, much less any points of H. Now we let ϕ denote the decomposition map of E^3 onto $G = E^3$ and we define $B_i^h = \phi^{-1}(C_i^h)$. It is easily verified that we are done. Obviously the conditions (1), (2), and (3) will be satisfied if the B^h are indeed cells. And this is easily seen as follows. Since the boundary, S_i^h, of B_i^h does not intersect any element of H, ϕ is a homeomorphism, when restricted to S_i^h, and hence S_i^h is a 2-sphere. Finally, we observe why it is a tame 2-sphere. The boundary of C_i^h can be homeomorphically approximated from each side by 2-spheres which contain no rational points, and hence no points of H. S_i^h is homeomorphically approximated by ϕ^{-1} of these 2-spheres, and thus by [3] it is tame.

Proof of sufficiency. Before giving the proof of this part of the theorem we give an outline of what will be done. The details appear in the following paragraphs. We shall construct a Cauchy sequence of homeomorphisms $f_1, f_2, \cdots$ so that $f = \text{limit}_{i\to\infty} f_i$ is a compact map of E^n onto itself, such that for each $p \in E^n$, $f^{-1}(p)$ is an element of G. Hence G will be homeomorphic to E^n. (f is compact if and only if for every compact set A, $f^{-1}(A)$ is also compact.) Let $\varepsilon_1, \varepsilon_2, \cdots$ be a sequence of positive numbers whose sum is finite, and $\varepsilon_i \leqq 1/i$. Let $G(\varepsilon_i)$ be the subcollection of G consisting of all elements of diameter $\geqq \varepsilon_i$. In order to get f_i we shall use a countable, locally finite collection 0_i of n-cells which cover $G(\varepsilon_i)$ and whose boundaries miss H^*. Then Lemma 1.3 will be used to obtain the homeomorphisms, f_i.

We shall now describe some sets $B'(k)$ which will be helpful in choosing the 0_i's. Let $B(k)$ be the ball of radius k with center at the origin. We shall enlarge $B(k)$ to get a set whose boundary misses H^*. Since $G(\varepsilon_i)^* \cap \dot{B}(k)$ is a compact set, it can be covered with a finite collection of n-cells each of whose boundaries miss H^*, and each of which is within ε_i of some element of $G(\varepsilon_i)$ that intersects $\dot{B}(k)$. The set $B'(k)$ is simply the union of $B(k)$ and the n-cells just described, finitely many for each integer i. It is easily shown that each $B'(k)$ is a compact set containing $B(k)$ whose boundary misses H^*. For this reason

$$G(\varepsilon_i)^* \cap (B'(k) - B'(k-1)) = G(\varepsilon_i)^* \cap \text{ interior } (B'(k) - B'(k-1))$$

is a compact set for each pair of integers i and k.

Now we describe the collection 0_1. Since $G(\varepsilon_1)^* \cap \text{interior } (B'(k) - B'(k-1))$ is compact, for each k, we cover it with a finite collection of n-cells, each of whose boundaries miss H^* and each of which is contained in interior $(B'(k) - B'(k-1))$. We also assume that each of these n-cells is contained in a neighborhood of radius 1 of some element of $G(\varepsilon_1)$. The collection 0_1 is obtained by taking the union of these finite collections, one collection for each integer k. It is possible that 0_1 is a finite collection.

Now we use Lemma 1.3 to obtain a homeomorphism g_1 of E^n onto E^n such that

(1) $g_1 \mid E^n - \bigcup_{C\in 0_1} C = \text{identity}$,

(2) $g_1 \mid \dot{C} = \text{identity}$ for each $C \in 0_1$, and

(3) diameter $g_1(h) < 1$ for each $h \in H$ such that $h \subseteq C$ for some $C \in 0_1$. In particular, for each $h \in G(\varepsilon_1)$ and since $\varepsilon_1 < 1$, diameter $g_1(h) < 1$ for each $h \in H$.

The elements of 0_2 must be chosen with more care than those of 0_1. For each integer k we cover $G(\varepsilon_2)^* \cap \text{interior } (B'(k) - B'(k-1))$ with a finite collection of n-cells, each of whose boundaries miss H^* and each of which is contained in interior $(B'(k) - B'(k-1))$. In addition we assume that if $D \in 0_2$ then

(1) D is contained in a $1/2$ neighborhood of some element of $G(\varepsilon_2)$,

(2) the diameter of $g_1(D) < 2$, and

(3) if $C \in 0_1$ and $D \cap C \neq \varnothing$, then $D \subseteq C$.

Each $D \in 0_2$ can be chosen so that the diameter of $g_1(D)$ is < 2 because for each $h \in G(\varepsilon_2)$ the diameter of $g_1(h) < 1$.

Now we use Lemma 1.3 again to obtain a homeomorphism, g_2, of E^n onto E^n such that

(1) $g_2 \mid E^n - \bigcup_{D \in 0_1} g_1(D) = \text{identity}$,

(2) $g_2 \mid g_1(\dot{D}) = \text{identity}$ for each $D \in 0_2$, and

(3) diameter $g_2 \circ g_1(h) < 1/2$ for each $h \in H$ such that $h \subseteq D$ for some $D \in 0_2$.

In particular for each $h \in G(\varepsilon_2)$ and since $\varepsilon_2 < 1/2$, diameter $g_2 \circ g_1(h) < 1/2$ for each $h \in H$.

The description of the collection 0_i is very similar to that of 0_2. For each integer k we cover $G(\varepsilon_i)^* \cap$ interior $(B'(k) - B'(k-1))$ with a finite collection of n-cells each of whose boundaries miss H^* and each of which is contained in interior $(B'(k) - B'(k-1))$. In addition we assume that if $E \in 0_i$ then

(1) E is contained in a $1/i$ neighborhood of some element of $G(\varepsilon_i)$,

(2) the diameter of $g_{i-1} \circ \cdots \circ g_2 \circ g_1(E) < 1/(i-2)$, and

(3) if $j < i$, $C \in 0_j$ and $E \cap C \neq \varnothing$ then $E \subseteq C$.

Each $E \in 0_i$ can be chosen so that the diameter of $g_{i-1} \circ \cdots \circ g_2 \circ g_1(E) < 1/(i-2)$ because for each $h \in G(\varepsilon_i)$ the diameter of $g_{i-1} \circ \cdots \circ g_2 \circ g_1(h) < 1/(i-1)$.

Next use Lemma 1.3 to get a homeomorphism, g_i, of E^n onto E^n such that

(1) $g_1 \mid E^n - \bigcup_{E \in 0_i} g_{i-1} \circ \cdots \circ g_2 \circ g_1(E) = \text{identity}$,

(2) $g_i \mid g_{i-1} \circ \cdots \circ g_2 \circ g_1(\dot{E}) = \text{identity}$ for each $E \in 0_i$, and

(3) diameter $g_i \circ g_{i-1} \circ \cdots \circ g_2 \circ g_1(h) < 1/i$ for each $h \in H$ such that $h \subseteq E$ for some $E \in 0_i$. In particular for each $h \in G(\varepsilon_i)$ and since $\varepsilon_i < 1/i$, the diameter $g_i \circ g_{i-1} \circ \cdots \circ g_2 \circ g_1(h) < 1/i$ for each $h \in H$.

Now define $f_k = g_k \circ g_{k-1} \circ \cdots \circ g_2 \circ g_1$ for each positive integer k. Then let $f = \text{limit}_{k \to \infty} f_k$. Clearly each f_k is a homeomorphism of E^n onto itself. The f_k's are a Cauchy sequence because if $p \in C \in 0_j$, then $f_i(p) \in f_{j-1}(C)$ for each $i \geqq j$ and the diameter of $f_{j-1}(C) < 1/(j-2)$. Hence the function f exists and is continuous.

Clearly f shrinks each element of G to a point. Also if $p \neq q$ are two points of E^3 which are not in the same element of G, then there exists an integer i such that no one element of 0_j contains both p and q for $j \geqq i$. If neither p nor q is in any element of any 0_j then $f(p) = p \neq q = f(q)$. Otherwise, suppose $p \in C \in 0_j$ and $q \notin C$. Then $f_j(p) \in f_j(C)$ and $f_j(q) \notin f_j(C)$. Finally it follows from this construction that $f(p) \in f_j(C)$ and that $f(q) \notin f_j(C)$. Hence $f(p) \neq f(q)$.

Finally we show that f is a compact and onto map. The compactness follows trivially from Theorem 1 of [4]. An elementary proof is easily obtained so we include it here. Let $g = \text{limit}_{n \to \infty}(g_n \circ \cdots \circ g_2)$. Then g moves no point more than 2 and hence g is both compact and onto. But $f = g \circ f_1$, hence f is also both compact and onto.

It is easy to see that in the proof of Lemmas 1.1, 1.2 and 1.3 we could have obtained an isotopy whose final stage was the homeomorphism f. Hence in the proof of Theorem 1.4 we could have obtained a pseudo isotopy which would shrink the elements of G to points.

The following is an easy corollary to Theorem 1. It also has been proven by S. Armentrout in [1].

COROLLARY 1.5. *Let G be a monotone upper semicontinuous decomposition of E^3, such that H is a countable collection, and G is a 3-manifold. Then G is E^3.*

Proof. In order to show that G satisfies the hypothesis of Theorem 1.4 pick a 3-cell $C_h \subseteq G$, with $h \in$ interior C_h, for each $h \in H$. C_h is the union of an uncountable collection of concentric 3-cells which have disjoint boundaries, and since H is countable, most of the cells have boundaries which miss H. The inverse images of these cells whose boundaries miss H are easily shown to be cells and hence Theorem 1.4 applies and $G = E^3$.

2. A necessary condition that the decomposition space be E^n.

THEOREM 2.1. *Let G be a cellular upper semicontinuous decomposition of E^n. Let ϕ be the decomposition map. Let U be an open subset of E^n such that $U = \phi^{-1}(\phi(U))$. Then $\phi(U)$ simply connected implies that U is simply connected.*

Proof. Let D be the standard 2-simplex. Let f be a map of Bd D into U. Let h be a map of D into $\phi(U)$ such that $h/\text{Bd } D = \phi \circ f$. Such an h exists since $\phi(U)$ is simply connected. Let $F = h(D)$. For each $g \in G$ with $g \subseteq \phi^{-1}(F)$ there exists an n-cell C_g such that $g \subseteq$ interior $C_g \subseteq C_g \subseteq U$. Let 0_g be the union of all elements of G contained in interior C_g. Then 0_g is open, contains g and $\phi(0_g)$ is also open. The sets $\phi(0_g)$ are an open cover of F. Let T be a triangulation of D that is so fine that for each 2-simplex, σ, there exists a $g \in G$ such that $h(\sigma) \subseteq \phi(0_g)$. For each 2-simplex, σ, pick one such 0_g which will be referred to as THE set associated with σ.

Let $v_1, \cdots, v_k$ be the vertices of T. We wish to define f on $v_1, \cdots, v_k$. If v_i is contained in Bd D then f is defined on v_i. If v_i is not in Bd D, then choose a point $p_i \in U$ such that $\phi(p_i) = h(v_i)$, and define $f(v_i) = p_i$. Now we shall define f on the 1-simplexes of T. Let τ be a 1-simplex. If τ is in the boundary of D then f is defined on τ. If τ is not in the boundary of D let σ and σ' be the 2-simplexes of T having τ as a common 1-simplex, and let v and v' be the vertices of τ. Let 0_g and $0_{g'}$ be THE sets associated with σ and σ'. Since the pre-image of the component of $\phi(0_g) \cap \phi(0_{g'})$ that contains $h(\tau)$ is arcwise connected, there exists an arc A from $f(v)$ to $f(v')$ such that $A \subseteq 0_g \cap 0_{g'}$. Extend f to take τ onto A. Finally we must define f on the 2-simplexes of T. Let σ be a 2-simplex. f is already defined on Bd σ and furthermore, if 0_g is THE set associated with σ, then $f(\text{Bd}\,\sigma) \subseteq 0_g \subseteq C_g$. But C_g is simply connected, so f can be extended to take σ into C_g which is contained in U. Thus f is now extended to all of D and hence U is simply connected.

THEOREM 2.2. *Let G be a cellular upper semicontinuous decomposition of E^n, n a positive integer with $n \neq 4$. Let ϕ denote the decomposition map of*

E^n onto G. If C' is an open subset of G such that C' is an open n-cell, then $\phi^{-1}(C')$ is also an open n-cell contained in E^n.

Proof. Let $C = \phi^{-1}(C')$. For $n = 1$ the result follows from the fact that C is connected and open. For $n = 2$ the result follows from the facts that C is open, connected, and does not separate the plane.

For $n = 3$ the result will follow from [5] when it is shown that C is a contractible 3-manifold which is 1-connected at infinity. Similarly for $n \geqq 5$ the result will follow from [7] when it is shown that C is a contractible piecewise linear manifold which is 1-connected at infinity. In either case the manifold, respectively, piecewise linear manifold, structure exists because C is an open subset of E^n.

To show that C is 1-connected at infinity, let A be a compact subset of C. Then $\phi(A)$ is a compact subset of C'. Hence there is an n-cell B such that $\phi(A) \subseteq B$ and $C' - B$ is simply connected. Hence by Theorem 2.1 $C - \phi^{-1}(B)$ is simply connected and $A \subseteq \phi^{-1}(B)$.

Finally it remains to be shown that C is contractible. By a theorem of J. H. C. Whitehead [8] it is sufficient to show that C has trivial homotopy groups. Hence it will be of the same homotopy type as E^n and will be contractible. That C is connected follows from well-known decomposition theorems. Theorem 2.1 implies that C is simply connected. It is possible, with somewhat more difficulty, to use the methods employed in the proof of Theorem 2.1 to show, constructively, that all the homotopy groups of C are trivial. Instead notice that ϕ is an acyclic map, i.e., $\phi^{-1}(g)$ has trivial Čech homology for each $g \in G$. Hence by a theorem of R. L. Wilder [9], it follows that ϕ induces an isomorphism from the Čech homology groups of C onto those of C'. Thus C has trivial Čech homology and hence trivial singular homology, and finally, by the Hurewicz Theorem, it has trivial homotopy groups.

COROLLARY 2.3. *If G is a cellular upper semicontinuous decomposition of E^n, n a positive integer not equal to 4, such that G is an n-manifold, then each $g \in G$, as a subset of E^n, has arbitrarily small neighborhoods which are the union of elements of G and are open n-cells, i.e., if $g \in G$ and U is an open subset of E^n with $g \subseteq U$ then there exists an open subset V with $g \subseteq V \subseteq U$ such that V is the union of elements of G and V is an open n-cell.*

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